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ON FIELDS FOR WHICH THE NUMBER OF ORDERINGS IS DIVISIBLE BY A HIGH POWER OF 2, II

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Abstract: We consider the pythagorean fields with n independent orderings; where the total number of orderings is $2^{n-2} (n \geq 4)$ or $3.2^{n-3} (n \geq 3)$. The Galois group of the maximal 2-extension over any such field F ; the stability index; Witt ring and the chain length of F itself is determined. Finally it is observed that the square class invariant classifies the quadratic forms over such fields. No proofs are presented.

INTRODUCTION. In this paper we keep to the notation used in [4], [6], [7]. We define F (or $K,L,\ldots$) to be a formally real field, T_F to be the set of elements in F that are the sum of the non-zero squares of elements of F, $\dot{F}$ is the multiplicative group of the field F; $[\dot{F}:T_F]$ is the group-index. V - a valuation on F. A_V - the valuation ring corresponding to V. U_V - the group of units of A_V. M_V - the maximal ideal of A_V, F_V - the residue field of V. V is fully compatible with T_F iff $1 + M_V \subset T_F$.

The field F is of type $(k,2^n)$ if $[\dot{F}:T_F] = 2^n$ and the number of orderings of the field F is k.

If n is any integer >1, then we define (see [6], Def. 2.4.1, Prop. 2.A.4)

$$u(n) = t + \sum_{i=0}^{t} \varepsilon_i \quad \text{where}$$

$$n = \sum_{i=0}^{t} \varepsilon_i 2^i \quad \text{with} \quad \varepsilon_i \in \{0,1\}, \ \varepsilon_t = 1 .$$

We let $u(1) = 0$ (in [6], $u(1) = 1$).

Definition. Let F be any field with $[\dot{F}:T_F] < \infty$. Define $t(F)$ to be the minimum value of $\log_2[\dot{F}_V:T_{F_V}]$ where V runs through all the valuations on the field F, which are fully compatible with T_F.

We note that the type of the field F does not necessarily determine $t(F)$. In fact from the Theorems 1, 2 [7] one can easily deduce the following.

Proposition 1. Suppose that F is a field of type $(2^{n-j}m, 2^n)$, where m is an odd number ≥ 3, $n > 2j-u(m)$. Then

A) If $j = u(m)$ then $t(F) = u(m)$. So we can determine $t(F)$.

B) If $m = 3$, $j = 4$, then $t(F) = 5$.

C) If neither $j = u(m)$ nor $j = 4$, $m = 3$ hold, then there exists a field K of the same type as F such that $t(F) \neq t(K)$. Hence $t(F)$ is not determined by the type of F.

Suppose now that F is a field of type $(2^{n-j}, 2^n)$ with $n > 2j$ then

D) If $j = 2$, then $t(F) = 4$

If $j = 3$, then $t(F) = 6$

E) If $j \neq 2, 3$ then there exists a field K of the same type as F such that $t(F) \neq t(K)$.

We may also state the following

Proposition 2. Suppose that F is of one of the following types:

$$(2^{n-1}, 2^n), n \geq 1, (2^{n-2}, 2^n), n \geq 4 \qquad (*)$$
$$(3 2^{n-3}, 2^n), n \geq 3$$

Let V be any valuation of F, fully compatible with T_F, such that $[\dot{F}_V : T_{F_V}]$ is minimal. Then F_V is a SAP field (i.e. F_V is of type $(t(F), 2^{t(F)})$). We note also that if F is of type $(2^{n-j}m, 2^n), n > 2j-u(m)$ and different from any types described in (*); then there exists a field K of the same type as the field F and a valuation V on K such that V is fully compatible with $T_K, [\dot{K}_V : T_{K_V}] = 2^{t(K)}$ and K_V is not a SAP field.

Using Proposition 2 and known facts about SAP and super-pythagorean fields, we can derive properties of pythagorean fields of the type $(2^{n-2}, 2^n), n \geq 4$ and $(32^{n-3}, 2^n), n \geq 3$.

§2. From here we understand field to mean formally real pythagorean field.

Theorem 1. Let F be a field of type $(2^{n-2}, 2^n), n \geq 4$ and $F(2)$ the maximal 2-extension of F. Then the Galois group $G(F(2)|F)$ is isomorphic to the semidirect product $\mathbb{Z}_2^{n-4} \rtimes H$ where H is the free product of the groups $\{1, \sigma_i\}$, $\sigma_i^2 = 1$, $i = 1,2,3,4$, in the category of pro-2-groups. $\mathbb{Z}_2^{n-4}$ denotes the direct product of $n-4$ additive groups of 2-adic integers. (If $n = 4$ we put $\mathbb{Z}_2^{n-4} = \{0\}$.)

The group $\mathbb{Z}_2^{n-4}$ is a normal subgroup of $G(F(2)|F)$ and the action of the group H on $\mathbb{Z}_2^{n-4}$ is described by $\sigma_i \alpha \sigma_i^{-1} = -\alpha$, $\alpha \in \mathbb{Z}_2^{n-4} - \{0\}$, $i = 1,2,3,4$. If F is a field of type $(3.2^{n-3}, 2^n)$ then $G(F(2)|F)$ is a semidirect product $\mathbb{Z}_2^{n-3} \rtimes L$ where L is the free product of the groups $\{1, \sigma_i\}$, $\sigma_i^2 = 1$, $i = 1,2,3$ in the category of pro-2-groups. $\mathbb{Z}_2^{n-3}$ is a normal subgroup of $G(F(2)|F)$ and the action of

L on $\mathbb{Z}_2^{n-3}$, is described by $\sigma_i \alpha \sigma_i^{-1} = -\alpha$ for $i = 1,2,3$ $\alpha \in \mathbb{Z}_2^{n-3} - \{0\}$.

The proof of Theorem 2 is based on results found in [2], [11] and basic facts about the Witt rings of quadratic extensions [see [3], page 25].

Using the generalisation of the Kurosch's theorem about subgroups of free-products to the profinite groups [1]; and the fact that the group $G(M(2)/M(\sqrt{-1}))$ is torsion-free for any field M we can deduce the following result:

Corollary 1. Let F be a field of type (i) $(3.2^{n-3}, 2^n), n \geq 3$, (ii) $(2^{n-2}, 2^n), n \geq 4$. Then the group $G(F(2)|F)$ is the semidirect product (i) $(U \times \mathbb{Z}_2^{n-3}) \rtimes \{1,\sigma\}$

(ii) $(U \times \mathbb{Z}_2^{n-4}) \rtimes \{1,\sigma\}$

where U is the free pro-2-group generated by

(i) elements $a = \sigma_1\sigma_2$, $\sigma_2\sigma_3 = b$

(ii) $\sigma_1\sigma_2 = a$, $\sigma_2\sigma_3 = b$, $\sigma_3\sigma_4 = c$,

and $U \times \mathbb{Z}_2^{n-3}$ ($U \times \mathbb{Z}_2^{n-4}$ resp.) is the direct product. (Here $\sigma_i^2 = 1$ and in fact σ_i can be chosen the same as in the Theorem 1.. σ can be identified with σ_1.)

Furthermore the action of σ on $U \times \mathbb{Z}_2^{n-3}$ is described by $\sigma\alpha\sigma^{-1} = -\alpha$ (i) $\alpha \in \mathbb{Z}_2^{n-3} - \{0\}$

(ii) $\alpha \in \mathbb{Z}_2^{n-4} - \{0\}$

and

(i) $\sigma a \sigma^{-1} = a^{-1}$, $\sigma b \sigma^{-1} = a.b^{-1}.a^{-1}$

(ii) $\sigma a \sigma^{-1} = a^{-1}$, $\sigma b \sigma^{-1} = a.b^{-1}.a^{-1}$, $\sigma c \sigma^{-1} = a.b.c^{-1}.b^{-1}.a^{-1}$

Alternatively we can deduce Theorem 1 from Corollary 1; and we can prove Corollary 1 using results in papers [10], [11].

Some details can be found in [8] where we use also cute argument from [9].

Corollary 2. Let F be a field. We denote the cohomological dimension of $G(F(2)|F(\sqrt{-1}))$ by $cd_2 G(F(\sqrt{-1}))$. If F is a field of type $(32^{n-3},2^n), n \geq 3$, then $cd_2(G(F(\sqrt{-1}))) = n-2$. If F is a field of type $(2^{n-2},2^n), n \geq 4$, then $cd_2(F(\sqrt{-1})) = n-3$.

It is very easy to compute the stability index, $st(F)$, of the fields under investigation. (For definition of $st(F)$ see e.g. [4]).

Theorem 2. Let F be a field of type $(2^{n-2},2^n), n \geq 4$ then $st(F) = n-3$. If F is of type $(3.2^{n-3},2^n), n \geq 3$ then $st(F) = n-2$.

Theorem 3. Let F be a field of type $(2^{n-2},2^n), n \geq 4$. Then $Witt(F) \cong (\mathbb{Z} + 2(\mathbb{Z}^{(4)}))[H]$, where $\mathbb{Z}+2(\mathbb{Z}^{(4)})$ is a subring of $\mathbb{Z}^{(4)}$, ($\mathbb{Z}$ is a diagonal of $\mathbb{Z}^{(4)}$ and H is the 2-elementary group of rank $n-4$) . The chain length invariant of F , $cl(F)$ equals 4 .

If F is a field of type $(3.2^{n-3},2^n), n \geq 3$, then $Witt(F) \cong (\mathbb{Z} + 2(\mathbb{Z}^{(3)}))[H]$, where H is the 2-elementary group of rank $(n-3)$. $cl(F) = 3$.

In both cases the square class invariant classifies the quadratic forms over F .

For definition of the chain length invariant see e.g. [4] Def. 8.4. Definition of the square class invariant is in [5].

References

1. E. Binz, J. Neukirch, G. H. Wenzel, A subgroup Theorem for free products of pro-finite groups, J. of Algebra 19, 104-109 (1971).

2. L. Van Den Dries: Model theory of field, Dissertation Amsterdam (1978).

3. R. Elman and T. Y. Lam, Quadratic forms under algebraic extensions, Math. Ann. 219, 21-42 (1976).

4. T. Y. Lam, Orderings, Valuations and Quadratic forms, CBMS Vol. 52, (1983).

5. T. Y. Lam and D. Shapiro, The square class invariant for Pythagorean fields, Contemporary Math., Vol. 8 (1982), p. 327-340.

6. J. Merzel, Quadratic forms over fields with finitely many orderings, contemporary Math., Vol. 8, (1982), p. 185-229.

7. J. Mináč, On fields for which the number of orderings is divisible by a high power of 2, I. (Will appear in C. R. Math Rep. Acad. Sci. Canada).

8. J. Mináč, T. M. Viswanathan, On the absolute Galois group of maximal fields with respect to extension of orderings (in preparation).

9. I. R. Šafarevič: On p-extensions, Amer. Math. Soc. Trans. Series 2, Vol. 4 (1956).

10. R. Ware, Quadratic forms and profinite 2-groups, Journal of Algebra, Vol. 58, No. 1, (1979).

11. R. Ware, Quadratic forms and pro-2-groups III Preprint No. 84019, Dept. of Math., Pennsylvania State University.

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δ-ANNEAUX ET λ-ANNEAUX

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Résumé

Nous montrons comment obtenir la théorie des λ-anneaux [3] à partir de celle des δ-anneaux [4]. La suite d'opérations $(\lambda^n | n\in N)$ est remplacée par une suite d'opérations $(\delta_p | p$ nombre premier) satisfaisant à des identités explicites. En conséquence, nous pouvons exhiber pour chaque $n\in N$ une base remarquable des caractères du groupe symétrique S_n.

1- δ-opérations sur un λ-anneau

Soit $(\Psi^n | n\geq 1)$ la suite d'opérations d'Adams en théorie des λ-anneaux. On sait que Ψ^n est un endomorphisme d'anneau et que

$$\Psi^n \circ \Psi^m = \Psi^{nm} \quad \text{pour tout } n,m\geq 1.$$

Pour chaque nombre premier p soit c^p l'opération de puissance cyclique d'ordre p [1]. On vérifie l'identité

$$pc_p(x) = x^p + (p-1)\Psi^p(x).$$

Posons

$$\delta_p(x) = \Psi^p(x) - c^p(x).$$

On a alors

$$\Psi^p(x) = x^p + p\ \delta_p(x)$$

Comme le λ-anneau libre sur un générateur est sans p-torsion et que Ψ^p est un endomorphisme, on conclut que l'opération δ_p donne à cet anneau une structure de δ-anneau [4]. Plus généralement, on voit que tout λ-anneau est un δ-anneau relativement à chaque nombre premier p. Nous désirons expliciter les relations entre δ_{p_1} et δ_{p_2} pour des nombres premiers distincts p_1 et p_2.

Soient $q_1 = p_1^{r_1}$ et $q_2 = p_2^{r_2}$ où p_1 et p_2 sont des nombres premiers distincts. Soit A un anneau commutatif muni de deux opérations unaires δ_1 et δ_2. On suppose que (A,δ_1) est un δ-anneau relativement à (p_1,q_1) et que (A,δ_2) est un δ-anneau relativement à (p_2,q_2).

DEFINITION Les opérations δ_1 et δ_2 sont <u>permutables</u> si l'identité suivante est vérifiée:

$$\delta_1(\delta_2(x)) + \frac{p_1^{q_2-1} - 1}{p_2}\, \delta_1(x)^{q_2} + \sum_{i=1}^{q_2-1} \frac{1}{p_2}\binom{q_2}{i} p_1^{i-1} \delta_1(x)^i x^{q_1(q_2-i)}$$

$$= \delta_2(\delta_1(x)) + \frac{p_2^{q_1-1} - 1}{p_1}\, \delta_2(x)^{q_1} + \sum_{i=1}^{q_1-1} \frac{1}{p_1}\binom{q_1}{i} p_2^{i-1} \delta_2(x)^i x^{q_2(q_1-i)}$$

La permutabilité des opérations δ_1 et δ_2 entraîne la commutation des endomorphismes de Frobenius $f_1(x) = x^{q_1} + p_1\delta_1(x)$ et $f_2(x) = x^{q_2} + p_2\delta_2(x)$. Réciproquement, si A est sans p_1p_2-torsion, la relation $f_1f_2 = f_2f_1$ entraîne la permutabilité de δ_1 et δ_2.

2- P-anneaux

Soit P un ensemble de nombres premiers.

DEFINITION Un P-anneau A est un anneau commutatif muni d'une opération unaire $\delta_p: A \to A$ pour chaque $p\varepsilon P$. On demande que soient réalisées les conditions suivantes:

i) A muni de δ_p est un δ-anneau relativement au couple (p,p)

ii) δ_p et δ_ℓ sont permutables pour tout couple d'éléments distincts $p,\ell\varepsilon P$

Exemple 1 Tout λ-anneau est naturellement muni d'une structure de P-anneau où P est l'ensemble de tous les nombres premiers.

Exemple 2 Soit n un entier ≥ 1 et soit $\zeta_n \varepsilon \mathbb{C}$ une racine primitive n-ième de l'unité. Soit R_n l'anneau des entiers du corps cyclotomique $Q(\zeta_n)$. Soit P_n l'ensemble des nombres premiers ne divisant pas n. Pour chaque $p\varepsilon P_n$, soit f_p l'unique automorphisme de R_n tel que $f(\zeta_n) = \zeta_n^p$. On a pour tout $x\varepsilon R_n$,

$$f_p(x) = x^p \mod pR_n$$

Ceci montre que R_n est un δ-anneau relativement au couple (p,p). Comme $f_p f_\ell = f_\ell f_p$ pour tout $p,\ell\varepsilon R_n$, on conclut que R_n est un P_n-anneau.

Un morphisme de P-anneaux est un homomorphisme d'anneaux qui préserve chaque opération $\delta_p(p\varepsilon P)$. On vérifie que chaque endomorphisme de Frobenius $f_p(p\varepsilon P)$ est un morphisme de P-anneaux.

Soit A un P-anneau et soit M(P) l'ensemble des suites finies d'éléments de P. Pour chaque $\sigma = (p_1,\ldots,p_n)\varepsilon M(P)$, posons

$$\delta_\sigma = \delta_{p_1} \circ \delta_{p_2} \circ \ldots \circ \delta_{p_n}$$

Désignons par $C(P) \subset M(P)$ le sous-ensemble des suites croissantes pour l'ordre naturel de P.

THEOREME 1 Soit $A = \mathbb{Z}[x_\sigma | \sigma \in C(P)]$ l'anneau des polynômes sur des indéterminés $x_\sigma(\sigma \in C(P))$. Il y a une seule structure de P-anneau sur A pour laquelle $\delta_\sigma(x_o) = x_\sigma$ pour tout $\sigma \in C(P)$. Muni de cette structure, A est le P-anneau libre sur x_o.

Désignons par $\underline{A}$ la catégorie des anneaux et par $\underline{PA}$ celle des P-anneaux.

THEOREME 2 Le foncteur oubliant $U: \underline{PA} \to \underline{A}$ possède un adjoint à droite $V: \underline{A} \to \underline{PA}$.

COROLLAIRE Le foncteur V est représentable par l'anneau $\mathbb{Z}[x_\sigma | \sigma \in C(P)]$. On a une bijection naturelle.

$$V(A) \simeq A^{C(P)}$$

3- Λ-anneaux

Dans ce qui suit nous supposerons que P est l'ensemble de tous les nombres premiers. Pour tout entier $n \geq 1$ on définit un endomorphisme de Frobenius

$$f_n = f_{p_1}^{r_1} \circ \ldots \circ f_{p_k}^{r_k}$$

où $p_1^{r_1} \ldots p_k^{r_k}$ est une décomposition de n en facteurs premiers.

PROPOSITION 1 On peut définir en théorie des P-anneaux une suite unique d'opérations unaires $\alpha_1, \alpha_2, \ldots$ vérifiant les identités

$$f_n(x) = \sum_{r|n} r\, \alpha_r(x)^{n/r} \quad (n \geq 1)$$

PROPOSITION 2 Il y a sur $Z[A_1,A_2,\ldots]$ une seule structure de P-anneaux pour laquelle $\alpha_n(A_1) = A_n$ pour tout $n\geq 1$. Muni de cette structure, $Z[A_1,A_2,\ldots]$ est un P-anneau libre sur A_1.

THEOREME 3 Lorsque P est l'ensemble de tous les nombres premiers, les concepts de P-anneaux et de λ-anneaux sont équivalents.

La démonstration de ce théorème utilise l'approche de Cartier [2].

Exemple 3 Soit R_n l'anneau de l'exemple 2. On peut munir $R_n[1/n]$ d'une structure de λ-anneau. En effet, il suffit de définir f_p = identité pour $p\notin P_n$. En particulier, $Z[i,\frac{1}{2}]$ est un λ-anneau.

Le λ-anneau $\Lambda[x]$ libre sur un générateur est un anneau de polynômes $Z[x, \lambda x, \lambda^2 x, \ldots]$. On introduit sur $\Lambda[x]$ une graduation naturelle en posant $\deg \lambda^n x = n$ pour tout $n\in N$. Cet anneau gradué peut encore se décrire en utilisant la théorie des caractères des groupes symétriques $S_n (n\geq 0)$ [1]. Soit $R(S_n)$ le groupe des caractères de S_n. Considérons l'inclusion $S_n\times S_m \subset S_{n+m}$. On définit une opération bilinéaire

$$R(S_n) \times R(S_m) \to R(S_n\times S_m) \to R(S_{n+m})$$

et par suite, une structure d'anneau gradué sur

$$\bigoplus_{n\geq 0} R(S_n)$$

Cet anneau est isomorphe à $\Lambda[x]$ [5]. Soit maintenant $Z[\delta_\sigma(x)|\sigma\in C(P)]$ l'anneau des polynômes mentionné dans le théorème 1. On introduit une graduation sur cet anneau en posant

$$\deg \delta_\sigma(x) = p_1 p_2 \ldots p_n$$

lorsque $\sigma = (p_1, p_2, \ldots, p_n)$.

COROLLAIRE <u>Les anneaux gradués</u> $\bigoplus_{n \geq 0} R(S_n)$ <u>et</u> $\mathbb{Z}[\delta_\sigma(x) | \sigma \in C(P)]$ <u>sont isomorphes</u>.

BIBLIOGRAPHIE

[1] M. Atiyah. Power Operations in K-theory. Quart. J. Math. (2) 17 (1966), 165-93.

[2] P. Cartier. Groupes formels associés aux anneaux de Witt généralisés. C.R. Acad. Sc. Paris, t.265, 1967, A-49-52.

[3] A. Grothendieck. Classes de Faisceaux et Théorème de Riemann-Roch. Lect. Notes in Math. No 225 (1972) Springer Verlag.

[4] A. Joyal. δ-anneaux et vecteurs de Witt. C.R. Math. Société Royale du Canada.

[5] D. Knutson. λ-rings and the representation theory of the symmetric group. Lect. Notes in Math. 308 (1973) Springer Verlag.

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LE GROUPE DES AUTOMORPHISMES ISOMETRIQUES DU GROUPE P_∞ DES TABLEAUX DE RANG INFINI

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Abstract: L.Kaloujnine has defined the wreath product of a denumerable sequence of groups [4]. Particularly, such a product of cyclic groups of order p, written P_∞, is the infinite analogue of p-groups of Sylow P_m of the symmetrical group S_{p^m} of degree p^m. P_∞ may be considered as the projective limit of the P_m $(m \in N)$ with the obvious homomorphisms. Kaloujnine defines there a certain ultrametric distance [3],[4] and we determine in this paper the automorphisms of P_∞ which are isometric in respect to this distance.

La présente note est consacrée à quelques généralisations de résultats exposés dans nos notes [5] et [6], où on a décrit le groupe des automorphismes du p-groupe de Sylow P_m du groupe symétrique S_{p^m} de degré p^m. Ces généralisations concernent les automorphismes du groupe P_∞ des tableaux de rang infini.

Le groupe P_m peut être et a été présenté comme le produit d'entrelacement (wreath product) de m-groupes cycliques d'ordre p, tel qu'il a été défini par L.Kaloujnine dans sa thèse [2]. En bref, P_m est l'ensemble des tableaux de m coordonnées (on dit de rang m) de la forme $A=[a_1,a_2(X_1),\dots,a_m(X_{m-1})]$, où X_i désigne le vecteur des variables $x_1,\dots,x_i$ et sa s-ième coordonnée $[A]_s=a_s(X_{s-1})$ est un polynôme en $x_1,x_2,\dots,x_{s-1}$ à coefficients dans le corps de p éléments F_p, dans lequel les exposants de chaque x_i ne sont pas supérieurs à p-1. Si X^A_{s-1} $(s<m)$ est le transformé du vecteur X_{s-1} par la transformation

$$x_i^A=x_i+a_i(X_{i-1}) \quad (1\le i\le s-1) ,$$

la loi de composition des tableaux de P_m est donnée par la formule

$[AB]_s = a_s(X^B_{s-1}) + [B]_s$ (1)

Par définition, le groupe P_∞ [3] est constitué par l'ensemble des tableaux

$A = [a_1, a_2(X_1), a_3(X_2), \ldots, a_m(X_{m-1}), \ldots; m \in N]$ (*)

(où le vecteur de variables X_i ainsi que la i-ème coordonnée $[A]_i = a_i(X_{i-1})$ ont le même sens que précédemment) avec une suite dénombrable de coordonnées; ces tableaux sont organisés par l'opération définie par la formule (1).

Pour tout $m \leq t \leq \infty$, on considère l'homomorphisme canonique $f_{m,t}$ du groupe P_t sur le groupe P_m qui applique le tableau $A \in P_t$ sur le tableau $B \in P_m$ qui vérifie:

$[B]_s = [A]_s$, pour tout $1 \leq s \leq m$.

Ce tableau B est appelé la m-projection de A et il sera désigné par $\bar{A}^m$.

Si $r \leq m \leq t$, on a $f_{r,t} = f_{r,m} \circ f_{m,t}$. Le groupe P_∞ peut être défini (d'une manière équivalente) comme la limite projective [1] $P_\infty = \varprojlim(P_m, f_{m,t})$ des groupes P_m à l'aide des homomorphismes $f_{m,t}$ pour $m \leq t < \infty$.

L'homomorphisme $f_{m,t}$ est de noyau

$\Delta_m^{(t)} = \{A \in P_t ; [A]_s = 0 \text{ pour tout } 1 \leq s \leq m\}$.

Il induit donc un isomorphisme $f_t^m : A\Delta_m^{(t)} \longrightarrow \bar{A}^m$ du quotient $P_t/\Delta_m^{(t)}$ ($m \leq t \leq \infty$) sur P_m. Les éléments $A\Delta_m^{(t)}$ et $\bar{A}^m$ seront désormais identifiés. On pose $\Delta_s = \Delta_s^{(\infty)}$.

Un automorphisme η de P_t ($t \leq \infty$) est dit isométrique, s'il préserve tout $\Delta_s^{(t)}$ ($s \leq t$) comme ensemble (le terme se justifie du fait qu'il est une isométrie par rapport à une distance ultramétrique très naturelle qu'on peut définir sur P_t [4]). Ces automorphismes forment un sous-groupe de $\mathrm{Aut}(P_t)$ ($t \leq \infty$), qu'on note $\mathrm{Aut}_{is}(P_t)$ et qu'on appelle le groupe des automorphismes isométriques. Un sous-groupe de P_t stable par tous les $\eta \in \mathrm{Aut}_{is}(P_t)$ sera dit isométriquement caractéristique. Si

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(*) On désigne par N l'ensemble des entiers strictement positifs t est fini et $p\neq 2$, tout $\eta\in \mathrm{Aut}(P_t)$ préserve $\Delta_s^{(t)}$ $(s\leq t)$ comme ensemble [2]. Par conséquent, si t est fini et $p\neq 2$, on a $\mathrm{Aut}_{is}(P_t)=\mathrm{Aut}(P_t)$. La manière dont on a procédé à la détermination de $\mathrm{Aut}(P_m)$ [5],[6] et surtout un résultat intermédiaire, qu'on formulera à la fin de cette note, nous ont permis de déterminer $\mathrm{Aut}_{is}(P_\infty)$.

Soit $\eta:A\longrightarrow\eta(A)$ un automorphisme $\in \mathrm{Aut}_{is}(P_t)$ $(m<t\leq\infty)$. Vu que $\Delta_m^{(t)}$ est stable (comme ensemble) par η, tout $\eta\in\mathrm{Aut}_{is}(P_t)$ induit au quotient par $\Delta_m^{(t)}$ un automorphisme $\bar\eta^m\in \mathrm{Aut}_{is}(P_m)$, qu'on appelle la m-projection de η. On définit $\bar\eta^m$ tout simplement par la correspondance

$$P_m\ni \bar{A}^m=A\Delta_m^{(t)}\xrightarrow{\bar\eta^m}\overline{\eta(A)}^m=\eta(A)\Delta_m^{(t)}\in P_m\quad .$$

L'application $\bar\eta^m$ est un homomorphisme à cause du fait que $\Delta_m^{(t)}$ est stable par η. Pour démontrer la bijectivité, il suffit de remarquer d'une part que

$$\bar\eta^m\,\overline{(\eta^{-1})}^m=\overline{\eta\eta^{-1}}^m=\bar1^m$$

$$\overline{(\eta^{-1})}^m\bar\eta^m=\overline{\eta^{-1}\eta}^m=\bar1^m$$

et d'autre part que la m-projection $(m\in N)$ de l'automorphisme identique de P_∞ est l'automorphisme identique de P_m.

Si $\alpha<\beta<t\leq\infty$ $(\alpha,\beta,t$ des entiers positifs) et si $\eta\in\mathrm{Aut}(P_t)$, alors on a la relation évidente

$$\overline{(\bar\eta^\beta)}^\alpha=\bar\eta^\alpha$$

L'application $\Phi_t^m:\eta\longrightarrow\bar\eta^m$ de $\mathrm{Aut}_{is}(P_t)$ $(m<t\leq\infty)$ dans $\mathrm{Aut}_{is}(P_m)$ est un homomorphisme. Chaque automorphisme f de P_m appartenant à l'image de Φ_t^m est appelé relevable à P_t et chaque $\eta\in\mathrm{Aut}_{is}(P_t)$ qui vérifie $\bar\eta^m=f$ est appelé un relèvement de f.

Si $f\in\mathrm{Aut}(P_{m-1})$ est relevable à P_t $(m\leq t\leq\infty)$, il est aussi relevable à tout P_s, pour $m\leq s\leq t$, et en particulier à P_m. En effet, si $f=\bar\eta^{m-1}$ pour un $\eta\in\mathrm{Aut}_{is}(P_t)$, l'automorphisme $\bar\eta^s$ $(m\leq s\leq t)$ est un relèvement de f à P_s.

On note W_α le groupe des automorphismes du groupe P_α qui sont relevables à P_t, pour tout $t>\alpha$ (on les dira infiniment relevables). Pour α,β entiers positifs tels que $\alpha\leq\beta$, on pose $\phi_{\alpha,\beta}$ l'homomorphisme de W_β sur W_α défini par $\phi_{\alpha,\beta}(\eta)=\eta^{-\alpha}$, si $\alpha<\beta$, et $\phi_{\alpha,\alpha}$ l'application identique sur W_α. Si α,β,γ sont des entiers positifs tels que $\alpha\leq\beta\leq\gamma$, alors on a $\phi_{\alpha,\gamma}=\phi_{\alpha,\beta}\phi_{\beta,\gamma}$. De ce qu'on a dit sur les projections des automorphismes du groupe P_∞, il résulte que $\mathrm{Aut}_{is}(P_\infty)$ est égal à la limite projective [1] $W=\varprojlim(W_\alpha,\phi_{\alpha,\beta})$ de la famille des groupes $(W_\alpha)_{\alpha\in N}$ à l'aide des homomorphismes $\phi_{\alpha,\beta}$.

Pour préciser W_{m-1} $(m\in N)$, il suffit de considérer l'image $\mathrm{Im}\Phi_m^{m-1}$ de l'homomorphisme Φ_m^{m-1}, c'est-à-dire les éléments de $\mathrm{Aut}_{is}(P_{m-1})$, qui se relèvent à P_m et de remarquer qu'ils se relèvent tous à P_∞. L'homomorphisme Φ_m^{m-1} est celui qui a été désigné dans les notes [5], [6] tout simplement par Φ. Comme il a été déjà énoncé dans [6],

$$\mathrm{Im}\Phi_m^{m-1}=\Omega_{m-1}\mathrm{Int}(P_{m-1})^{(*)}\ ,$$

où $\mathrm{Int}(P_{m-1})$ est le groupe des automorphismes intérieurs et Ω_{m-1} est le groupe constitué par les automorphismes ω de P_{m-1}, dont chacun est défini, à l'aide de m-1 éléments $(\omega_1,\dots,\omega_{m-1})$ de F_p non nuls, par la correspondance

$$\omega:[a_1,a_2(x_1),\dots,a_{m-1}(x_1,\dots,x_{m-2})]\longrightarrow$$
$$\longrightarrow[w_1a_1,w_2a_2(w_1^{-1}x_1),\dots,w_{m-1}a_{m-1}(w_1^{-1}x_1,w_2^{-1}x_2,\dots,w_{m-2}^{-1}x_{m-2})].$$

Les groupes Ω_{m-1} et $\mathrm{Int}(P_{m-1})$ commutent entre eux et il en est de même pour Ω_∞ et $\mathrm{Int}(P_\infty)$, où Ω_∞ est l'analogue infini de Ω_{m-1} contenant les automorphismes ω de P_∞ définis comme suit à l'aide de vecteurs infinis $(\omega_1,\omega_2,\dots,\omega_t,\dots;t\in N)$, où tous les $\omega_t\in F_p$ sont $\neq0$:

$$\omega:[a_1,a_2(x_1),\dots,a_t(x_1,\dots,x_{t-1}),\dots;t\in N]\longrightarrow$$
$$\longrightarrow[w_1a_1,w_2a_2(w_1^{-1}x_1),\dots,w_ta_t(w_1^{-1}x_1,\dots,w_{t-1}^{-1}x_{t-1}),\dots;t\in N]$$

de sorte que

$$\mathrm{Aut}_{is}(P_\infty)=\Omega_\infty\mathrm{Int}(P_\infty)\ .$$

(*) La démonstration de cette formule étant assez longue ne peut pas être présentée dans cette note.

BIBLIOGRAPHIE

[1] Bourbaki: Théorie des ensembles,chap.III,Hermann,Paris (1970)

[2] Leo Kaloujnine: Structures des p-groupes de Sylow des groupes symétriques finis, Annales Scientifiques de l'Ecole Normale Supérieure (1948), pp.239-271

[3] Leo Kaloujnine: Sur le groupe P_∞ des tableaux infinis, Comptes Rendus de l'Académie des Sciences, tome 224 (1947),pp.1097-1099

[4] Leo Kaloujnine: A propos d'une généralisation des p-groupes de Sylow des groupes symétriques (en russe), Acta Math.Acad.Scient. Hungaricae, tom.II(1951), fasc.3-4, pp.198-221

[5] Paul Lentoudis: Détermination du groupe des automorphismes du p-groupe de Sylow du groupe symétrique de degré p^m: l'idée de la méthode, Comptes Rendus Mathématiques de l'Académie des Sciences, La société Royale de Canada, Vol. VII (1985), pp. 67-72.

[6] Paul Lentoudis: Le groupe des automorphismes du p-groupe de Sylow du groupe symétrique de degré p^m:résultats, Comptes Rendus Mathématiques de l'Académie des Sciences, La société Royale de Canada, Vol. VII (1985), pp. 133-136.

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THE NEUMANN PROBLEM IN WEIGHTED SOBOLEV SPACES[1)]

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Presented by L. Lorch, F.R.S.C.

1. Introduction. Let us consider a formal differential operator $\mathcal{L}$ of order $2k$, given by the formula

$$(1.1) \qquad (\mathcal{L}u)(x) = \sum_{|\alpha|,|\beta|\leq k} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x)\, D^{\beta}u(x))$$

defined on a domain $\Omega \subset \mathbb{R}^N$. We associate the operator $\mathcal{L}$ with a bilinear form $a(u,v)$ by the formula

$$(1.2) \qquad a(u,v) = \sum_{|\alpha|,|\beta|\leq k} \int_{\Omega} a_{\alpha\beta}(x)\, D^{\beta}v(x)\, D^{\alpha}u(x)\, dx ,$$

assuming that the coefficients of the operator satisfy

$$a_{\alpha\beta} \in L^{\infty}(\Omega)$$

and that the operator $\mathcal{L}$ is elliptic in the following sense: There exists a positive constant c such that for all functions u from the Sobolev space $W^{k,2}(\Omega)$ the following inequality holds:

$$(1.3) \qquad a(u,v) \geq c\|u\|^2_{W^{2,k}(\Omega)} .$$

In [1], it is shown that under certain conditions on the domain Ω, on the set $M \subset \partial\Omega$ and on the weight function $s = s(t)$ $(t > 0)$, the Dirichlet problem for the operator $\mathcal{L}$ is (weakly) solvable in the weighted Sobolev space $W^{k,2}(\Omega;\, s(d_M))$ defined as the set of all functions $u = u(x)$ such that

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$$(1.4)\qquad \|u\|_{k,2;s(d_M)} = \Big(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^2\, s(d_M(x))dx\Big)^{1/2} < \infty$$

with $d_{M}(x) = \mathrm{dist}(x,M)$.

The aim of this short communication (?) paper is to extend the existence and uniqueness result in weighted spaces to the case of the Neumann problem for the operator $\mathcal{L}$. For simplicity, we shall consider the case of a second order operator $\mathcal{L}$, i.e., the case $k = 1$. We shall deal with a power weight only, considering the case

$$s(t) = t^{\varepsilon}\ ,\ \varepsilon \in \mathbb{R}\ .$$

The corresponding weight space will be denoted by

$$W^{1,2}(\Omega;\ d_{M,\varepsilon})\ .$$

2. Formulation of the problem. Let f , g be continuous linear functionals from $[W^{1,2}(\Omega;\ d_{M,\varepsilon})]^*$, g such that $\langle g,v\rangle = 0$ for $v \in C_0^\infty(\Omega)$. We shall say that the function $u \in W^{1,2}(\Omega;\ d_{M,\varepsilon})$ is a weak solution of the formal Neumann problem

$$(2.1)\qquad \begin{cases} \sum\limits_{|\alpha|,|\beta|=1} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u(x)) = f(x) & \text{on } \Omega\subset \mathbb{R}^N\ , \\ \sum\limits_{|\alpha|,|\beta|=1} a_{\alpha\beta}\, \nu_\alpha\, D^\beta u = g & \text{on } \partial\Omega\ , \end{cases}$$

where ν_α are the components of the outer normal ν to the boundary $\partial\Omega$ of Ω , if

$$(2.2)\qquad a(u,v) = \langle f,v\rangle + \langle g,v\rangle \quad \text{for every } v \in C^\infty(\overline{\Omega})\ .$$

3. Remark. In most applications, the functionals f and g are of the form

$$\langle f,v\rangle = \int_\Omega fv\,dx\ ,\ \langle g,v\rangle = \int_{\partial\Omega} gv\,ds$$

with suitable f and g (e.g., $f \in L^2(\Omega;\ d_M,-\varepsilon)$, $g \in L^2(\partial\Omega;\ d_M,\eta)$ with η depending on ε and M) .

4. Main existence and uniqueness theorem. Let Ω be a bounded domain in $\mathbb{R}^N$ with a locally Lipschitzian boundary $\partial\Omega$. Let $M \subset \partial\Omega$ be an m-dimensional manifold and let

$$N - m \geqq 3\ . \tag{4.1}$$

Let $\mathcal{L}$ be the differential operator of order 2 from (2.1) and let $\mathcal{L}$ be elliptic in the sense of (1.3) (with $k = 1$) . Then there exists an open interval I containing zero and such that for $\varepsilon \in I$ there is one and only one weak solution $u \in W^{1,2}(\Omega;\ d_M,\varepsilon)$ of the Neumann problem (2.1). Further, there is a positive constant $c = c(\varepsilon)$ such that

$$\|u\|_{1,2;d_M,\varepsilon} \leqq c(\varepsilon)(\|f\|_* + \|g\|_*)\ .$$

5. Remarks. (i) The proof of the foregoing theorem is analogous to that of the existence and uniqueness of the Dirichlet problem in [1], Chapter 14: Using properties of the weighted space $W^{1,2}(\Omega;\ d_M,\varepsilon)$ one can show, via the ellipticity condition (1.3), that the bilinear form $a(u,v)$ from (1.2) (with $k = 1$) is (H_1,H_2) - elliptic with $H_1 = W^{1,2}(\Omega;\ d_M,\varepsilon)$, $H_2 = W^{1,2}(\Omega;\ d_M,-\varepsilon)$ (for a definition see e.g. [1], Remark 13.7), and then use a generalized form of the Lax-Milgram lemma due to J. Nečas (see e.g. [1], Lemma 13.6).

(ii) The most important property of the weighted spaces used in the proof is the imbedding

$$(5.1) \qquad W^{1,2}(\Omega;\ d_M,\eta) \hookrightarrow L^2(\Omega;\ d_M,\eta-2)\ .$$

This imbedding holds for

$$(5.2) \qquad \eta > 2 + m - N$$

(see [1], Sections 8.19 and 8.20, and the paper of J. Rakosnik mentioned there; see also [2]). In the proof of Theorem 4, we need the imbedding (5.1) for $\eta = \varepsilon$ as well for $\eta = -\varepsilon$, which leads, via (5.2), to the inequalities

$$2 + m - N < \varepsilon < N - m - 2\ ,\ \text{i.e.,}\ N - m > 2\ .$$

This fact requires the imposition of the rather restrictive condition (4.1).

(iii) Obviously, the Neumann problem as well as the existence and uniqueness theorem can be formulated easily for $k > 1$, too. In this case, we have to use imbedding (5.1) repeatedly, for $\eta = \pm\varepsilon\ ,\pm\varepsilon - 2\ ,\dots,\pm\varepsilon - 2(k-1)$, which leads to a more restrictive condition $N - m > 2k$, i.e. $N - m \geqq 2k + 1$ - as an extension of condition (4.1).

(iv) Let us note that the case $M = \{x_0\}$ with $x_0 \in \partial\Omega$ is dealt with in [3], Chapter 6, for $N \geqq 3$. Since in this case $m = 0$, condition (4.1) is satisfied and our result contains the result from [3].

6. Concerning the interval I in the Theorem of §4. The main theorem states only the existence of the interval I of admissible values of the power ε in the weight function d_M^ε. In every concrete case, this important interval can be at least estimated: clearly, it depends on the geometry of Ω and of M as well as on the properties of the coefficients $a_{\alpha\beta}$. Let us note that, e.g., in the case of the operator $-\Delta u + c(x)u$ with $c(x) \geqq c_0 > 0$

one can show that $I \subset (-1,1)$ and, in some cases, $I = (-1,1)$.

7. Concerning condition (4.1). This condition excludes the cases $N - m = 2$ as well as $N - m = 1$, e.g., the interesting cases $M = \{x_0\}$ for $N = 2$ $(m = 0)$ or $M = \partial\Omega$ for every N $(m = N - 1)$. Let us mention that the authors are able to derive results analogous to that of Theorem 4 also in some of these exceptional cases, but only for very special domains Ω and sets M (e.g. for $\Omega = (0,1)^N$, $M = \{x \in \partial\Omega , x_N = 0\}$ - i.e. $d_M(x) = x_N$, where for $\mathcal{L}u = -\Delta u + cu$ again $I = (-1,1)$) .

REFERENCES

[1] Kufner, A.: Weighted Sobolev spaces. J. Wiley & Sons, Chichester - New York - Brisbane - Toronto - Singapore, 1985.

[2] Edmunds, D.E.; Kufner, A.; Rakosnik, J.: Embeddings of Sobolev spaces with weights of power type. Zeitschrift Analysis Anwend, 1985 (to appear).

[3] Nečas, J.: Les methodes directes en theorie des equations elliptiques. Academia, Prague & Masson et C^{ie}, Paris, 1967.

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ESTIMATES FOR INTEGRAL MODULI OF CONTINUITY FOR SOLUTIONS OF ELLIPTIC EQUATIONS[1]

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Presented by L. Lorch, F.R.S.C.

1. INTRODUCTION

In the present paper we consider the linear elliptic equation

(1) $$L(u) \equiv \sum_{|k|\leq 2m} a_k(x)\, D^k u = f(x) ,$$

in a bounded region Ω of n-dimensional Euclidean space E_n. Here $x = (x_1,\ldots,x_n)$ is a point of the region Ω ; $k = (k_1,\ldots,k_n)$ is a multi-index, $k_i = 0,1,2,\ldots$, $i = 1,\ldots,n$, $|k| = k_1+\ldots+k_n$ is its length;

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} ,$$

and $m = 0,1,2,\ldots$.

In what follows, we assume everywhere that the problem satisfies the condition for ellipticity of the equation (1):
$\forall x \in \Omega$, $\forall \ell = (\ell_1,\ldots,\ell_n) \in E_n$, $\ell^k = \ell_1^{k_1}\ldots\ell_n^{k_n}$

(2) $$\Lambda(\ell) \equiv \sum_{|k|=2m} a_k(x)\, \ell^k \geq \lambda|\ell|^{2m} , \lambda = \text{const.} > 0 .$$

The solutions $u(x)$ of the equation (1) are understood to be generalized: $u(x)$ belongs to the Sobolev space $W_p^{2m}(\Omega)$, $p > 1$; see [8; Chapter IV] or [9; §7] and satisfies the equation (1) almost everywhere in Ω .

Let Ω' be a region whose closure is contained in Ω . We shall say that the n-dimensional vector h is <u>admissible</u> if for any $x \in \Omega'$ all the vectors $(x + th) \in \Omega$, where $0 \leq t \leq 1$.

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It is known (see [2; §3], [3; §7] and [7; §2]) that if in (1) the function $f(x) \in L_p(\Omega)$, $p > 1$, and the coefficients $a_k(x)$, $|k| \leq 2m$, are continuous in Ω , then the leading derivatives $D^k u$, $|k| = 2m$ of the generalized solutions of the equation (1) belong to the space $L_p(\Omega^\delta)$, $p > 1$, in any interior subregion $\Omega^\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$, $\delta > 0$, and their norms in $L_p(\Omega^\delta)$ may be estimated in terms of the norm in $L_p(\Omega)$ of the function $f(x)$ and the norm in $L_p(\Omega)$ of the same solution $u(x)$.

In the present paper under the assumptions that the function $f(x) \in L_p(\Omega)$, $p > 1$, satisfies a Hölder condition in $L_p(\Omega)$, $p > 1$, with exponent α , $0 < \alpha < 1$ (see [6; §1]).

$$H^{\Omega}_{\alpha,p}(f) \equiv \sup \left(\int_{\Omega'} \left| \frac{f(x+h) - f(x)}{|h|^{\alpha}} \right|^p dx \right)^{1/p} < \infty ,$$

where the supremum is taken for all $\Omega' \subset \Omega$ and all admissible h , and that the coefficients $a_k(x)$ satisfy the usual Hölder condition with exponent α , we establish a priori estimates for $H^{\Omega^\delta}_{\alpha,p}(D^k u)$, $|k| = 2m$, in terms of $H^{\Omega}_{\alpha,p}(f)$, the norm in $L_p(\Omega)$ of the function $f(x)$ and the norm in $L_p(\Omega)$ of the same solution $u(x)$.

On the other hand, if the function $f(x) \in L_p(\Omega)$, $p > 1$, $H^{\Omega}_{\alpha,s}(f) < \infty$, $1 < s < p$, $0 < \alpha < 1$, if the coefficients $a_k(x)$, are continuous in Ω and $H^{\Omega}_{\alpha,q}(a_k) < \infty$, $q = \frac{ps}{p-s}$, $|k| \leq 2m$, then we shall establish a priori estimates for $H^{\Omega^\delta}_{\alpha,s}(D^k u)$, $|k| = 2m$ in terms of $H^{\Omega}_{\alpha,s}(f)$, the norm in $L_p(\Omega)$ of the function $f(x)$ and the norm in $L_p(\Omega)$ of the same solution $u(x)$.

We denote by $L_{p,\alpha}(\Omega)$ $(W^{2m}_{p,\alpha}(\Omega))$ the set of all functions $u(x) \in L_p(\Omega)$ $(W^{2m}_p(\Omega))$ having respective finite norms

(4) $$\|u\|^{\Omega}_{0,\alpha,p} = \|u\|_{L_p(\Omega)} + H^{\Omega}_{\alpha,p}(u) ,$$

(5) $$\|u\|^{\Omega}_{2m,\alpha,p} = \sum_{|k| \leq 2m} \|D^k u\|^{\Omega}_{0,\alpha,p} , \quad D^0 u \equiv u .$$

We make use also of the following known norms for the functions $u(x)$ defined in the region Ω and satisfying the usual Hölder condition with exponent β , $0 < \beta < 1$:

$$(6) \qquad |u|_{0,\beta}^{\Omega} = |u|_{0,0}^{\Omega} + H_{\beta}^{\Omega}(u) ,$$

where

$$|u|_{0,0}^{\Omega} = \sup_{x\in\Omega} |u(x)| , \quad H_{\beta}^{\Omega}(u) = \sup_{\substack{x\in\Omega \\ (x+\Delta x)\in\Omega}} \frac{|u(x+\Delta x) - u(x)|}{|\Delta x|^{\beta}} .$$

With regard to the coefficients of the equation (1) we assume that one of the following assumptions holds:

$$\sum_{|k|\le 2m} |a_k|_{0,\alpha}^{\Omega} = B_1 < \infty , \quad \sum_{|k|\le 2m} |a_k|_{0,\alpha,q}^{\Omega} = B_2 < \infty .$$

The symbol K denotes any constant which depends on δ , n , p , λ , α , $\operatorname{diam}\Omega$, m , B_1 (or B_2) .

2. INNER BOUNDS FOR SOLUTIONS OF ELLIPTIC EQUATIONS WITH CONSTANT COEFFICIENTS.

Let Q be the unit cube in n-dimensional space E_n :

$$0 \le x_j \le 1 \quad (j = 1,\dots,n) .$$

We consider in the interior of Q the differential equation of order $2m$ and of elliptic type

$$(7) \qquad L_0(u) \equiv \sum_{|k|=2m} a_k D^k u = f(x) ,$$

where the form

$$\Lambda_0(\ell) \equiv \sum_{|k|=2m} a_k \ell^k$$

is positive definite, that is, it satisfies the inequality (2).

Lemma 1. Suppose that, in the equation (7), $f(x) \in L_{p,\alpha}(Q)$, $\sum_{|k|=2m} |a_k| = B_1 < \infty$. Then, for any generalized solution $u(x) \in W^{2m}_{p,\alpha}(Q)$, $p > 1$, $0 < \alpha < 1$, of the equation (7), the following inequalities hold:

(8) $$\|u\|_{W_p^{2m}(Q^\delta)} \le K[\|f\|_{L_p(Q)} + \frac{1}{\delta^{2n+2m+1}} \|u\|_{L_p(Q)}] ,$$

(9) $$\sum_{|k|\le 2m} H_{\alpha,p}^{Q^\delta} (D^k u) \le K[\frac{\|f\|_{L_p(Q)}}{\delta^{n+\alpha}} + H_{\alpha,p}^{Q}(f) + \frac{\|u\|_{L_p(Q)}}{\delta^{2n+2m+1+\alpha}}] ;$$

here K is independent of δ.

In the proof of this lemma we obtain, with the help of the regularizing function (see [3; §2] or [1; §3]).

(10) $$\mathcal{G}(\varepsilon ; \ell', z) = \frac{1}{[1 - i\,\varepsilon\,(|\ell'|i + z)]^{n+1}} ;$$

here $\ell' = (\ell_1, \ldots, \ell_n)$, $|\ell'| = (\sum_{j=1}^{n-1} \ell_j^2)^{1/2}$, ε is a positive number and z is a complex variable; a representation for the functions

(11) $$u_\varepsilon(x) \equiv \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathcal{G}(\varepsilon ; \ell', \ell_n)(\widetilde{\eta_\delta u})(\ell) e^{i(\ell,x)} d\ell_1 \ldots d\ell_n$$

where $x \in Q^\delta$, η_δ is an auxiliary function in Q; see [7, Appendix], $\widetilde{\eta_\delta u}$ is the Fourier transform of the function $\eta_\delta u$; in terms of $\widetilde{\eta_\delta f}$, $\mathcal{G}(\varepsilon ; \ell', z)$, $\Lambda_0(\ell', z)$ and the integral

(12) $$G(\varepsilon ; x , \xi) = \frac{1}{(2\pi)^n} \int_{|\ell'|\ge 1} \int_{-\infty}^{+\infty} \frac{\mathcal{G}(\varepsilon ; \ell', z)}{\Lambda_0(\ell', z)} e^{i(\ell,x-\xi)} d\ell' dz +$$
$$+ \frac{1}{(2\pi)^n} \int_{|\ell'|<1} \int_{\gamma} \frac{\mathcal{G}(\varepsilon ; \ell', z)}{\Lambda_0(\ell', z)} e^{i(\ell,x-\xi)} d\ell' dz ;$$

here $\ell = (\ell', z)$, $\xi \in Q$, γ is a contour in the complex plane $\{z\}$, consisting of the following parts: the intervals of the real axis $(-\infty, -\delta_0)$ and $(\delta_0, +\infty)$ (δ_0 is a positive number), and the semi-circle $|z| = \delta_0$ lying in the upper half-plane.

By virtue of theorems of S.G. Mihlin (see [4] or [5]), of Sobolev [9, p.36] and of the properties of the functions n_δ, $\mathcal{G}(\varepsilon\,;\ell',z)$ and $G(\varepsilon\,;x,\xi)$ we obtain the inequality (8).

Let Q_r be the cube $0 \le x_j \le r < 1$ $(j = 1,\dots,n)$.

Lemma 2. Suppose that in the equation (7) $f(x) \in L_{p,\alpha}(Q_r)$, $\sum_{|k|=2m} |a_k| = B_1 < \infty$. Then, for any generalized solution $u(x) \in W^{2m}_{p,\alpha}(Q_r)$, $p > 1$, $0 < \alpha < 1$, of the equation (7), the following inequalities hold:

$$(13) \qquad \|u\|_{W_p^{2m}(Q_r^\delta)} \le K\Big[\|f\|_{L_p(Q_r)} + \frac{1}{r^{2m}\delta^{2n+2m+1}}\|u\|_{L_p(Q_r)}\Big],$$

$$(14) \qquad \sum_{|k|\le 2m} H^{Q^\delta}_{\alpha,p}(D^k u) \le K\Big[\frac{\|u\|_{L_p(Q_r)}}{r^\alpha\,\delta^{n+\alpha}} + H^{Q_r}_{\alpha,p}(f) + \frac{\|u\|_{L_p(Q_r)}}{r^{2m+\alpha}\,\delta^{2n+2m+1+\alpha}}\Big].$$

Here the constant K is independent of δ and of r.

3. INNER BOUNDS FOR SOLUTIONS OF ELLIPTIC EQUATIONS WITH VARIABLE COEFFICIENTS.

The following inner bounds are established.

Theorem 1. Let, in equation (1), $f(x) \in L_{p,\alpha}(\Omega)$, $\sum_{|k|\le 2m} |a_k|^\Omega_{0,\alpha} = B_1 < \infty$, $p > 1$, $0 < \alpha < 1$.
Then for any generalized solution $u(x) \in W^{2m}_{p,\alpha}(\Omega)$ of (1) the following inequality is satisfied.

$$(15) \qquad \|u\|^{\Omega^\delta}_{2m,\alpha,p} \le K\big[\|f\|^\Omega_{0,\alpha,p} + \|u\|_{L_p(\Omega)}\big],$$

Theorem 2. Let in equation (1), $f(x) \in L_p(\Omega) \cap L_{s,\alpha}(\Omega)$, $1 < s < p$, the coefficients $a_k(x)$, $|k| \le 2m$, be continuous in Ω and $\sum_{|k|\le 2m} \|a_k\|^\Omega_{0,\alpha,q} = B_2 < \infty$, $q = \dfrac{sp}{p-s}$. Then for any generalized solution $u(x) \in (W_p^{2m}(\Omega) \cap W^{2m}_{s,\alpha}(\Omega))$ of (1) the following inequality is satisfied

$$(16) \qquad \|u\|_{2m,\alpha,s} \le K\big[\|f\|_{L_p(\Omega)} + H^\Omega_{\alpha,s}(f) + \|u\|_{L_p(\Omega)}\big],$$

where

$$K = K(\delta, n, m, p, s, \alpha, \lambda, B_2, \operatorname{diam}\Omega).$$

The proofs of these theorems will be published later.

References

1. S. Bochner, Zeta functions and Green's functions for linear partial differential operators of elliptic type with constant coefficients. Ann. of Math. (2) 57, (1953), 32-56.

2. F.E. Browder, On the regularity properties of solutions of elliptic differential equations. Comm. Pure Appl. Math., 9 (1956), 351-361.

3. A.I. Košelev, A priori estimates in L_p and generalized solutions of elliptic equations and systems. Uspehi Mat. Nauk 13, No. 4 (1958), 29-88; English transl. Amer. Math. Soc. Translations (2) 20 (1962), 105-171.

4. S.G. Mihlin, On the multipliers of Fourier integrals. Dokl. Akad. Nauk. SSSR, 109, (1956), 701-703 (Russian).

5. _____________, Fourier integrals and multiple singular integrals. Vestnik Leningrad. Univ. 12 (1957), No. 7, 143-155 (Russian, English summary).

6. S.M. Nikol'skii, Extension of functions of several variables preserving differential properties. Mat. Sb. 40 (82), 243-268 (1956). English transl. Amer. Math. Soc. Transl. (2), 83 (1969), 159-188.

7. L. Nirenberg, Estimates and existence of solutions of elliptic equations. Comm. Pure Appl. Math., 9,(1956), 509-530.

8. V.I. Smirnov, A Course of Higher Mathematics, vol. V, Moscow. Fizmatgiz, 1960 [Russian]. English translation: Pergamon Press. International Series of Monographs in Pure and Applied Mathematics, vol. 62, 1964.

9. S.L. Sobolev, Applications of Functional Analysis in Mathematical Physics, Leningrad: Izdat. Leningrad, Gos. Univ. 1950 [Russian]. English translation: Translations of Mathematical Monographs, vol. 7, Providence, R.I.: American Mathematical Society, 1963.

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A NOTE ON STRONG EXTREME POINTS

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Abstract: The interrelationship between two notions of strong extreme points of convex sets in a Banach space is examined. A question of Kunen-Rosenthal is answered.

1. Several variations and specializations of the concept of an extreme point of a bounded convex set in a Banach space are known. Of these, there are two notions of strong extreme points introduced by R. McGuigan [7] and more recently by K. Kunen-H.P. Rosenthal [5]. This note is concerned with the inter-relations between these two notions; a question posed by Kunen-Rosenthal is answered.

2. Let X be a Banach space and K a bounded subset of X. Given $\varepsilon > 0$, a point $x \in X$ is called an ε-strong extreme point of K if there exists a $\delta > 0$ so that, if y, z belong to K and the line segment joining y and z contains u with $||u - x|| < \delta$, then either $||u - y|| < \varepsilon$ or $||u - z|| < \varepsilon$. This definition is due to K. Kunen and H.P. Rosenthal, who studied, among other things, the connection between its existence and δ-trees, i.e., dyadic martingales with differences everywhere δ-bounded away from zero.

<u>Definition (K-R)</u>: A point $x \in K$ is a strong extreme point of K, if it is an ε-strong extreme point for every $\varepsilon > 0$.

It should be noted that the set K is not assumed to be necessarily convex.

The following consequences of this definition are given in [5]:

(i) Let K be bounded and convex. Then x is an ε-strong extreme point iff there is $\delta > 0$ so that if y and z are in K and $||(y+z)/2 - x|| < \delta$ and $||y-z|| < 2\varepsilon$.

(ii) A strong extreme point is an extreme point; a denting point of a convex set is a strong extreme point of the set.

(iii) A point x in a convex bounded set K is a strong extreme point of K iff the following property (*) holds:

(*) for every pair of sequences $\{u_n\}$ and $\{v_n\}$ of K, with $u_n + v_n \to 2x$ in norm, $||u_n - v_n|| \to 0$.

Every locally uniformly rotund (LUR) Banach space has the property that every point of its unit sphere is a strong extreme point of its unit ball.

Kunen and Rosenthal conclude their paper with the question: Is a Banach X with this property LUR?

3. A Banach space is midpoint locally uniformly rotund (MLUR) if, whenever $x \in X$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $||x|| = 1$, $||x_n|| \to 1$, $||y_n|| \to 1$ and $||2x - (x_n + y_n)|| \to 0$ then $||x_n - y_n|| \to 0$. This generalization of LUR was introduced by K.W. Anderson [1], and has been of recent interest; M.A. Smith [9, 10] and M.I. Kadec [4].

Examples of MLUR spaces which are not LUR are known; for example, Smith [9] gives the following renorming of ℓ_2-space: for $x = (x_1, x_2, \ldots)$ in ℓ_2, let $x' = (0, x_2, \ldots)$ and define the equivalent norm $||x|| = |x_1| + ||x'||_2$. Choosing a sequence $\{\alpha_n\}$

of reals, decreasing to zero, let T be the continuous linear operator on ℓ_2 defined by $T(x_1, x_2, \ldots) = (\alpha_2 x_2, \alpha_3 x_3, \ldots)$. Then for $x \in \ell_2$, let $|||x||| = (||x||^2 + ||Tx||_2^2)^{1/2}$. It is shown in [9] that $(\ell_2, |||\cdot|||)$ is MLUR but not LUR.

From (iii) of §2 and the definition of MLUR, one can easily show that a Banach space X is MLUR iff every point of its unit sphere $S(X)$ is a strong-extreme point of the unit ball $U(X)$.

Then the example above answers the question of Kunen-Rosenthal.

4. A Banach space X has property H if whenever $x \in X$ and $\{x_n\}$ is a sequence in X such that $\{x_n\}$ converges to x weakly and $||x_n|| \to ||x||$ then $\{x_n\} \to x$.

If X has H and is also rotund (strictly convex) then X is HR. Mark Smith [10] has renormed the space c_0 so that it is MLUR but not HR in the equivalent norm. On the other hand, a space can be HR without being MLUR, again due to another example of Smith [9]. However, M.I. Kadec [4] has proved that if X is HR and does not contain ℓ_1, then X is MLUR. Therefore, if X is HR and does not contain ℓ_1, then every point on the unit sphere $S(X)$ is a strong-extreme point of $U(X)$.

5. Let us now consider the definition of a strong extreme point considered by R. McGuigan [7].

Let X be a Banach space and K a convex subset of X. Let $||x - K|| = \inf(||x - y||: y \in K\}$ and $d_\alpha(x,\alpha) = \inf\{\max[||(x+\alpha y)-K||, ||(x - \alpha y) - K||]: ||y|| = 1\}$. If $K = S(X)$, the unit sphere of X, write $d_K(x,\alpha) = d_X(x,\alpha) = \inf\{\max||x \pm \alpha y||: ||y|| = 1\} - 1$.

The function $d_K(x,\alpha)$, introduced first by Milman [8], is the modulus of extremeness of the point x .

Definition M: A point $x \in K$ is a strong extreme point of K iff, for every $\alpha > 0$, $d_K(x,\alpha) > 0$.

To distinguish between the two definitions, we shall denote a strong extreme point according Definition (M) by strong extreme (M) .

An equivalent form of the definition (M) is: A point x on the unit sphere $S(X)$ of X is a strong extreme point (M) iff, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{||x+y|| , ||x-y||\} \le 1+\delta \Rightarrow ||y|| < \varepsilon .$$

J. Cima and J. Thompson [2] proved that a function f in the Hardy space $H^1(U^n)$; where U^n is the unit polydisc, is an extreme point of the unit sphere Σ of $H^1(U^n)$ iff it is a strong extreme point (M) of Σ . This is true of Hardy spaces H^1 on Riemann surfaces also [3] .

6. A point $x \in S(X)$ which is strongly extreme (M) need not be strongly extreme (in the sense of K-R) .

To see this, consider the renorming of the ℓ_1-space, defined by M.A. Smith [9], so that the resulting space is HR but not MLUR. This space is separable and conjugate and the norm and weak* sequences agree on the unit sphere and therefore, every extreme point of $S(X)$ is strongly extreme (M). But since it is not MLUR, it follows that there exist strong extreme (M) points which are not strong extreme (K - R) .

Finally, let x be a denting point of the unit sphere $S(X)$ of X, i.e., for every $\varepsilon > 0$, x does not belong to the convex closure of the complement in $S(X)$ of the set of all y such that $||x - y|| \leq \varepsilon$. Then a theorem of McGuigan [7] says that x is strongly extreme (M) in $S(X)$, and it is known [6] that if every point of $S(X)$ is a denting point of $S(X)$, X is MLUR .

Thus, when x is a denting point of $S(X)$ the two notions of strong-extremeness agree.

References

(1) K.W. Anderson, Midpoint local uniform convexity, Ph.D. Dissertation, Univ. of Illinois, (1960).

(2) J.A. Cima and J. Thompson, Strong-extreme points in H^p, Duke Math J. 40, (1973), 529-532.

(3) M. Goldstein and S. Swaminathan, Pseudo-uniform convexity of the Hardy Class H^1 on Riemann surfaces Math. Ann. 226, (1977), 271-273.

(4) M.I. Kadec, Relations between some properties of convexity of the unit ball in a Banach space, Functional Analysis and its Appl. 16, 1982, 204-206.

(5) K. Kunen and H.P. Rosenthal, Martingale proofs of some geometrical results in Banach space theory, Pacific J. Math. 100, (1982), 153-175.

(6) Bor-Luh Lin and Pei-Kee Lin, Property (H) in Lebegue-Bochner Function spaces, (preprint).

(7) Robert McGuigan, Strongly extreme points in Banach spaces, Manus. Math. 5, (1971), 113-122.

(8) V.D. Milman, Infinite-dimensional geometry of the unit sphere in a Banach space, Soviet Math. Dokl. 8, (1967), 1440-1444.

(9) Mark A. Smith, Some examples concerning rotundity in Banach spaces, Math. Ann. 233, (1978), 155-161.

(10) Mark A. Smith, A Banach space that is MLUR but not HR, Math. Ann. 256, (1981), 277-279.

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A GENERALIZATION OF AFFINE FUNCTIONS

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Abstract. All 3-place real functions h are determined, for which the existence of a 2-place function s with values strictly between 0 and 1 and of a continuous 1-place function ϕ, satisfying

$$\phi\left[(1-s(x,y))x+s(x,y)y\right] = h\left[\phi(x),\phi(y);s(x,y)\right]$$

on the cartesian square of an interval, implies that

$$\phi\left[(1-\lambda)x+\lambda y\right] = h\left[\phi(x),\phi(y);\lambda\right] \quad \text{for all } \lambda\in[0,1].$$

We determine also all such functions ϕ.

1. Introduction. By the definition of convex functions (see e.g. [5]), the general solution $\phi: D\to\mathbb{R}$ (D is a real interval) of

$$(1)\qquad \phi\left[(1-\lambda)x+\lambda y\right] = (1-\lambda)\phi(x)+\lambda\phi(y) \quad \text{for all } x,y\in D,\ \lambda\in[0,1]$$

is affine ($\phi(x)=\alpha x+\beta$ on D, where α and β are real constants). But the following is also true (see [5] again): If there exists a function $s: D^2\to\,]0,1[$ (no other restrictions) and a continuous $\phi: D\to\mathbb{R}$ such that

$$(2)\qquad \phi\left[(1-s(x,y))x+s(x,y)y\right] = (1-s(x,y))\phi(x)+s(x,y)\phi(y) \quad \text{for all } x,y\in D,$$

then ϕ is affine again. This fact has been used in [3] for another proof of the fact (cf. [4]) that the Korovkin closure of $\{p_1,p_2\}$, where $p_1(x)=x$, $p_2(x)=x^2$, is the space spanned by these two functions.

This situation raises the following questions.

(i). Let J be a real interval. Take a function $h: J^2\times[0,1]\to\mathbb{R}$. For what continuous functions $\phi: D\to J$ (D is an interval) does the following implication hold :

If there exists a function $s: D^2\to\,]0,1[$ such that

$$(3)\qquad \phi\left[(1-s(x,y))x+s(x,y)y\right] = h\left[\phi(x),\phi(y);s(x,y)\right] \quad \text{for all } (x,y)\in D^2,$$

then ϕ satisfies also

$$(4)\qquad \phi\left[(1-\lambda)x+\lambda y\right] = h\left[\phi(x),\phi(y);\lambda\right] \quad \text{for all } (x,y)\in D^2,\ \lambda\in[0,1].$$

(ii). We have again a real interval J. Characterize all functions $h : J^2 \times [0,1] \to \mathbb{R}$ such that every continuous function $\phi : D \to J$ which satisfies (3) with some $s : D^2 \to]0,1[$, satisfies also (4) on $D^2 \times [0,1]$. We will denote the set of all function h having this property by H .

The purpose of this paper is to answer these questions. We will see that H has to be rather special : it is composed of quasilinear means (see e.g. [1]).

Offhand, this seems rather obvious : if ϕ has an inverse f , then (4) can be written as

$$h(u,v;\lambda) = f^{-1}\left[(1-\lambda)f(u)+\lambda f(v)\right] \qquad \text{on } \phi(D)^2 \times [0,1] ,$$

that is, h will indeed be a quasilinear mean there. But, in order that ϕ should have an inverse, we have to prove that it is strictly monotonic (it is already continuous by definition). Also the domains, on which these representations hold, need some clarification and specification.

2. Preliminaries. Let the real interval D_ϕ be the domain of a continuous function ϕ and $I_\phi = \phi(D_\phi)$ its range (image; also an interval). Suppose that this ϕ satisfies the equation (4), more exactly

$$(4') \quad \phi\left[(1-\lambda)x+\lambda y\right] = h\left[\phi(x),\phi(y);\lambda\right] \qquad \text{for all } (x,y)\in D_\phi^2 , \lambda\in[0,1] .$$

We prove that ϕ is strictly monotonic if it is not constant (we use an argument similar to [2]). Indeed, if ϕ is not strictly monotonic then, by the continuity, there exist two points $\lambda_1 \neq \lambda_2$ in D_ϕ where ϕ assumes the same value. We choose an arbitrary μ in the interior of D_ϕ and then a λ so close to 1 that both $\mu + (1-\lambda)(\lambda_2 - \lambda_1)$ and $[\mu - (1-\lambda)\lambda_1]/\lambda$ are still in D_ϕ . By (4'),

$$\phi\left[(1-\lambda)(\lambda_2-\lambda_1)+\mu\right] = \phi\left[(1-\lambda)\lambda_2 + \lambda\,\frac{\mu-(1-\lambda)\lambda_1}{\lambda}\right] =$$

$$= h\left[\phi(\lambda_2),\phi\left(\frac{\mu-(1-\lambda)\lambda_1}{\lambda}\right);\lambda\right] = h\left[\phi(\lambda_1),\phi\left(\frac{\mu-(1-\lambda)\lambda_1}{\lambda}\right);\lambda\right] =$$

$$= \phi\left[(1-\lambda)\lambda_1 + \lambda\frac{\mu-(1-\lambda)\lambda_1}{\lambda}\right] = \phi(\mu) \quad .$$

Thus, starting with any μ in the interior of D_ϕ , the function ϕ is periodic with arbitrary small periods (because λ may be chosen as close to 1 as we wish). Since ϕ is continuous, it has to be constant (also at any endpoint of D_ϕ belonging to D_ϕ, again by the continuity). So ϕ is indeed either strictly monotonic or constant, as asserted.

If $\phi : D_\phi \to I_\phi$ is continuous and strictly monotonic, then it has an inverse $f : I_\phi \to D_\phi = \phi^{-1}(I_\phi)$ and, for all $u,v \in I_\phi$, there exist $x,y \in D_\phi$ such that $\phi(x)=u$, $\phi(y)=v$. So (4') goes over into

$$(5) \qquad h(u,v,\lambda) = f^{-1}\left[(1-\lambda)f(u)+\lambda f(v)\right] \quad \text{for all } (u,v)\in I_\phi^2 \text{ , } \lambda\in[0,1] \quad .$$

If, on the other hand, ϕ is constant, then I_ϕ is a singleton and (5) is trivially satisfied for any continuous strictly monotonic f.

If I_ϕ is not a singleton (ϕ is not constant), then f _in_ (5) _is determined up to an affine tranformation_. Indeed, if

$$f^{-1}\left[(1-\lambda)f(u)+\lambda f(v)\right] = \tilde f^{-1}\left[(1-\lambda)\tilde f(u)+\lambda\tilde f(v)\right] \quad \text{for all } (u,v)\in I_\phi^2, \lambda\in[0,1],$$

then, with $f(u)=x\in D_\phi$, $f(v)=y\in D_\phi$, $\tilde\phi := \tilde f f^{-1}$,

$$\tilde\phi\left[(1-\lambda)x+\lambda y\right] = (1-\lambda)\tilde\phi(x) + \lambda\tilde\phi(y) \quad \text{for all } (x,y)\in D_\phi^2 \text{ , } \lambda\in[0,1] \text{ ,}$$

so that (see [1]) $\tilde f f^{-1}(x) = \tilde\phi(x) = \alpha x+\beta$ for some constants α ($\neq 0$, because $\tilde f f^{-1}$ is strictly monotonic) and β , that is , $\tilde f(u) = \alpha f(u) + \beta$ for all $u\in I_\phi$.

We have proved the following result which may be interesting in itself.

Proposition. If D_ϕ _is a real interval and_ $\phi : D_\phi \to I_\phi = \phi(D_\phi)$ _a continuous function which satisfies_ (4'), _then the restriction of_ h _to_ $I_\phi^2 \times [0,1]$ _is of the form_ (5), _where_ f _is continuous and strictly monotonic. In particular, every continuous non constant solution_ ϕ _of_ (4') _is strictly monotonic. In this case_ $f = \alpha\phi^{-1} + \beta$ ($\alpha \neq 0$, β _arbitrary constants) in_ (5).

Conversely, let I _be a real interval,_ $f : I \to R$ _a continuous and strictly monotonic function and_

$$(6) \qquad h(u,v;\lambda) = f^{-1}\left[(1-\lambda)f(u)+\lambda f(v)\right] \quad \underline{\text{for all}} \ (u,v)\in I^2 \text{ , } \lambda\in[0,1] \quad ,$$

then (4) has a solution and its general continuous solution whose range is a subset of I is given by $\phi(x) = f^{-1}(\alpha x+\beta)$, where α and β are arbitrary constants.

In particular, if (4') has a continuous solution ϕ, then $h|_{I_\phi^2 \times [0,1]}$ is continuous in all variables, strictly increasing in u and v and, if $u \neq v$, then it is also strictly monotonic in λ.

3. Main results. We have obtained a representation of h on I_ϕ^2 (for the time being, we ignore λ which is always in $[0,1]$). We try now to find representations of h on more general domains of which the above cartesian squares will be subsets. It may happen that I_ϕ is a subinterval of an $I_{\phi'}$. Then, in view of the result in Section 2 on the uniqueness of f in the representation (5), we may choose α and β so that ϕ is a restriction of ϕ' and f a restriction of f'. More generally, if $I_\phi \cap I_{\phi'}$ is not empty, then, by the same uniqueness consideration, there exists a continuous and strictly monotonic f defined on $I_\phi \cup I_{\phi'}$ by which h can be represented as

$$h(u,v,\lambda) = f^{-1}\left[(1-\lambda)f(u)+\lambda f(v)\right] \quad \text{for all } (u,v)\in I_\phi^2 \cup I_{\phi'}^2,\ \lambda\in[0,1]$$

(this blending of f anf f' can be done even if $I_\phi \cap I_{\phi'}$ is a singleton).

Finally, if $I_\phi \cap I_{\phi'} = \emptyset$ and cannot be connected by the above procedure, then there may be two equations of the form (5) with different f and f' which need not be restrictions of the same continuous strictly monotonic function: take for instance $I_\phi =]-\pi/2,\pi/2[$, $I_{\phi'} =]\pi/2,3\pi/2[$, $f(u) = \tan u$.

So we take an $h : J^2 \times [0,1] \to \mathbb{R}$ and denote by $\Omega_h \subseteq J^2$ the union of all squares I^2 such that $h|_{I^2 \times [0,1]}$ has a representation of the form (6). (Some I^2's may be points; Ω_h may even be empty).

Further we denote by Φ_h the class of all continuous funtions $\phi : D_\phi \to I_\phi$ for which there exists an $s : D_\phi^2 \to]0,1[$ such that

$$(3')\quad \phi\left[(1-s(x,y))x + s(x,y)y\right] = h\left[\phi(x),\phi(y);s(x,y)\right] \quad \text{for all } (x,y)\in D_\phi^2$$

is satisfied.

If $\phi \in \Phi_h$ is such that $I_\phi^2 \subseteq \Omega_h$ then

$f\phi\left[(1-s(x,y))x+s(x,y)y\right] = (1-s(x,y))f\phi(x)+s(x,y)f\phi(y)$ for all $(x,y)\in D_\phi^2$, that is (cf.(2)), $f\phi(x) = \alpha x+\beta$ on D_ϕ and thus

$f\phi\left[(1-\lambda)x+\lambda y\right] = (1-\lambda)f\phi(x) + \lambda f\phi(y)$ for all $(x,y)\in D_\phi^2$, $\lambda\in[0,1]$,

$\phi\left[(1-\lambda)x+\lambda y\right] = f^{-1}\left[(1-\lambda)f\phi(x)+\lambda f\phi(y)\right] = h\left[\phi(x),\phi(y);\lambda\right]$ on $D_\phi^2\times[0,1]$,

that is, then also (4') is satisfied.

Conversely, if $\phi\in\Phi_h$ and (4') is satisfied, then, by the Proposition, we have $I_\phi^2\subseteq\Omega_h$. So we have the following answer to our original questions:

Theorem. Let J be an interval and $h: J^2\times[0,1]\to R$ a function. A function $\phi: D_\phi\to I_\phi=\phi(D_\phi)$ belonging to Φ_h (i.e. a continuous solution of (3')) is a solution of (4') if and only if $I_\phi^2\subseteq\Omega_h$ (the union of all squares on which the restriction of h is of the form (6)).

Furthermore $h\in H$ if and only if $I_\phi^2\subseteq\Omega_h$ for all $\phi\in\Phi_h$.

Remark. The theorem evidently holds also in the case where $\Phi_h=\emptyset$, that is for h with which (3) has no continuous solutions at all.

The following example shows that there may exist a subset Ω of J^2 such that, for all $I^2\subseteq\Omega$ (I an interval), h has the representation (6) and still there are solutions of (3') which are not solutions of (4').

Take the following subintervals of $J=[0,1]$: $I_1=[0,3/4]$, $I_2=[1/4,1]$. Let $\Omega=I_1^2\cup I_2^2$. Define $h: J^2\times[0,1]\to\mathbb{R}$ by

$$h(u,v;\lambda) = \begin{cases} (1-\lambda)u+\lambda v & \text{for } (u,v)\in\Omega,\ \lambda\in[0,1], \\ 1/2 & \text{for } (u,v)\in J^2\setminus\Omega,\ \lambda\in[0,1]. \end{cases}$$

Clearly this is of the form (6) on Ω. Take, however, $\phi(x)=x$ $(x\in J)$. This function satisfies (3') on J^2 with

$$s(x,y) = \begin{cases} 1/2 & \text{for } (x,y)\in\Omega, \\ (1-2x)/2(y-x) & \text{for } (x,y)\in J^2\setminus\Omega. \end{cases}$$

Obviously $s(x,y)\in]0,1[$. But this $\phi(x)=x$ does not satisfy (4') on all of $J^2\times[0,1]$. - In this case, of course, $I_\phi^2\not\subseteq\Omega$.

However, if we do not require in advance that h be defined <u>on all of</u> $J^2 \times [0,1]$, then the following holds with essentially the same proof.

<u>Theorem 2. Let</u> $A \neq \emptyset$ <u>be an index set,</u> $\Omega = \bigcup_{\alpha \in A} I_\alpha^2$ (<u>all</u> I_α <u>are intervals, some may degenerate to points</u>) <u>and</u> h <u>a real valued function defined now on</u> $\Omega \times [0,1]$ <u>so that</u> $h|_{I_\alpha^2 \times [0,1]}$ <u>is of the form</u> (6) <u>for all</u> $\alpha \in A$. <u>Then every continuous</u> ϕ <u>which satisfies</u> (3) <u>is also a solution of</u> (4).

References

[1] Aczél,J.,Lectures on Functional Equations and Their Applicaions. Academic Press,New Jork-London,1966.

[2] Aczél,J.,On strict monotonicity of continuous solutions of certain types of functional equations. Canad.Math.Bull.9 (1966), 229-232.

[3] Aczél,J.,Related functional equations applied to Korovkin approximation and to characterization of Rényi entropies - links to the uniqueness theory. C.R.Math.Rep.Acad.Sci.Canada 6 (1984), 319-330.

[4] Donner,K.,Extensions of Positive Operators and Korovkin Theorems. (Lecture Notes in Mathematics, vol. 904).Springer,Berlin-Heidelberg-New York, 1982.

[5] Hardy,G.H.-Littlewood,J.E.-Pólya,G., Inequalities. Cambridge University Press, Cambridge, 1934,1952.

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A Note on Skew Transformations

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0 Let E be an n - dimensional vector space over R with an euclidean inner product (,). Let L(E), Sk(E), Sym (E) denote the space of linear, skew, self - adjoint, transformations of E, respectively. For a fixed $\varphi \in L(E)$, denote the characteristic coefficients of φ by $\gamma_p = \gamma_p(\varphi)$:-

(1)... $\det(\varphi + \lambda\iota) = \sum_{p=0}^{n} \gamma_p \lambda^{n-p}$, $\lambda \in R$.

Finally, denote the characteristic polynomials in φ by φ_p :-

(2)... $\varphi_0 = \iota$, $\varphi_p = \gamma_p \iota - \varphi \circ \varphi_{p-1}$, $p = 1, 2, \ldots$.

Observe that the Cayley - Hamilton theorem says that $\varphi_n = 0$.

1 Now suppose that $\varphi \in Sk(E)$. It follows from (1) that $\gamma_p = 0$ for p odd. It follows from (2) that $\varphi_p \in Sk(E)$, for p odd and $\varphi_p \in Sym(E)$, for p even. Since φ commutes with all its characteristic polynomials, (2) also implies that

$$\varphi_p^2 = (\gamma_p \iota - \varphi_{p-1} \circ \varphi) \circ \varphi_p = \gamma_p \varphi_p - \varphi_{p-1} \circ (\gamma_{p+1} \iota - \varphi_{p+1});$$

i.e.,

(3)... $$\varphi_p^2 = \gamma_p \varphi_p - \gamma_{p+1} \varphi_{p-1} + \varphi_{p-1} \circ \varphi_{p+1}$$

Next, let $q = q(\varphi)$ be the largest non - negative integer for which $\gamma_q \neq 0$ and put $\pi = (\gamma_q)^{-1} \varphi_q$. Then q is even and (3) implies

(4) (a) $\varphi_p^2 = \varphi_{p-1} \circ \varphi_{p+1}$, $p > q$;

(b) $\varphi_q^2 = \gamma_q \varphi_q + \varphi_{q-1} \circ \varphi_{q+1}$.

We are now in a position to state our result.

<u>Proposition</u> : <u>With the above hypotheses on</u> $\varphi \in Sk(E)$, <u>we have</u>

(a) $\varphi_p = 0$, $p > q$;

(b) π <u>is the orthogonal projection of E onto the kernel of</u> φ.

<u>Proof:</u> First, recall that the mapping $L(E) \times L(E) \longrightarrow R$, given by

$(\psi_1, \psi_2) = tr(\psi_1 \circ \psi_2^*)$, $\psi_1, \psi_2 \in L(E)$, defines an euclidean inner product on L(E) and, hence, on the subspaces Sk(E) and Sym(E). It follows that

$$(\psi, \psi) = \begin{cases} -tr\,\psi^2, & \text{if } \psi \in Sk(E) \\ tr\,\psi^2, & \text{if } \psi \in Sym(E). \end{cases}$$

In either case, $\psi^2 = 0$ implies that $\psi = 0$.

Therefore part (i) of the Proposition will be proved if we can show that, for p > q, either $\varphi_p = 0$ or $\varphi_p^2 = 0$. But $\varphi_n = 0$. Thus (4)(a) shows that $\varphi_{n-1}^2 = 0$ and hence $\varphi_{n-1} = 0$. An easy induction, based on (4)(a), completes the proof of part (a) of the Proposition.

As regards part (b), we note that (4)(b) shows that $\pi^2 = \pi$ and the fact that q is even implies $\pi^* = \pi$. Thus

π is an orthogonal projection and it only remains to show that $\operatorname{Im}\pi = \ker\varphi$. But the relations

$$\varphi\circ\pi = \pi\circ\varphi = 0 \qquad \text{and} \qquad \gamma_q(\iota - \pi) = \varphi\circ\Phi_{q-1}$$

are immediate consequences of the definition of q and π. These imply that, for $x \in E$,

$$\pi(x) = x \iff \varphi(x) = 0.$$

$\underline{\underline{2}}$ The integer q determined by $\varphi \in Sk(E)$ is the rank of φ (and hence is even).

To see this, first note that

$$rk\,\varphi = n - \dim\ker\varphi = n - \dim\operatorname{Im}\pi = n - tr\,\pi$$

and that $tr\,\pi = n - q$.

The latter relation is a special case of a formula which relates characteristic coefficients and polynomials for <u>any</u> $\varphi \in L(E)$; namely,

$$tr\,\Phi_p = (n-p)\,\gamma_p(\varphi), \qquad p = 0,1,\ldots,n.$$

In fact, this formula is valid when E is a vector space over any field of characteristic 0 (cf. Proposition 7.6.2. of <1>).

References

<1> Greub, W., Multilinear Algebra, Springer, 1978.

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VARIATIONAL INEQUALITIES RELATED WITH A SIGNORINI PROBLEM

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Presented by G.deB. Robinson, F.R.S.C.

ABSTRACT: In this paper, using the fixed point technique, we prove the existence of a unique solution for a class of variational inequalities related with a Signorini Problem.

INTRODUCTION

Variational inequalities have appeared as a natural generalization of the Signorini problem with friction in elastostatics. It is well known that the general problem of equilibrium of elastic bodies in contact with rigid foundations on which frictional forces are developed remains one of the most difficult and interesting problems in solid mechanics. We here consider a variational principle governing a class of contact problems with friction, which involves a class of variational inequalities defined on a set of admissible displacements satisfying the unilateral contact conditions. Formulation of such problems was originally considered and studied by Duvaut and Lions [1], but they were unable to prove the existence of solution of such problems except in special cases.

In this paper, using the fixed point technique of Glowinski, Lions and Tremolieres [2], we prove the existence of a unique solution of a more general class of variational inequalities related with a Signorini problem. We also discuss several special cases which can be derived from this general class of variational inequalities.

PRELIMINARIES

Let H be a Hilbert space with its dual H', whose norm and inner product are denoted by $||\cdot||$ and $(.,.)$ respectively. We denote by $<.,.>$, the pairing between H' and H. In fact, $<f,u> = (\Lambda f,u)$ for all $f \in H'$ and $v \in H$, where Λ is the canonical isomorphism from H' onto H, see [3,7]. Let M be a closed convex set of H. If a (u,v) is a coercive continuous bilinear form on H, that is there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$a(v,v) > \alpha||v||^2, \quad \text{for all } v \in H \qquad (1)$$

and

$$a(u,v) < \beta||u|| \, ||v||, \text{ for all } u, v \in H \qquad (2)$$

Finally, let $j:H \to R$ be a convex, lower semi-continuous and proper functional. If f is a differentiable nonlinear continuous functional on H, then one can show, see Noor [5], that the minimum of $I[v]$ defined by

$$I[v] = \tfrac{1}{2}\, a(v,v) + j(v) - f(v) \tag{3}$$

on M in H can be characterized by a class of variational inequalities of the form

$$a(u,v-u)+j(v)-j(u) \geq \langle f'(u), v-u\rangle, \quad \text{for all } v\in M, \tag{4}$$

where $f'(u)$ is the Frechet derivative of f at $u\in H$.

REMARKS: The variational inequality (4) characterizes the Signorini problem with non-local friction. If the interior of an elastic body is subjected to external forces depending upon the displacement u and part of the body may come into contact with a rigid foundation, then the inequality (4) is simply a statement of the principle of virtual work for an elastic body restrained by frictional forces, assuming that a non-local law of friction holds, see [7]. The $a(u,v-u)$ is the work produced by the stresses through strains caused by the virtual displacement $v-u$. $f'(u)$ is the Frechet differential of f, where f represents the work done by the external forces and the functional $j(v)$ is the work done by the frictional forces. The functional $I[v]$ defined by (3) represents the potential energy associated with the static friction problem for the coulomb law.

The main motivation of this paper is to prove that under certain conditions there does exist a unique solution of a more general variational inequality of which (4) is a special case.

PROBLEM 1: Find $u\in M$ such that

$$a(u,v-u)+j(v)-j(u) \geq \langle A(u), v-u\rangle, \quad \text{for all } v\in M, \tag{5}$$

where A is a nonlinear operator such that $A(u)\in H'$.

We also defined some notions.

Definition: An operator A: $M \to H'$ is called

i. Antimonotone, if $\langle Au-Av,u-v\rangle < 0$, for all $u,v\epsilon M$.

ii. Lipschitz continuous, if there exists a constant $\gamma>0$ such that

$$||Au-Av|| < \gamma\, ||u-v||, \quad \text{for all } u,v\epsilon M.$$

Since $a(u,v)$ is a continuous bilinear form on H, therefore by the Riesz-Frechet representation theorem, we have

$$a(u,v) = \langle Tu,v\rangle, \quad \text{for all } v\epsilon H.$$

It has been shown that $||T|| < \beta$,see [4]. Finally, we define a canonical isomorphism Λ from H' onto H by

$$\langle f,v\rangle = (\Lambda f,v), \quad \text{for all } v\epsilon H,\ f\epsilon H'. \tag{7}$$

Then $||\Lambda||_{H'} = ||\Lambda^{-1}||_H = 1$.

We make the following standard hypothesis, see [6], for details.

Condition N: We assume that $\gamma<\alpha$, where γ is the Lipschitz constant of the nonlinear operator A and α is the coercivity constant of $a(u,v)$.

MAIN RESULT

THEOREM 1: Let $a(u,v)$ be a coercive continuous bilinear form and $j(v)$ be a convex, lower semi continuous functional. If A is Lipschitz continuous antimonotone operator and condition N holds, then there exists a unique solution $u\epsilon M$ such that (5) holds.

PROOF: Uniqueness;

Let $u_1,u_2\epsilon M$ be two solutions of (5), then

$$a(u_1,v-u_1)+j(v)-j(u_1) > \langle A(u_1),v-u_1\rangle \quad \text{for all } v\epsilon M \tag{8}$$

and

$$a(u_2,v-u_2)+j(v)-j(u_2) > \langle A(u_2),v-u_2\rangle \quad \text{for all } v\epsilon M \tag{9}$$

We note that $j(u_i)$, $i=1,2$ is finite, since $j(.)$ being proper, we have $j(u_i) > -\infty$ and moreover that (8) and (9) imply for

$i=1,2$ that

$$j(u_i) \leq a(u_i, v-u_i)+j(v) - \langle A(u_i), v-u_i\rangle \quad \text{for all } v\in M.$$

Since $j \not\equiv \infty$, there exists $v_0 \in M$ such that $j(v_0)<\infty$; taking $v=v_0$ in the above inequality, we deduce that $j(u_i)<\infty$. Under these conditions, we can take $u=u_2$ in (8) and $v=u_1$ in (9) and adding these inequalities, we have

$$a(u_1-u_2, u_1-u_2) \leq \langle A(u_1)-A(u_2), u_1-u_2\rangle \leq 0, \text{ by using the}$$

the antimonotonicity of A.
Thus by the coercivity of $a(u,v)$, we obtain $\alpha||u_1-u_2||^2 \leq 0$,

from which, the uniqueness of the solution of (5) follows.

Existence; We now use the fixed point technique of Glowinski, Lions and Tremolieres [2] to prove the existence of a solution of (5). For given $u\in M$, we consider the auxiliary problem of finding $w\in M$, see [2], satisfying the variational inequality

$$(P) \left\{ \begin{array}{l} (w,v-w)+\rho j(v)-\rho j(w) \geq (u,v-w)+\rho\langle A(u),v-w\rangle- \rho a(u,v-w) \\ \text{for all } v\in M \text{ and some positive } \rho. \end{array}\right.$$

Let w_1,w_2 be two solutions of (P) related to $u_1,u_2 \in M$ respectively. It is enough to show that the mapping $u \to w$ has a fixed point belonging to the closed convex set M in H satisfying (5). In other words, we have to show that for ρ well chosen

$$||w_2-w_1|| \leq \theta\, ||u_2-u_1|| \,,$$

with $0<\theta<1$, where θ is independent of u_1 and u_2.
Taking $v=w_2$(respectively w_1) in (P) related to u_1(respectively u_2), we get

$$(w_1,w_2-w_1)+\rho j(w_2)-\rho j(w_1) \geq (u_1,w_2-w_1)+\rho\langle A(u_1),w_2-w_1\rangle -\rho a(u_1,w_2-w_1)$$

$$(w_2,w_1-w_2)+\rho j(w_1)-\rho j(w_2) \geq (u_2,w_1-w_2)+\rho\langle A(u_2),w_1-w_2\rangle -\rho a(u_2,w_1-w_2)$$

Adding these inequalities and using (6), we obtain

$$(w_1-w_2,w_1-w_2) \leq (u_1-u_2-\rho\Lambda(Tu_1-Tu_2),w_1-w_2)+\rho(\Lambda A(u_1)-\Lambda A(u_2),w_2-w_1)$$

Now using the coercivity, continuity of a(u,v) and Lipschitz continuity of A, we obtain

$$||w_2-w_1||^2 < ||u_2-u_1-\rho A(Tu_2-Tu_1)||\ ||w_2-w_1|| + \rho\gamma||u_2-u_1||\ ||w_2-w_1||,$$

from which, it follows that, see Noor [6] for full details,

$$||w_2-w_1|| < \{\sqrt{(1-2\alpha\ \rho+\rho^2\beta^2)} + \rho\gamma\}\ ||u_2-u_1|| = \theta||u_2-u_1||,$$

where $\theta=\{\sqrt{(1-2\alpha\rho + \rho^2\beta^2)}+\rho\gamma\}< 1$ for $0<\rho< \dfrac{2(\alpha-\gamma)}{\beta^2-\gamma^2}$ and $\rho\gamma<1$

by condition N, which shows that the mapping $u \to w$ defined by (P) has a fixed point, which is a solution of (5), the required result.

Special Cases:

i. It is obvious that for Au=f'(u), the existence of a unique solution of variational inequality (4) follows under the assumptions of theorem 1.

ii. If A is independent of u, that is A(u)=f,(say), then Lipschitz constant γ is zero and problem 1 becomes:

For given $f\epsilon H'$, find $u\epsilon M$ such that

$$a(u,v-u)+j(v)-j(u) > \langle f,v-u\rangle, \quad \text{for all } v\epsilon M,$$

a problem originally considered and studied by Duvaut and Lions.

iii. If the frictional forces, that is j(v)=0, then theorem 1 is exactly the same as proved by Noor [4,5], where one can also find the error estimates for the finite element approximation of a class of variational inequality

$$a(u,v-u) > \langle A(u), v-u\rangle, \qquad \text{for all } v\epsilon M.$$

iv. If j(v)=0 and A is independent of u, i.e., A(u)=f (say), then problem 1 reduces to the classical Signorini problem of elastostatics, that is the analysis of a linear elastic body in contact with a rigid frictionless foundation, a problem studied and investigated by Lions and Stampachhial [3].

Furthermore, if $j(v)=0$ and $M=H$, then we have the classical problem of elasticity and problem 1 becomes:

For given $A(u) \in H'$, find $u \in H$ such that
$a(u,v) = \langle A(u),v \rangle$, for all $v \in H$,

a problem considered and studied by Noor and Noor [6].

REFERENCES

1. Duvaut, G. & Lions, J.L. Les inequations en mecanique et en physique, Dunod, Paris, 1972.
2. Glowinski, R., Lions, J. & Tremolieres, R. Numerical analysis of variational inequalities, North-Holland, 1981.
3. Lions, J. and Stampachhia, G., Variational inequalities Comm. Pure Appl. Math. 20(1967), 493-519.
4. Noor, M.A., Variational inequalities and approximations, Punj. Univ. J. Math. 8(1975), 24-40.
5. Noor, M.A., Mildly nonlinear variational inequalities, Mathematics (Cluj), 24(1982), 99-110.
6. Noor, K.I. and Noor, M.A., A generalization of the Lax-Milgram Lemma, Canad. Math. Bull. 23(1980), 179-184.
7. Oden, J. and Carey, G., Finite Elements, V, Prentice-Hall, N.J. 1984.

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Further Inequalities For Generalized Laguerre Polynomials

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In an earlier paper [3] we established the inquality

$$|L_n^{(\alpha)}(x)| \leq 2^{-\alpha} e^{\frac{1}{2}x}, x \geq 0,\ \alpha \leq 0, \tag{1}$$

for generalized Laguerre polynomials $L_n^{(\alpha)}$. In this paper we establish the inequalities

$$|L_n^{(\alpha)}(x)| \leq q_n\, 2^{-\alpha} e^{\frac{1}{2}x},\ x \geq 0,\ \alpha \leq -\tfrac{1}{2}, \tag{2}$$

and

$$|L_n^{(\alpha)}(x)| \leq 2^{\frac{1}{2}}\, q_n (\alpha+1)_n e^{\frac{1}{2}x}/(\tfrac{1}{2})_n,\ x \geq 0,\ \alpha \geq -\tfrac{1}{2}, \tag{3}$$

where

$$q_n = ((2n)!)^{\frac{1}{2}}/(2^{n+\frac{1}{2}} n!). \tag{4}$$

Since $q_0 = 2^{-\frac{1}{2}}$ and, as is easy to see, $q_{n+1} < q_n$, (2) is stronger than (1) if $\alpha \leq -\frac{1}{2}$. For large n, (2) is much stronger than (1) since from Stirling's formula $q_n \sim (4\pi n)^{-\frac{1}{4}}$ as $n \to \infty$. For large n, (3) is stronger than (1) if $-\frac{1}{2} \leq \alpha < -\frac{1}{4}$ since $2^{\frac{1}{2}} q_n (\alpha+1)_n/(\frac{1}{2})_n \sim (2\pi)^{\frac{1}{2}} n^{\alpha+\frac{1}{4}}/\Gamma(\alpha+1)$ as $n \to \infty$ but for $-\frac{1}{4} \leq \alpha \leq 0$, (3) is weaker than (1), while for $\alpha > 0$ it is weaker than the inequality

$$|L_n^{(\alpha)}(x)| \leq (\alpha+1)_n e^{\frac{1}{2}x}/n! \tag{5}$$

given in [2;10.18(14)], since $(\alpha+1)_n/n! \sim n^{\alpha}/\Gamma(\alpha+1)$ as $n \to \infty$.

Before proving (2) and (3), we wish to make two remarks concerning results used in [3] and which will be used in this paper. The first remark concerns the formula

$$(\Gamma(\mu))^{-1} \int_x^{\infty} (t-x)^{\mu-1} e^{-t} L_n^{(\nu)}(t)\,dt = e^{-x} L_n^{(\nu-\mu)}(x),\ x \geq 0,\ \mathrm{Re}\,\mu > 0,\ \nu \in \mathbb{C}, \tag{6}$$

of which we gave a proof in [3]. We have been informed that, as we suspected, this formula is known; a proof is given in [1; pp, 682-3]. Our second remark concerns the inequality

$$|H_{2m}(x)| \leq k\, 2^m ((2m)!)^{\frac{1}{2}} e^{\frac{1}{2}x^2},\ x \in \mathbb{R} \tag{7}$$

where k is a constant of approximate value 1.086435. We have been informed that this inequality has been improved so that k can be taken to be 1; thus we have

$$|H_{2m}(x)| \leq 2^m ((2m)!)^{\frac{1}{2}} e^{\frac{1}{2}x^2}\ \ x \in \mathbb{R}. \tag{8}$$

Inequalities equivalent to (8) can be found in [4;p. 134] or [5;p.190].

To prove (2) we first note that from [2;10.13(2)]

$$L_n^{(-\frac{1}{2})}(y^2) = (-1)^n H_{2n}(y)/(2^{2n}n!), \tag{9}$$

and thus from (8), if $x \epsilon R$

$$|L_n^{(-\frac{1}{2})}(y^2)| = |H_{2n}(y)|/(2^{2n}n!) \leq ((2n)!)^{\frac{1}{2}} e^{\frac{1}{2}y^2}/(2^n n!) = q_n 2^{\frac{1}{2}} e^{\frac{1}{2}y^2}$$

so that letting $y^2 = x$, if $x \geq 0$

$$|L_n^{(-\frac{1}{2})}(x)| \leq q_n 2^{\frac{1}{2}} e^{\frac{1}{2}x}, \tag{10}$$

which is (2) for $\alpha = -\frac{1}{2}$. Now from (6) with $\mu = -(\alpha+\frac{1}{2})$, $\nu = -\frac{1}{2}$, if $\alpha < -\frac{1}{2}$

$$e^{-x} L_n^{(\alpha)}(x) = (\Gamma(-(\alpha+\tfrac{1}{2})))^{-1} \int_x^\infty (t-x)^{-(\alpha+\frac{1}{2})-1} e^{-t} L_n^{(-\frac{1}{2})}(t)\,dt, \quad x \geq 0 \tag{11}$$

so that using (10), if $x \geq 0$

$$\begin{aligned} |e^{-x} L_n^{(\alpha)}(x)| &\leq (\Gamma(-(\alpha+\tfrac{1}{2})))^{-1} \int_x^\infty (t-x)^{-(\alpha+\frac{1}{2})-1} e^{-t} |L_n^{(\frac{1}{2})}(t)|\,dt \\ &\leq q_n 2^{\frac{1}{2}} (\Gamma(-(\alpha+\tfrac{1}{2})))^{-1} \int_x^\infty (t-x)^{-(\alpha+\frac{1}{2})} e^{-\frac{1}{2}t}\,dt \\ &= q_n 2^{\frac{1}{2}} (\Gamma(-(\alpha+\tfrac{1}{2})))^{-1} e^{\frac{1}{2}x} \int_0^\infty t^{-(\alpha+\frac{1}{2})-1} e^{-\frac{1}{2}t}\,dt \\ &= q_n 2^{-\alpha} e^{\frac{1}{2}x} \end{aligned} \tag{12}$$

and (2) is proved.

To prove (3) we use [2; 10.12(30)] in the form

$$L_n^{(\mu+\nu)}(x) = x^{-(\mu+\nu)} \Gamma(\mu+\nu+n+1)(\Gamma(\nu+n+1)\Gamma(\mu))^{-1} \int_0^x (x-t)^{\mu-1} t^\nu L_n^{(\nu)}(t)\,dt, \tag{13}$$

$$\mathrm{Re}\ \mu > 0,\ \mathrm{Re}\ \nu > -1,\ x > 0.$$

Setting $\mu = \alpha+\frac{1}{2}$, $\nu = -\frac{1}{2}$, we thus have

$$L_n^{(\alpha)}(x) = x^{-\alpha} \Gamma(\alpha+n+1)(\Gamma(n+\tfrac{1}{2}))\Gamma(\alpha+\tfrac{1}{2}))^{-1} \int_0^x (x-t)^{\alpha-\frac{1}{2}} t^{-\frac{1}{2}} L_n^{(-\frac{1}{2})}(t)\,dt, \tag{14}$$

$$\mathrm{Re}\alpha > -\tfrac{1}{2},\ x > 0$$

and thus using (10), if $x > 0$, $\alpha > -\frac{1}{2}$

$$\begin{aligned} |L_n^{(\alpha)}(x)| &\leq x^{-\alpha} \Gamma(\alpha+n+1)(\Gamma(n+\tfrac{1}{2})\Gamma(\alpha+\tfrac{1}{2}))^{-1} \int_0^x (x-t)^{\alpha-\frac{1}{2}} t^{-\frac{1}{2}} |L_n^{(-\frac{1}{2})}(t)|\,dt \\ &\leq x^{-\alpha} q_n 2^{\frac{1}{2}} \Gamma(\alpha+n+1)(\Gamma(n+\tfrac{1}{2})\Gamma(\alpha+\tfrac{1}{2}))^{-1} e^{\frac{1}{2}x} \int_0^x (x-t)^{\alpha-\frac{1}{2}} t^{-\frac{1}{2}}\,dt \end{aligned}$$

$$= q_n 2^{\frac{1}{2}}\Gamma(\alpha+n+1)(\Gamma(n+\tfrac{1}{2})\Gamma(\alpha+\tfrac{1}{2}))^{-1}e^{\frac{1}{2}x}\int_0^1 (1-t)^{\alpha-\frac{1}{2}}t^{-\frac{1}{2}}dt$$

$$= q_n 2^{\frac{1}{2}}\Gamma(\alpha+n+1)\Gamma(\tfrac{1}{2})(\Gamma(n+\tfrac{1}{2})\Gamma(\alpha+1))^{-1}e^{\frac{1}{2}x}$$

$$= 2^{\frac{1}{2}}q_n(\alpha+1)_n e^{\frac{1}{2}x}/(\tfrac{1}{2})_n,$$

so that (3) is proved for $x>0$ and follows by continuity for $x = 0$.

In conclusion we mention that using the methods above we can prove inequalities similar to (2) and (3) for complex α satisfying $\operatorname{Re}\alpha \leq -\frac{1}{2}$ and $\operatorname{Re}\alpha \geq -\frac{1}{2}$ respectively, but as noted in [3], such inequalities are of little interest.

References

1. R. Askey, Dual equations and classical orthogonal polynomials, J. Math. Anal. & App. 24(1968), 677-685.

2. A. Erdelyi et al., Higher transcendental functions II, New York (McGraw-Hill), 1953.

3. P. G. Rooney, an inequality for generalized Laguerre polynomials, C. R. Math. Reports Roy. Soc. Can. 6(1984), 361-4.

4. O. Szasz, On the relative extrema of the Hermite orthogonal functions, J. Ind. Math. Soc. 25(1951), 129-134.

5. G. Szego, Orthogonal polynomials (4th edition), A.M.S. Colloquium Publication XXIII, Providence (A.M.S.), 1975.

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LA CONDITION DES CHAINES ASCENDANTES POUR LES IDEAUX RADICAUX.

par Paulo Ribenboim, F.R.S.C.

Dédié à la mémoire de mon ami Marc KRASNER

Résumé : On donne un exemple pour l'illustrer que la condition des chaînes ascendantes pour les idéaux radicaux peut être vraie pour un anneau, mais fausse pour l'anneau de séries formelles.

Dans son livre "An Introduction to Differential Algebra", Kaplansky démontre le théorème de Ritt & Raudenbush, et il affirme qu'on peut tirer de la démonstration le résultat suivant :

Si A est un anneau commutatif avec unité satisfaisant la condition des chaînes ascendantes pour les idéaux radicaux, alors il en est de même pour $A[X]$.

Dans un travail scolaire, proposé par J.P. Soublin, A. Benhissi (1985) donne une démonstration détaillée de ce fait.

La terminologie suivante est naturelle.

Un idéal I d'un anneau A est <u>radicalement de type fini</u> s'il existe des éléments $a_1,\dots,a_n \in A$ tels que

$I = \sqrt{\sum_{i=1}^{n} Aa_i}$ (le radical de l'idéal engendré par $\{a_1,\dots a_n\}$).

On voit sans peine que A satisfait la condition des chaînes ascendantes pour les idéaux radicaux si et seulement si tout idéal radical de A est radicalement de type fini.

Le but de cette note est de donner un exemple d'un anneau A qui a seulement deux idéaux radicaux, mais dont l'anneau des séries formelles $A[[X]]$ ne satisfait pas la condition des chaînes ascendantes pour les idéaux radicaux.

<u>Exemple</u>. Soit A l'anneau de la valuation v de hauteur 1, telle que :

1) v prolonge la valuation triviale de $\mathbb{Q}$.

2) le groupe de valeurs Γ est un sous-groupe de $\mathbb{R}$ ayant rang rationnel infini (c'est-à-dire $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ est un $\mathbb{Q}$-espace vectoriel de dimension infinie).

Par exemple, on peut prendre A égal à l'anneau des séries formelles généralisées à coefficients dans $\mathbb{Q}$ et à exposants dans $\Gamma = \mathbb{R}$, v étant la valuation définie par l'ordre de la série formelle.

Notons P l'unique idéal premier non nul de A. A a seulement deux idéaux premiers, donc il satisfait la condition de chaînes ascendantes pour les idéaux radicaux.

Dans $A[[X]]$ soit l'idéal $P[[X]]$ de toutes les séries formelles $s = \sum_{n=0}^{\infty} a_n X^n$ avec $a_n \in P$ pour tout $n > 0$. Il est facile de vérifier que $P[[X]]$ est un idéal premier de $A[[X]]$.

Nous introduisons la notation suivante :

Si $s = \sum_{n=0}^{\infty} a_n X^n \in A[[X]]$ soit $v_n(s) = \min_{0 < i < n} \{v(a_i)\}$.

Donc $v_o(s) > v_1(s) > v_2(s) > \ldots \ldots > 0$ et si $s \in P[[X]]$ alors $v_n(s) > 0$ pour tout $n > 0$.

Remarquons : Si $t_1,\ldots,t_h \in P[[X]]$ et $s_1,\ldots,s_h \in A[[X]]$ alors pour tout $n > 0$ on a

$$v_n\left(\sum_{i=1}^{h} s_i t_i\right) > \min_{1<i<h} \{v_n(t_i)\} .$$

En effet, soit $s_i = \sum_{j=0}^{\infty} c_{ij} X^j$, $t_i = \sum_{j=0}^{\infty} d_{ij} X^j$,

donc $\sum_{i=1}^{h} s_i t_i = \sum_{i=1}^{h} \left(\sum_{j+k=m} c_{ij} d_{ik} \right) X^m$, et alors

$$v\Big(\sum_{j+k=m}\ \sum_{i=1}^{h} c_{ij}d_{ik}\Big) \geq \min_{\substack{1\leq i\leq h \\ j+k=m}} \{v(c_{ij}d_{ik})\} \geq \min_{\substack{1\leq i\leq h \\ 0\leq k\leq m}} \{v(d_{ik})\} ;$$

par conséquent $v_n\Big(\sum_{i=1}^{h} s_i t_i\Big) \geq \min_{1\leq i\leq h} \{v_n(t_i)\}$.

Nous voulons montrer que $P[[X]]$ n'est pas radicalement de type fini, c'est-à-dire si $t_1,\dots,t_h \in P[[X]]$ il existe $s \in P[[X]]$ tel que pour tout entier $n \geq 1$ on ait $s^u \notin \sum_{i=1}^{h} A[[X]]t_i$.

Soit $\alpha_n = \min_{1\leq i\leq h} \{v_n(t_i)\}$, donc $\alpha_n > 0$ et $(\alpha_n)_{n\geq 0}$ est une suite décroissante. Choisissons dans Γ une suite $\gamma_0 > \gamma_1 > \gamma_2 > \dots > 0$ avec les propriétés suivantes :

1) $0 < \gamma_0$ et $0 < \gamma_n < \frac{1}{n}\alpha_{n^2}$ pour tout $n \geq 1$

2) les nombres réels γ_n $(n \geq 0)$ sont rationnellement indépendants.

Pour tout $n \geq 0$ soit $c_n \in A$ tel que $v(c_n) = \gamma_n$, et soit $s = \sum_{n=0}^{\infty} c_n X^n \in P[[X]]$.

Soit $u \geq 1$ arbitraire et $s^u = \sum_{m=o}^{\infty} c'_m X^m$, donc

$$c'_m = \sum k_{e_o,\dots,e_m}\, c_o^{e_o} c_1^{e_1} \cdots c_m^{e_m}$$

avec $k_{e_o,\dots,e_m}$ entier positif et

$$(*)\quad \begin{cases} e_o + e_1 + \dots + e_m = u \\ e_1 + 2e_2 + \dots + me_m = m \end{cases} .$$

Remarquons que si $(e_o,\dots,e_m) \neq (e'_o,\dots,e'_m)$ satisfont les conditions indiquées, on a

$$v(k_{e_o,\ldots,e_m} c_o^{e_o} \ldots c_m^{e_m}) \neq v(k_{e'_o,\ldots,e'_m} c_o^{e'_o} \ldots c_m^{e'_m})$$

sinon, puisque $v|\mathbb{Q}$ est la valuation triviale, on aurait

$$\sum_{i=0}^{m} e_i v(c_i) = \sum_{i=0}^{m} e'_i v(c_i) \quad \text{, donc}$$

$$\sum_{i=0}^{m} (e_i - e'_i) v(c_i) = 0 \quad \text{avec} \quad e_i - e'_i \in \mathbb{Z}$$, non tous nuls, ce qui

contredit l'hypothèse d'indépendance rationnelle des nombres réels γ_n (pour $n > 0$) .

Donc $v(c'_m) = \min_{*} \{ \sum_{i=0}^{m} e_i v(c_i) \}$.

Soit $m = u^2$ et $e_u = u$, $e_i = 0$ pour tout $i \neq u$, $0 < i < u^2$.
Alors $(e_o,\ldots,e_{u^2})$ satisfait la condition (*) donc

$$v(c'_{u^2}) < e_u v(c_u) = u\gamma_u < \alpha_{u^2} .$$

Donc

$$v_{u^2}(s^u) < v(c'_{u^2}) < \alpha_{u^2} .$$

Ceci montre que $s^u \notin \sum_{i=1}^{h} A[[X]]t_i$ et conclut la démonstration.

Références.

1. A. Benhissi, Le théorème de base de Hilbert pour les idéaux radicaux, Mémoire de D.E.A., Université de Provence, 1985 (en forme manuscrite).

2. I. Kaplansky, An Introduction to Differential Algebra, Hermann, Paris, 1957.

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