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## Memoirs - Mémoires

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## LIKELIHOOD AND MAXIMUM LIKELIHOOD ESTIMATION

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**Abstract.** The likelihood function is introduced as a measure of plausibility or uncertainty applicable to the estimation of an unknown parameter  $\theta$  on the basis of observations  $X$  whose probabilities are functions of  $\theta$ . Likelihood is contrasted with probability as a measure of uncertainty. The properties, uses, and interpretation of likelihood are discussed and exemplified. The method of maximum likelihood estimation is described and interpreted in terms of the likelihood function.

**Key Words.** Conditional likelihood, marginal likelihood, maximised or profile likelihood, plausibility, sufficient statistic.

## 1. Introduction

The theory of statistical estimation deals with the problem of making statements of uncertainty about unknown quantities or parameters  $\theta$ . If  $\theta$  could be measured directly without error, then, after measurement,  $\theta$  would change from being unknown to being known exactly. This, however, seldom happens in practice. Most commonly quantities  $X = x_1, x_2, \dots, x_n$  can be measured that are related to  $\theta$  only stochastically via a probability function  $f(X; \theta)$ . Then, after observing  $X = X_0$ ,  $\theta$  will change from being unknown to being known with some degree of uncertainty. The purpose of statistical estimation is to quantify the objective amount of uncertainty about  $\theta$  entailed by knowledge of the numerical values of the observations  $X_0$ .

Since probability is historically the oldest measure of uncertainty, it might be thought that the answer to the problem of estimating  $\theta$  would be to obtain a probability density function of  $\theta$  in the light of the observed  $X_0$ . Unfortunately this cannot often be done. For example, if  $x$  is the number of successes in  $k$  independent trials, each of which can result in a success or a failure with probabilities  $\theta$  and  $1 - \theta$ , respectively, then  $f(x; \theta)$  is the binomial probability function,  $f(x; \theta) = \binom{k}{x} \theta^x (1 - \theta)^{k-x}$ ,  $x = 0, 1, 2, \dots, k$ ,  $0 \leq \theta \leq 1$ . Given any numerical value of  $\theta$ , probability statements about  $X$  can be made. However, for fixed  $X_0$ ,  $f(X_0; \theta)$  is *not* by itself a probability function for  $\theta$ , and statements of uncertainty about  $\theta$  using only  $f$  can not be made in terms of probabilities.

## 2. Bayes' theorem, probability statements about $\theta$

If, in addition to  $f(X; \theta)$ , there is given a probability density function  $\pi(\theta)$  for  $\theta$ , called a prior distribution, a probability density function for  $\theta$  on the basis of an observed  $X_0$ , called the posterior distribution, can be found by Bayes' theorem

$$p(\theta | X_0) = \pi(\theta) f(X_0; \theta) / \int_{\Omega} \pi(\theta) f(X_0; \theta) d\theta \quad (1)$$

where  $\Omega$  is the parameter space. The notation  $p(\theta | X)$  means the conditional probability of  $\theta$  given  $X$ , and in (1)  $f(X; \theta)$  is interpreted as the conditional probability of  $X$  given  $\theta$  (that is, as synonymous with  $f(X | \theta)$ ).

This method was first put forward by Bayes (1763), see Barnard (1958), and is discussed extensively by Fisher (1973). The problem of estimation in this case is solved, since (1) is a probability density function for  $\theta$  incorporating all the information about  $\theta$  contained in  $X_0$ . The probability that  $\theta$  lies in any given range  $(a, b)$  can then be determined numerically by integrating  $p(\theta | X_0)$  from  $a$  to  $b$ . Unfortunately, in scientific research the prior distributions required in (1) are rarely known.

### 3. Likelihood

The *likelihood* of  $\theta$  was defined by Fisher (1921) to be a quantity proportional to the probability of observing  $X_0$  from a population specified by  $\theta$ . That is, the likelihood function of  $\theta$  produced by the observed  $X = X_0$  is defined by

$$L(\theta; X_0) = C(X_0)f(X_0; \theta), \quad (2)$$

where  $C(X_0)$  is an arbitrary positive function not involving  $\theta$ . Thus defined, the likelihood function is the contribution of the observed sample  $X_0$  to the determination of the posterior distribution (1).

As emphasized by Fisher, likelihood and probability are quantities of an entirely different nature. The likelihood function is an equivalence class of functions such that two functions  $h(\theta)$  and  $k(\theta)$  are equivalent if and only if their ratio is not a function of  $\theta$ . Since, unlike probabilities, likelihoods are not standardized, only ratios of likelihoods are meaningful.

It is convenient to obtain a unique representation by means of the *relative likelihood function*

$$R(\theta; X_0) = L(\theta; X_0) / \sup_{\theta} L(\theta; X_0) = L(\theta; X_0) / L(\hat{\theta}; X_0), \quad (3)$$

which varies between 0 and 1. The quantity  $\hat{\theta} = \hat{\theta}(X_0)$  which maximizes  $L(\theta; X_0)$  is called the *maximum likelihood (ml) estimate* of  $\theta$ . Since  $f(X; \theta)$  is a probability function, it is necessarily bounded, and so the denominator of (3) exists. For example, if  $x$  has the binomial probability function, the likelihood of  $\theta$  for a specified  $X = x$  is  $L(\theta; x) = C(x)\theta^x(1-\theta)^{n-x}$ . The ml estimate is  $\hat{\theta} = x/n$ , and the resulting relative likelihood is

$$R(\theta; x) = (\theta/\hat{\theta})^x [(1-\theta)/(1-\hat{\theta})]^{n-x} = n^n \theta^x (1-\theta)^{n-x} / x^x (n-x)^{n-x}.$$

To avoid the use of Bayes' theorem with its requirement of a prior distribution, Fisher based his theory of estimation on the likelihood function. This he did initially using maximum likelihood estimation (Sections 10, 11), and later using (3) directly (Section 7). The likelihood function is the basic concept underlying all of non-Bayesian estimation theory. Its shape determines the kind of inferences possible and whether maximum likelihood estimation can be applied.

#### 4. Likelihoods arising from continuous observations

Since measurements can only be made with finite precision, all observations are essentially discrete. However, if the precision of measurement is sufficiently high, it is convenient to represent the resulting observations by a continuous random variate. If  $x$  is a continuous random variate with probability density function  $f(x; \theta)$ , then the contribution of  $x$  to the likelihood function of  $\theta$  is usually taken as proportional to the density function  $f(x; \theta)$ . For example, if the time to failure  $t$  of an item has the exponential distribution with mean  $\theta$ , its density function is  $f(t; \theta) = [\exp(t/\theta)]/\theta$ ,  $\theta > 0$ ,  $t > 0$ . If  $t$  is measured sufficiently precisely, the likelihood function of  $\theta$  based on  $n$  independent observed failure times  $t_1, t_2, \dots, t_n$ , is proportional to (Section 6.1)

$$\prod_{i=1}^n \{[\exp(t_i/\theta)]/\theta\} = [\exp(-T/\theta)]/\theta^n, \quad T = \sum t_i. \quad (4)$$

However, use of the density function involves an approximation to the probability that the observation lies within a small neighbourhood of  $x$ . Failure to appreciate the approximation entailed by the use of continuous random variables has led to misapprehensions. For example,  $n$  observations from the 3-parameter log normal distribution yield a likelihood function proportional to

$$L(\phi, \theta, \sigma) = (1/\sigma)^n \exp\{-\sum [\log(x_i - \phi) - \theta]^2 / 2\sigma^2\} \prod_{i=1}^n (x_i - \phi), \quad (x_i > \phi),$$

which has a singularity at the maximum likelihood estimate  $\hat{\phi} = x_{(1)}$ , the smallest observation,  $\hat{\theta} = \sum \log(x_i - \hat{\phi})/n$ ,  $\hat{\sigma}^2 = \sum [\log(x_i - \hat{\phi}) - \hat{\theta}]^2/n$ . This has been thought to cause theoretical difficulties in the interpretation of likelihoods, and in the definition of the relative likelihood (3) in particular. However, the behaviour near this singularity can be seen by letting  $x_{(1)} - \phi = \epsilon$ . For sufficiently small  $\epsilon$ ,  $L \doteq 1/\epsilon(\log \epsilon)^n$ . This

function increases very slowly, being only 12.6 when  $\epsilon = 10^{-9}$  for  $n = 5$ . Since scientific measurements can rarely be made with that precision, it is merely necessary to take  $\hat{\phi} = x_{(1)} - 10^{-9}$  and there will be no problem with the singularity. See Barnard (1967).

## 5. Likelihood as information

Up to now the likelihood function  $L(\theta; X)$  has been regarded as a function of  $\theta$  for a fixed value of  $X$ . In this Section, it will be considered as a function of  $X$ , for all  $X$  in the sample space  $S$ . Since  $X$  is a random variable, the frequency properties of the likelihood can be examined.

A function  $T(X)$  of the observations is called a *statistic*. Considered as a function of the random variable  $X$ ,  $L(\theta; X)$  is the likelihood function statistic. Its observed value at  $X_0$  is the equivalence class of functions of  $\theta$  (Section 3) to which  $L(\theta; X_0)$  belongs. Thus the set of all  $X$  for which  $L(\theta; X) = L(\theta; X_0)$  identically in  $\theta$  give the same value  $L(\theta; X_0)$  to the likelihood statistic. Denote this set by  $S(X_0)$ .

The likelihood function statistic contains all the information in  $X$  about  $\theta$  in the following sense: the conditional distribution of  $X$  given  $X \in S(X_0)$  is independent of  $\theta$ . For, if  $X \in S(X_0)$ , then  $L(\theta; X) = L(\theta; X_0)$  identically in  $\theta$ , so that, from (2),  $C(X)f(X; \theta) = C(X_0)f(X_0; \theta)$ . Thus the conditional distribution of  $X$  given  $L(\theta; X) = L(\theta; X_0)$  is

$$\begin{aligned} f(X; \theta) / \sum f(X; \theta) &= [C(X_0)f(X_0; \theta) / C(X)] / [\sum C(X_0)f(X_0; \theta) / C(X)] \\ &= [1/C(X)] / [\sum 1/C(X)], \end{aligned}$$

which is independent of  $\theta$ . The summation is over all  $X \in S(X_0)$ . If it is agreed that any probability function  $g(X)$  that does not involve  $\theta$  contains no information about  $\theta$ , then this conditional distribution contains no information about  $\theta$ ; all the sample information about  $\theta$  is contained in the likelihood function statistic. Such a statistic is said to be *sufficient* for the estimation of  $\theta$ .

The argument reverses, showing that  $L(\theta; X)$  must take the same value for any two observations giving the same value to any other sufficient statistic. Thus  $L(\theta; X)$  is minimal sufficient. Any statistic from which  $L$ , together with its distribution, can be determined is sufficient.

The likelihood function, together with its distribution, thus contains all the sample information about  $\theta$ . It represents a complete summary of the evidence in  $X$  about  $\theta$ . A first step in making inferences about  $\theta$  is to examine the likelihood function.

## 6. Properties of likelihood

### 6.1 Combination of independent likelihoods

Since the joint probability of independent events is the product of their individual probabilities, the likelihood of  $\theta$  based on two independent data sets  $X_1, X_2$ , is, from (2), the product of the individual likelihoods based on each data set separately, as in (4). The log likelihoods based on independent data sets are combined by addition to obtain the combined log likelihood based on all the data. For this reason the log likelihood is used more than the likelihood itself.

### 6.2 Functional invariance

Likelihood is functionally invariant (as is probability). By this is meant that any quantitative statement about  $\theta$  implies a corresponding statement about any 1-1 function  $g(\theta)$  by algebraic substitution. For example, whatever is the numerical measure of uncertainty concerning the proposition  $\theta = a$ , exactly the same uncertainty applies to the proposition  $\theta^3 = a^3$ . Hence, if  $\phi = g(\theta)$  and  $R(\theta)$ , is the relative likelihood of  $\theta$ , then solving  $\theta = g^{-1}(\phi)$  and substituting into  $R$  gives  $R[g^{-1}(\phi)]$  as the relative likelihood of  $\phi$ . This simplifies the problem of inference considerably.

### 6.3 Nonadditivity of likelihoods

Likelihood is a point function. This is in sharp distinction to probability, which is a set function. The likelihood of a disjunction " $\theta = a$  or  $\theta = b$ " is not defined. Thus, if a problem contains two parameters  $\theta_1, \theta_2$ , the joint likelihood of  $\theta_1$  and  $\theta_2$  can be obtained as above, since in (2)  $\theta$  need not be a scalar. But the likelihood of  $\theta_1$  alone is not defined. The use of likelihoods to make inferences about subsets of the parameters is thus more difficult, and special techniques must be used. These will be outlined briefly in Section 8.

## 7. The interpretation of likelihood

The likelihood  $L(\theta; X_0)$  is a measure of the *relative* plausibility of  $\theta$  in the light of the observed  $X_0$ . An interpretation of the statement " $L(\theta'; X_0)/L(\theta''; X_0) = 4$ " is that the observed result  $X_0$  would be obtained four times more often if  $\theta'$  were the true value of  $\theta$  than if  $\theta''$  were the true value. In this sense  $\theta'$  is four times more plausible than  $\theta''$ . Similarly, if  $L(\theta'; X_0)/L(\theta''; X_0) = 4$ , and

$$L(\theta'; Y_0)/L(\theta''; Y_0) = L(\theta'; Z_0)/L(\theta''; Z_0) = 1/2,$$

then, from Section 6, on the combined data  $(X_0, Y_0, Z_0)$  the two values  $\theta'$  and  $\theta''$  are equally plausible in the sense that the combined data are equally probable under  $\theta'$  as under  $\theta''$ . Note that the likelihood measures the plausibility based on the observed data only. There may be other reasons for preferring  $\theta'$  over  $\theta''$  or conversely.

The ml estimate  $\hat{\theta}$  is the most plausible value of  $\theta$  in the sense that it makes the observed  $X_0$  most probable. The relative likelihood function  $R(\theta; X_0)$  of (3) measures the plausibility of all specific values of  $\theta$  relative to  $\hat{\theta}$ . In this way  $R(\theta; X_0)$  gives a plausibility ranking varying from zero to unity of all values of  $\theta$ . A graph of  $R(\theta)$  gives a visual summary of the data bearing on values of  $\theta$ .

*Example 1.* Consider the likelihood (4) arising from the exponential distribution. The corresponding relative likelihood (3) is

$$R(\theta) = (\theta/\hat{\theta})^n \exp\{n[1 - (\hat{\theta}/\theta)]\}, \quad \hat{\theta} = T/n. \quad (5)$$

Suppose  $n = 2$  and  $T = 12$ . The following table gives the relative likelihoods of various values of  $\theta$ .

|             |       |       |       |       |       |       |       |
|-------------|-------|-------|-------|-------|-------|-------|-------|
| $\theta$    | 2.0   | 4.0   | 6.0   | 10.0  | 20.0  | 30.0  | 35.0  |
| $R(\theta)$ | 0.164 | 0.828 | 1.000 | 0.801 | 0.365 | 0.198 | 0.154 |

Values of  $\theta$  in the range (1.942, 36.158) have  $R(\theta) \geq .15$ . This is *not* a measure of plausibility or uncertainty attached to the range (1.942, 36.158), since likelihood is not additive (Section 6.3). It is a measure of plausibility attached to *individual* values of  $\theta$  in the range. It is a 15% likelihood interval, and asserts that each individual value of  $\theta$  in the range has a relative likelihood larger than .15, and each individual value of  $\theta$  outside the range has a relative likelihood less than .15.

Further, from the functional invariance of likelihoods (Section 6.2), similar statements can be made about all 1-1 functions of  $\theta$ . For example, the relative likelihoods of the survivor function  $S(\tau) = \exp(-\tau\theta)$ , which is the probability a given item has a lifetime exceeding  $\tau$ , can be immediately obtained. In particular, all values of  $S(\tau)$  between  $\exp(-\tau/1.942)$ ,  $\exp(-\tau/36.158)$  have relative likelihoods of .15 or more. For  $\tau = 50$ , this range is  $0 \leq S(50) \leq .25$ , so that probabilities of more than .25 of a lifetime exceeding 50 have relative likelihoods less than 15%.

A first step in summarizing the evidence in terms of quantitative measures of plausibility of  $\theta$  is therefore to plot  $R(\theta)$ , or to give sets of (nested) likelihood intervals at various levels of relative likelihoods .10,.15,.2,...,1.00 as exemplified above. This will exhibit the changes of plausibility with changes in  $\theta$ . In the above exponential example there is a high degree of asymmetry, values of  $\theta$  to the left of  $\hat{\theta} = 6.0$  being much less plausible than values of  $\theta$  an equal distance to the right of  $\theta$ .  $R(\theta)$  is thus skewed to the right. This complicates the presentation of the evidence, since simple estimation statements of the form  $\theta = \hat{\theta} \pm a$  standard error (or in terms of a "point" estimate and a variance) conceal this asymmetry and are misleading. Only a graph of  $R(\theta)$  or a set of nested intervals converging on  $\hat{\theta}$  will exhibit all the relevant information.

Fisher (1973) and Barnard, Jenkins, and Winsten (1962) gave the first examples of the direct use of likelihoods as a measure of plausibility. The importance of the shape of the likelihood function on the nature of the inference possible has continually been underlined by Barnard, cf. Barnard et. al. (1962), Barnard (1962). For further discussion and examples, see also Barnard (1967), Sprott and Kalbfleisch (1965, 1969), Sprott (1970, 1973a) and Edwards (1972).

## 8. Elimination of parameters

Suppose there are two (or more) parameters  $\theta_1$  and  $\theta_2$  and it is required to make inferences about  $\theta_1$  independently of  $\theta_2$ . If a prior distribution  $\pi(\theta_2)$  for  $\theta_2$  can be assumed (Section 2), a likelihood for  $\theta_1$  alone can be obtained merely by integrating  $\theta_2$  out of the resulting joint density  $f(X; \theta_1, \theta_2)\pi(\theta_2)$  of  $X$  and  $\theta_2$ . When such an assumption is not warranted, so that only the joint likelihood of  $\theta_1, \theta_2$  is available, making inferences about  $\theta_1$  independently of  $\theta_2$  is complicated by the fact that, unlike probabilities, likelihoods are not additive (Section 6.3).

### 8.1 Conditional likelihood

If there exists a 1-1 mapping of the observations  $X \longleftrightarrow (S, T)$  such that the conditional probability of  $S$  given  $T$  is  $g(S; \theta_1 | T)$  independent of  $\theta_2$ , then the probability function of the observations factors

$$f(X; \theta_1, \theta_2) = g(S; \theta_1 | T)h(T; \theta_1, \theta_2). \quad (6)$$

$T$  is a sufficient statistic for  $\theta_2$  for given  $\theta_1$  (Section 5). If the marginal distribution  $h(T; \theta_1, \theta_2)$  of  $T$  can be said to contain no information about  $\theta_1$  in the absence of knowledge of  $\theta_2$ , then  $h$  can be neglected and inferences about  $\theta_1$  alone can be based on  $g$ . The conditional likelihood of  $\theta_1$  is then defined as proportional to  $g(S; \theta_1 | T)$ .

### 8.2 Marginal likelihood

If in the above mapping  $X \longleftrightarrow (S, T)$  it is the marginal probability of  $T$  that is independent of  $\theta_2$ , then the probability function of the observations factors

$$f(X; \theta_1, \theta_2) = g(S; \theta_1, \theta_2 | T)h(T; \theta_1). \quad (7)$$

$T$  is said to be an ancillary statistic for  $\theta_2$  for given  $\theta_1$ . If the information in  $g$  about  $\theta_1$  can similarly be neglected then inferences about  $\theta_1$  can be based on  $h$ . The marginal likelihood of  $\theta_1$  is then defined as proportional to  $h(T; \theta_1)$ .

*Example 2. 2x2 contingency table.* Suppose  $X = (x, y)$  where  $x$  and  $y$  are independent binomial variates with probability function

$$f(x, y, \theta_1, \theta_2) = \binom{m}{x} p_1^x (1-p_1)^{m-x} \binom{n}{y} p_2^y (1-p_2)^{n-y},$$

where  $\theta_1 = \log[p_1/(1-p_1)] - \log[p_2/(1-p_2)]$  is the difference of the log odds and  $\theta_2$  is the corresponding sum of the log odds. Letting  $X \longleftrightarrow (x, t = x + y)$ , this probability function can be factored as in (6) with

$$g(x; \theta_1 | t) = c_x / \sum c_x, \quad c_x = \binom{m}{x} \binom{n}{t-x} \exp(x\theta_1).$$

The summation is over all  $x$  such that  $c_x > 0$ .

Observations quite commonly occur in the form of  $2 \times 2$  tables, as for example in medical trials where the observations are "success" or "failure" of a treatment, and two treatments  $A$  and  $B$  are to be compared on the basis of  $m$  and  $n$  independent trials, respectively. There is an extensive and controversial literature surrounding this problem, mostly centering around the conditioning on  $t$  and the information thereby ignored by neglecting the marginal distribution of  $t$ . See, for example, Yates (1984). However, in practice such information, if any, is negligible, and the above conditional distribution is proportional at least to an approximate conditional likelihood of  $\theta_1$ .

The use of marginal and conditional likelihoods is dependent on the parametric functions being estimated. In the  $2 \times 2$  table for example, the existence of a conditional likelihood depends on measuring the difference between treatments by  $\theta_1$  above. A conditional likelihood cannot be obtained for  $\log p_1/p_2$  or for  $p_1 - p_2$ ; if the difference between treatments is measured by these parameters a different procedure will have to be used. It is noteworthy that the range of  $\theta_1$  is the whole of the real line.

The conditions (6) and (7) for the existence of marginal and conditional likelihoods are highly restrictive. When these are not present, a more generally applicable likelihood is the maximized relative likelihood, or profile likelihood

$$R_M(\theta_1) = f[X; \theta_1, \hat{\theta}_2^*(\theta_1)] / f(X; \hat{\theta}_1, \hat{\theta}_2),$$

where  $\hat{\theta}_2^*(\theta_1)$  is the ml estimate of  $\theta_2$  given a specified fixed  $\theta_1$ . While this exists more generally than do marginal or conditional likelihoods, problems arise with the interpretation of  $R_M(\theta_1)$ , particularly when  $\theta_2$  is a vector containing a large number of unknown parameters. In fact, it must be recognized that in some cases it may be impossible to make inferences about  $\theta_1$  without regard to  $\theta_2$ .

Marginal, conditional and maximized likelihoods, and the elimination of parameters in general, have been discussed and exemplified by Fraser (1968, 1979), Spratt and Kalbfleisch (1965, 1969), Spratt (1970) and Kalbfleisch and Spratt (1970, 1973). The question of the information ignored by the use of  $g$  and  $h$  of (6) and (7) for inferences

about  $\theta_1$  has been examined by Barnard (1963), Barndorff-Nielsen (1973) and Spratt (1975a).

### 9. Approximations to the likelihood function

Reference was made at the end of Section 7 to the effect of the shape of the likelihood function on inferences. The shape is usually determined by the first few terms of its Taylor expansion. Let  $\theta$  be a single scalar parameter with relative likelihood function  $R(\theta; X)$  given by (3). Using (2), the ml estimate  $\hat{\theta}$  is usually a solution of the equation

$$S(X; \theta) = \partial \log f(X; \theta) / \partial \theta = \partial \log R(\theta; X) / \partial \theta = 0. \quad (8)$$

$S(X; \theta)$  is called the score function.

Assuming  $\hat{\theta}$  satisfies (8), and that  $R(\theta)$  satisfies appropriate regularity conditions, which it usually does in practice, expanding  $\log R(\theta)$  in a Taylor series about  $\hat{\theta}$  up to the quartic term gives the approximation

$$\log R(\theta; X) \doteq -(1/2)u_\theta^2 [1 + (u_\theta/3)F_3(\hat{\theta}) - (u_\theta^2/12)F_4(\hat{\theta})], \quad (9a)$$

where

$$u_\theta = (\hat{\theta} - \theta)I_\theta^{1/2}, \quad I_\theta = -\partial^2 \log f(X; \theta) / \partial \theta^2, \quad (9b)$$

$$F_3(\hat{\theta}) = [\partial^3 \log f(X; \theta) / \partial \theta^3] I_\theta^{-3/2}, \quad (9c)$$

$$F_4(\hat{\theta}) = [\partial^4 \log f(X; \theta) / \partial \theta^4] I_\theta^{-2}, \quad (9d)$$

Spratt and Viveros (1984).

If (9c) and (9d) are sufficiently small, then  $R(\theta; X) \doteq \exp(-u_\theta^2/2)$ . This means that  $\theta$  has approximately a normal likelihood centred at  $\hat{\theta}$  with precision or "variance" specified by  $I/I_\theta$  of (9b), that is, a likelihood of the form  $\exp[-I_\theta(\theta - \hat{\theta})^2/2]$ . In this case, the likelihood of  $\theta$  can be represented by a point estimate  $\hat{\theta}$  and its precision  $I_\theta$ . The quantity  $I_\theta$  of (9b) is called the observed Fisher information; the larger it is, the more precise is  $\hat{\theta}$  as an estimate of  $\theta$  in the approximating normal likelihood function of  $\theta$ .

If (9c) and (9d) are not small, there is appreciable asymmetry and thickness of tails (kurtosis) in the likelihood  $R$ . Then the likelihood cannot be adequately summarized by  $(\hat{\theta}, I_{\theta})$ , and the inferences are correspondingly more complicated, as in Example 1.

## 10. Maximum likelihood estimation

The preceding has concentrated on the likelihood function as a whole. Historically the likelihood function arose almost exclusively through maximum likelihood estimation, which was developed by Fisher in the 1920's, cf. Fisher (1925). Maximum likelihood theory deals with the frequency properties of the ml estimate  $\hat{\theta}$ . It can easily be shown that, under suitable regularity conditions, the mean and variance of the score function given by (8) are  $E[S(X; \theta)] = 0$  and

$$\begin{aligned} \text{var}[S(X; \theta)] &= \mathbf{I}(\theta) = E[\partial \log f(X; \theta) / \partial \theta]^2 \\ &= -E[\partial^2 \log f(X; \theta) / \partial \theta^2]. \end{aligned} \quad (10a)$$

The ml equation (8) is called an unbiased estimating equation, since it is of the form  $S(X; \theta) = E[S(X; \theta)]$ . Further, it can be shown that if  $u(X; \theta) = 0$  is any other unbiased estimating equation, standardized to have  $|E[\partial u(X; \theta) / \partial \theta]| = |E[\partial S(X; \theta) / \partial \theta]|$ , then  $\text{var}[u(X; \theta)] \geq \mathbf{I}(\theta)$ , Godambe (1960). The quantity  $\mathbf{I}(\theta)$  of (10a) is called the Fisher information function, and can be compared to the observed Fisher information (9b). These results are exact for samples of any size, and serve to focus attention on ml estimation (8) as the locally most precise method of estimation.

The central limit theorem implies that asymptotically  $S(X; \theta) / \mathbf{I}(\theta)^{1/2}$  is a  $N(0, 1)$  variate, that is,  $S(X; \theta) / \mathbf{I}(\theta)^{1/2}$  converges in distribution to a variate that is normally distributed with mean 0 and variance 1. Further, under suitable regularity conditions on the first three derivatives of the log likelihood, standard text books show that  $S(X; \theta) / \mathbf{I}(\theta)^{1/2}$  is asymptotically equivalent to  $(\hat{\theta} - \theta) \mathbf{I}(\theta)^{1/2}$ , so that as  $n \rightarrow \infty$ ,

$$(\hat{\theta} - \theta) \mathbf{I}(\theta)^{1/2} \rightarrow N(0, 1). \quad (10b)$$

This is the standard textbook theorem on ml estimation, the main purpose of which is to interpret  $\hat{\theta}$  as a point estimate with variance  $1/\mathbf{I}(\theta)$ . These last results are strictly asymptotic. However, the practical use of ml estimation depends on its application to *finite* samples. Asymptotic behaviour gives little guidance about this.

## 11. The interpretation of maximum likelihood estimation

Because of (10b), ml estimation is often interpreted in terms of point estimates and their variances. It is justified as producing point estimates  $\hat{\theta}$  which are asymptotically unbiased, that is  $E(\hat{\theta}) \rightarrow \theta$ , with asymptotically minimum variance. It is not entirely clear to what use such estimates can be put unless they are normally distributed.

Assuming  $u(\hat{\theta}, \theta)$  of (10b) is exactly a  $N(0,1)$  variate, then  $\text{Prob}(-u \leq u(\hat{\theta}, \theta) \leq u)$  can be obtained numerically for any value of  $u$ . Solving the inequality for  $\theta$  gives an interval which contains the true value  $\theta$  with probability determined by the numerical value of  $u$ . If  $u = 1.96$  this probability is .95. That is, in repeated samples the intervals obtained by solving  $-1.96 \leq u(\hat{\theta}, \theta) \leq 1.96$  for  $\theta$  will contain the true value of  $\theta$  with probability .95. Intervals with this coverage frequency property are called *confidence intervals*. However, the above is true only asymptotically, so that its accuracy and validity when applied to finite samples remain to be discussed.

Functions  $u(X; \theta)$  that have a known distribution (like  $N(0,1)$ ) were called *pivotal quantities* by Fisher. Pivotal quantities play a large role in Statistical Inference, Barnard (1977), Barnard and Sprott (1983). In fact, the solution to the problem of estimating  $\theta$  may be said to be a pivotal  $u(X; \theta)$  that is linear in some parametric function  $\phi = \phi(\theta)$  and that is efficient in the sense of reproducing the likelihood function of  $\phi$ , and hence also that of  $\theta$ . For then probability statements about  $u$  can be inverted to produce corresponding statements about  $\phi$  and hence about  $\theta$ . Further, if the probability intervals in  $u$  are shortest (intervals of highest density  $f(u)$ ), the corresponding intervals in  $\phi$  will be likelihood intervals in the sense of Section 7, and hence the resulting intervals in  $\theta$ , or any  $1-1$  function of  $\theta$ , will also be likelihood intervals.

Efficient linear pivotal quantities thus combine the coverage frequency property of confidence intervals with the likelihood property of Section 7, and can thus be called likelihood-confidence intervals. These give measures of plausibility both to the individual points in the interval and to the interval as a whole.

From this viewpoint, a more fruitful interpretation of ml estimation is as a method of producing approximate efficient linear pivotal quantities. Not only does this give ml an operational meaning in terms of likelihood-confidence intervals, it can extend its domain of application to small samples. For there are many functions asymptotically equivalent to (10b), such as  $u_f$  of (9b), but with different statistical behaviours in finite samples. Unlike (10b) and like  $u_f$ , few of them will be expressible as a point estimate and a variance. Further, other functions can be obtained by transforming parameters

$\phi = \phi(\theta)$ . By functional invariance, (Section 6.2), this does not change the estimation problem. But it does change the statistical behaviour of functions like (10b) and  $u_\phi$ , since the information measures (9b) and (10a) are not functionally invariant,  $I_\phi = I_\theta(d\theta/d\phi)^2$ . Some of the resulting functions can be accurate pivotal quantities, even in samples of size  $n = 2$ . The following example illustrates these points.

*Example 3.* In Example 1 the likelihood  $R(\theta)$  is very skewed, and (10a) will not have an approximate  $N(0,1)$  distribution in a sample of size  $n = 2$ .

Consider, however, the parameter  $\phi = \theta^{-1/3}$ . It can easily be verified from (9c) and (9d) that  $F_3(\hat{\phi}) = 0$  and  $F_4(\hat{\phi}) = -2/9n = -.11$ . Thus the relative likelihood of  $\phi$  is approximately normal. The quantity (9b) is  $I_\phi = 9n/\hat{\phi}^2 = 18/\hat{\phi}^2$ , where  $\hat{\phi} = \hat{\theta}^{-1/3} = (T/2)^{-1/3}$ , so that from (9a)  $u_\phi = (\hat{\phi} - \phi)3(2^{1/2})/\hat{\phi}$ . Applying ml estimation to  $\phi$  suggests taking  $u_\phi$  as a  $N(0,1)$  pivotal. This yields approximate confidence intervals  $\phi = \hat{\phi}[1 + u/3(2^{1/2})]$ , which in terms of  $\theta$  are  $\theta = \hat{\theta}[1 + u/3(2^{1/2})]^{-3}$ , where  $u$  is a  $N(0,1)$  variate. Taking  $u = 1.645, 1.960, 2.575$  with  $\hat{\theta} = 6$  gives the .90, .95, and .99 approximate likelihood-confidence intervals as (2.245, 26.141), (1.92, 38.52), and (1.45, 49.40). Similarly, the .15 likelihood interval of Example 1 is an approximate 94.57% confidence interval. Clearly, as in Example 1, corresponding likelihood-confidence intervals can be obtained directly for the survivor function  $S(\tau) = \exp(-\tau/\theta)$ .

Since the exact distribution of  $v = T/\theta$  has density function  $f(v) = v \exp(-v)$  the accuracy of these confidence intervals can be assessed. Based on  $f(v)$ , the exact confidence levels of the above intervals are .8917, .9465, and .9908.

Thus, applying asymptotic ml theory to  $\phi$ , the likelihood of which is approximately normal, can produce highly accurate results even in samples as small as  $n = 2$ . In fact, a large sample can be defined as one for which the resulting likelihood functions, suitably parameterized, are approximately normal. Further, use of the function  $u_\phi$  as an approximate  $N(0,1)$  pivotal is preferable to the use of the estimate  $\hat{\phi}$  and its distribution.

Transformations that normalize the likelihood, like  $\phi$  above, were developed by Anscombe (1964). Their application to ml estimation was examined by Sprott and Kalbfleisch (1969), and Sprott (1973b, 1975b, 1980). The interpretation of ml as a procedure for obtaining approximate pivotal quantities has been discussed and exemplified by Sprott and Viveros (1984, 1985). They also examine the possibility of pivots

having other standard distributions such as Student's  $t$  and log  $F$  to accommodate deviations from normality. This would further extend the use of maximum likelihood to small samples.

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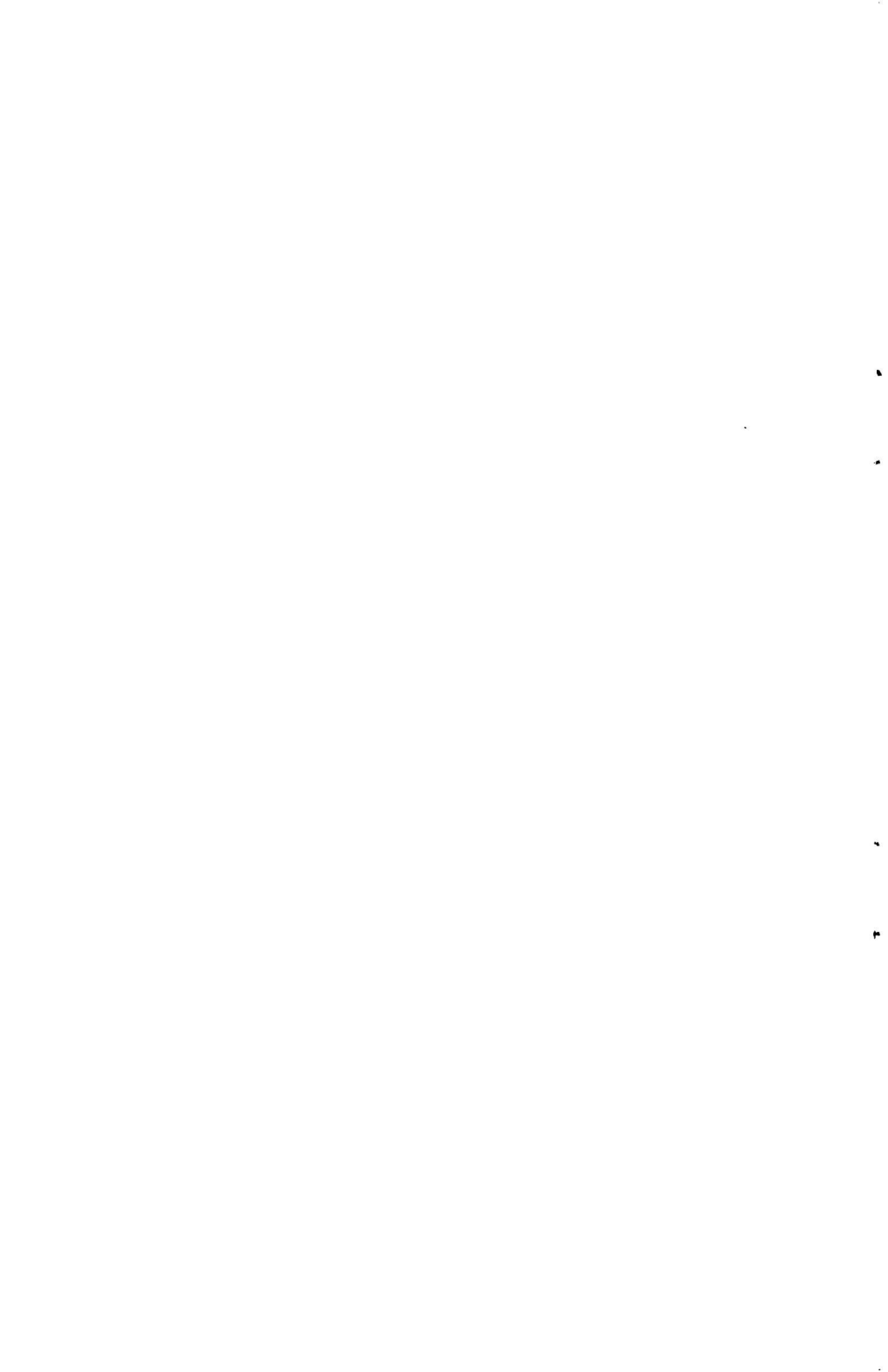
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ZEROS OF CERTAIN TRINOMIALS

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*Presented by P. Ribenboim, F.R.S.C.*

1. Introduction. The zero distribution of the trinomials

$$P_n(z) = \lambda z^n + 1 - \lambda - (\lambda z + 1 - \lambda)^n, \quad 0 < \lambda < 1,$$

depends on  $\lambda$  in an unexpected way.

Theorem. If  $\lambda$  is fixed and  $n$  is sufficiently large, the trinomial  $P_n(z)$  has a zero strictly inside the unit circle  $U$  if and only if  $\lambda$  is not the reciprocal of a positive integer.

2. The most difficult case. This occurs when  $\lambda = 1/q$  with  $q$  an integer greater than 2; we outline how this is proved. By the principle of the argument it suffices to show that the image of  $U$  under  $P_n(z)$  intersects the non-positive real axis  $L$  only when  $z = e^{i\theta} = 1$ , in which case  $P_n(\theta) : = P_n(z) = 0$ . Set

$$(2.1) \quad \rho = \rho(\theta) = |\lambda z + 1 - \lambda|^n, \quad \sigma = \sigma(\theta) = n \arg(\lambda e^{i\theta} + 1 - \lambda),$$

so

$$(2.2) \quad P_n(\theta) = \lambda e^{in\theta} + 1 - \lambda - \rho(\theta)e^{i\sigma(\theta)}.$$

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Whenever  $\rho(\theta) < 1 - 2\lambda$ , the curve  $p_n(\theta)$  is confined inside a circle of radius  $1 - \lambda$  centered at  $(1 - \lambda, 0)$ , so only  $\rho(\theta) \geq 1 - 2\lambda$  concerns us. We need some estimates; in the following and throughout, each occurrence of  $C$  shall represent a possibly different positive constant. From

$$(2.3) \quad n^{-1} \ln(1 - 2\lambda) \leq \ln \rho^{1/n} \leq \frac{1}{2} \ln(1 - C\theta^2) \leq -C\theta^2/2$$

we deduce

$$(2.4) \quad \theta \leq C/\sqrt{n}, \quad \rho(\theta) \leq e^{-Cn\theta^2}.$$

For small values of  $\theta$  (in particular, for large  $n$ ) the arc-tangent expansion and (2.4) yield

$$(2.5) \quad \sigma(\theta) = n\lambda\theta + O(\theta^3 n).$$

3. Idea of the proof. When  $\theta$  is very small the image of  $U$  under  $P_n(z)$  is "well approximated" by the epitrochoid  $E$  defined by

$$(3.1) \quad f(t) = f(t; \rho) = \lambda e^{it/\lambda} + 1 - \lambda - \rho e^{it}, \quad 0 \leq \rho < 1.$$

Since  $\lambda^{-1}$  is integral, we may restrict  $t$  to  $[0, 2\pi]$ . It is easily seen that  $E$  does not intersect  $L$  (if  $\sin t \neq 0$  and  $\text{Im } f(t) = 0$ , then  $\rho = (\sin qt)/q \sin t$ ; hence  $\text{Re } f(t) > 0$  follows from  $\sin(q-1)t/\sin t < q-1$ ).

Lemma. There is a constant  $C = C(\lambda)$  such that

$$(3.2) \quad |f(n\lambda\theta; \rho(\theta)) - p_n(\theta)| \leq Cn^{-1/2}(1 - \rho(\theta))^{3/2}, \quad n \text{ large.}$$

Proof. Clearly

$$|f - p_n| \leq \rho(\theta) \left| \int_{n\lambda\theta}^{\sigma(\theta)} e^{it} dt \right| \leq \rho(\theta) |\sigma(\theta) - n\lambda\theta|,$$

so by (2.5) it suffices to show  $n^2\theta^6 \leq Cn^{-1}(\rho^{-2/3} - \rho^{1/3})^3$ .

By (2.3),

$$\rho^{-2/3} - \rho^{1/3} \geq e^{2Cn\theta^2/3} - e^{-Cn\theta^2/3} \geq Cn\theta^2. \quad \square$$

Next, some rather careful analytic geometry establishes that no element of  $L$  is closer to  $E$  than the origin. Thus, since  $n^{-1/2} \rightarrow 0$ , we only need the following

Theorem. There is a constant  $C = C(q)$  such that

$$(3.3) \quad |f(t; \rho)| \geq C(1 - \rho)^{3/2}, \quad 0 \leq t \leq 2\pi.$$

This can be reduced to three cases:

$$(3.4) \quad (A) \quad 0 \leq t \leq C_1(1 - \rho)^{1/2}, \quad (B) \quad C_1(1 - \rho)^{1/2} \leq t \leq \pi/[2(q - 1)],$$

and

$$(C) \quad \pi/[2(q - 1)] \leq t \leq \pi/2; \quad \text{here } C_1 = (20q)^{-1}.$$

On (A), (C) we can establish respectively

$$(3.5) \quad x = \operatorname{Re} f(t) \geq .5(1 - \rho), \quad y = \operatorname{Im} f(t) \geq C > 0.$$

For (B), observe that

$$(3.6) \quad q(q - 1)^{-1}t^{-3}[x \sin t - y \cos t] \\ = t^{-3}\sin t - t^{-3}(q - 1)^{-1}\sin(q - 1)t =: t^{-3}h(t).$$

On the union of (A) and (B) we have  $h(0) = 0$ ,  $h(t)$

increasing, and  $t^{-3}h(t)$  is continuous but never zero. Hence

$$(3.9) \quad (x^2 + y^2)^{1/2} \geq \max(|x|, |y|) \geq Ct^3 \geq C(1 - \rho)^{3/2}$$

is established for the most critical interval (B). Next, ordinary calculus shows that if  $t > 0$  and  $|f(t)|^2$  is minimal then

$$(3.10) \quad \rho = \frac{\sin qt}{\sin(q-1)t + \sin t} = \frac{\cos(qt/2)}{\cos[(q-2)t/2]}.$$

Thus

$$(3.11) \quad \lambda \sin qt - \rho \sin t = \lambda \rho [\sin(q-1)t - (q-1)\sin t]$$

and

$$(3.12) \quad |y| \geq \lambda \rho |(q-1)\sin \frac{\pi}{2(q-1)} - 1| \geq C$$

must hold on (C). For the interval (A), the power series for the exponential yields

$$(3.13) \quad f(t) = 1 - \rho + i(1 - \rho)t - (q - \rho)t^2/2 + R^*$$

where

$$(3.14) \quad |R^*| \leq \frac{1}{q} \left| \frac{(igt)^3}{3!} + \frac{(igt)^4}{4!} + \dots \right| + \left| \frac{(it)^3}{3!} + \dots \right|.$$

Majorization via geometric series and  $|qt| < .1$  yields

$$(3.15) \quad |R^*| \leq 2q^2t^3 + t^3 \leq 2qt^2/10 + qt^2/10 \leq 3qt^2/10.$$

Hence

$$(3.16) \quad x \geq 1 - \rho - C_1^2(q - \rho)(1 - \rho)/2 - .3qC_1^2(1 - \rho)$$

and (3.5) is proved.

4. Remarks. The "Q-analogue" of the parameter  $\lambda$  is  $A_Q(\lambda) = (Q^\lambda - 1)/(Q - 1)$ ; clearly  $A_Q(\lambda) \rightarrow \lambda$  as  $Q \rightarrow 1$ . This concept arises, for example, in the theory of basic hypergeometric functions, and the "classical inequalities" (e.g. Hölder's) can be derived from relations between  $\lambda$  and  $A_Q(\lambda)$ . Thus it seems natural to ask when  $A_Q(\lambda) = \lambda$ . For  $Q = e^u$  this is

$$e^{\lambda u} = 1 - \lambda + \lambda e^u.$$

Since  $e^x$  is  $\lim(1 + x/n)^n$ , "removal of limits" and the change of variable  $z = 1 + u/n$  leads to our trinomial.

For literature on related equations, see [1-4].

Computer calculations suggest that aside from  $z = 1$ , every zero  $\alpha$  of  $P_n(z)$  satisfies

$$|\alpha| \geq 1 + \frac{.0147}{n} + O\left(\frac{1}{n^2}\right).$$

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FINITE ELEMENT ANALYSIS OF A CLASS OF CONTACT PROBLEMS

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*Presented by G. de B. Robinson, F.R.S.C.*ABSTRACT:

In this paper, we characterize the Signorini problem with friction by a class of variational inequalities. Using the piece-wise linear finite elements, we show that the error estimate for the approximate solution of variational inequalities with friction is of order  $O(h^{\frac{1}{2}})$  in the energy norm. We also discuss several special cases, which can be derived from our main results.

INTRODUCTION:

In many engineering and physical situations, one meets problems, when one deformed body may come into contact with another. It is well known that a mathematical formulation of these problems leads to the use of variational inequalities, which characterize a Signorini problem with friction. Inherent in the friction problem is the free-surface problem of indentifying a priori the unknown contact surface. It is also known that in most cases, the existence of solution to these problems is an open problem. In order to overcome these difficulties, we apply the technique of finite elements coupled with variational inequalities to a class of contact problems with friction in elasto-statics. We remark that the complete study of boundary value problems arising in the formulation of Signorini problem with friction is an interesting and difficult problem from both mathematical and solid mechanics point of view.

2. VARIATIONAL INEQUALITY FORMULATION

Let  $H$  be a real Hilbert space with its dual  $H'$ , whose inner product and norm are denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$  respectively. Let  $M$  be a closed convex set in  $H$  and  $\langle f, v \rangle$ , be a pairing between  $f \in H'$  and  $v \in H$ . Infact,  $\langle f, v \rangle = (Af, v)$ , for all  $f \in H'$  and  $v \in H$ , where  $A$  is the canonical isomorphism from  $H'$  onto  $H$ , see [8,9]. If  $a(u, v)$  is a coercive continuous

bilinear form,  $j(v)$  is a proper, lower semi-continuous convex non-differentiable functional, and  $f$ , a differentiable nonlinear continuous functional on  $H$ , then it is known [1,2] that the minimum of  $I[v]$ , where

$$I[v] = \frac{1}{2}a(v,v) + j(v) - f(v) \quad (1)$$

on  $MCH$  can be characterized by a class of variational inequalities of the type:

$$a(u,v-u) + j(v) - j(u) \geq \langle f'(u), v-u \rangle \quad \text{for all } v \in M, \quad (2)$$

where  $f'(u)$  is the Fréchet differential of the nonlinear functional  $f$  at  $u \in H$ . For physical and mathematical formulation of such problems, see Oden and Pires [3] and Noor [4].

Remark:

It is important to note that the inequality (2) characterizes a Signorini problem with non-local friction. The strain energy of the body corresponding to an admissible displacement  $v$  is  $\frac{1}{2}a(u,v)$ .  $a(u,v-u)$  is the work produced by the stresses through strains caused by the virtual displacement  $v-u$ . Here the functional  $j(v)$  represents the work-done by the frictional forces obeying Coulomb's law, see [3,5] and  $f$  is the external force depending upon the displacement  $u$ . In fact,  $I[v]$  defined by (1) represents the potential energy associated with the statics friction problem for the Coulomb law.

SPECIAL CASES:

- i. If  $f'(u)$  is independent of  $u$ , that is  $f'(u) = F$  (say), then inequality (2) becomes

$$a(u,v-u) + j(v) - j(u) \geq \langle F, v-u \rangle, \quad \text{for all } v \in M,$$

which characterizes a Signorini problem originally formulated and studied by Duvaut and Lions [5].

- ii. If the frictional forces are zero, that is  $j(v) = 0$ , then inequality (2) becomes:

$$a(u,v-u) \geq \langle f'(u), v-u \rangle \quad \text{for all } v \in M, \quad (3)$$

a problem originally introduced and studied by Noor [1] in studying a

class of mildly nonlinear elliptic problems satisfying certain constraint conditions. It is now clear that the classical Signorini problem of elastostatics, that is the analysis of deformation of an elastic body in contact with rigid frictionless foundation can be characterized by a class of variational inequalities of type (3). Error estimates for (3), using piece-wise linear finite elements have been derived by Noor [6] and Janovsky and Whiteman [7].

iii. For  $F \in H'$ , and  $j(v) = 0$ , we get a class of variational inequalities of the type:

$$a(u, v-u) \geq \langle F, v-u \rangle, \quad \text{for all } v \in M,$$

originally studied by Lions and Stampachia [8]. Furthermore, if  $M = H$ , then we have the classical problem of linear elasticity.

It is obvious that the variational inequality (2) introduced in this paper is the most general and unifies all the previous known classes of variational inequalities, which is the main motivation of this paper.

### 3. FINITE ELEMENT APPROXIMATION

In order to derive the error estimates for the approximate solution for variational inequalities, we consider the approximate form of (2). Thus, let  $S_h \subset H$  be a finite dimensional subspace and  $M_h \subset H$  be a finite dimensional convex set. An approximate form of (2) is that of finding  $u_h \in M_h$  such that

$$a(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq \langle f', v_h - u_h \rangle, \quad \text{for all } v_h \in M_h. \quad (4)$$

We also suppose that there exists a Hilbert space  $U$  which is densely and continuously embedded in the dual space  $H'$ . It is then possible to identify  $H$  with a subspace of  $U'$ , dense in  $U'$  by a continuous injection.

With these hypothesis and preliminaries established, we now derive the general error estimate for  $u - u_h$ . Using the technique of [2,3,4], we can prove the following result.

#### THEOREM 1

Let  $u \in M$  and  $u_h \in M_h$  be respectively solutions of (2) and (4). The

mapping  $T:H \rightarrow H'$  is defined for all  $u \in H$  by  $a(u,v) = \langle Tu, v \rangle$ , for all  $v \in H$ , where  $a(u,v)$  is a coercive continuous bilinear form. If the Fréchet differential  $f'(u)$  is antimonotone and Lipschitz continuous and  $Tu - f \in U$ , then

(a) for  $M_h \subset M$ ,

$$\begin{aligned} \|u - u_h\| \leq C [ \|u - u_h\|_H + \|v - u_h\|_H + \|v_h - v\|_H + \{ \|f'(u) - Tu\|_U \|u - v_h\|_U \\ + \|f'(u_h) - Tu_h\|_U \|u_h - v\|_U \}^{\frac{1}{2}} + \{ \|v - u_h\|_H + \|v_h - u\|_H \}^{\frac{1}{2}} ], \\ \text{for all } v \in M, v_h \in M_h. \end{aligned}$$

(b) for  $M_h \subset M$ ,

$$\|u - u_h\| \leq C [ \|u - v_h\|_H + \{ \|f'(u_h) - Tu\|_U \|u - v_h\|_U \}^{\frac{1}{2}} + \{ \|u - v_h\|_H \}^{\frac{1}{2}} ] \\ \text{for all } v_h \in M_h.$$

**PROOF:** Its proof is similar to the one given in [2,3].

#### 4. APPLICATIONS

We consider the following Signorini type contact problem:

$$\left. \begin{aligned} \sigma_{ij}(u)_{,j} &= f_i(u), \quad \sigma_{ij}(u) = E_{ijkl} u_{k,l} \quad \text{in } \Omega \\ u_i &= 0 \text{ on } T_D, \quad \sigma_{ij} n_j = t_i \text{ on } T_F \\ \sigma_n &= 0, \quad \sigma_T = 0, \quad \text{if } u_n < 0 \\ v_F | F_n &= g, \quad |\sigma_T| < g \implies u_T = 0, \\ |\sigma_T| = g &\implies \text{there exists } \lambda > 0 \text{ such that } \\ u_T &= -\lambda \sigma_T \end{aligned} \right\} \quad (5) \quad \text{on } T_C$$

Here  $\Omega$  is the elastic body in a bounded open domain in  $\mathbb{R}^n$  with Lipschitz boundary  $\bar{\Omega} = T_D \cup T_F \cup T_C$ , where  $T_D$  ( $T_F$ ) are portion of  $T$  on which the displacements (tractions) are prescribed and  $T_C$  is the contact surface on which the body may come in contact with the foundation upon the application of loads. It is assumed through that  $\bar{T}_C \cap \bar{T}_D = \emptyset$ .  $u = (u_1, u_2, \dots, u_N)$  is the displacement vector;  $u = u(x)$ , where  $x = (x_1, x_2, \dots, x_N)$  is a point in  $\Omega$ .

- $\sigma_{ij}$  are components of the stress tensor; its value at a displacement  $u$  is  $\sigma_{ij}(u) \stackrel{\text{def}}{=} E_{ijkl} u_{k,l}$ .
- $E_{ijkl}$  are the elasticities of the material of which body is composed, satisfying the usual ellipticity and symmetry conditions.
- $f_i(u)$  are components of body force depending upon the displacement vector  $u$ , assumed to be given in  $L_2(\Omega)$ .
- $t_i$  are components of surface traction, assumed to be given in  $L_\infty(T_F)$ .
- $\sigma_n$  is the normal stress on the boundary, that is
 
$$\sigma_n = \sigma_n(u) = \sigma_{ij}(u) n_i n_j = E_{ijkl} u_{k,l} n_i n_j$$
- $u_n = u \cdot n$  = normal displacement of particles on the boundary  $T$ .
- $s$  is the normalized initial gap between the body  $\Omega$  and the foundation prior to the application of loads.
- $v_F$  is the coefficient of friction, assumed to be a given strictly positive constant.

Then it is clear that solution of (5) can be characterized by a class of variational inequalities of type (2), see [3,4].

Here

$$\begin{aligned}
 H &= \{v \in (H^1(\Omega))^N : \gamma(v) = 0 \text{ a.e. on } T_D\}, \\
 K &= \{v \in H, \gamma(v_i) n_i - s \leq \text{ on } H^{\frac{1}{2}}(T_C)\} \\
 a(u, v) &= \int_{\Omega} E_{ijkl} u_{k,l} v_{ij} d\Omega \\
 f(v) &= \int_{\Omega} f(\eta) d\eta d\Omega + \int_{T_F} t_i v_i ds + \int_{T_C} F_n v_n ds \\
 j(v) &= \int_{T_C} g |v_T| ds = \int_{T_C} v_F |F_n| ds,
 \end{aligned}$$

where  $\gamma$  is the trace operator mapping  $H^1(\Omega)$  onto  $H^{\frac{1}{2}}(T)$  and  $v_T$  is the tangential component of  $v$  on  $T_C$ . For notations and conventions, see [9].

If the coefficient of friction  $v_F = 0$ , then problem (5) is the classical unilateral contact problem without friction, which can be characterized by the variational inequalities of type(3). For more details, see [3,4,9].

If  $s = 0$  and  $T_C^h$  is defined by a connected set of straight line segments in a convex polygon  $\Omega$ , and let  $\{T_h\}_{h>0}$  be a regular family of triangulation  $\Omega$ , see Ciarlet [10] and define

$$S_h = \{b \in \bar{\Omega} : b \text{ is a nodal point on } T_C^h\},$$

where all nodes are situated at the endpoints of the elements. In this case, if the solution  $u$  to the problem (5) is in  $(H^2(\Omega))^2$ , we can take

$$\|u - u_h\|_{r, \Omega} \leq Ch^{2-r}, \quad r = 0, 1$$

and

$$M_h = \{v_h \in S_h : v_h / T_C^h(b) \leq 0, b \in S_h\}.$$

Taking  $U = U' = (L_2(\Omega))^2$ , we obtain  $\|u - u_h\| = O(h^{\frac{1}{2}})$

Furthermore, in the absence of the friction forces, that is  $j(v) = 0$ , we obtain

$$\|u - u_h\| = O(h),$$

the well known results for variational inequalities.

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HALF HENSELIAN VALUATIONS

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*Presented by P. Ribenboim, F.R.S.C.*

**Abstract:** Characterizations of nonhenselian valuations admitting finite-dimensional henselian extensions are given.

The restriction of a henselian valuation of a field (for basic terminology, see [5]) to a subfield of finite codimension need not be henselian. For example, any proper valuation of the complex number field is trivially henselian, but its restriction to the real number field is not [4, Satz 2.2]. This suggests the problem of characterizing valuations  $v$  that are not henselian but have a henselian finite-dimensional extension. The problem has been discussed by Kaplansky and Schilling [3, Theorem 4], Bourbaki [1, Exercise 17, §8, Ch. 6, p. 194], Ribenboim [6, C], and Endler [2]. The results obtained thus far concern only rank one valuations (Bourbaki's exercise is in error) and are subsumed by Endler's theorem [2].

If  $u$  and  $v$  are valuations of a field,  $u < v$  means that the valuation ring of  $u$  strictly contains that of  $v$ . If  $K'$  is an algebraic extension of  $K$ ,  $[K':K]_s$  denotes the dimension over  $K$  of the subfield of  $K'$  consisting of elements separable over  $K$ .

Theorem 1. Let  $v$  be a nonhenselian valuation of a field  $K$ ,  $v'$  a henselian valuation extending  $v$  to an algebraic extension of  $K$  such that  $[K':K]_s < +\infty$ . Then  $K'$  contains a root  $i$

of  $X^2 + 1$ ,  $i \notin K$ , and the restriction of  $v'$  to  $K(i)$  is henselian (and thus is a henselization of  $v$ ). Moreover, there exists a compatible total ordering of  $K$  and a (possibly improper) henselian valuation  $u$  of  $K$  such that  $u < v$ , the restriction of  $u$  to the multiplicative group of all strictly positive elements of  $K$  is a decreasing epimorphism to the value group of  $u$ , the residue field of  $u$  is real-closed, and the residue field of the unique extension  $u'$  of  $u$  to  $K'$  is an algebraically closed field of characteristic zero.

Fields of characteristic zero that admit a complete, rank one valuation  $v$  such that the restriction of  $v$  to every subfield of finite codimension is henselian may be simply characterized:

Theorem 2. Let  $v$  be a complete rank one valuation of a field  $L$  of characteristic zero. The following assertions are equivalent:

- 1° The restriction of  $v$  to every finite-codimensional subfield of  $L$  is henselian.
- 2° Every finite-codimensional subfield of  $L$  is closed.
- 3° The closed subfields  $K$  of  $L$  such that  $L$  is an algebraic extension of  $K$  are precisely the finite-codimensional subfields.
- 4°  $L$  is not algebraically closed.

Definition. A valuation  $v$  of a field  $K$  is half henselian if  $v$  has precisely two extensions to the algebraic closure of  $K$ .

Here are some characterizations of half henselian valuations:

Theorem 3. Let  $v$  be a valuation of a field  $K$ , let  $\Omega$  be an algebraic closure of  $K$ , and let  $v_H$  be a henselization of  $v$  defined on a subfield  $H$  of  $\Omega$ . The following statements are equivalent:

- 1°  $v$  is half henselian.
- 2°  $v$  is not henselian but has only finitely many extensions to  $\Omega$ .
- 3°  $v$  is not henselian, but there is a henselian valuation extending  $v$  to a subfield  $L$  of  $\Omega$  such that  $[L:K] < +\infty$ .
- 4°  $1 < [H:K] < +\infty$ .
- 5°  $K$  does not contain a root  $i$  of  $X^2 + 1$ , and  $H = K(i)$ .
- 6° There is a (possibly improper) henselian valuation  $u$  of  $K$  such that  $u < v$ , the residue field  $k_u$  of  $u$  is a real-closed field, and the valuation induced on  $k_u$  by  $v$  is not henselian.
- 7° There is a real-closed subfield of  $\Omega$  that contains  $K$ ; if  $R$  is any such field,  $v$  has a unique extension  $v_R$  to  $R$ , and  $v_R$  is not henselian.

The existence of a half henselian valuation of a field is equivalent to the existence of a certain kind of henselian valuation:

Theorem 4. A field  $K$  admits a half henselian valuation if and only if it admits a henselian valuation whose residue field is real-closed.

For example, let  $k$  be a real-closed field,  $G$  a totally ordered abelian group, and let  $K = S(k, G)$ , the field of all

functions from  $G$  to  $k$  whose support is a well-ordered subset of  $G$ , with convolution as multiplication. The order valuation  $u$  of  $K$  (if  $f \neq 0$ ,  $u(f)$  is the smallest  $\alpha \in G$  such that  $f(\alpha) \neq 0$ ) is a maximal and hence henselian valuation of  $K$  whose value group is  $G$  and whose residue field is canonically isomorphic to  $k$ . Thus by Theorem 4,  $K$  admits a half henselian valuation  $v$ , and by Theorem 1,  $v$  has a henselian extension  $v'$  to  $K(i)$ . But if  $G$  is not divisible,  $K$  cannot be real-closed nor can  $K(i)$  be algebraically closed. In particular, there exist fields admitting half henselian valuations that are not real-closed; moreover, the incorrectness of [1, Exercise 17, §8, Ch. 6, p. 194] is established.

By the interval topology on a totally ordered field we mean the topology generated by the bounded open intervals.

Theorem 5. If  $R$  is a real-closed field, every topology on  $R$  determined by a proper valuation is also determined by a half henselian valuation. If  $K$  is a field that is not real-closed, the topology determined by a half henselian valuation of  $K$  is the interval topology determined by a compatible total ordering.

We conclude with conditions on a half henselian valuation implying that its underlying field is real-closed:

Theorem 6. If  $v$  is a half henselian valuation of a field  $K$ , then  $K$  is real-closed if  $v$  has rank one, or if the topology determined by  $v$  is not the interval topology determined by a compatible total ordering on  $K$ , or if the value group of  $v$  is divisible.

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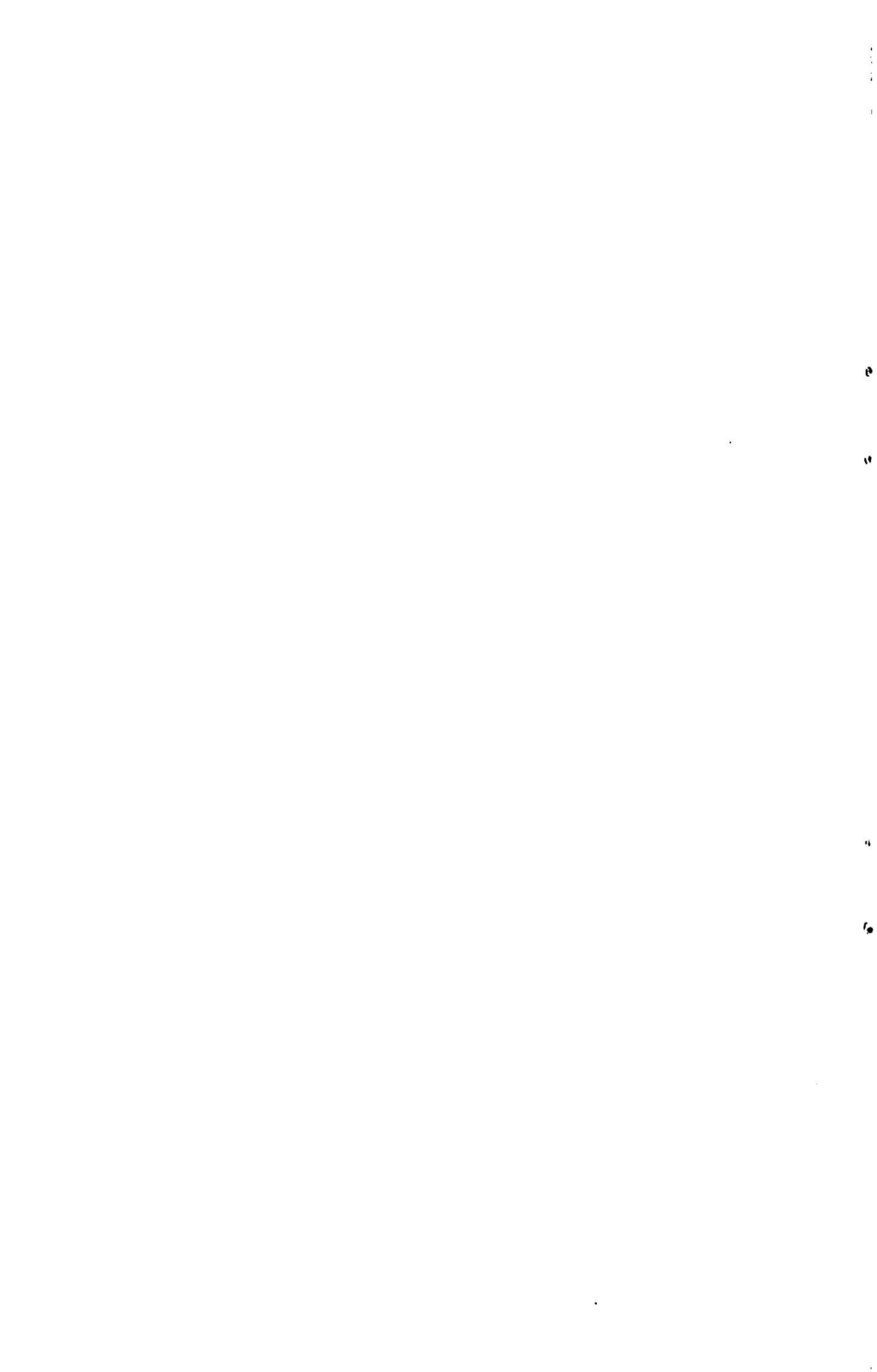
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$H^i(BG^{\text{top}}; \mathbb{Z}/n)$  DOES NOT ALWAYS INJECT INTO  $H^i(BG^\delta; \mathbb{Z}/n)$ 

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*Presented by J.G. Arthur, F.R.S.C.*§1: Introduction

A remarkable result of Suslin [Su 1; Su 2] states that the natural map  $H^i(B(\text{GL}\mathbb{C})^{\text{top}}; \mathbb{Z}/n) \rightarrow H^i(B(\text{GL}\mathbb{C})^\delta; \mathbb{Z}/n)$  is an isomorphism. Here  $(\text{GL}\mathbb{C})^{\text{top}}$  is the infinite general linear group over the complex numbers and  $(\text{GL}\mathbb{C})^\delta$  is the same group with the discrete topology. From this result Karoubi and Jardine (independently) have derived, as a corollary, similar results for the orthogonal and symplectic groups  $[K 1; K 2; J]$ . Further results of this type have been obtained by Karoubi (unpublished) for topological groups related to  $C^*$ -algebras. In fact, it has been conjectured [M 1] that  $BG^\delta \rightarrow BG^{\text{top}}$  should induce isomorphisms on  $H^*(-; \mathbb{Z}/n)$  for an arbitrary Lie group  $G$ . This is supported by some simple cases in [M 1] and by results of Sah in low dimensions.

The question arises: for what compact topological groups,  $G$ , might one expect  $BG^\delta \rightarrow BG^{\text{top}}$  to induce isomorphisms on  $H^*(-; \mathbb{Z}/n)$ ? At once one sees that this map is not always bijective. For example, if  $G = \prod_1^\infty (\mathbb{Z}/p)$  with the product topology then  $H^1(G^{\text{top}}; \mathbb{Z}/p)$  consists of continuous maps  $G^{\text{top}} \rightarrow \mathbb{Z}/p$  which is much smaller than  $\text{Hom}(G; \mathbb{Z}/p) = H^1(G^\delta; \mathbb{Z}/p)$ . Eric Friedlander suggested this example to me. In fact, almost all profinite groups exhibit this phenomenon.

Amending the question next one naturally asks: Is  $H^i(BG^{\text{top}}; \mathbb{Z}/n) \rightarrow H^i(BG^\delta; \mathbb{Z}/n)$  injective for all compact groups,  $G$ ? This note arose in response to Max Karoubi's putting this question to me - the counterexamples below are perhaps not quite what he anticipated!

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1.1: Theorem: There exist profinite groups  $G$  (one such being  $\text{Gal}(\bar{Q}/Q)$ ) such that  $H^2(\text{BG}^{\text{top}}; \mathbb{Z}/2) \rightarrow H^2(\text{BG}^{\delta}; \mathbb{Z}/2)$  is not injective.

I apologise for the arithmetical nature of the counterexample - which may be considered beyond the pale by those wishing for a more topological example.

The proof of Theorem 1.1 is given in §3, §2 being devoted to recalling the arithmetical background we will need.

§2: 2.1: In this section, we will deal with non-singular, symmetric bilinear forms over a field  $k$  with  $\text{char}(k) \neq 2$ . Let  $G(\bar{k}/k)$  denote the profinite Galois group, given by

$$G(k/k) = \varprojlim_{\substack{N/k \\ \text{finite, Galois}}} \text{Gal}(N/K).$$

$H^1(\text{BG}(\bar{k}/k)^{\text{top}}; \mathbb{Z}/2)$  consists of the continuous maps,  $f: G(\bar{k}/k) \rightarrow \mathbb{Z}/2$ , that is, the  $f$  such that  $f(\text{Gal}(\bar{k}/N)) = 0$  for some finite  $N/k$ . We may identify  $H^i(\text{BG}(\bar{k}/k)^{\text{top}}; \mathbb{Z}/2)$  as  $H^i(k; \mathbb{Z}/2)$ , the  $i$ -th Galois cohomology group [Ser 2].

There is an isomorphism

$$\ell: k^*/k^{**} \xrightarrow{\cong} H^1(k; \mathbb{Z}/2) \quad (k^* = k - 0; k^{**} = \text{squares in } k^*)$$

given by  $\ell(a)(g) = g(\sqrt{a})/\sqrt{a} \quad (g \in G(\bar{k}/k), a \in k^*)$ .

2.2: If  $\psi: V \times V \rightarrow k$  is a symmetric, non-singular bilinear form over  $k$ , it may be diagonalised  $(V, \psi) \sim \langle a_1 \rangle \oplus \dots \oplus \langle a_m \rangle$  ( $m = \text{rank } V$ ) where  $\langle a \rangle: k \times k \rightarrow k$  is given by  $(x, y) \mapsto axy$ . Following Delzant [D; Ser 1; Sn] define the  $i$ -th Stiefel-Whitney class of  $(V, \psi)$  by  $w_i(V, \psi) = (\text{the } i\text{-th elementary symmetric function in } \ell(a_1), \dots, \ell(a_m))$ , which lies in  $H^i(k; \mathbb{Z}/2)$ . This definition is independent of the chosen diagonalisation.

2.3: A symmetric bilinear form  $(V, \psi)$ , as in §2.2, is classified by an element of  $H^1(k; O_m \bar{k})$  where  $O_m \bar{k} = \{X \in \text{GL}_m \bar{k}; XX^T = 1\}$ . A 1-cocycle for this group corresponds to a family of homomorphisms ([Ser 3; Sn])

$$\hat{f}_{N,L}: G(N/k) \rightarrow G(N/k) \times O_m(L) \quad (2.4)$$

(for  $k \subset L \subset N$  finite Galois extensions) compatible under the inverse system over  $N$  and the direct system over  $L$  and having the form  $\hat{f}(g) = (g, f(g))$ . If we disregard the topology on  $G(\bar{k}/k)$ , then  $(V, \psi)$  gives rise in this way to a homomorphism  $\hat{f} = (1, f) : G(\bar{k}/k) \rightarrow G(\bar{k}/k) \times O_m(\bar{k})$ . The equivalence relation on cocycles corresponding to boundaries translates into the relation of being conjugate by an element of  $O_m(\bar{k})$  in (2.4).

2.5: If  $L/k$  is a finite separable extension, the non-singular form given by  $\psi : L \times L \rightarrow k$ ,  $\psi(x, y) = \text{Trace}_{L/k}(xy)$  is called the Trace Form of  $L/k$ . By the normal basis theorem, we have a map  $\lambda : G(\bar{k}/k) \rightarrow \Sigma_n$  ( $n = [L:k]$ ) with image  $\pi \cong \text{Gal}(N/k)$ , where  $N$  is the normal closure of  $L/k$ . Of course  $\Sigma_n$  may be considered as a subgroup of  $O_n(\bar{k})$ .

Let  $\langle L \rangle$  denote the trace form of  $L/k$ , then we have

2.6: Lemma [Sn].

In §§2.4/2.5, the trace form,  $\langle L \rangle$ , is represented by

$$\hat{f} = (1, \lambda) : G(\bar{k}/k) \rightarrow G(\bar{k}/k) \times \Sigma_n \rightarrow G(\bar{k}/k) \times O_n(\bar{k}).$$

§3: The proof of Theorem 1.1

We consider trace forms,  $L/k$ , as in §2.5. From [Ser 3; Sn] we have a formula in  $H^2(k; \mathbb{Z}/2)$ ,

$$w_2 \langle L \rangle = \lambda^*(w_2) + \ell(2) \cdot \lambda^*(w_1),$$

where  $w_1, w_2$  are the topological Stiefel-Whitney classes in  $H^*(\Sigma_n; \mathbb{Z}/2)$ .

However, we will show below that the image of  $w_2 \langle L \rangle$  in  $H^2(G(\bar{k}/k)^\delta; \mathbb{Z}/2)$  is  $\lambda^*w_2$  so that  $\ell(2) \cdot \lambda^*w_1$  is in the kernel of  $H^2(G(\bar{k}/k)^{\text{top}}; \mathbb{Z}/2) \rightarrow H^2(G(\bar{k}/k)^\delta; \mathbb{Z}/2)$ . However,  $\lambda^*w_1 = d_L$ , the discriminant of  $L/k$  [Ser 3; Sn] and it is well-known (for example [M 2]) that  $\ell(2)d_L = 0 \in H^2(k; \mathbb{Z}/2)$  if and only if  $d_L$  is a norm from  $k(\sqrt{2})$ . There exist many examples with  $L/k$  number fields (and  $k = \mathbb{Q}$ ) for which this condition is not satisfied (see [C-P; Ser 3]).

Consider the Serre spectral sequence for  $G(\bar{k}/k) \ltimes O(\bar{k})$ , as a discrete group,

$$E_2^{p,q} = H^p(G(\bar{k}/k); H^q(O(\bar{k}); \mathbb{Z}/2)) \rightarrow H^{p+q}(G(\bar{k}/k) \ltimes O(\bar{k}); \mathbb{Z}/2).$$

By [Su 1, Su 2; K 1; K 2; J],  $H^*(O(\bar{k}); \mathbb{Z}/2)$  is detected in the diagonal group  $\prod_{t=1}^{\infty} (\pm 1)^t$ .

Hence the  $E_2$ -term is  $H^p(G(\bar{k}/k); \mathbb{Z}/2) \otimes H^q(O(\bar{k}); \mathbb{Z}/2)$  and, by comparison with  $G(\bar{k}/k) \times (\prod_{t=1}^{\infty} (\pm 1)^t)$ , the spectral sequence collapses. Hence there exists a

unique class  $\hat{w}_2 \in H^2(G(\bar{k}/k) \times O(\bar{k}); \mathbb{Z}/2)$  restricting to  $1 \otimes w_2$  on  $G(\bar{k}/k) \times (\prod_{t=1}^{\infty} (\pm 1)^t)$ .

Clearly, the image of  $w_2 \langle L \rangle$  in  $H^2(G(\bar{k}/k); \mathbb{Z}/2)$  may be defined by  $\hat{f}^*(\hat{w}_2)$ , where  $\hat{f}$  classifies  $\langle L \rangle$ , as in §2.4. Therefore, by §2.6,  $w_2 \langle L \rangle = \lambda^*(w_2) \in H^2(G(\bar{k}/k); \mathbb{Z}/2)$  and the proof is complete.

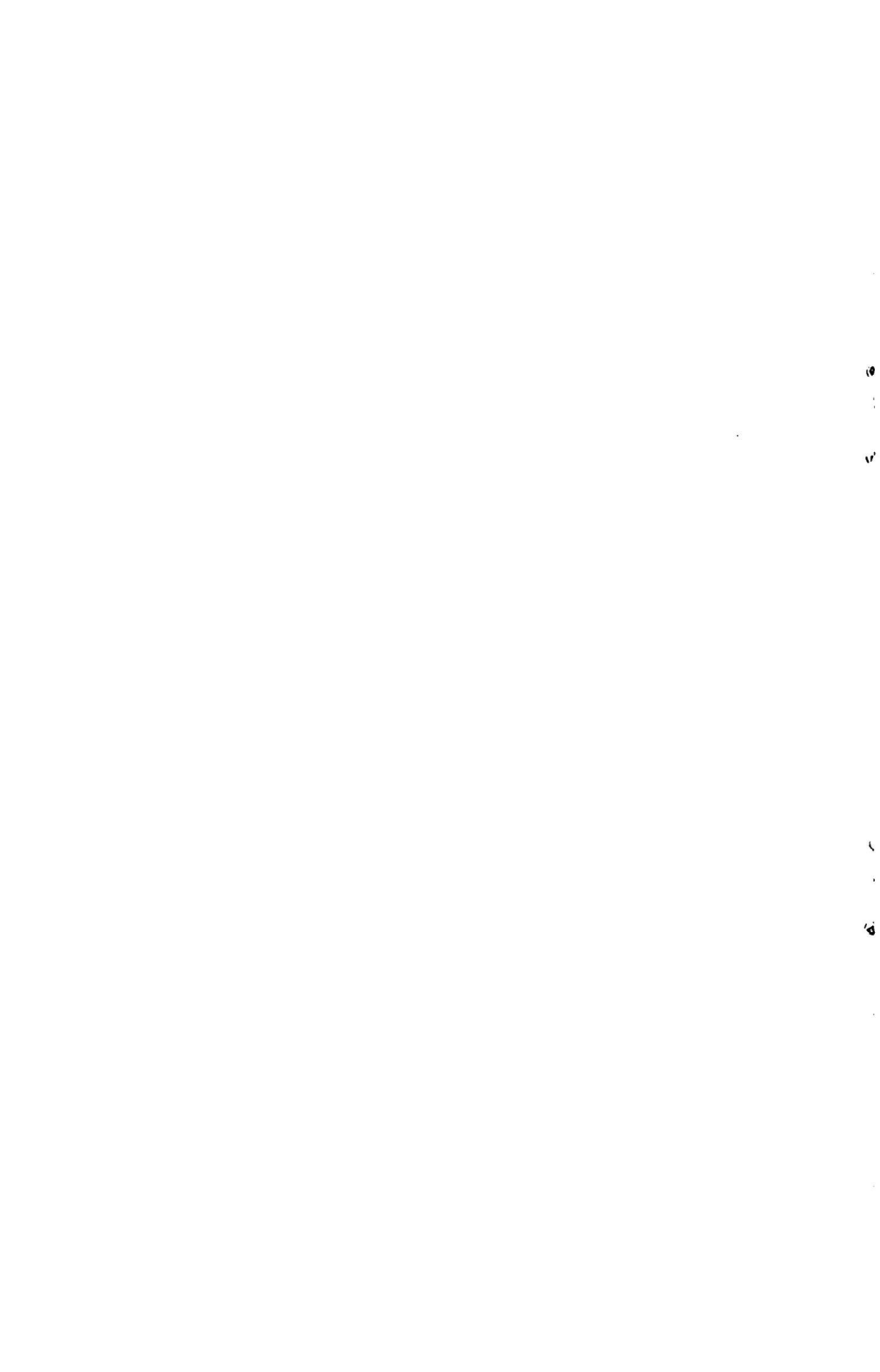
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## ON THE ASYMPTOTIC BEHAVIOR OF THE LEBESGUE CONSTANTS FOR JACOBI SERIES

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*Presented by L.A. Lorch, F.R.S.C.*

**ABSTRACT.** Explicit expressions are obtained for the implied constants in the two  $O$ -terms in Lorch's asymptotic expansions of the Lebesgue constants associated with Jacobi series [Amer. J. Math., 81(1959), 875-888]. Our analysis differs from that of Lorch, and makes use of recently obtained uniform asymptotic expansions for the Jacobi polynomials and their zeros.

The  $n$ -th partial sum of the expansion of an arbitrary function in terms of Jacobi polynomials can be written as an integral involving a kernel analogous to the well-known Dirichlet kernel in the theory of Fourier series. The integral of the absolute value of this kernel is known as the  $n$ -th Lebesgue constant, and has the explicit representation

$$(1) \quad L_n(\alpha, \beta) = \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(n + \beta + 1)} \int_0^\pi (\sin \frac{\theta}{2})^{2\alpha+1} (\cos \frac{\theta}{2})^{2\beta+1} \left| P_n^{(\alpha+1, \beta)}(\cos \theta) \right| d\theta.$$

The behaviour of the sequence  $\{L_n(\alpha, \beta)\}$  is closely connected with the convergence and divergence properties of Jacobi polynomial expansions, and the importance of this sequence has led many mathematicians to be concerned not only with just the asymptotic formula of  $L_n(\alpha, \beta)$ , as  $n \rightarrow \infty$ , but also with its asymptotic expansion.

Rau [7] was the first to show that for  $\alpha > -\frac{1}{2}$  and  $\beta > -1$

$$(2) \quad L_n(\alpha, \beta) = A_{\alpha\beta} n^{\alpha+\frac{1}{2}} + o(n^{\alpha+\frac{1}{2}}), \quad n \rightarrow \infty,$$

where

$$(3) \quad A_{\alpha\beta} = \frac{2}{\pi^{\frac{1}{2}}} \frac{\Gamma(\frac{\alpha}{2} + \frac{1}{4})\Gamma(\frac{\beta}{2} + \frac{3}{4})}{\Gamma(\alpha + 1)\Gamma(\frac{\alpha+\beta}{2} + 1)}.$$

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Later Szegő [11] had an alternative proof of (2), and furthermore showed that

$$(4) \quad L_n\left(-\frac{1}{2}, \beta\right) = \frac{4}{\pi^2} \log n + o(\log n)$$

for  $\beta > -1$ , and that for  $-1 < \alpha < -\frac{1}{2}$  and  $\beta > -1$ ,

$$(5) \quad L_n(\alpha, \beta) = \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\infty \theta^\alpha |J_{\alpha+1}(\theta)| d\theta + o(1),$$

where  $J_{\alpha+1}(\theta)$  is the Bessel function of the first kind.

The above results have been sharpened by Lorch [4,5] particularly in the cases  $\alpha = -\frac{1}{2}$  and  $-\frac{1}{2} < \alpha < \frac{1}{2}$ . For  $-\frac{1}{2} < \alpha < \frac{1}{2}$ ,  $\alpha - \beta < 1$ , and  $\beta > -1$ , Lorch's result can be stated as follows:

$$(6) \quad L_n(\alpha, \beta) = A_{\alpha\beta} n^{\alpha+\frac{1}{2}} + B_\alpha + O(n^{\alpha-\frac{1}{2}}) + O(n^{\alpha-\beta-1}),$$

where

$$(7) \quad B_\alpha = \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \left\{ -M_1(\alpha) + \int_0^{j_1} x^\alpha J_{\alpha+1}(x) dx \right. \\ \left. + 2\alpha \sum_{k=1}^{\infty} (-1)^k \int_{j_k}^{j_{k+1}} x^{\alpha-1} J_\alpha(x) dx + 2 \sum_{k=1}^{\infty} \left[ M_k(\alpha) - \frac{\sqrt{2}}{\pi^{\frac{1}{2}}} \int_{j_{k-1}}^{j_k} x^{\alpha-\frac{1}{2}} dx \right] \right\},$$

both infinite series being absolutely convergent. (Equation (4) in [5] contains two misprints;  $M_{k+1}(\alpha)$  should be replaced by  $M_k(\alpha)$  and  $M_1(\alpha)$  should have a minus sign.) In (7),  $j_n = j_{\alpha+1, n}$  is the  $n$ -th positive zero of  $J_{\alpha+1}(x)$ ,  $n = 1, 2, \dots$ ,  $j_0 = 0$ , and

$$(8) \quad M_k(\alpha) \equiv (-1)^k (j_{\alpha+1, k})^\alpha J_\alpha(j_{\alpha+1, k}) > 0, \quad k = 1, 2, \dots$$

For  $\alpha = -\frac{1}{2}$  and  $\beta > -1$ , Lorch also obtained the result that as  $n \rightarrow \infty$ ,

$$(9) \quad L_n\left(-\frac{1}{2}, \beta\right) = \frac{4}{\pi^2} \log n + C_\beta + O(n^{-1} \log n) + O(n^{-\beta-\frac{3}{2}}),$$

where

$$(10) \quad C_\beta = \frac{8}{\pi^2} \log 2 + \frac{2}{\pi} \int_0^1 \theta^{-1} \sin \theta d\theta \\ - \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} \left( \frac{1 - (\cos \theta)^{\beta+\frac{1}{2}}}{\sin \theta} \right) d\theta - \frac{2}{\pi} \int_1^\infty \theta^{-1} \left\{ \frac{2}{\pi} - |\sin \theta| \right\} d\theta,$$

the last integral being convergent. (There is a typographical error in [5,(8)]; the factor in front of  $\log 2$  should be  $8/\pi^2$ , not  $4/\pi^2$ ).

Lorch's investigation [6] was motivated by a question raised by Szegő concerning the asymptotic monotonicity of the sequence  $\{L_n(0,0)\}$ ; see also the editor's comments at the end of [8]. The result in [5,(6)], however, fails to answer the question of Szegő. Lorch thus posed to us in 1980 the problem of replacing the  $O$ -terms in (6) and (9) by explicitly determined constants plus terms of lower asymptotic order, and the purpose of this note is to present a solution to his problem. The fact that  $L_n(0,0)$  is an asymptotically increasing sequence is an immediate consequence of the result given in (15) below.

In the statement of results which follows, we have kept several  $O$ -terms which can actually be included in others in order to indicate the appropriate orders of succeeding terms in the asymptotic expansions of the Lebesgue constants.

First, we sharpen the result in (6) for the restricted range  $-\frac{1}{2} < \alpha < \frac{1}{2}$  and  $-\frac{1}{2} < \beta < \frac{1}{2}$ . In this case, we have

$$(11) \quad L_n(\alpha, \beta) = A_{\alpha\beta} n^{\alpha+\frac{1}{2}} + B_\alpha + C_{\alpha\beta} A_{\alpha\beta} n^{\alpha-\frac{1}{2}} + \frac{1}{\Gamma(\alpha+1)} D_\beta n^{\alpha-\beta-1} \\ + O(n^{\alpha-\frac{3}{2}}) + O(n^{-2}) + O(n^{\alpha-\beta-2}),$$

where

$$(12) \quad C_{\alpha\beta} = \frac{(\alpha + \beta + 2)(\alpha + \frac{1}{2})}{2}$$

and

$$(13) \quad D_\beta = 2^{-\beta} \sum_{k=1}^{\infty} \left[ \widehat{M}_k(\beta) - \frac{\sqrt{2}}{\pi^{\frac{3}{2}}} \int_{j_{k-1}^{j_k}} x^{\beta+\frac{1}{2}} dx - \frac{\beta + \frac{1}{2}}{\sqrt{2\pi}} \int_{j_{k-1}^{j_k}} x^{\beta-\frac{1}{2}} dx \right].$$

In (13),  $j_n^i = j_{\beta,n}$  is the  $n$ -th positive zero of  $J_\beta(x)$ ,  $n = 1, 2, \dots$ ,  $j_0^i = 0$  and

$$(14) \quad \widehat{M}_k(\beta) \equiv (-1)^k (j_{\beta,k})^{1+\beta} J_{\beta-1}(j_{\beta,k}).$$

In the important particular case of Laplace series (i.e., series in terms of Legendre polynomials at the end point  $x = 1$ ), we have  $\alpha = \beta = 0$ , and (11) becomes

$$(15) \quad L_n(0,0) = \frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} n^{\frac{1}{2}} + \left[ 1 + 2 \sum_{k=1}^{\infty} \left\{ M_k(0) - \frac{2^{\frac{3}{2}}}{\pi} \left[ k^{\frac{1}{2}} - (k-1)^{\frac{1}{2}} \right] \right\} \right] \\ + \sqrt{\frac{2}{\pi}} n^{-\frac{1}{2}} + n^{-1} \sum_{k=1}^{\infty} \left\{ \widehat{M}_k(0) - \frac{2^{\frac{3}{2}}}{3} \left[ k^{\frac{3}{2}} - (k-1)^{\frac{3}{2}} \right] - 2^{-\frac{3}{2}} \left[ k^{\frac{1}{2}} - (k-1)^{\frac{1}{2}} \right] \right\} \\ + O(n^{-\frac{3}{2}}) + O(n^{-2}).$$

The principal term in (15) was first given by Gronwall [2,3], and later by Szegő [9,10] with simpler proofs.

An improved version of (9) is

$$(16) \quad L_n(-\frac{1}{2}, \beta) = \frac{4}{\pi^2} \log n + C_\beta + E_\beta n^{-1} + \frac{1}{\sqrt{\pi}} D_\beta n^{-\beta-\frac{1}{2}} \\ + O(n^{-2} \log n) + O(n^{-2}) + O(n^{-\beta-\frac{1}{2}}),$$

valid for  $-\frac{1}{2} < \beta < \frac{1}{2}$ , where

$$(17) \quad E_\beta = \frac{4}{\pi^2} \left( \frac{\beta + \frac{3}{2}}{2} \right)$$

and  $D_\beta$  is the same constant as given in (13).

Putting  $\alpha = \beta = -\frac{1}{2}$  in (1) gives rise to the case of Tchebycheff polynomials. In [4, p. 757] and [5, p. 877], Lorch has remarked that the  $n$ -th Lebesgue constant in this case has not only the same principal term, but also the same constant term as the corresponding Lebesgue constant in Fourier series; cf. [13, (5)]. In fact, the two Lebesgue constants are identical for all  $n$  and consequently have, term by term, identical asymptotic expansions.

Finally, we turn to the case  $-1 < \alpha < -\frac{1}{2}$ . Under the additional restrictions  $\alpha - \beta > -1$  and  $-\frac{1}{2} < \beta < \frac{1}{2}$ , we have the following sharpened form of (5):

$$(18) \quad L_n(\alpha, \beta) = C_0 + C_1 n^{\alpha+\frac{1}{2}} + C_2 n^{\alpha-\frac{1}{2}} + C_3 n^{\alpha-\beta-1} + O(n^{-2}) + O(n^{\alpha-\frac{3}{2}}),$$

where

$$(19) \quad C_0 = \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha |J_{\alpha+1}(x)| dx,$$

$$(20) \quad C_1 = \frac{4}{\Gamma(\alpha+1)\pi^{\frac{1}{2}}} \left\{ \frac{1}{\alpha + \frac{1}{2}} \left( \frac{\pi}{2} \right)^{\alpha+\frac{1}{2}} + \int_0^{\frac{\pi}{2}} [(\sin \theta)^{\alpha-\frac{1}{2}} (\cos \theta)^{\beta+\frac{1}{2}} - \theta^{\alpha-\frac{1}{2}}] d\theta \right\},$$

and

$$(21) \quad C_2 = \frac{1}{2}(\alpha + \beta + 2)(\alpha + \frac{1}{2})C_1, \quad C_3 = \frac{1}{\Gamma(\alpha+1)} D_\beta,$$

with  $D_\beta$  again being the same constant as given in (13).

Lorch's method essentially consists of replacing the Jacobi polynomial in (1) by its asymptotic formula of "Hilb's type" [12, p. 242], and splitting the interval of integration  $(0, \pi)$  at the points  $j_{\alpha+1, k}/N$ ,  $k = 1, 2, \dots, [N]$ , where  $N = n + \frac{1}{2}(\alpha + \beta + 2)$ . Our approach differs from that of Lorch. We first split the interval  $(0, \pi)$  at the exact zeros of the Jacobi polynomial and then apply recently obtained uniform asymptotic expansions for the Jacobi polynomials and their zeros [1]. Our method may also be extended to give higher order approximations when desired. The details of our analysis will appear elsewhere.

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IRREDUCIBILITY AND ZEROS  
OF GENERALIZED BERNOULLI POLYNOMIALS

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**Abstract.** It is shown that generalized Bernoulli polynomials belonging to a certain class of quadratic characters are irreducible. The distribution of real zeros of generalized Bernoulli polynomials is studied; this is applied to investigate a class of Diophantine equations.

1. **Definitions.** Let  $\chi$  be a primitive (residue class) character with conductor  $f$ . The complex numbers  $B_{\chi}^n$ , defined by

$$\sum_{a=1}^f \frac{\chi(a) t e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} \frac{B_{\chi}^n}{n!} t^n,$$

are called generalized Bernoulli numbers belonging to  $\chi$ . The polynomials

$$B_{\chi}^n(x) = \sum_{s=0}^n \binom{n}{s} B_{\chi}^s x^{n-s}$$

are the generalized Bernoulli polynomials belonging to  $\chi$ .

These numbers and polynomials have properties similar to those of the (classical) Bernoulli numbers and polynomials; the latter can be seen (with some minor differences) as a special case of the generalized numbers and polynomials for  $\chi = \chi_0$ , the principal character. For basic properties, see e.g. [1] or [2].

2. Irreducibility. The question of irreducibility of Bernoulli polynomials is a difficult and in general still unsolved problem. However, some classes of generalized Bernoulli polynomials behave more nicely.

If  $\chi$  is a quadratic character then all  $B_{\chi}^n(x)$  have rational coefficients, and it makes sense to talk about irreducibility over  $\mathbb{Q}$ . In this case we have

Proposition 1: Let  $\chi$  be a quadratic character with conductor  $f = p$  (prime), where  $p \equiv 3 \pmod{8}$ . Then

- (a)  $B_{\chi}^n$  is irreducible for odd  $n \geq 1$ ,  
 (b)  $x^{-1} B_{\chi}^n(x)$  is irreducible for even  $n \geq 2$ .

Sketch of proof: (a) Using the recursion formula for generalized Bernoulli numbers (see e.g. [2, Th.1.1]), we find

$$(i) \quad B_{\chi}^1 \not\equiv 0 \pmod{2}$$

$$(ii) \quad B_{\chi}^3 \equiv B_{\chi}^5 \equiv B_{\chi}^7 \equiv B_{\chi}^9 \equiv B_{\chi}^{11} \equiv 2 \pmod{4}$$

Using a generalized analogue to the classical Kummer congruences (see [2, Th.1.8]), we get

$$3 \cdot B_{\chi}^n = 3 \cdot B_{\chi}^{n+8} \pmod{4} \quad \text{for } n \geq 4.$$

Hence with (ii) we get  $B_{\chi}^k = 2 \pmod{4}$  for all odd  $k \geq 3$ .

Now we apply Eisenstein's irreducibility criterion to

$$B_{\chi}^n(x) = \binom{n}{1} B_{\chi}^1 x^{n-1} + \binom{n}{3} B_{\chi}^3 x^{n-3} + \dots + \binom{n}{n-2} B_{\chi}^{n-2} x^2 + B_{\chi}^n$$

(note that  $B_{\chi}^j = 0$  for  $j$  even if  $\chi$  is an odd character). Since  $\binom{n}{1} = n$  is odd, the leading coefficient of  $B_{\chi}^n(x)$  is not divisible

by 2. All the other coefficients are, and  $B_x^n \equiv 2 \pmod{4}$ . Hence  $B_x^n(x)$  is irreducible. The proof of (b) is similar.

3. Zeros. The real zeros of the (classical) Bernoulli polynomials have been studied by J. Lense, D.H. Lehmer, and by K. Inkeri. Although Inkeri's results are somewhat stronger, I shall quote Lehmer's main result [4].

$B_{2k}(x)$  has two zeros  $\alpha_k, \alpha_k'$  in the interval  $(0, 1)$ , where  $\alpha_k' = 1 - \alpha_k$ , and

$$\frac{1}{4} - \frac{1}{2^{2k+1}\pi} < \alpha_k < \frac{1}{4}.$$

The generalized Bernoulli polynomials have similar properties. Let  $\delta = 1$  if  $\chi$  is odd, and  $\delta = 0$  if  $\chi$  is even. Then the analogue to  $B_m(x)$  having even, respectively odd index is  $B_x^m(x)$  with  $m \equiv \delta \pmod{2}$ , resp.  $m \not\equiv \delta \pmod{2}$ . First we regard the case  $m \equiv \delta \pmod{2}$ .

Proposition 2: Let  $\chi$  be a primitive quadratic character with odd conductor  $f$ , and let  $m = 2k - 1$  if  $\chi$  is odd,  $m = 2k$  if  $\chi$  is even. Then there is a zero  $\alpha_x^m$  of  $B_x^m(x)$  such that (for  $m$  sufficiently large)  
 (a) if  $\chi$  is odd and  $f \equiv 7 \pmod{8}$ , or  $\chi$  is even and  $f \equiv 1 \pmod{8}$ , then

$$\frac{f}{4} - \frac{f}{\pi \cdot 2^{m+1}} < \alpha_x^m < \frac{f}{4};$$

(b) if  $\chi$  is odd and  $f \equiv 3 \pmod{8}$ , or  $\chi$  is even and  $f \equiv 5 \pmod{8}$ , then

$$\frac{f}{4} < \alpha_x^m < \frac{f}{4} + \frac{f}{\pi \cdot 2^{m+1}}.$$

Sketch of proof: We use the Fourier expansion for the generalized

Bernoulli polynomials (see [1, p.421]; Berndt uses it as definition) to show that  $B_x^m(f/6)$  and  $B_x^m(f/4)$  in case (a), or  $B_x^m(f/4)$  and  $B_x^m(f/3)$  in case (b) have different signs if  $m$  is large enough. Hence there is a zero in the interval  $(f/6, f/4)$ , resp.  $(f/4, f/3)$ . Now we follow Lehmer's method in [4] to get the result.

While the (classical) Bernoulli polynomials are symmetric (or anti-symmetric) around  $1/2$ , the generalized Bernoulli polynomials are symmetric (or anti-symmetric) around 0. So the analogue to  $B_{2k-1}(x)$  having zeros at  $x=0$  and  $x=1$  would be the vanishing of  $B_x^m(x)$ , for  $m \not\equiv \delta \pmod{2}$ , at  $f/2$  and at  $-f/2$ . However, this is not quite the case; instead, we have

Proposition 3: Let  $\chi$  be a primitive quadratic character with odd conductor  $f$ , and let  $m=2k$  if  $\chi$  is odd,  $m=2k-1$  if  $\chi$  is even. Then, if  $m$  is sufficiently large, there are zeros  $\beta_x^m$  of  $B_x^m(x)$  such that

$$(a) \quad \frac{f}{2} - \frac{\pi^{m-1} f^{3/2}}{24(m-1)!} < \beta_x^m < \frac{f}{2} \quad \text{if } k \text{ is even,}$$

$$(b) \quad \frac{f}{2} < \beta_x^m < \frac{f}{2} + \frac{\pi^{m-1} f^{3/2}}{24(m-1)!} \quad \text{if } k \text{ is odd.}$$

Sketch of proof: We first show that there is a zero between  $f/3$  and  $f/2$  if  $k$  is even, and between  $f/2$  and  $2f/3$  if  $k$  is odd, again using the Fourier expansions. Then we apply the method of Lehmer again.

Finally, we prove a statement about the uniqueness of the zeros in Propositions 2 and 3.

Proposition 4: Let  $\chi$  be a primitive quadratic character with odd conductor  $f$ . Then, if  $m$  is sufficiently large, the only zeros of  $B_x^m(x)$  in the interval  $[-2f/3, 2f/3]$  are

$$\begin{aligned} & \pm \alpha_x^m && \text{if } m \equiv \delta \pmod{2}, \\ & 0 \text{ and } \pm \beta^m && \text{if } m \not\equiv \delta \pmod{2}. \end{aligned}$$

These are simple zeros.

4. Diophantine equations. Using properties of the zeros of Bernoulli polynomials, and some deep general theorems on Diophantine equations, K. Györy, R. Tijdeman, and M. Voorhoeve [5] proved the following

Theorem: Let  $R(x)$  be a fixed polynomial in  $\mathbb{Z}[x]$ . Let  $b \neq 0$  and  $k \geq 2$  be fixed integers such that  $k \notin \{3, 5\}$ . Then the equation

$$1^k + 2^k + \dots + x^k + R(x) = b \cdot y^z$$

has only finitely many solutions in integers  $x, y \geq 1$ , and  $z > 1$ .

Using the method of [5], we can prove

Proposition 5: Let  $\chi$  be a primitive quadratic character with odd conductor  $f$ . If  $b \neq 0$  is an integer, and  $k$  is a sufficiently large integer with  $k \equiv \delta \pmod{2}$ , then the equation

$$\chi(1)1^k + \chi(2)2^k + \dots + \chi(xf)(xf)^k = b \cdot y^z \quad (1)$$

has only a finite number of solutions in integers  $x, y, z > 1$ .

Sketch of proof: The generalized Bernoulli polynomials satisfy the following functional equation (c.f. [1, Prop.6.3]).

$$B_x^n(x+f) - B_x^n(x) = n \sum_{a=1}^f \chi(a)(a+x)^{n-1}.$$

We set  $n = k+1$ ; if we add up the  $k$  equations obtained by setting  $x = 0, f, \dots, (k-1)f$ , we get

$$\sum_{a=1}^{xf} \chi(a)a^k = \frac{1}{k+1} (B_x^{k+1}(xf) - B_x^{k+1}(0)).$$

But  $B_x^{k+1}(0) = B_x^{k+1} = 0$  for  $k+1 \not\equiv \delta \pmod{2}$  (see [2, Th.1.2]).

Hence the left-hand side of (1) is equal to  $B_x^{k+1}(xf)/(k+1)$ . To be able to apply two theorems on the Diophantine equation  $P(x) = by^z$  (see [5]), we need to know that  $P(x) = B_x^{k+1}(xf)/(k+1)$  has at least three simple zeros. But this is the case by Prop. 4, for  $k+1 \not\equiv \delta \pmod{2}$  and if  $k$  is large enough. This completes the proof of Prop. 5.

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COMPOSITION OPERATORS AND THE INVARIANT  
SUBSPACE PROBLEM

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The Hardy space  $H^2$  can be described as the set of all functions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  such that  $\sum_{n=0}^{\infty} |a_n|^2$  is finite;  $f$  is, in particular, analytic in the open unit disk. If  $\phi$  is an analytic function mapping the disk into itself then the operator  $C_\phi$  is defined by  $(C_\phi)(z) = f(\phi(z))$  for  $f \in H^2$ . Such composition operators have been extensively studied - see [5] for a survey of all but recent work and see [3] for recent results. In this note we describe part of a forthcoming study [6] of composition operators induced by linear fractional transformations of the disk onto itself.

In [6] we treat all linear fractional transformations; in the present note we only discuss the hyperbolic ones. Up to similarity, such have the form  $\phi(z) = \frac{z-r}{1-rz}$  for  $r$  a non-zero real number between  $-1$  and  $1$ . The composition operators induced by such functions turn out to be surprisingly complicated.

A Hilbert-space operator is called universal if every operator is similar to a multiple of a "part" of it. That is,  $T$  is universal if for any bounded linear operator  $A$  on a separable Hilbert space there is a positive real number  $t$ , an invariant subspace  $M$  of  $T$  and an invertible linear operator  $S$  such that

$$S^{-1}(T|M)S = tA .$$

A universal operator, then, "contains" all operators, up to similarity and positive multiples.

Rota [7] gave the first example of a universal operator, the backwards unilateral shift of infinite multiplicity. Caradus [1] showed that every operator that is onto and has an infinite-dimensional nullspace is universal.

Theorem ([6]) If  $\phi$  is a hyperbolic linear fractional transformation mapping the unit disk onto itself and  $\lambda$  is in the interior of the spectrum of  $C_\phi$ , then  $C_{\phi-\lambda}$  is universal.

The spectrum of  $C_\phi$  is an annulus containing the unit circle in its interior [4].

If every operator has a non-trivial invariant subspace then  $(T|M)$  has a non-trivial invariant subspace whenever  $M$  is invariant under  $T$  and the dimension of  $M$  is greater than 1. If  $T$  is universal then the converse holds. Since translates and multiples of  $T$  have the same invariant subspaces as  $T$ , and since existence of non-trivial invariant subspaces is preserved by similarity, the above Theorem has the following Corollary.

Corollary [6] Fix any hyperbolic  $\phi$ . Then every operator has a non-trivial invariant subspace if and only if the minimal non-zero invariant subspaces of  $C_\phi$  are all one-dimensional.

The corollary can be re-phrased as follows. Fix any hyperbolic  $\phi$ . Then the invariant subspace problem is equivalent to the

question: is it true that for every  $f \in H^2$  not an eigenvector of  $C_\phi$  there exists a  $g \in \bigvee_{n=0}^{\infty} \{C_\phi^n f\}$  such that  $g \neq 0$  and  $\bigvee_{n=0}^{\infty} \{C_\phi^n g\} \neq \bigvee_{n=0}^{\infty} \{C_\phi^n f\}$ ? (The notation  $\bigvee$  denotes "smallest closed subspace containing"). This seems to be a very concrete, tractable problem, but we have been unable to solve it.

The proof that  $C_\phi - \lambda$  is universal is more complicated than might be expected. We use Caradus' theorem. It's not too hard to see that the nullspace of  $C_\phi - \lambda$  is infinite-dimensional, but it seems hard to show that  $C_\phi - \lambda$  is onto. We obtain this in [6] as a consequence of the structure theorem for  $C_\phi$  that is outlined below. Let  $K_0 = \bigvee_{n=-\infty}^{\infty} \{\phi^{(n)}\}$ , where  $\phi^{(n)}$  denotes the  $n$ -fold composition of  $\phi$  with itself if  $n$  is positive, the  $(-n)$ -fold composition of  $\phi^{-1}$  with itself if  $n$  is negative, and  $\phi^{(0)}(z) = z$ . Then  $K_0$  is an invariant subspace of  $C_\phi$ , and the fact that  $C_\phi \phi^{(n)} = \phi^{(n+1)}$  suggests that  $C_\phi|_{K_0}$  is like a shift operator. There is a problem, however:  $\phi^{(n)}$  approaches a constant function as  $n \rightarrow \infty$ . The constants are invariant under  $C_\phi$ ; if we let  $\mathbb{C}$  denote the constants and  $L = K_0 \ominus \mathbb{C}$ , then the matrix of  $C_\phi|_{K_0}$  with respect to  $\mathbb{C} \oplus L$ , has the form  $\begin{pmatrix} 1 & X \\ 0 & W \end{pmatrix}$  for suitable  $X$  and  $W$ . The operator  $W$  is similar to a bilateral weighted shift. The only way we can prove this is by using Shapiro and Shields' [8]  $H^2$  version of Carleson's interpolation theorem. Given this, many results about  $C_\phi|_{K_0}$  can be extracted from the known facts about weighted shifts described in Shields' excellent survey [9]. In particular, it follows that  $(C_\phi - \lambda)K_0 = K_0$  for  $\lambda$  in

the interior of  $\sigma(C_\phi)$ . To lift such information from  $K_0$  to  $H^2$  we need additional structure.

It can be shown that the sequence of zeroes of the  $\phi^{(n)}$  form a Blaschke sequence; the Blaschke product  $B$  formed from that sequence plays an important role in understanding  $C_\phi$ . Let  $M$  denote the isometry on  $H^2$  consisting of multiplication by  $B$ . Then  $MC_\phi = -C_\phi M$ . Thus the subspace  $K = K_0 + BK_0$  is also invariant under  $C_\phi$ . Moreover, the isometry  $M^2$  commutes with  $C_\phi$ , so for each positive integer  $\ell$  the subspace  $(M^2)^\ell K$  is invariant under  $C_\phi$  and the restriction of  $C_\phi$  to  $(M^2)^\ell K$  is unitarily equivalent to its restriction to  $K$ . In addition,  $(M^2)^{\ell_1} K$  can be shown to be orthogonal to  $(M^2)^{\ell_2} K$  if  $\ell_1$  and  $\ell_2$  differ by more than 1. Let  $M = \sum_{n=0}^{\infty} \oplus (M^2)^{2n} K$  and  $N = \sum_{n=0}^{\infty} \oplus (M^2)^{2n+1} K$ . Then  $C_\phi|_M$  and  $C_\phi|_N$  are unitarily equivalent to "inflations" (= "ampliations") of  $C_\phi|_{K_1}$  and  $M+N = H^2$ .

The above structure of  $C_\phi$  is the main contribution of [6]; it has a number of consequences. In particular, knowing  $(C_\phi - \lambda)K_0 = K_0$  then easily implies  $(C_\phi - \lambda)K = K$ ,  $(C_\phi - \lambda)M = M$ ,  $(C_\phi - \lambda)N = N$  and  $(C_\phi - \lambda)H^2 = H^2$ , so the universality of  $C_\phi - \lambda$  follows. In addition to related spectral results, the structure theorem also implies that the strongly closed algebras generated by compositions with disk automorphisms are reflexive. Full details are contained in [6], which also contains a simplification and generalization of the theorem of Cima-Wogen [2] describing the algebra generated by the set of composition operators induced by disk automorphisms.

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CRITERES POUR L'INDEPENDANCE  
ALGEBRIQUE DE FAMILLES DE NOMBRES

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*Presented by P. Ribenboim, F.R.S.C.*

**Résumé:** Nous énonçons un critère d'indépendance algébrique améliorant et unifiant les différents critères déjà connus. Nous mentionnons également quelques applications à la théorie des nombres algébriquement indépendants et des mesures d'indépendance algébrique.

**1-Présentation:** L'introduction de critères pour démontrer l'indépendance algébrique de deux nombres est due à A.O.Gel'fond (voir [3], Lemme VII, p.148). Des raffinements et extensions du critère de Gel'fond ont été obtenus par W.D.Brownawell et M.Waldschmidt (cf.[1], par exemple). G.V.Chudnovsky (cf.[2]), puis E.Reyssat[7], M.Waldschmidt et Zhu Yao Chen[9] ont étendu le critère de Gel'fond pour l'indépendance algébrique de plusieurs nombres. Le théorème suivant généralise ou améliore essentiellement tous les critères précédents.

**Théorème 1:** Soient  $\underline{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ ,  $k \in \{0, \dots, n-1\}$  et  $a, b$  deux nombres réels vérifiant  $0 < b < a < b(1 + 1/k + 1)$ . Soient  $\tau(N) > \delta(N)$  deux fonctions croissantes de  $N$  dans l'ensemble des réels  $> 1$  tendant vers l'infini avec  $N$ . S'il existe une suite  $(I_N = (P_{1,N}, \dots, P_{m(N),N}))_{N \geq N_0}$  d'idéaux de  $\mathbb{Z}[x_1, \dots, x_n]$  dont l'ensemble des zéros dans la boule  $B(\underline{\theta}, \exp(-\tau(N)\delta(N)^{k+a}))$  de  $\mathbb{C}^n$  est discret et telle que pour tout  $N \geq N_0$  et tout  $i=1, \dots, m(N)$  on ait  $d^{oP_{i,N}} < \delta(N)$ ,  $t(P_{i,N}) < \tau(N)$  et

$$0 < \max_{1 \leq i \leq m(N)} (|P_{i,N}(\underline{\theta})|) < \exp(-\tau(N)\delta(N)^{k+b}),$$

alors au moins  $k+1$  des nombres  $\theta_1, \dots, \theta_n$  sont algébriquement indépendants. On a noté  $d^{\circ}P$  et  $t(P)$  respectivement le degré total et la somme de  $d^{\circ}P$  et du maximum des logarithmes des valeurs absolues des coefficients du polynôme  $P$  de  $\mathbb{Z}[X_1, \dots, X_n]$ .

La démonstration fait l'objet de [5]. Ce théorème permet d'améliorer les résultats déjà connus sur les grands degrés de transcendance. Nous donnons au paragraphe suivant un exemple d'une telle amélioration dans le cadre des puissances du groupe multiplicatif. M. Waldschmidt a établi dans [8], en s'appuyant sur le théorème 1, de nombreux autres cas de familles de nombres de grands degrés de transcendance.

2-Applications: Soient  $x_1, \dots, x_\ell$  (resp.  $y_1, \dots, y_m$ ) des nombres complexes linéairement indépendants sur  $\mathbb{Q}$  et vérifiant de plus la condition d'approximation suivante:

pour tout  $\epsilon > 0$ , tout  $X > X(\epsilon)$  et tout  $\ell$ -uplet  $(\lambda_1, \dots, \lambda_\ell)$  (resp. tout  $m$ -uplet  $(\mu_1, \dots, \mu_m)$ ) d'entiers rationnels, non tous nuls, de valeurs absolues  $\leq X$  on a

$$\left| \sum_{i=1}^{\ell} \lambda_i x_i \right| \quad (\text{resp.} \quad \left| \sum_{j=1}^m \mu_j y_j \right|) > \exp(-X^\epsilon) .$$

On peut alors énoncer le théorème suivant, où  $\text{degtr}(\dots)$  désigne le degré de transcendance sur  $\mathbb{Q}$  de la famille de nombres indiquée.

Théorème 2: Sous les hypothèses précédentes on a

- (i)  $\text{degtr}(x_i, y_j, \exp(x_i y_j); i=1, \dots, \ell, j=1, \dots, m) > \ell m / (\ell + m)$
- (ii)  $\text{degtr}(x_i, \exp(x_i y_j); i=1, \dots, \ell, j=1, \dots, m) > (\ell m - m) / (\ell + m)$
- (iii)  $\text{degtr}(\exp(x_i y_j); i=1, \dots, \ell, j=1, \dots, m) > (\ell m / (\ell + m)) - 1 .$

La démonstration de ce théorème repose sur le théorème 1, via les techniques standard de transcendance (cf. [5], paragraphe 2 de la seconde partie). Enfin du cas (ii) du théorème ci-dessus on déduit le résultat suivant sur le problème de Gel'fond-Schneider.

Corollaire 3: Soient  $\alpha$  un nombre algébrique distinct de 0 et 1, et  $\beta$  un nombre algébrique de degré  $d \geq 2$  sur  $\mathbb{Q}$ . Alors  $[d/2]$  des nombres

$$\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}$$

sont algébriquement indépendants.

Démonstration: Il suffit de prendre  $l=m=d$  et  $x_i = y_i / \log \alpha = \beta^{i-1}$  pour  $i=1, \dots, d$  où  $\log \alpha$  est une détermination du logarithme de  $\alpha$ . La condition d'approximation résulte de l'inégalité de la taille dans  $\mathbb{Q}(\beta)$

3-Mesures d'indépendance algébrique: Le théorème 1 contient le critère de Gel'fond. Nous allons maintenant donner un énoncé qui en renforçant quelque peu les hypothèses du théorème 1 permet d'établir des mesures d'indépendance algébrique de familles de nombres. Un énoncé de ce type avait déjà été donné par Y.V. Nesterenko dans [4]. Avant de formuler le théorème nous introduisons quelques notations et définitions qui sont à la base des démonstrations de [5] et [6].

Si  $v$  est une place de  $\mathbb{Q}$  on note  $\mathbb{C}_v$  le complété d'une clôture algébrique de  $\mathbb{Q}_v$ . Pour un polynôme  $P$  à coefficients dans  $\mathbb{Q}$  (ou  $\mathbb{C}_v$ ) on note  $M_v(P)$  le maximum des valeurs absolues  $v$ -adiques des coefficients de  $P$  si  $v$  est finie et la mesure de Mahler de  $P$  si  $v$  est infinie. On a défini dans [5] les notions de hauteur et hauteur invariante de  $P$  (à coefficients dans  $\mathbb{Q}$ ) par:

$$\bar{h}(P) = \sum_v \max\{0, \log(M_v(P))\}$$

$$h(P) = \sum_v \log(M_v(P)) \quad .$$

Soit  $I$  un idéal homogène de  $\mathbb{Q}[X_0, \dots, X_n]$  de codimension  $n+1-r$  et  $\underline{d} \in \mathbb{N}^r$ . On a encore défini dans [5] (cf. Définition 1.14) les quantités

$$Ht_{\underline{d}}(I) =: h(f) \quad (\text{dite hauteur d'indice } \underline{d} \text{ de } I)$$

$$\text{Deg}_{\underline{d}}(I) =: d \cdot f \quad (\text{dite degré d'indice } \underline{d} \text{ de } I) \quad ,$$

où  $f$  est une forme U-éliminante d'indice  $\underline{d}$  de  $I$  quelconque (voir [5]).

Si  $\underline{x} \in \mathbb{P}_n(\mathbb{C}_v)$  on a défini dans [5] (cf. Définition 1.15) les morphismes

$\tilde{d}_{\underline{x}, \underline{d}}$  et la quantité

$$\|I\|_{\underline{x}, v, \underline{d}} =: M_v(\tilde{d}_{\underline{x}, \underline{d}}(f)) / M_v(f) \quad (\text{dite valeur absolue d'indice } \underline{d} \text{ de } I \text{ en } \underline{x}).$$

Si  $I$  est engendré par des polynômes  $Q_1, \dots, Q_m$  les hauteur et degré d'indice  $\underline{d}$  de  $I$  sont liés aux tailles (cf. théorème 1) des polynômes

$Q_j$  pour  $j=1, \dots, m$  tandis que la valeur absolue d'indice  $\underline{d}$  de  $I$  en  $\underline{x}$  dépend également de la quantité  $\max_{1 \leq j \leq m} \{|Q_j(\underline{x})|\}$ .

Pour un idéal  $J$  pur de codimension  $n+1-r$  de  $\mathbb{Q}[X_1, \dots, X_n]$  on pose

$$\|J\|_{\underline{x}} = \|{}^h J\|_{\underline{x}, v, \underline{1}} \quad \text{lorsque la place } v \text{ est infinie et } T(J) = Ht_{\underline{1}}({}^h J) + \text{Deg}_{\underline{1}}({}^h J)$$

où  $\underline{1} = (1, \dots, 1) \in \mathbb{N}^r$  et  ${}^h J$  est l'idéal homogénéisé de  $J$  dans  $\mathbb{Q}[X_0, \dots, X_n]$

engendré par les homogénéisés des éléments de  $J$ .

Avec les notations ci-dessus on peut énoncer :

**Théorème 4 :** Soient  $\underline{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n, k \in \{0, \dots, n-1\}$ ,  $u$  une fonction strictement croissante de  $\mathbb{N}$  dans l'ensemble des réels  $> 0$ , tendant vers l'infini à l'infini et  $0 < \epsilon < 1$  un réel. On suppose qu'il existe une suite  $(I_N)_{N > N_0}$  d'idéaux de  $\mathbb{Z}[X_1, \dots, X_n]$  telle que l'ensemble des zéros de  $I_N$  dans la boule  $B(\underline{\theta}, \exp(-N^{k+1} u(N)^{1+\epsilon/(k+1)}))$  de  $\mathbb{C}^n$  est vide

et que  $I_N$  soit engendré par des polynômes  $Q_{1,N}, \dots, Q_{m(N),N}$  de tailles  $< N$ , vérifiant pour  $j=1, \dots, m(N)$

$$|Q_{j,N}(\underline{\theta})| < \exp(-N^{k+1}u(N)) .$$

Alors il existe un réel  $C=C(n)$  tel que pour tout idéal  $J$  de  $\mathbb{Q}[x_1, \dots, x_n]$  de codimension  $> n-k$  on a, en notant  $w$  la fonction inverse de  $u$ ,

$$\|J\|_{\underline{\theta}} > \exp(-CT(J)^{1+\varepsilon} w(T(J))^{k+1}) .$$

Ce théorème se déduit du théorème 2 de [6] qui introduit un raffinement supplémentaire. On trouvera également dans [6] un énoncé effectif correspondant au corollaire 3 et qui découle du théorème 4 ci-dessus.

4-Commentaires: Nous voudrions terminer cette note en signalant qu'une notion de taille d'un idéal  $J$ , proche de notre quantité  $T(J)$  du paragraphe précédent, avait déjà été introduite par Y.V.Nesterenko (se reporter par exemple à [4]). Par ailleurs l'auteur avait obtenu auparavant un critère d'indépendance algébrique semblable au théorème 1 mais en supposant que les homogénéisés  $h_{I_N}$  des idéaux  $I_N$  étaient de dimension projective zéro, généralisant ainsi un résultat antérieur de R.Dvornicich qui supposait de plus les idéaux  $h_{I_N}$  premiers, alors que nous rappelons que les critères de G.V.Chudnovsky, E. Ressat, M. Waldschmidt et Zhu Yao Chen mentionnés en introduction utilisent des suites d'idéaux  $I_N$  principaux (i.e.  $m(N)=1$ ). Le théorème 1 rassemble ces deux aspects de la théorie de l'indépendance algébrique.

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CONFLUENCE AND STABILITY OF ORBITS OF QUADRATIC POLYNOMIALS

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In recent years, there has been a great deal of interest in the iteration of quadratic polynomials, more precisely, in the way in which the iterative behavior of the functions in one-parameter families of restrictions of such polynomials to real intervals depends on the parameter (see [1,2,3,4,6]). In this note we discuss two important aspects of this behavior. These are the orderly flow of cyclic orbits from the complex plane into the real line and the existence of maximally attractive ("super-stable") orbits for parameter values arbitrarily close to the limiting Čebyšev value. Proofs and further results will appear elsewhere.

## 1. Preliminaries

Given a quadratic polynomial  $p$ , where  $p(z) = az^2 + bz + c$ , we define its iterative discriminant  $\Delta(p)$  by

$$(1) \quad \Delta(p) = (b-1)^2 - 4ac.$$

The iterative discriminant is a complete invariant for linear conjugacy of quadratic polynomials [5]: Two quadratic polynomials  $p$  and  $q$  are linearly conjugate, i.e., there exists a non-constant linear function  $f$  such that

$$(2) \quad p(z) = f^{-1}(q(f(z))) \text{ for all } z,$$

if and only if  $\Delta(p) = \Delta(q)$ . In particular, any quadratic polynomial  $p$  is linearly conjugate to a quadratic polynomial  $q$  of the form

$$(3) \quad q(z) = z^2 - \frac{\Delta(p)-1}{4} = z^2 - \frac{\Delta(q)-1}{4}.$$

And any proper restriction of  $p$  is linearly conjugate to a corresponding proper restriction of  $q$ . Thus, for  $0 < \lambda \leq 2$ , the restriction of  $2\lambda z(1-z)$  to the unit interval  $[0,1]$  is linearly conjugate to  $z^2 - \lambda(\lambda-1)$  restricted to the interval  $[-\lambda, \lambda]$ .

An n-cycle of a function  $g$  is a set  $Z_n$  of  $n$  distinct points  $\{z_1, \dots, z_n\}$  in the domain of  $g$  such that

$$(4) \quad g(z_1) = z_2, \dots, g(z_{n-1}) = z_n, \quad g(z_n) = z_1.$$

The single point in a 1-cycle of  $g$  is a fixed-point of  $g$ , while any point in an  $n$ -cycle of  $g$  is a fixed-point of  $g^n$ , the  $n^{\text{th}}$  iterate of  $g$ . If  $g$  is differentiable, then the multiplier of an  $n$ -cycle  $Z_n = \{z_1, \dots, z_n\}$  of  $g$  is the number

$$(5) \quad M_g(Z_n) = (g^n)'(z_m) = g'(z_1)g'(z_2) \cdots g'(z_n),$$

where  $z_m$  is any point in the  $n$ -cycle. An  $n$ -cycle  $Z_n$  of  $g$  is repulsive if  $|M_g(Z_n)| > 1$ , attractive if  $|M_g(Z_n)| < 1$ , and maximally attractive if  $M_g(Z_n) = 0$ .

It is immediate from (2) that, if  $p$  and  $q$  are linearly conjugate, then  $Z_n$  is an  $n$ -cycle of  $p$  if and only if  $f(Z_n)$  is an  $n$ -cycle of  $q$ , and

$$(6) \quad M_p(Z_n) = M_q(f(Z_n)).$$

From (6) in turn it follows that, if  $J_p$  is the Julia set of  $p$ , i. e., the closure of the set of repulsive cycles of  $p$  (see [2], Theorem 4.1), then  $f(J_p) = J_q$ , the Julia set of  $q$ .

## 2. Inflow of the cycles

Our interest is in the iteration of quadratic polynomials with real iterative discriminants. The invariance of iterative behavior under linear conjugation therefore allows us to confine

our attention to the family  $\{q_\delta\}$  of functions given by

$$(7) \quad q_\delta(z) = z^2 - \frac{\delta-1}{4}$$

with  $\delta = \Delta(q_\delta)$  real, and particularly to the subfamily of  $\{q_\delta\}$  with  $\delta$  in the interval  $[0,9]$ .

The Julia set of  $q_1$  is the unit circle; for  $n > 1$ , the points in the  $n$ -cycles of  $q_1$  are precisely the primitive  $(2^n-1)^{\text{th}}$  roots of unity, and the multiplier of each  $n$ -cycle is  $2^n$ . The Julia set of  $q_9$  is the interval  $[-2,2]$ ;  $q_9$  is linearly conjugate to the Čebyšev polynomial  $2x^2-1$ , and for  $n \geq 1$ , the points in the  $n$ -cycles of  $q_9$  are of the form  $2 \cos \left( 2\pi \frac{m}{2^n \pm 1} \right)$ . As  $\delta$  increases from 1 to 9, the  $n$ -cycles gradually detach themselves from the unit circle and wander in to the real axis. Once a cycle is on the axis, it stays there.

If any point in an  $n$ -cycle of  $q_\delta$  is real, then all are. Therefore any  $n$ -cycle is either entirely real or entirely complex. In the latter case, since a point in an  $n$ -cycle is a root of the polynomial equation  $q_\delta^n(z) - z = 0$  with real coefficients, any  $n$ -cycle, as a set of complex numbers, must either be self-conjugate, or be paired with a conjugate complex  $n$ -cycle. Both cases occur.

The function  $q_\delta$  has 1-cycles at the points  $\frac{1}{2}(1 \pm \sqrt{\delta})$ . Thus as  $\delta$  increases from negative to positive values, the two 1-cycles of  $q_\delta$  first approach one another along the vertical line  $x = \frac{1}{2}$ , merge for an instant at  $x = \frac{1}{2}$  when  $\delta = 0$ , then separate again but remain on the real axis. Correspondingly, for  $\delta \neq 4$ ,  $q_\delta$  has a single 2-cycle consisting of the points  $-\frac{1}{2}(1 \pm \sqrt{\delta-4})$ . As  $\delta$  increases, the points in the cycle first approach one another along the vertical line  $x = -\frac{1}{2}$ , merge, when  $\delta = 4$ , with the

fixed-point  $-\frac{1}{2}$  (thus, for an instant, disappearing as a 2-cycle), then, for  $\delta > 4$ , reappear as a 2-cycle that remains on the real axis.

The behavior of the two 3-cycles of  $q_5$  is qualitatively similar to that of the two 1-cycles. For  $\delta < 8$ , the two 3-cycles are complex conjugates; when  $\delta = 8$ , they merge into a single 3-cycle whose points are the 3 real roots of the cubic equation  $x^3 + \frac{1}{2}x^2 - \frac{9}{4}x - \frac{1}{8} = 0$ , and for  $\delta > 8$ , separate into two 3-cycles on the real axis.

For  $\delta < 6$ , one of the three 4-cycles of  $q_6$  is self-conjugate (the abscissae of the 4 points in this cycle being the roots of the quadratic equation  $x^2 + tx - \frac{1}{4} = 0$ , where  $t$  in turn is the single real root of the cubic equation  $t^3 - \frac{\delta-4}{4}t - \frac{1}{2} = 0$ ). At  $\delta = 6$ , this 4-cycle merges with the single 2-cycle  $\left\{ \frac{1}{2}(1 \pm \sqrt{2}) \right\}$  of  $q_6$ . For  $\delta > 6$ , the 4-cycle and 2-cycle separate again but stay on the real axis. The remaining two 4-cycles are complex conjugates for  $\delta < 3 \cdot 4^{1/3} + 4$ , merge into a single 4-cycle on the real axis when  $\delta = 3 \cdot 4^{1/3} + 4 \approx 8.7622$ , and split into two 4-cycles on the real axis for  $\delta > 8.7622$ .

Returning to the general situation, we see that an  $n$ -cycle can appear on the real axis only by merging for an instant either with another  $n$ -cycle or (if  $n$  is even) with an  $(n/2)$ -cycle. At the instant of merger, the multiplier of each  $n$ -cycle is 1, while the multiplier of the  $(n/2)$ -cycle, if any, is -1. As  $\delta$  increases beyond the merger value, the merged cycles separate; each stays on the real axis, with one (the  $(n/2)$ -cycle, if there is one) becoming repulsive, and the other (an  $n$ -cycle) attractive, as its multiplier decreases. When the multiplier of the attractive  $n$ -cycle reaches -1, it merges with a  $2n$ -cycle that has just entered

the real axis.

The order of appearance of cycles on the real axis is intricate: nevertheless, some simple patterns emerge. Let  $\alpha_n$  be the value of  $\delta$  for which an  $n$ -cycle first appears on the real axis, and  $\omega_n$  the value of  $\delta$  for which an  $n$ -cycle last appears on the real axis. Thus  $\alpha_1 = 0$ ,  $\alpha_2 = 4$ ,  $\alpha_3 = 8$ ,  $\alpha_4 = 6$ , etc., and  $\omega_1 = 0$ ,  $\omega_2 = 4$ ,  $\omega_3 = 8$ ,  $\omega_4 = 8.7622$ , etc. Thus the  $\alpha_n$ 's are clearly in reverse Šarkovskij order, i. e.,

$$\alpha_1 < \alpha_2 < \alpha_4 < \cdots < \alpha_{20} < \alpha_{12} < \cdots < \alpha_{10} < \alpha_6 < \cdots < \alpha_5 < \alpha_3,$$

while the  $\omega_n$ 's are in natural order, i. e.,

$$\omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5 < \omega_6 < \cdots .$$

In short, what appears (if one confines one's attention to the real line) as a somewhat chaotic sequence of apparitions and bifurcations, is in fact an orderly parade of confluences.

### 3. Stability

It has been conjectured (cf. [6], p. 193) that for a set of values of  $\delta$  that is open and dense in the interval  $[0,9]$ , the polynomial  $q_\delta$  has an attractive cycle. As a step towards proving this conjecture, we have shown that there is a strictly increasing sequence  $\{\delta_n\}$  with  $\lim_n \delta_n = 9$ , such that for each  $n \geq 1$ ,  $q_{\delta_n}$  has a maximally attractive  $n$ -cycle in the interval  $[-\frac{1}{2}(1+\sqrt{\delta_n}), \frac{1}{2}(1+\sqrt{\delta_n})]$ . The first few terms of the sequence are:  $\delta_1 = 1$ ,  $\delta_2 = 5$ ,  $\delta_3 = 4\xi_3 + 1$ , where  $\xi_3$  is the single real root of the cubic equation  $\xi^3 - 2\xi^2 + \xi - 1 = 0$ , and  $\delta_4 = 4\xi_4 + 1$ , where  $\xi_4$  is the larger of the 2 real roots of the sextic equation  $\xi^2(\xi - 1)(\xi - 2)(\xi^2 + 1) + 1 = 0$ . The sequence  $\{\delta_n\}$  is related to the sequence  $\{\omega_n\}$  of the preceding section by

$\omega_n < \delta_n < \omega_{n+1}$  for all  $n \geq 1$ .

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A GENERALIZATION IN SEVERAL VARIABLES

OF A TRANSCENDENCE CRITERION OF GEL'FOND ( II )

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*Presented by P. Ribenboim, F.R.S.C.*

**Abstract.** M. Waldschmidt and the author [6] gave a generalization in several variables of a transcendence criterion of Gel'fond. In the present paper an improvement of this result is proved.

1. Introduction

In 1949 A. O. Gel'fond proved a transcendence criterion for a complex number ([3], Ch.3, §4, Lemma 7). A number of authors considered its generalization in several variables (see [1],[2],[4],[5], etc.). M. Waldschmidt and the author [6] also gave a form of its generalization. The present paper improves the result of [6].

For  $P \in \mathbb{C}[x_1, \dots, x_n]$ ,  $H(P)$  denotes the height of  $P$ , i.e. the maximum absolute value of the coefficients of  $P$ , and  $d(P)$  denotes the maximum of the degrees of  $P$  in  $x_i (i=1, \dots, n)$ . We put  $t(P) = \max(\log H(P), 1 + d(P))$ .

**Theorem.** Let  $(\theta_1, \theta_2) \in \mathbb{C}^2$  and  $\eta > 0$ . If there are constants  $N_0 > 0$  and  $c_0 > c_1 > 0$  such that for all reals  $N \geq N_0$  and all  $(z_1, z_2) \in \mathbb{C}^2$  with  $|z_i - \theta_i| < \exp(-c_0 N^\eta) (i=1,2)$  there exists a polynomial  $P_N \in \mathbb{Z}[x_1, x_2]$  with  $t(P_N) \leq N$  satisfying  $0 < |P_N(z_1, z_2)| < \exp(-c_1 N^\eta)$ , then  $\eta \leq 2 + \sqrt{3}$ .

**Corollary.** Let  $n \geq 2$ ,  $(\theta_1, \dots, \theta_n) \in \mathbb{C}^n$  and  $\eta > 0$ . If  $N_0 > 0$  and  $a_1 > a_2 > 0$  are constants such that for any integer  $N \geq N_0$  there is a polynomial  $P_N \in \mathbb{Z}[x_1, \dots, x_n]$  with  $t(P_N) \leq N$  and

$$\exp(-a_1 N^\eta) \leq |P_N(\theta_1, \dots, \theta_n)| \leq \exp(-a_2 N^\eta),$$

then  $\eta \leq (2+\sqrt{3})2^{n-2}$ .

The proof is different from [6], some new technique is introduced in the present paper, which originates from the idea of coloured sequence of Čudnovskiĭ [2].

It is a pleasure to acknowledge my indebtedness to Professor Michel Waldschmidt for his helpful suggestions.

## 2. Auxiliary Lemmas

Lemma 2.1. ([2]) Let  $(\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ . If there is a polynomial  $P \in \mathbb{Z}[x_1, \dots, x_n]$  such that  $|P(\theta_1, \dots, \theta_n)| < \epsilon < 1$  and  $t(P) \leq T$ , then either (i) there is a polynomial  $R \in \mathbb{Z}[x_1, \dots, x_n]$  with  $R \neq 0$ ,  $R|P$  such that

$$|R(\theta_1, \dots, \theta_n)| < \epsilon^{1/3}, \quad t(R) \leq 2T,$$

with  $R(x_1, \dots, x_n) = (S(x_1, \dots, x_n))^s$  where  $S \in \mathbb{Z}[x_1, \dots, x_n]$  is irreducible, and  $t(S) \leq 2T/s$  with  $s \geq 1$ ; or (ii) there are two polynomials  $P_1, P_2 \in \mathbb{Z}[x_1, \dots, x_n]$  with  $P_1|P, P_2|P$  such that

$$\text{g.c.d.}(P_1, P_2) = 1,$$

$$\max(|P_1(\theta_1, \dots, \theta_n)|, |P_2(\theta_1, \dots, \theta_n)|) < \epsilon^{1/3},$$

$$t(P_1) \leq 2T, \quad t(P_2) \leq 2T.$$

Lemma 2.2. ([4]) Let  $(\theta_1, \theta_2) \in \mathbb{C}^2$ . There is a constant  $C > 0$  such that if there exists a constant  $N_0 > 0$  having the following property: for any integer  $N \geq N_0$  there are polynomials  $F_N^{(1)}, \dots, F_N^{(m)} \in \mathbb{Z}[x, y]$ , where  $m = m(N) \geq 2$ , such that

$$\text{g.c.d.}(F_N^{(1)}, \dots, F_N^{(m)}) = 1,$$

$$|F_N^{(j)}(\theta_1, \theta_2)| < \exp(-CN^3), \quad t(F_N^{(j)}) \leq N \quad (j=1, \dots, m),$$

then  $(\theta_1, \theta_2) \in \mathbb{Q}^2$ .

3. The set  $A^*(\theta_1, \dots, \theta_n)$ 

Let  $n \geq 1$  and  $(\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ . We generalize the definitions of the set  $A(\theta_1, \dots, \theta_n)$  and the number  $J(\theta_1, \dots, \theta_n)$  of [6].

**Definition 3.1.**  $A^*(\theta_1, \dots, \theta_n)$  denotes the set of all reals  $\eta > 1$  having the following property: there are reals  $c_0 > c_1 > 0$  and  $N_0 > 0$  such that for all reals  $N \geq N_0$  and all  $(z_1, \dots, z_n) \in \mathbb{C}^n$  with  $|\theta_i - z_i| < \exp(-c_0 N^\eta)$  ( $i=1, \dots, n$ ) there is a polynomial  $F_N \in \mathbb{Z}[x_1, \dots, x_n]$  with  $t(F_N) \leq N$  satisfying  $0 < |F_N(z_1, \dots, z_n)| < \exp(-c_1 N^\eta)$ . We put  $J^*(\theta_1, \dots, \theta_n) = \sup A^*(\theta_1, \dots, \theta_n)$  if  $A^*(\theta_1, \dots, \theta_n)$  is nonempty; otherwise let  $J^*(\theta_1, \dots, \theta_n) = 1$ .

**Proposition 3.2.** Both sets  $A(\theta_1, \dots, \theta_n)$  and  $A^*(\theta_1, \dots, \theta_n)$  are either empty or an interval. Furthermore  $J(\theta_1, \dots, \theta_n) = J^*(\theta_1, \dots, \theta_n)$ .

**Proposition 3.3.** Let  $n \geq 2$ . If  $\eta \in A^*(\theta_1, \dots, \theta_n)$  with  $\eta > 2$ , and  $\{i_1, \dots, i_{n-1}\}$  are arbitrary distinct  $n-1$  numbers in the set  $\{1, \dots, n\}$ , then  $\eta/2 \in A^*(\theta_{i_1}, \dots, \theta_{i_{n-1}})$ . Therefore  $J^*(\theta_1, \dots, \theta_n) \leq 2J^*(\theta_{i_1}, \dots, \theta_{i_{n-1}})$ .

**Proposition 3.4.** Let  $(\theta_1, \dots, \theta_n) \in \mathbb{C}^n$  and  $\eta > 1$ . If there are constants  $N_0 > 0$  and  $a_1 > a_2 > 0$  such that for any integer  $N \geq N_0$  there is a polynomial  $P_N \in \mathbb{Z}[x_1, \dots, x_n]$  with  $t(P_N) \leq N$  and

$$\exp(-a_1 N^\eta) \leq |P_N(\theta_1, \dots, \theta_n)| \leq \exp(-a_2 N^\eta),$$

then  $\eta \in A^*(\theta_1, \dots, \theta_n)$ .

## 4. Proof of Theorem

Without loss of generality we assume that  $\theta_1$  and  $\theta_2$  are transcendental. Use reduction to absurdity. Assume that  $\eta > 2 + \sqrt{3}$ . Suppose that there are constants  $c_0 > c_1 > 0$  and  $M_0 > 0$  such that for any real  $M \geq M_0$

and  $(z_1, z_2) \in \mathcal{C}^2$  with  $|\theta_i - z_i| < \exp(-c_0 M^\eta)$  ( $i=1,2$ ) there is a polynomial  $P = P_M \in \mathcal{Z}[x, y]$  with  $t(P) \leq M$  and  $0 < |P(z_1, z_2)| < \exp(-c_1 M^\eta)$ . Thus there is a polynomial  $P = P_M \in \mathcal{Z}[x, y]$  with  $t(P) \leq M$  and  $0 < |P(\theta_1, \theta_2)| < \exp(-c_1 M)$ . It is clear we can assume that  $M_0 \geq \frac{4 + \log((|\theta_1|+1)(|\theta_2|+1))}{\min(c_1, c_0 - c_1)} + 1$ .

We put  $M_0^* = 6M_0^2 \frac{2c_0}{c_1}$ , thus if  $M \geq M_0^*$ , then  $M \geq M_0$ . By Lemma 2.1, we consider two cases.

Case I. There are two polynomials  $p^{(i)} = p_M^{(i)} \in \mathcal{Z}[x, y]$  ( $i=1,2$ ) with  $\text{g.c.d.}(p^{(1)}, p^{(2)}) = 1$  and for  $i = 1, 2$

$$t(p^{(i)}) \leq 2M \quad \text{and} \quad 0 < |p^{(i)}(\theta_1, \theta_2)| < \exp(-\frac{1}{3}c, M^\eta).$$

Case II. There are polynomials  $R = R_M$  and  $S = S_M \in \mathcal{Z}[x, y]$  with  $R(x, y) = (S(x, y))^s$  where  $S$  is irreducible, and  $t(S) \leq \frac{2M}{s}$  with  $s \geq 1$  and  $|R(\theta_1, \theta_2)| < \exp(-\frac{1}{3}c, M^\eta)$ ,  $t(R) \leq 2M$ .

(IIa)  $S$  is a polynomial of one variable, say,  $x$ . Thus there is a zero  $z$  of  $S$  with  $|z - \theta_1| < \exp(-\frac{1}{6}c, M^{\eta-1})$ .

(IIb)  $S$  is dependent on  $x$  and  $y$ . Thus we get

$$|a_0(\theta_1) \prod_{k=1}^d (\theta_2 - z_k)|^s < \exp(-\frac{1}{3}c, M^\eta),$$

where  $z_k \in \mathcal{C}$  ( $k=1, \dots, d$ ) are the zeros of  $S(\theta_1, y)$ .

(IIb-1) There is a zero  $z'$  of  $S(\theta_1, y)$  with  $|\theta_2 - z'| < \exp(-\frac{1}{6sd}c, M^\eta)$ .

(IIb-2)  $|a_0^s(\theta_1)| < \exp(-\frac{1}{6}c, M^\eta)$ .

For every case, we can find two polynomials  $f^{(i)} = f_M^{(i)} \in \mathcal{Z}[x, y]$  ( $i=1,2$ ) with  $\text{g.c.d.}(f^{(1)}, f^{(2)}) = 1$ , and holds either

$$(A) \quad \begin{cases} |f^{(i)}(\theta_1, \theta_2)| < \exp(-\frac{1}{18} c_1 M^\eta), \\ \tau(f^{(i)}) \leq 4M \quad (i=1,2); \end{cases}$$

or

$$(B) \quad \begin{cases} |f^{(1)}(\theta_1, \theta_2)| < \exp(-\frac{c_1^2}{72c_0} M^{\eta-1}), \\ |f^{(2)}(\theta_1, \theta_2)| < \exp(-\frac{1}{6} c_1 M^\eta), \\ \tau(f^{(1)}) \leq (\frac{c_1}{6c_0})^{1/\eta} M^{1-1/\eta}, \quad \tau(f^{(2)}) \leq 2M. \end{cases}$$

Let  $\tau = [M^{1/\eta}] + 1$ , and put  $N = 4M$ ,  $N_0^* = \max(4M_0^*, (C/c)^{\frac{1}{\theta-3}})$ , where  $C$  is the constant of Lemma 2.2 and  $c = c_1^2 / (4^\theta \cdot 72c_0)$  with  $\theta = \min(\eta, \eta + \frac{1}{\eta} - 1) > 3$ . For any real  $N \geq N_0^*$ , if (A) occurs, we put  $F_N^{(i)} = f^{(i)}$  ( $i=1,2$ ); otherwise let  $F_N^{(1)} = (f^{(1)})^\tau$ ,  $F_N^{(2)} = f^{(2)}$ .

Thus we have

$$\text{g.c.d. } (F_N^{(1)}, F_N^{(2)}) = 1$$

$$|F_N^{(i)}(\theta_1, \theta_2)| < \exp(-CN^3), \quad \tau(F_N^{(i)}) \leq N(i=1,2).$$

By Lemma 2.2,  $(\theta_1, \theta_2) \in \bar{Q}^2$ . This contradicts the hypothesis. Q.E.D.

The Corollary in §1 is a consequence of Propositions 3.3 and 3.4.

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A SHEAF PROPERTY FOR EXCESSIVE FUNCTIONS OF RIGHT PROCESSES

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This note essentially summarizes the results of [8] which again generalize those of [7]. We study the following question: given a right process in the sense of [4], with respect to which topology on its state space is excessivity of measurable functions a local property?

A result of Shur [6] establishes a sheaf property for excessive functions of a standard process with respect to the initial topology, which of course remains no longer true for general right processes. In the following we define another topology (a certain Choquet topology in the sense of [2]) on the state space of a right process, which seems to be better adapted to the spatial behaviour of the process. The main theorem establishes the sheaf property of a suitably defined local structure with respect to this Choquet topology under some additional topological assumptions (which generalize the local compactness and the existence of a countable base in the context of standard processes in [6]). This result is of some interest within the potential theory of standard H-cones.

To begin with let  $(E, \underline{E})$  be the state space of a right process  $X = (X_t)_{t \geq 0}$  with semigroup  $(P_t)_{t \geq 0}$ . We adopt the notation of [4]. Moreover  $\zeta$  denotes the life time of the process, and for a universally measurable set  $U \in \underline{E}^*$  we define  $\Pi_U$  to denote the exit kernel from  $U$  which is the kernel associated with the entry time

$D_{tU}$  of  $\{U\}$ . Furthermore let  $\bar{E}$  be any Ray-Knight compactification of  $E$  and let  $(\bar{P}_t)_{t \geq 0}$  denote the associated Ray semigroup on  $\bar{E}$ .

**DEFINITION:** A set  $A \subset E$  is said to be absorbing if  $\bar{P}_0(x, A) = 1$  for all points  $x$  in the Ray closure of  $A$  in  $\bar{E}$ .

The absorbing sets in  $E$  actually form the closed sets of a topology on  $E$ . This topology, which is apparently weaker than the Ray topology and in general not separated, can be characterized as a Choquet topology in the sense of [2] with respect to a certain cone of continuous supermedian functions on  $\bar{E}$  (cf.[8]). To distinguish it from the Ray topology for what follows we call it the Choquet topology on  $E$ . It coincides with the Ray topology if the process is Feller.

However, the universal Borel structures generated by both topologies are the same (cf.[8]). Thus considering  $X$  as a process in  $E$  equipped with the Choquet topology makes sense. One obtains the following properties of  $X$  applying results of [4] (cf.[8]):

PROPOSITION 1: With respect to the Choquet topology, the paths of  $X$  are a.s. right continuous with left limits in  $E$  and  $X$  is quasi-left-continuous.

PROPOSITION 2: The Choquet topology does not depend on the choice of the Ray-Knight compactification of the state space  $E$ .

In analogy to Shur's result for standard processes (cf.[6]) we want to establish a sheaf property for locally excessive functions suitably defined with respect to the Choquet topology on  $E$ .

To this end let  $\emptyset \subset E$  be a Choquet open set and let  $\mathcal{U}(\emptyset)$  denote the collection of all Choquet open subsets  $U$  of  $\emptyset$  which are Choquet relatively quasi-compact in  $\emptyset$  (i.e. for which there exists a Choquet quasi-compact set  $C$  such that  $U \subset C \subset \emptyset$ ).

$S^*(\emptyset)$  is defined to be the class of nearly Ray Borel functions  $f \in \mathbb{E}_+^*$  on  $E$  which satisfy the inequalities  $\Pi_U f \leq f$  for all sets  $U \in \mathcal{U}(\emptyset)$ .  $E(\emptyset)$  is defined to be the class of those functions  $f \in S^*(\emptyset)$  which are finely lower semi-continuous in  $\emptyset$ .

The fine lower semi-continuity in the definition of  $E(\emptyset)$  could be replaced by some other local regularity condition, but of course this does not affect the validity of a sheaf property.

PROPOSITION 3: Assume the Choquet topology admits a quasi-compact exhaustion of  $E$ . Then the class  $S^*(E)$  consists of all strongly supermedian functions of  $X$  (in the sense of [5]).

Prop.3 of course yields a characterization of excessivity in terms of the exit distributions from Choquet open sets. The assumption is rather weak, it is trivially satisfied e.g. if  $E$  consists of all non-branch points of Markovian  $(\bar{P}_t)_{t \geq 0}$ . The proof of Prop.3 is based mainly on the path properties of  $X$  stated in Prop.1.

The main theorem is the following:

THEOREM: Suppose the Choquet topology on  $E$  is locally quasi-compact and admits a quasi-compact exhaustion of the state space  $E$ . Then  $E(E)$  is exactly the class of excessive functions on  $E$ , and the classes  $E(O)$ , where  $O$  is a Choquet open set in  $E$ , form a sheaf, i.e.: if  $O = \bigcup_{i \in I} O_i$  for  $O_i$  Choquet open in  $E$ , then  $E(O) = \bigcap_{i \in I} E(O_i)$ .

Sketch of proof: What essentially is to be done is to prove the inclusion  $\bigcap_{i \in I} S^*(O_i) \subset S^*(O)$ . Because of the assumption of local quasi-compactness one can moreover restrict oneself to the case  $I = \{1, 2\}$ . Hence let  $O = O_1 \cup O_2$  and let  $U \in \mathcal{U}(O)$ . The main step is to construct a suitable decomposition of  $U$  into a union of sets  $U_i \in \mathcal{U}(O_i)$ , which is done by the following lemma. Its proof uses essentially the local quasi-compactness of the Choquet topology.

LEMMA: Given the above situation there exist sets  $U_i \in \mathcal{U}(O_i)$  for  $i = 1, 2$  such that  $U = U_1 \cup U_2$  and such that the following convergence property holds:

If a sequence  $(x_n)_{n \in \mathbb{N}}$  of points in  $U$  converges with respect to the Ray topology in  $\bar{E}$ , then this sequence lies finally either in  $U_1$  or in  $U_2$ .

Having obtained this decomposition we define iterated exit times as follows:  $T_0 := 0$ ,  $T_1 := D_{U_1}$ , and for  $n \geq 1$

$$T_{2n} := T_{2n-1} + D_{U_2} \circ \theta_{T_{2n-1}},$$

$$T_{2n+1} := T_{2n} + D_{U_1} \circ \theta_{T_{2n}}, \text{ and } T := \sup_k T_k,$$

where  $(\theta_t)_{t \geq 0}$  denotes the shifts (cf. [4]).

By the second part of the above lemma it is seen that the convergence of the sequence  $(T_n)_n$  to  $T$  is a.s. trivial, which means that a.s.  $\bigcup_n (T_n = T) \cap \{T < \zeta\} = \{T < \zeta\}$ . This implies in particular that a.s. on  $\{T < \zeta\}$   $(T_n)_n$  converges to the exit time  $D_{\{U\}}$ .

Thus for  $x \in E$  and bounded  $f \in S^*(O_1) \cap S^*(O_2)$ , applying the strong Markov property we get the following inequalities:

$$\begin{aligned} \Pi_U f(x) &= E^x [ f \circ X_{D_{\{U\}}}, D_{\{U\}} < \zeta ] = E^x [ \lim_n f \circ X_{T_{2n}}, D_{\{U\}} < \zeta ] \\ &\leq \liminf_n E^x [ f \circ X_{T_{2n}}, T_{2n} < \zeta ] \leq \liminf_n (\Pi_{U_2} \Pi_{U_1})^n f(x) \leq f(x). \end{aligned}$$

□

- REMARKS:** 1.) The result can be extended to classes of not necessarily positive functions which satisfy certain lower boundedness conditions (cf.[8]).
- 2.) In the context of standard-H-cones (cf.[3]) the Choquet topology turns out to be just the facial topology on the extreme points of the representing simplex and the above notion of 'locally excessive' leads to some sheaf property for standard-H-cones which is apparently different from the one considered in [3]. In particular in the case where the standard H-cone can be represented on a Bauer simplex which means that it can be associated with a standard balayage space in the sense of [1], one gets back the sheaf property of [1].
- 3.) In order to establish a more general result which besides the one stated above reproduces Shur's sheaf property for standard processes, it is possible to define the notions in a more general abstract setting, which is done in [8].

4.) There are many examples of right processes for which the assumptions of the theorem on the associated Choquet topology are fulfilled, in particular those given in [4].

On the other hand some examples show that one should be able to weaken these assumptions and still obtain a sheaf property (cf. [7] or [8] for a discussion).

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THE SPACE OF SMOOTH ISOMETRIC IMMERSIONS OF A COMPACT MANIFOLD INTO AN EUCLIDEAN SPACE IS A FRECHET MANIFOLD

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Abstract Let  $M$  be a compact, connected and smooth manifold.  $\langle, \rangle$  denotes a fixed scalar product on  $\mathbb{R}^n$ . The map  $m$  sends any smooth immersion  $j$  of  $M$  into  $\mathbb{R}^n$  to a smooth Riemannian metric  $m(j)$  on  $M$ , given by  $j^*\langle, \rangle$ . Fix the immersion  $i$ . Those  $j$  with  $m(i)=m(j)$  form a Fréchet manifold.

1) The relative description of the differentials of immersions

Let  $M$  be any compact smooth manifold of dimension  $r$ . The collection of all smooth immersions of  $M$  into the Euclidean space  $(\mathbb{R}^n, \langle, \rangle)$  is denoted by  $I(M, \mathbb{R}^n)$ . This set is open in  $C^\infty(M, \mathbb{R}^n)$  with respect to Whitney's  $C^\infty$ -topology. Fix an immersion  $i \in I(M, \mathbb{R}^n)$  and denote its connected component in  $I(M, \mathbb{R}^n)$  by  $O_i$ . Given  $j \in O_i$ . The differential  $dj$ , the principal part of the tangent map  $T_j = (j, dj)$ , can be expressed via  $di$  as

$$1) \quad dj = g \cdot di \cdot f .$$

$f$  is a strong smooth bundle isomorphism (cf. [G,H,K] of  $TM$  which restricted to each tangent space of  $M$  is selfadjoint with respect to  $m(i)$  and  $g \in C^\infty(M, SO(n))$ . Equation (1) reads as  $dj(v_p) = g(p)(di f(v_p))$  when applied to  $v_p \in T_p M$  with  $p \in M$ . We refer to [Bi] for a verification of (1).

Next let  $m: I(M, \mathbb{R}^n) \rightarrow M(M)$  be the map sending each  $j \in I(M, \mathbb{R}^n)$  into  $m(j) = j^*\langle, \rangle$ . By  $M(M)$  we denote the Fréchet manifold

of all smooth Riemannian metrics on  $M$ . Again it carries Whitney's  $C^\infty$ -topology. Certainly  $j \in m^{-1}(m(i))$  iff  $f = \text{id}_{TM}$ . Thus

$$dj = g \cdot di \text{ for } j \in m^{-1}(m(i)).$$

Denote  $\{dj | j \in m^{-1}(m(i))\}$  by  $m^{-1}(m(i))/\mathbb{R}^n$ .

2)  $T_i m^{-1}(m(i)), T_{di} m^{-1}(m(i))/\mathbb{R}^n$  and the normal form of  $dh$

$m : I(M, \mathbb{R}^n) \rightarrow M(M)$  is a smooth map between smooth Fréchet manifolds. Here smoothness is meant in the sense of [Gu]. Let

$$T_i m^{-1}(m(i)) := \{h \in C^\infty(M, \mathbb{R}^n) | Dm(i)(h) = 0\}$$

where  $Dm(i)(h)$  is the derivative of  $m$  at  $i$  applied to

$h \in C^\infty(M, \mathbb{R}^n)$ . . . Refer to  $\{dh | h \in T_i m^{-1}(m(i))\}$  by  $T_{di} m^{-1}(m(i))/\mathbb{R}^n$ .

$Dm(i)(h)$  evaluated at any two members  $X, Y$  in the collection  $\text{sec } TM$  of all smooth vector fields on  $M$  yields

$$Dm(i)(h)(X, Y) = \langle dhX, diY \rangle + \langle dhY, diX \rangle.$$

Clearly  $Dm(i)(h) = 0$  iff  $\langle dhX, diY \rangle$  is skew-symmetric in  $X$  and  $Y$ . Let  $\omega_h$  therefore be the two-form

$$\omega_h(X, Y) = \langle dhX, diY \rangle$$

associated with  $h \in T_i m^{-1}(m(i))$ . This form is exact.

There is a strong bundle map  $C : TM \rightarrow TM$ , skew-symmetric with respect to  $m(i)$ , for which

$$\omega_h(X, Y) = m(i)(CX, Y).$$

The differential  $dh$  of any  $h \in T_i m^{-1}(m(i))$  splits therefore into

$$2) \quad dh = c \cdot di + di \cdot C$$

with  $c \in C^\infty(M, \text{so}(n))$  where  $\text{so}(n)$  denotes the Lie algebra of  $SO(n)$ . Without loss of generality we can in addition assume that

for each  $p \in M$

$$c(p)(di T_p M) \subset v_p(i) \quad \text{and} \quad c(p)(v_p(i)) \subset di T_p M .$$

$v_p(i)$  is the normal space to  $di(T_p M) \subset \mathbb{R}^n$ , defined via the Gauss map of  $i$ . We refer to (2) as the normal form of  $dh$ .

### 3) On the structure of the normal form

Throughout the section let  $h \in T_i m^{-1}(m(i))$  whose differential is given in the normal form  $dh = c \cdot di + di \cdot C$ . There is an  $s \in C^\infty(M, so(n))$  with

$$3) \quad s \cdot di = c \cdot di + di \cdot C .$$

Observe that for all  $X, Y, Z \in TM$ .

$$4) \quad \langle ds(X) \cdot diY, diZ \rangle = 0 .$$

This, together with the fact that  $S\omega_h = 0$  yields

$$\langle c S(i)(X, Y), diZ \rangle = \langle c S(i)(X, Z), diY \rangle .$$

$S(i)(X, Y)$  stands for the pointwise normal component of  $d(diY)(X)$ .

To interpret  $C$  decompose  $h$  into

$$h = di X_h + h^\perp ,$$

where  $X_h \in \text{sec } TM$  and  $h^\perp$  is the pointwise normal component.

$$\text{Then } 5) \quad CX = \nabla(i)_{X_h} X_h + BX ,$$

where  $B : TM \rightarrow TM$  is a strong bundle endomorphism given by the equation

$$m(i)(BX, Y) = -\frac{1}{2} L_{X_h} (m(i))(X, Y) .$$

$$6) \quad L_{X_h} (m(i))(X, Y) = 2 \langle h^\perp, S(i)(X, Y) \rangle .$$

Equation (6) shows that in case of a sufficiently high co-dimension  $T_i m^{-1}(m(i))$  is infinite dimensional. It also describes under which circumstances  $X_h$  is an infinitesimal isometry.

4) The integration of  $h \in T_i m^{-1}(m(i))$

We will show that

$$(\exp \circ s) di$$

(with  $s$  introduced as in equation (3)) is the differential of an immersion  $j \in m^{-1}(m(i))$ . By

$$\exp : so(n) \longrightarrow SO(n)$$

we denote the exponential map. By [D] its differentiated  $d(\exp)$  at  $\varphi$  in the direction of  $\psi$  is determined by

$$(\exp - \varphi) d \exp(\varphi)\psi = \left( \sum_0^{\infty} \frac{1}{(m+1)!} (\text{ad}(-\varphi))^m \right) \cdot \psi.$$

Given  $s : M \longrightarrow so(n)$  and  $X \in \text{sec } TM$ , the chain rule then yields

$$\exp(-s) \cdot d(\exp \circ s)X = \left( \sum_0^{\infty} \frac{1}{(m+1)!} (\text{ad}(-s))^m \right) \cdot ds(X)$$

or for the sake of simplicity,

$$d(\exp \circ s)X = a(s) \cdot ds(X)$$

for some  $a(s) \in \text{End } so(n)$ . Since  $s \cdot di = dh$ , the exterior differential  $\delta(s \cdot di)$  has to vanish:

$$\delta((\exp \circ s) \cdot di)(X, Y) = 0.$$

Observe furthermore that  $di$  and  $(\exp \circ s) \cdot di$  belong to the same cohomology class if  $s$  is near enough to the constant map  $\overline{id} : M \longrightarrow SO(n)$  assuming  $id$  as its value. Thus we have:

Theorem: Given any  $h \in T_i m^{-1}(m(i))$ , there is a unique  $s \in C^\infty(M, so(n))$  such that  $s \cdot di$  represents the normal form of  $dh$  and  $(\exp \circ s) \cdot di$  is the differential of an immersion in  $m^{-1}(m(i))$  if  $h$  is close enough to  $o \in T_i m^{-1}(m(i))$ .

5)  $m^{-1}(m(i))$ , a Fréchet manifold

Again let  $h \in T_i m^{-1}(m(i))$  and let  $s \cdot di$  represent the normal form of  $dh$  given by  $dh = c \cdot di + di \cdot C$ . To construct a chart at  $di$  in  $m^{-1}(m(i))/\mathbb{R}^n$  choose any neighbourhood  $V$  of zero in  $so(n)$  which is mapped diffeomorphically under  $\exp$  onto a neighbourhood of  $id \in SO(n)$ . Then set

$$U := \{s \in C^\infty(M, so(n)) \mid s(M) \subset V\}.$$

The set

$$V := \{s \cdot di \mid s \in U\}$$

is open in  $T_{di} m^{-1}(m(i))/\mathbb{R}^n$ . To show that

$$W := \{(\exp \circ s) \cdot di \mid s \in U\}$$

is open in  $m^{-1}(m(i))/\mathbb{R}^n$ , observe that the notions introduced so far make sense in the  $C^k$ -setting for any  $k > 2, 3 \dots$ . Due to the implicit function theorem in Banach spaces the set  $W_k$  corresponding to  $W$  is open in the set corresponding to  $m^{-1}(m(i))/\mathbb{R}^n$ . Since  $W = \bigcap_2^\infty W_k$  we know that  $W$  is an open chart in  $m^{-1}(m(i))/\mathbb{R}^n$  of  $di$  with the chart map

$$(\exp^{-1})_* : W \longrightarrow T_{di} m^{-1}(m(i))/\mathbb{R}^n,$$

sending  $(\exp \circ s) \cdot di$  to  $s \cdot di$ .

Finally we observe that the atlas on  $m^{-1}(m(i))/\mathbb{R}^n$  given by the above construction is smooth in the sense of [Gu].

So we may state

Theorem:  $m^{-1}(m(i))/\mathbb{R}^n$  and in turn  $m^{-1}(m(i))$  are smooth Fréchet manifolds

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