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1 May/84	On bounded holomorphic reductions of homogeneous spaces B. Gilligan	175
8 May/84	Radar reception and nilpotent harmonic analysis VI W. Schempp	179
18 May/84	Trigonometrically well-bounded operators E. Berkson and T.A. Gillespie	183
23 May/84	On a fixed-point property of reflexive locally uniform convex Banach spaces. M. Edelstein	189
25 May/84	A characterization of complex projective space S.I. Goldberg	193
29 May/84	A non-Archimedean Mittag-Leffler Theorem Y. Fenevrol-Perrin and L. Haddad	199
7 June/84	A convex energy integral for the Navier-Stokes equations in three space dimensions G.F.D. Duff F.R.S.C.	205
11 June/84	Tensor products of $C^*$ -algebra fibre bundles M-D. Jean	211
12 June/84	More new identities in mixed exterior algebra J.R. Vanstone	217
	Mailing addresses	221

ON BOUNDED HOLOMORPHIC REDUCTIONS OF HOMOGENEOUS SPACES

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*Presented by P. Ribenboim, F.R.S.C.*

**Abstract.** In this note we point out that as a straightforward consequence of a recent result of K. Nakajima [4] it follows that the base of the bounded holomorphic reduction of any homogeneous complex manifold is biholomorphic to a homogeneous bounded domain.

**1. Introduction.** Given a complex space  $X$ , there may exist a complex space  $Y$  and a surjective holomorphic map  $\pi: X \rightarrow Y$  such that the induced map  $\pi^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  is an isomorphism of  $\mathcal{O}(Y)$  onto  $\mathcal{O}(X)$ . The pair  $(Y, \pi)$  is called a holomorphic reduction of  $X$ , see [5]. In the present note we restrict our attention to bounded holomorphic reductions of homogeneous complex manifolds and we also require the base  $Y$  be maximal in the sense that it is separable with respect to its algebra  $\mathcal{B}(Y)$  of bounded holomorphic functions. In this setting one has the following.

**Theorem.** Suppose  $X$  is a connected homogeneous complex manifold. Then the base of its bounded holomorphic reduction is biholomorphic to a homogeneous bounded domain.

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This characterizes homogeneous bounded domains as those homogeneous complex manifolds having the property that bounded holomorphic functions separate points. It also means that  $\mathcal{B}(X)$  is a Stein algebra for any homogeneous complex manifold. For the full algebra  $\mathcal{O}(X)$  there are similar results in certain cases, but the situation is much more complicated. As examples, there is not much known about holomorphic reductions of solv-manifolds [2] and the question of the existence of a Stein envelope of holomorphy in the general case seems to be still open (see also [1]).

Bounded holomorphic reductions of open orbits of real forms of semi-simple groups in homogeneous rational manifolds were considered by J. Wolf [7, p. 1148]. In this setting he showed that the fiber is compact and the base is a bounded symmetric domain. And under the assumption that the homogeneous space carries an invariant Kähler metric there are some results which relate to the bounded holomorphic reduction, e.g., see [4] and the references listed there.

2. Proof of the theorem. Suppose  $X$  is a connected complex manifold homogeneous under the action of its automorphism group  $\text{Aut}(X)$  and let  $\mathcal{B}(X)$  denote the algebra of bounded holomorphic functions on  $X$ . For  $x_1, x_2 \in X$  define  $x_1 \sim x_2$  whenever  $f(x_1) = f(x_2)$  for every  $f \in \mathcal{B}(X)$ . It is easy to see that this equivalence relation is an open analytic equivalence relation on  $X$ . Hence, by a result of B. Kaup [3], the quotient space  $Y := X/\sim$  has a

natural complex structure and the canonical map  $\pi: X \rightarrow Y$  is holomorphic. The existence of the bounded holomorphic reduction was also noted in [6], but under the unnecessary assumption that a Lie group is acting transitively.

Now since  $Y$  is separable with respect to  $B(Y)$ , its Carathéodory pseudodistance is a distance and hence  $Y$  is hyperbolic. Thus  $\text{Aut}(Y)$  acts as a group of isometries and is, in fact, a Lie group. Also it is clear that any biholomorphic map of  $X$  onto itself maps any two points which are equivalent under  $\sim$  into equivalent points. Thus the map  $\pi$  is  $\text{Aut}(X)$ -equivariant and so  $Y$  is homogeneous. The result now follows, since K. Nakajima [4] has recently proved that a homogeneous hyperbolic manifold is biholomorphic to a homogeneous bounded domain. □

3. Remarks. Since the base  $Y$  of the bounded holomorphic reduction is topologically a cell, it follows that the fibers of the map  $\pi: X \rightarrow Y$  are connected. And because  $X$  is homogeneous and  $\pi$  is  $\text{Aut}(X)$ -equivariant, these fibers are also biholomorphic to one another and themselves homogeneous.

Now, at least in the case where a connected Lie group is acting holomorphically and transitively on  $X$ , a reasonable conjecture is that the fibers themselves have no non-constant bounded holomorphic functions. We intend to return to this at a later time.

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RADAR RECEPTION AND NILPOTENT HARMONIC ANALYSIS VI

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Since 1953 when Woodward [6] introduced the notion of radar auto-ambiguity function  $H(f; \dots)$  with respect to a square-integrable signal envelope  $f$  on the real line  $\mathbb{R}$ , the following problem was left open: Give a complete characterization of those complex-valued functions  $F$  on the time-frequency plane  $\mathbb{R} \otimes \mathbb{R}$  which occur as radar auto-ambiguity functions  $H(f; \dots)$  with respect to complex envelopes  $f \in L^2(\mathbb{R})$  of signals with finite energy. P.M. Woodward in his 1953 classic cited above, L. Auslander in his 1982 address 'Generalized special functions and the Wigner transform' at the 797th Meeting of the American Mathematical Society at the University of Maryland, College Park, M.R. Feix (oral communication, Université d'Orléans) and numerous other research workers in the field of quantum mechanics, signal theory, radar detection, and electro-acoustics emphasized the importance of this characterization problem from the mathematical as well as from the electrical engineering point of view. - The aim of the present part of this series of papers devoted to radar reception and nilpotent harmonic analysis is to announce a solution of the problem. As in the preceding parts of the series the basic tool is the harmonic analysis on the three-dimensional real Heisenberg nilpotent group  $\tilde{A}(\mathbb{R})$ .

Definition 1. Let the non-degenerate skew symmetric bilinear form on  $\mathbb{R} \otimes \mathbb{R}$

$$B(v_1, v_2) = B\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = x_1 y_2 - x_2 y_1$$

denote the standard symplectic form of the time-frequency plane  $R \otimes R$  - then a continuous complex-valued function  $F$  on  $R \otimes R$  is said to be of **positive type** on the two-dimensional real symplectic vector space  $(R \otimes R; B)$  if, for any finite sequence of vectors  $(v_j)_{1 \leq j \leq m}$  of  $R \otimes R$ , the matrix

$$(e^{-\pi i B(v_j, v_k)} F(v_j - v_k))_{\substack{1 \leq j \leq m \\ 1 \leq k \leq m}}$$

is positive hermitean.

**Definition 2.** A continuous complex-valued function  $F$  of positive type on the two-dimensional real symplectic vector space  $(R \otimes R; B)$  is said to be **minimal** (or **pure**) if in every decomposition

$$F = F_1 + F_2$$

of  $F$  into a sum of two continuous functions  $F_1$  and  $F_2$  of positive type on  $(R \otimes R; B)$  the functions  $F_1$  and  $F_2$  are necessarily proportional to the given function  $F$ .

Now we are in a position to announce the aforementioned characterization result.

**Theorem.** Let  $\mathcal{D}(R \otimes R)$  denote the Schwartz space of rapidly decreasing complex-valued  $\mathcal{C}^\infty$ -functions on the time-frequency plane. Let the function  $F \in \mathcal{D}(R \otimes R)$  be given - then there exists a complex signal envelope function  $f \in \mathcal{D}(R)$  such that

$$F = H(f; \dots)$$

holds if and only if  $F$  is a minimal function of positive type on the two-dimensional real symplectic vector space  $(R \otimes R; B)$ . Moreover, the function  $F$  determines  $f$  uniquely up to a phase factor  $c \in \mathbb{C}$ ,  $|c| = 1$ , i.e., every other envelope function  $g \in \mathcal{D}(R)$  such that the identity

$$F = H(g; \dots)$$

holds satisfies  $g = cf$ .

Corollary. A function  $G \in \mathcal{A}(\mathbb{R} \oplus \mathbb{R})$  can be represented as a Wigner quasi-probability distribution function

$$G = P(g; \dots)$$

with respect to a state  $g \in \mathcal{A}(\mathbb{R})$  if and only if the Euclidean Fourier cotransform  $\mathcal{F}_{\mathbb{R}^2} G$  of  $G$  is a minimal function of positive type on the two-dimensional real symplectic state space  $(\mathbb{R} \oplus \mathbb{R}; B)$ .

The proof of the preceding characterization theorem depends upon the strong Stone-von Neumann theorem which on its part is a consequence of the Kirillov theory for simply connected nilpotent real Lie groups admitting generically flat coadjoint orbits in the vector space dual of their Lie algebras (cf. Howe [1] and Moore-Wolf [2]), and upon the classical Gelfand-Segal construction. Replacing  $\tilde{A}(\mathbb{R})$  by the  $(2n+1)$ -dimensional real Heisenberg nilpotent group  $\tilde{A}(\mathbb{R}^n)$ , an extension of the theorem to general  $2n$ -dimensional real symplectic vector spaces  $\mathbb{R}^n \oplus \mathbb{R}^n$  can be easily done.

The Definitions 1 and 2 supra suggest to introduce the notion of minimal distribution of positive type on the two-dimensional real symplectic vector space  $(\mathbb{R} \oplus \mathbb{R}; B)$ . These complex distributions which can be proved to be automatically tempered on  $\mathbb{R} \oplus \mathbb{R}$  form a link to the theory of Schwartz kernels. Since the minimal distributions of positive type on  $(\mathbb{R} \oplus \mathbb{R}; B)$  include also the limits of the Gabor functions they represent a natural framework for establishing an extended version of Hudson's theorem concerning the positivity of the Wigner quasi-probability distribution functions.

For some further details the reader is referred to the articles [3], [4] and [5]. Proofs and additional material can be found in the author's monograph on 'Harmonic analysis on the Heisenberg group with applications to signal theory' which is presently under preparation.

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## TRIGONOMETRICALLY WELL-BOUNDED OPERATORS

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**Abstract:** Operators of the form  $e^{iA}$  with  $A$  well-bounded of type (B) occur naturally in analysis and are associated with useful spectral decompositions. We characterize such operators (called "trigonometrically well-bounded") in terms of an intrinsic functional calculus defined for absolutely continuous functions on the circle. A second characterization of trigonometrically well-bounded operators, in terms of a suitable "unitary-like" Cartesian decomposition, is also described.

1. **Introduction:** An operator  $T$  on a Banach space  $X$  is well-bounded ([10], [11]) provided that for some compact interval  $J = [a, b]$  of the real line  $\mathbb{R}$ ,  $T$  has an  $AC(J)$ -functional calculus  $\psi$ , where  $AC(J)$  denotes the Banach algebra of absolutely continuous functions on  $J$ . A valuable subclass consisting of the well-bounded operators of type (B) can be defined by further requiring that  $\psi$  be weakly compact [4]. A type (B) operator  $T$  can be characterized by the property that for a suitably determined (and necessarily unique) spectral family of projections  $E(\cdot) : \mathbb{R} \rightarrow \mathcal{B}(X)$  we have  $T = aE(a) + \int_J \lambda dE(\lambda)$ , the integral existing as a strong limit of Riemann-Stieltjes sums (see the discussion in [1, §2]). In recent years operators  $S$  of the form  $S = e^{iA}$  where  $A$  is of type (B), which we

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shall call trigonometrically well-bounded, have been shown to occur naturally in analysis. For example, if  $1 < p < \infty$ , then all translation operators on  $L^p(G)$  ( $G$  an arbitrary locally compact abelian group) and all surjective isometries of the Hardy space  $H^p$  are trigonometrically well-bounded (see, respectively, [9] and the discussion in [2], [3]). Moreover, with the aid of Theorem (2.1) below, it can be shown that the ergodic flows on  $L^p$  spaces considered in [7] consist of trigonometrically well-bounded operators and satisfy the hypotheses of the (Banach space) generalization of Stone's Theorem in [1, Theorem (4.20)]. More generally, it can be seen from (2.1), [8], and [6, Theorem 3.15] that the (abelian) group representations underlying transference methods in reflexive  $L^p$  spaces [6, §3] consist of trigonometrically well-bounded operators. In a forthcoming paper outlined here [5], we obtain an intrinsic characterization of trigonometrically well-bounded operators in terms of an  $AC(\mathbb{T})$ -functional calculus ( $\mathbb{T}$  denotes the unit circle). This characterization is described in Theorem (2.1) below. In this regard we mention (see [10, §8.1]) that attempts to extend the notion of well-boundedness by means of a functional calculus, even to operators with spectra filling out a closed curve, face serious obstacles. The sufficiency proof in Theorem (2.1) automatically provides the spectral family of projections associated with the (normalized) logarithm of a trigonometrically well-bounded operator. With obvious modifications this sufficiency demonstration affords a simpler, more self-contained existence proof than hitherto available for the spectral family of a type (B) operator. A second characterization of trigonometrically well-bounded operators from [5] is described in Theorem (2.5) below. This is stated in terms of a "unitary-like" Cartesian decomposition.

2. The AC( $\mathbb{T}$ )-Functional Calculus. AC( $\mathbb{T}$ ) is a complex Banach algebra under the norm  $\|f\|$ , where  $\|f\| = |f(1)| + \text{var}(f, \mathbb{T})$ . Let  $B(X)$  denote the algebra of bounded operators on a Banach space  $X$ . An AC( $\mathbb{T}$ )-functional calculus for  $S \in B(X)$  is a norm-continuous homomorphism  $\phi$  of AC( $\mathbb{T}$ ) into  $B(X)$  such that  $\phi(e_0) = I$ ,  $\phi(e_1) = S$ , where  $e_0(z) \equiv 1$ ,  $e_1(z) \equiv z$ . We call  $\phi$  weakly compact provided that for each bounded set  $M$ ,  $w\text{-cl. } \phi(M)$  is compact in the weak operator topology ("w-cl." denotes closure in that topology).

(2.1) Theorem. Let  $S \in B(X)$ . In order that  $S$  be trigonometrically well-bounded it is necessary and sufficient that  $S$  have a weakly compact AC( $\mathbb{T}$ )-functional calculus  $\phi$ .

Sketch of Proof. After attention to some technical details, the necessity proof follows readily from the AC( $[0, 2\pi]$ )-functional calculus for a suitable  $A$  such that  $S = e^{iA}$ . In order to set the stage for sufficiency, three lemmas are required.

(2.2) Lemma. Suppose  $A$  is an algebra over  $\mathbb{R}$  with identity  $I$ , and  $K$  a subset of  $A$  such that each of  $K$ ,  $I - K$  is closed under multiplication. Then every extreme point of  $K$  is idempotent.

(2.3) Lemma. Suppose  $\psi$  is an AC( $\mathbb{T}$ )-functional calculus for  $U \in B(X)$ . Then  $\sigma(U)$ , the spectrum of  $U$ , is a subset of  $\mathbb{T}$ . If  $g \in AC(\mathbb{T})$  and vanishes on an open set (in the topology of  $\mathbb{T}$ ) that contains  $\sigma(U)$ , then  $\psi(g) = 0$ .

(2.4) Lemma. If, under the hypotheses of (2.3),  $\sigma(U) = \{\lambda_0\}$ , then  $U = \lambda_0 I$ .

The first lemma is a result of C. K. Fong and L. Lam [8, proof of Prop. 2.2]. Lemma (2.3) is established by use of Liouville's Theorem. Lemma (2.4) is obtained (for  $\lambda_0 = 1$ ) by constructing a sequence  $\{f_n\}$  in  $AC(\mathbb{T})$  such that  $\|e_1 - e_0 - f_n\| \rightarrow 0$  and each  $f_n$  vanishes on an open arc containing 1. The proof of (2.4) is completed by invoking (2.3) to get  $\psi(f_n) = 0$ . Returning to the sufficiency proof, we associate with each  $f \in AC(\mathbb{T})$  the function  $\tilde{f}$ , where  $\tilde{f}(t) = f(e^{it})$  for  $0 \leq t \leq 2\pi$ . For  $\lambda \in [0, 2\pi)$  and  $0 < \delta < 2^{-1}(2\pi - \lambda)$ , let  $F_{\lambda, \delta}$  be the set of all real-valued  $f \in AC(\mathbb{T})$  such that  $\tilde{f} = 1$  on  $[0, \lambda]$ ,  $\tilde{f} = 0$  on  $[\lambda + \delta, 2\pi - \delta]$ ,  $\tilde{f}$  is decreasing (resp., increasing) on  $[\lambda, \lambda + \delta]$  (resp.,  $[2\pi - \delta, 2\pi]$ ). Let  $K_{\lambda, \delta} = \text{w-cl. } \{\phi(f) : f \in F_{\lambda, \delta}\}$ , and put  $K_\lambda = \bigcap_{\delta} K_{\lambda, \delta}$ . Each  $K_\lambda$  is weakly compact, convex, and non-void. If  $E$  and  $F$  are idempotents in  $K_\lambda$ , a technical argument gives  $\sigma(S|F(I - E)X) \subseteq \{e^{i\lambda}, 1\}$ . With the aid of Lemma (2.4) it can be shown from this that  $E = F$ . By Lemma (2.2) and the Krein-Milman Theorem we now see that  $K_\lambda$  consists of a single projection, denoted  $E(\lambda)$ . Further arguments establish that  $E(\cdot)$  gives rise to a spectral family and that  $S = e^{iA}$ , where  $A = \int_{[0, 2\pi]} \lambda dE(\lambda)$ .

(2.5) Theorem. Let  $S \in B(X)$ . Then  $S$  is trigonometrically well-bounded if and only if there are commuting type (B) operators  $A, B$  such that  $T = A + iB$  and  $A^2 + B^2 = I$ . If this is the case,  $A$  and  $B$  are uniquely determined.

Comments on the Proof. If  $S$  is trigonometrically well-bounded, then  $S = e^{iC}$ , where  $C$  is of type (B) with  $\sigma(C) \subseteq [0, 2\pi]$ ,  $\ker(C - 2\pi I) = 0$  (see [1, §3]). It can be shown that  $\cos C$  and  $\sin C$  satisfy the requirements for  $A$  and  $B$ . The proof of the converse, though more lengthy and

complicated, proceeds from the simple observation that  $A = \cos D$  for a suitable  $D$  of type (B). We omit further details.

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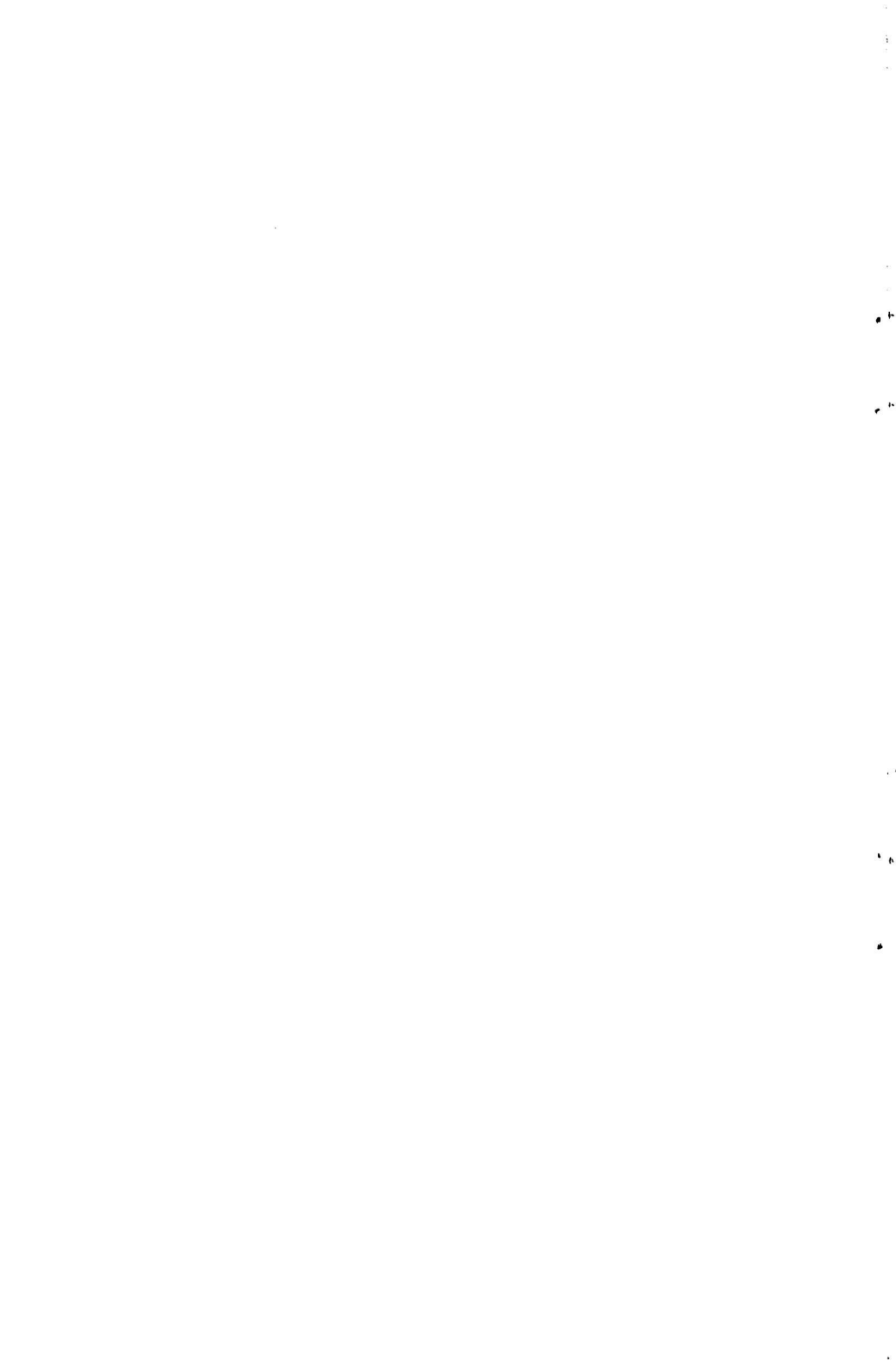
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ON A FIXED-POINT PROPERTY OF REFLEXIVE LOCALLY  
UNIFORM CONVEX BANACH SPACES

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1. In [2] we discussed certain fixed-point properties of strictly convex Banach spaces  $X$  under nonexpansive mappings  $f$  having the property that, for some  $x \in X$ , the sequence of iterates  $\{f^n(x)\}$  has a cluster point. If  $y \in X$  is such a cluster point and  $\{f^n(y)\}$  is bounded then  $\overline{\text{co}}\{f^n(y)\}$ , the closed convex hull of the orbit was shown to contain a fixed point under  $f$ . In this note we are concerned with the much wider class of mappings  $f$  satisfying the following condition:

(\*) For every  $x, y \in X$  and for each  $\eta > 0$  there is an  $n_0$  such that

$$\|f^n(x) - f^n(y)\| \leq (1 + \eta)\|x - y\| \quad (n \geq n_0).$$

Fixed point theorems for such mappings were given in [3] where they were first introduced and termed ultimately nonexpansive. The assumption on the existence of a precompact orbit, which was used in [3], is replaced here by the weaker requirement of [2], i.e. that some sequence of iterates have a cluster point generating a bounded orbit. Specifically, we prove the following:

Theorem. Let  $X$  be a reflexive locally uniformly convex Banach space and let  $f: X \rightarrow X$  be a continuous ultimately nonexpansive mapping (i.e. such that (\*) is satisfied). Suppose, further, that a  $y \in X$  exists such that  $y$  is a cluster point of a sequence of iterates  $\{f^n(x)\}$  for some  $x \in X$ , and  $\{f^n(y)\}$  is bounded. Then a  $\xi \in \overline{\text{co}}\{f^n(y)\}$  exists such that  $f(\xi) = \xi$ . Furthermore, if

$\xi' \neq \xi$  and  $\xi' \in \overline{\text{co}}\{f^n(y)\}$  then  $f(\xi') \neq \xi'$ .

Thus, while the setting of local uniform convexity of  $X$  is more restrictive than that of [2], the conclusion is essentially the same for the larger class of mappings dealt with here.

2. Lemma 1. Let  $X$  be a reflexive locally uniformly convex Banach space and  $f: X \rightarrow X$  an ultimately nonexpansive mapping. Let  $x_1, x_2 \in X$  have the property that  $f^{n_i}(x_1) \rightarrow x_1, f^{n_i}(x_2) \rightarrow x_2$  as  $i \rightarrow \infty$ , where  $\{n_i\}$  is an increasing sequence of positive integers. If  $x = \lambda x_1 + (1-\lambda)x_2$  for some  $\lambda, 0 \leq \lambda \leq 1$  then  $f^{n_i}(x) \rightarrow x$  as  $i \rightarrow \infty$ .

Proof. Given  $\eta > 0$ , it follows from (\*), that there is an  $i_0$  such that, for all  $i \geq i_0$ ,

$$\|f^{n_i}(x) - f^{n_i}(x_1)\| \leq (1+\eta)\|x - x_1\|$$

and  $\|f^{n_i}(x) - f^{n_i}(x_2)\| \leq (1+\eta)\|x - x_2\|$ .

By reflexivity of  $X$  there exists a subsequence  $\{f^{n_{i_j}}(x)\}$  of  $\{f^{n_i}(x)\}$  which converges weakly to some  $w \in X$ . Since norms are weakly lower semicontinuous, it follows from the preceding inequalities (when applied to  $f^{n_{i_j}}(x)$ ) that

$$\|w - x_1\| \leq (1+\eta)\|x - x_1\|$$

and  $\|w - x_2\| \leq (1+\eta)\|x - x_2\|$ .

Hence,  $\eta > 0$  being arbitrary,

$$\|w - x_1\| \leq \|x - x_1\|, \|w - x_2\| \leq \|x - x_2\|.$$

Now  $\|x_1 - x_2\| = \|x - x_1\| + \|x - x_2\| \geq \|w - x_1\| + \|w - x_2\| \geq \|x_1 - x_2\|$ , implying equality throughout. Thus  $\|x - x_1\| = \|w - x_1\|$ ,  $\|x - x_2\| = \|w - x_2\|$ , so that  $w$  belongs to both  $B_1 = \{y \in X: \|y - x_1\| \leq \|x - x_1\|\}$  and  $B_2 = \{y \in X: \|y - x_2\| \leq \|x - x_2\|\}$ . By strict convexity of  $X$  the only point in common to  $B_1$  and  $B_2$  is  $x$ . Hence  $w = x$ . This being true for every weakly convergent subsequence of  $\{f^{n_i}(x)\}$  it readily follows that the entire sequence weakly converges to  $x$ . Now, for  $i \geq i_0$ , the sequence  $\{(1+n)^{-1}(f^{n_i}(x) - f^{x_i}(x_1))\}$ ;  $i = 1, 2, \dots$  lies in the closed ball  $B = \{y: \|y\| \leq \|x - x_1\|\}$  and is weakly convergent to the point  $x - x_1$  on its boundary. By local uniform convexity this is known to imply that the above sequence converges (in the norm topology) to the same point; i.e.

$$\lim_{i \rightarrow \infty} [(1+n)^{-1}(f^{n_i}(x) - f^{x_i}(x_1))] = x - x_1.$$

It readily follows that  $\lim_{i \rightarrow \infty} f^{n_i}(x) = x$ , as claimed.

### 3. Proof of the theorem.

The hypothesis that  $y$  is a cluster point of  $\{f^n(x)\}$  implies that  $f^{n_i}(y) \rightarrow y$  for a suitable increasing sequence of positive integers  $\{n_i\}$ . (A proof of this fact for nonexpansive mappings was given in [1]; with minor changes it applies to ultimately nonexpansive

mappings.) By the lemma the set  $S = \{z \in X: f^{n_i}(z) \rightarrow z\}$  is convex. Further,  $f|_S$  the restriction of  $f$  to  $S$ , is clearly nonexpansive. Indeed,

$$\|f(x_1) - f(x_2)\| = \|\lim_{i \rightarrow \infty} (f^{n_i+1}(x_1) - f^{n_i+1}(x_2))\| \leq \|x_1 - x_2\|.$$

Proposition 1 of [2] applies to the effect that  $f[S] \subset S$  and  $f|_S$  is an isometry of  $S$  into itself. Since isometries in strictly convex Banach spaces are always affine,  $f|_S$  is an affine isometry. By Proposition 6 of [2]  $f|_S$  extends to an affine isometry on the closure  $L$  of the affine hull of  $S$ . Propositions 8 and 9 of [2] now apply to guarantee the existence of a  $\xi$  such as in the conclusion of the theorem.

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A CHARACTERIZATION OF COMPLEXPROJECTIVE SPACE

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ABSTRACT. Complex projective space with the Fubini metric is characterized by the spectrum of the Laplacian on 2-forms in all dimensions. This was obtained by Chen and Vanhecke except in complex dimensions 2 and 8.

1. Introduction. The implications of the spectrum of the Laplacian of a smooth compact Riemannian manifold on its geometry have been extensively studied during the past twenty years. For Kaehler manifolds  $(M, g)$ , Gilkey and Sacks [5] showed that it is possible to determine whether  $g$  is of constant holomorphic curvature from the spectra of its complex Laplacian on forms of bidegree  $(q, r)$ . Since the only compact manifold with a metric of constant positive holomorphic curvature is complex projective space  $\mathbb{C}P_n$  with the Fubini-Study metric  $g_0$  this proves that if  $(M, g)$  and  $(\mathbb{C}P_n, g_0)$  are isospectral, then they are holomorphically isometric.

Let  $(M, J, g)$  be a compact connected Kaehler manifold with complex structure  $J$  and Kaehler metric  $g$ , and denote by  $\Delta = -(dd^* + d^*d)$ , where  $d$  is the operator of exterior differentiation and  $d^*$  is its adjoint with respect to the Riemannian metric  $g$ , the real Laplacian acting on  $p$ -forms. Then, for each  $p = 0, 1, 2, \dots$ , we have the spectrum of  $\Delta$ :

$$\text{Spec}^p(N, g) = \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \leq \dots \leq \lambda_{k,p} \leq \dots \rightarrow -\infty\},$$

each eigenvalue being repeated as many times as its multiplicity. Hodge

theory implies that  $0 \in \text{Spec}^p(M, g)$  if and only if the  $p$ th Betti number  $b_p(M)$  is not zero, and its multiplicity is then  $b_p(M)$ .

The following question is asked in [4]: If  $\text{Spec}^p(M, g) = \text{Spec}^p(\mathbb{C}P_n, g_0)$  for a fixed value of  $p$ , is  $(M, J, g)$  holomorphically isometric with  $(\mathbb{C}P_n, J_0, g_0)$ , where  $J_0$  is the standard complex structure of  $\mathbb{C}P_n$ ? The answer is yes in the following cases:

1.  $(M, J) = (\mathbb{C}P_n, J_0)$  and  $p = 0$  [1], [8];
2.  $p = 0$  and  $n \leq 6$  [11];
3.  $p = 1$  and  $8 \leq n \leq 51$  [12];
4.  $p = 2$  and  $n \neq 2, 8$  [4]<sup>1</sup>.

The last case is a vast improvement of the theorem of Gilkey and Sacks. In this paper, we show that the answer is yes in the exceptional dimensions of case 4 as well, thereby completely characterizing  $(\mathbb{C}P_n, J_0, g_0)$  by  $\text{Spec}^2(M, g)$  for all  $n$ . This is the only case known where the geometry of  $(M, g)$  is completely determined by  $\text{Spec}^p(M, g)$  for some fixed  $p$  and in all dimensions.

Theorem. Let  $(M, J, g)$  be a compact Kaehler manifold with  $\text{Spec}^2(M, g) = \text{Spec}^2(\mathbb{C}P_n, g_0)$ . Then,  $(M, J, g)$  is holomorphically isometric to  $(\mathbb{C}P_n, J_0, g_0)$  in all dimensions.

2. Kaehler geometry. Let  $(M, J, g)$  be a Kaehler manifold. In terms of local complex coordinates,  $g = 2 \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}$  and the Ricci tensor  $S = 2 \sum R_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}$ ,  $i, j, \dots = 1, \dots, n$ ,  $i^* = i + n$ , where the coefficients  $R_{i\bar{j}}$  are given in terms of the components  $R^1_{j\bar{k}l}$  of the Riemann curvature tensor by  $-\sum R^k_{i\bar{k}j}$  (see [5]). Denote the scalar

<sup>1</sup>The proof of Theorem 1 in [4] breaks down in dimension 2.

curvature by  $\rho = 2 \sum R_{ij}^*$ , and by  $|R|$  and  $|S|$  the norms of  $R$  and  $S$ , respectively, that is,  $|R|^2 = 4 \sum |R_{jkl}^*|^2$  and  $|S|^2 = 2 \sum |R_{ij}^*|^2$ .

Define the scalars  $\rho_1, \dots, \rho_n$  by

$$\det(\delta_{ij} + tR_{ij}^*) = \sum_{k=0}^n \binom{n}{k} \rho_k t^k.$$

Then,  $\rho_0 = 1$ ,  $\rho_1 = \rho/2n$ ,  $\rho_2 = (\rho^2 - 2|S|^2)/4n(n-1), \dots, \rho_n = \det(R_{ij}^*)$ .

Let  $\omega$  and  $c_1$ , respectively, be the fundamental and first Chern classes of  $M$ . Then,

Lemma [9]. If  $M$  is cohomologically Einstein, that is, if  $c_1 = a\omega$  for some real number  $a$ ,

$$\int_M \rho_k * 1 = (2\pi a)^k \int_M * 1,$$

where  $*1$  is the volume element.

3. Proof of Theorem. For  $p = 2$ , the Minakshisundaram-Pleijel formula is given by

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,2} t) = \frac{1}{(4\pi t)^n} \sum_{i=0}^N a_{i,2} t^i + O(t^{N-n+1}), \quad t > 0,$$

where (cf. Patodi [10]) the coefficients  $a_{i,2}$ ,  $i = 0, 1, 2$ , are given by

$$(3.1) \quad a_{0,2} = n(2n-1) \int_M * 1, \quad a_{1,2} = \frac{2n^2 - 13n + 12}{6} \int_M \rho * 1,$$

and

$$(3.2) \quad a_{2,2} = \frac{1}{360} \int_M \{5(2n^2 - 25n + 60)\rho^2 - 2(2n^2 - 181n + 540)|S|^2 + 2(2n^2 - 31n + 120)|R|^2\} * 1.$$

Since  $\text{Spec}^2(N, g) = \text{Spec}^2(\mathbb{C}P_n, g_0)$ , we have

$$(3.3) \quad \dim_{\mathbb{C}} M = n, \quad b_2(M) = b^2(\mathbb{C}P_n) = 1,$$

$$(3.4) \quad \int_M \star 1 = \int_{\mathbb{C}P_n} \star 1, \quad \int_M \rho \star 1 = \int_{\mathbb{C}P_n} \rho' \star 1, \quad a_{2,2} = a'_{2,2},$$

where the prime indicates the corresponding quantities in  $(\mathbb{C}P_n, J_0, g_0)$ .

The second part of (3.3) says that  $M$  is cohomologically Einstein. Thus, from the Lemma,

$$(3.5) \quad (n-1) \left( \int_M \rho \star 1 \right)^2 = n \left( \int_M \star 1 \right) \int_M (\rho^2 - 2|S|^2) \star 1.$$

Since  $(\mathbb{C}P_n, J_0, g_0)$  has constant holomorphic curvature, say  $H$ ,

$$\rho' = n(n+1)H, \quad |S'|^2 = n(n+1)^2 H^2/2, \quad |R'|^2 = 2n(n+1)H^2.$$

The following formula is obtained in [4]:

$$(3.6) \quad (n^2 - 1)(n - 8)(2n - 15) \int_M \left( |R|^2 - \frac{2}{n(n+1)} \rho^2 \right) \star 1 \\ + 4(2n^4 + 16n^3 - 44n^2 - 91n + 120) \int_M \left( |S|^2 - \frac{\rho^2}{2n} \right) \star 1 = 0.$$

Consider the projective curvature tensor  $W$

$$W^i_{jkl} = R^i_{jkl} + \frac{1}{n+1} (R_{jl} \delta^i_k + R_{kl} \delta^i_j).$$

This tensor vanishes if and only if the Kaehler metric has constant holomorphic curvature. Setting  $|W|^2 = 4 \sum |W^i_{jkl}|^2$ , we obtain

$$0 \leq |W|^2 = |R|^2 - \frac{4}{n+1} |S|^2,$$

so since  $|S|^2 \geq \rho^2/2n$ , it follows that  $|R|^2 \geq 2\rho^2/n(n+1)$ , and (3.6) then yields  $|S|^2 = \rho^2/2n$ ,  $|R|^2 = 2\rho^2/n(n+1)$  for  $n \neq 2, 8$ . Thus,  $(M, J, g)$  has constant holomorphic curvature  $H_1$ . From (3.4),  $H_1 = H$ , so, for  $n \neq 1, 2, 8$ ,  $(M, J, g)$  is holomorphically isometric with  $(\mathbb{C}P_n, J_0, g_0)$ .

The case  $n = 2$ : Since  $b_2(M) = 1$ , the dimension of the space  $H^{1,1}(M, \mathbb{R})$  of harmonic forms of bidegree (1,1) is 1. Multiplying the Kähler metric  $g$  by some constant, if necessary, we may therefore assume that  $g$  is a Hodge metric. Thus,  $\omega$  is a positive element of  $H^{1,1}(M, \mathbb{Z})$ , and since  $c_1 = ((n+1)H/4\pi)\omega$ , the first Chern class is positive. Consequently, by a Theorem of Kobayashi [6],  $b_1(M) = 0$ . It follows that the Euler characteristic  $\chi(M) = \chi(\mathbb{E}P_2)$ . From the Gauss-Bonnet theorem

$$\chi(M) = \frac{1}{32\pi^2} \int_M (|R|^2 - 4|S|^2 + \rho^2) *1.$$

Thus,

$$\int_M (|R|^2 - 4|S|^2 + \rho^2) *1 = \int_{\mathbb{E}P_2} (|R'|^2 - 4|S'|^2 + \rho'^2) *1$$

from which

$$\int_M |R|^2 *1 = \int_{\mathbb{E}P_2} |R'|^2 *1, \quad \int_M |S|^2 *1 = \int_{\mathbb{E}P_2} |S'|^2 *1.$$

We conclude that  $(M, J, g)$  is holomorphically isometric with  $(\mathbb{E}P_2, J_0, g_0)$ .

The case  $n = 8$ : From (3.6),

$$\int_M (|S|^2 - \frac{\rho^2}{2n}) *1 = 0$$

for  $n = 8$ , so since  $|S|^2 \geq \rho^2/2n$ , equality holds, and consequently  $g$  is an Einstein metric with positive scalar curvature by (3.4). By Theorem 5 of Kobayashi [7], a principal circle bundle  $P$  over  $M$  with projection  $\pi$  may be constructed having an Einstein metric with positive scalar curvature  $k = (2n+1)\rho/(2n+2)$ . From Cor. 4 on p. 256 in [2] the volume of  $P$  is bounded above by the volume of the sphere  $S^{2n+1}(k/2n)$  having constant curvature  $k/2n$ , that is,

$$(3.16) \quad v(P) \leq v(S^{2n+1}(k/2n)) ,$$

with equality attained only if  $P$  has constant curvature, that is, only if  $P$  is isometric with  $S^{2n+1}(k/2n)$ .

Relating  $v(M)$  and  $v(P)$  (see [3]), the metric of  $P$  being given by  $G = \pi^*g + b^2\beta^2$ , where  $\beta$  is the 1-form on  $P$  defining a connection in  $P$  and  $b$  is a positive number, we get

$$v(M) \leq v(\mathbb{E}P_n(H)) .$$

Equality is attained only if  $P$  is isometric with  $S^{2n+1}(k/2n)$ , and hence only if  $(M, g)$  is isometric with  $\mathbb{E}P_n(H)$ , so by (3.4),  $(M, J, g)$  is holomorphically isometric with  $(\mathbb{E}P_n, J_0, g_0)$ .

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## A NON-ARCHIMEDEAN MITTAG-LEFFLER THEOREM

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In the well known theorem of Mittag-Leffler on meromorphic functions of a complex variable, as well as in the corresponding result of M. Lazard [7] for meromorphic functions over (real-valued) valued fields, the sole condition bears on the distribution of poles and not on the principal parts themselves.

That the situation might be different for valued fields which are not real valued, was already suspected by the first author Y. Feneysel-Perrin, [3], [4], [5], when she gave sufficient conditions on a family of rational fractions to be the principal parts of a meromorphic function, for a special class of valued fields, (see definition below).

Here we give a condition, necessary and sufficient, involving both the distribution of poles and the principal parts, thus showing that the situation is different.

I. Notations and definitions. - Let  $K$  be an algebraically closed, complete valued field, and let  $A$  be its valuation ring. We suppose

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that  $A$  contains a decreasing sequence of prime ideals  $P_n$  whose intersection is zero. Clearly then, the height of the valuation is greater than 1. For every prime ideal  $P$  in  $A$ , let  $A_P$  denote the corresponding local ring of fractions. For the definitions of analytic elements, analytic functions, meromorphic functions, and principal parts of meromorphic functions over  $K$ , see [6], [3], [4], [5].

Definitions. - Let  $P$  be a nonzero prime ideal in  $A$ . A subset  $B$  of  $K$  is said to be  $P$ -open whenever  $B \subset A_P$  and  $a + P \subset B$  for each  $a \in B$ . So a  $P$ -open set is an "ouvert borné, d'épaisseur non nulle" in the sense of [3], [5]. An element  $b$  of  $K$  is said to be  $P$ -near an element  $a$  of  $K$  whenever  $b - a$  belongs to  $P$ .

The notion of a  $P$ -fraction we now introduce seems to play a central part in the theory of analytic and meromorphic functions over  $K$ .

A  $P$ -fraction is any rational fraction  $f \in K(X)$  such that

$$(1) \quad f(x) \in A_P \quad \text{for each } x \in P.$$

It can be shown that  $f \in K(X)$  is a  $P$ -fraction iff  $f$  is of the form

$$(2) \quad f(X) = p(X) / \prod_{i \in I} (1 - X/a_i) \quad \text{where } p(X) \in K[X]$$

and  $a_i \notin P$  for every  $i \in I$ .

## II. Properties of $P$ -fractions.

Theorem 1. - The principal part  $R$  of a  $P$ -fraction  $f$  in a  $P$ -open set  $V$  is itself a  $P$ -fraction.

For the proof, we use the theory of resultants in arbitrary commutative rings with unity as developed in [1].

Let  $(a_i)_{i \in I}$  be the family of the poles of  $f$  inside  $V$ , and let

$(b_j)_{j \in J}$  be the family of the poles of  $f$  outside  $V$ . Set

$$S(X) = \prod_{i \in I} (X - a_i) \quad \text{and} \quad T(X) = \prod_{j \in J} (1 - X/b_j).$$

So that  $f(X) = p(X)/S(X)T(X)$  with  $p(X) \in K[X]$ .

The resultant of S and T is equal to

$\prod_{i \in I} \prod_{j \in J} (1 - a_i/b_j)$  and, therefore, has its inverse in  $A_p$ . Hence

there are polynomials  $F \in A_p[X]$  and  $G \in A_p[X]$  such that  $1 = FS + GT$ .

Then  $f = (pF/T) + (pG/S)$ , and R is the fractional part of the fraction  $pG/S$ . Euclidean division of  $pG$  by the unitary polynomial S gives the result.

Turn now to the following two approximation lemmas for P-fractions which are essential in the proof of the "Mittag-Leffler theorem" below.

Lemma 1. - Given a  $\in K$ , let P be the biggest prime ideal in A not containing  $1/a$ . Let f be a P-fraction whose poles are all P-near a. Then, for any other prime ideal Q properly containing P, there exists a polynomial  $g \in K[X]$  such that

$$(3) \quad f(x) - g(x) \in Q \quad \text{for each } x \in A_Q.$$

Lemma 2. - Let P and R be prime ideals in A such that  $R \not\supseteq P$ . Given  $a \in A_p \setminus R$  and a P-fraction f whose poles are all P-near a, there must exist a prime ideal Q' properly containing P with the following property : for each prime ideal Q in A such that  $Q' \supset Q \not\supseteq P$ , and each  $c \in Q \setminus P$ , there exists a rational fraction g with a unique pole  $b = a - c$  and such that

$$(4) \quad f(x) - g(x) \in P \quad \text{for each } x \in A_Q \setminus (a+Q)$$

The proof of lemma 1 (resp. lemma 2) is first reduced to the case of a P-fraction of the form  $d/(1 - X/a)^m$  (resp. the form  $d/(X-a)^m$ ) with  $d \in A_p$  which are then developed as formal power series in  $K[[X]]$ .

III. Mittag-Leffler theorem. - In [3],[4],[5] , the first author has given sufficient conditions for a sequence of rational fractions to be the sequence of principal parts of a meromorphic function on an open set  $D$  in  $K$ . We give below a "Mittag-Leffler type" theorem with a set of three (equivalent) conditions each being both necessary and sufficient.

Given a subset  $D$  of  $K$ , by definition, the  $P$ -interior of  $D$  (to be denoted  $D(P)$ ) is the biggest  $P$ -open set contained in  $D$ . For each  $n \in \mathbb{N}$ , we write  $D_n$  for the  $P_n$ -interior of  $D$ .

Theorem 2. - Let  $D$  be an open set in  $K$ , and  $Z$  a subset of  $K$ .

For each  $a \in Z$ , let  $R_a$  be a rational fraction whose unique pole is  $a$  and having a numerator whose degree exceeds that of the denominator. For each finite subset  $F$  of  $Z$ , let  $R_F = \sum_{a \in F} R_a$ .

Suppose  $0 \in D \cap Z$ . The following are equivalent :

- (i) There exists a meromorphic function  $f$  on  $D$  whose set of poles is  $Z$  and having principal part  $R_a$  at the pole  $a$ , for each  $a \in Z$ .
- (ii) There is a cofinite subset  $T$  of  $Z$  such that, for every nonzero prime ideal  $P$  in  $A$ , and for every  $P$ -open subset  $V$  of  $D$ , the subset  $F = T \cap V$  is finite and  $R_F$  is a  $P$ -fraction.
- (iii) There is a cofinite subset  $T$  of  $Z$  such that, for each pair of nonzero prime ideals  $P$  and  $Q$ , with  $Q \subsetneq P$ , the subset  $F = T \cap (D(Q) \setminus D(P))$  is finite, and there exists a rational fraction  $S$  having no poles inside  $D$  and such that

$$R_F(x) - S(x) \in P \quad \text{for each } x \in D(P).$$

- (iv) There is a cofinite subset  $T$  of  $Z$  such that, for each  $n \in \mathbb{N}$ , the subset  $F(n) = T \cap (D_{n+1} \setminus D_n)$  is finite, and there exists a rational fraction  $S_n$  having no poles inside  $D$  and such that

$$R_{F(n)}(x) - S_n(x) \in P_n \quad \text{for each } x \in D_n.$$

Sketch of the proof. To prove (i) implies (ii), first  $f$  is suitably approximated by a  $P$ -fraction then theorem 1 is used. To prove (ii) implies (iii) use is made of lemmas 1 and 2. Clearly (iii) implies (iv). Finally, to prove (iv) implies (i), a meromorphic function  $f$  on  $D$  is explicitly constructed using  $S_n$  and  $R_{F(n)}$ .

IV. Properties of the ring  $F(D)$  of analytic functions over an open set  $D$  in  $K$ .

Theorem 3. - Let  $D$  be any open set in  $K$ .

1° If  $D$  is  $P$ -open for some nonzero prime ideal  $P$ , then  $F(D)$  is a principal ideal ring.

2° Otherwise,  $F(D)$  is not a Bezout ring, not even a Prüfer ring.

Proof. The first assertion is proved in [3],[4],[5]. To prove the second assertion, we construct two analytic functions  $f$  and  $g$  "very near to each other" over  $D$  such that  $f$  (resp.  $g$ ) has an infinite sequence  $(a_n)$  (resp.  $(b_n)$ ) of simple poles in  $D$ ,  $a_n$  being  $P_n$ -near  $b_n$ . It is then shown that the ideal generated by  $f$  and  $g$  were it principal would be equal to the ring  $F(D)$  itself, and there would exist analytic functions  $u$  and  $v$  such that  $uf + vg = 1$ , that is  $1/fg = u/g + v/f$ . Computing the principal part of  $v/f$ , it is ultimately shown that it is not a  $P_n$ -fraction thus contradicting "Mittag-Leffler theorem". above. (See [2] for the notion of Bezout and Prüfer rings).

Finally, as a consequence of the "Weierstrass type" theorem on zeros of analytic functions (see [3],[4],[5]) we obtain the following result. The proofs are expected to appear in a forthcoming paper.

Theorem 4. - The ring  $F(D)$  is pseudo-principal, and, therefore, pseudo-Bezout. It is also completely integrally closed. It is a unique factorization ring iff  $D$  is  $P$ -open for some nonzero prime ideal  $P$ .

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## A CONVEX ENERGY INTEGRAL FOR THE NAVIER-STOKES EQUATIONS IN THREE SPACE DIMENSIONS

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Respectfully dedicated to Professor J. Leray fifty years after the publication of his remarkable memoir in Acta Mathematica.

### Abstract

The energy integral and the inequality for the rate of viscous energy dissipation are linearized by an exchange of independent and dependent variables, and integrated. The resulting energy relation contains a convex function of the energy variable, and permits an elementary analysis of the singularities and asymptotic properties of Navier Stokes flows.

### 1 The Navier Stokes equations and the energy estimates.

Let  $\Omega$  be a suitable smooth domain in  $\mathbb{R}^3$ . The initial boundary value problem for the Navier-Stokes flows of a viscous, incompressible fluid in  $\Omega$ , with velocity vector  $u_i(x,t)$ ,  $x \in \Omega$ ,  $t > 0$ , is given by

$$(1.1) \quad \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = - \frac{\partial p}{\partial x_i} + \nu \Delta u_i \quad i,k=1,2,3$$

where  $p(x,t)$  is the pressure function,  $\Delta$  the Laplace operator in three space dimensions, and  $\nu$  the viscosity. The condition of incompressibility is

$$(1.2) \quad \frac{\partial u_i}{\partial x_i} = 0,$$

the boundary condition

$$(1.3) \quad u_i = 0 \quad \text{on } \partial\Omega, t > 0,$$

and the initial condition

$$(1.4) \quad u_i(x,0) = u_{i0}(x) \in L^2(\Omega).$$

We shall assume no external forces. Let

$$(1.5) \quad \|u\|_2 = \left[ \int_{\Omega} |u|^2 dV \right]^{1/2}$$

this norm being a function of the time variable, with a corresponding definition for the gradient norm  $\|\nabla u\|_2$  in the appropriate Hilbert space of vector functions square-integrable over the space domain  $\Omega$ . We refer to Leray [8] as well as later sources [1,3,4,6,7,11,12,13] for precise descriptions of the solutions of the initial boundary value problem and their properties. We shall assume the solution vector  $u_i(x,t)$  is a smooth function in  $\Omega \times [0, \infty)$  except for a possible singular set of dimension at

most one [1,4,9,10].

The basic energy integral is obtained on multiplying (1.1) by  $u_t(x,t)$  and integrating over  $\Omega$ , in the form

$$(1.6) \quad \frac{d}{dt} \|u\|_2^2 = -2\nu \|\nabla u\|_2^2$$

The rate of viscous energy dissipation satisfies an inequality obtained by multiplication of (1.1) by  $\Delta u_t$ , or  $Au_t$ , the solenoidal Laplacian [3,4,7,12,13] followed by integration over  $\Omega$  and elementary estimations of the non-linear terms, in the form [2,6,8,12,13]

$$(1.7) \quad \frac{d}{dt} \|\nabla u\|_2^2 + \nu \|\Delta u\|_2^2 \leq K \|\nabla u\|_2^2,$$

Adjoining the inequality

$$(1.8) \quad \begin{aligned} \|\nabla u\|_2^2 &= (\nabla u, \nabla u) = -(u, \Delta u) \\ &= -(u, \Delta u) \leq \|u\|_2 \|\Delta u\|_2 \end{aligned}$$

we obtain also the modified inequality

$$(1.9) \quad \frac{d}{dt} \|\nabla u\|_2^2 + \nu \frac{\|\nabla u\|_2^4}{\|u\|_2^2} \leq K \|\nabla u\|_2^2.$$

After division by  $\|\nabla u\|_2^4$  we may write this relation in the form

$$(1.10) \quad \frac{-d}{dt} \frac{1}{\|\nabla u\|_2^2} + \frac{\nu}{\|u\|_2^2} \leq K \|\nabla u\|_2^2.$$

Upon integration over an interval  $(t_0, t)$  this yields

$$(1.11) \quad \frac{1}{\|\nabla u_0\|_2^2} + \nu \int_{t_0}^t \frac{dt'}{\|u\|_2^2} \leq K \int_{t_0}^t \|\nabla u\|_2^2 dt' + \frac{1}{\|\nabla u\|_2^2}$$

## 2. Linearization and Integration

Relations (1.6) and (1.9) together form a single differential inequality of second order, which may be integrated as follows: let  $y = \|u\|_2^2$  so that, with a prime denoting  $d/dt$ ,

$$(2.1) \quad y' = \frac{dy}{dt} = -2\nu \|\nabla u\|_2^2 < 0.$$

Then (1.9) yields

$$(2.2) \quad -\frac{1}{2\nu} \frac{dy'}{dt} + \frac{\nu}{y} \left( \frac{y'}{-2\nu} \right)^2 \leq K \left( \frac{y'}{-2\nu} \right)^3.$$

However,

$$(2.3) \quad \frac{dt}{dy} = \frac{1}{y'}, \quad \frac{d^2t}{dy^2} = \frac{-1}{(y')^3} y''$$

so that if we divide (2.2) by  $(-y'/2\nu)^3$  there is found

$$(2.4) \quad -4\nu^2 \frac{d^2 t}{dy^2} - 2\nu^2 \frac{1}{y} \frac{dt}{dy} \leq K.$$

To simplify further we introduce the variable  $z$ , where  $z^2 = y = \|u\|_2^2$ , and (2.4) becomes

$$(2.5) \quad \nu^2 \frac{d^2 t}{dz^2} + Kz^2 \geq 0.$$

In the following statement we summarize deductions that can now be made by elementary means.

**Theorem** Let  $z = \|u\|_2$  where  $u(x,t)$  is a Navier Stokes flow in  $\Omega$  for  $t > 0$  with  $u(x,0) = u_0(x) \in L^2(\Omega)$ . Then

(a) the expression

$$(2.6) \quad t + cz^4 = C(z)$$

is a continuous convex function of  $z$ , where  $c = K/12\nu^2$ , and  $0 \leq z_\infty < z \leq z_0 = \|u_0\|$ ,  $C(z_0) = cz_0^4$ .

(b) the derivative

$$(2.7) \quad \frac{dt}{dz} + 4cz^3 = C'(z).$$

is an increasing function of  $z$ , and any singularities of the solution lie on the  $z$ -interval where  $C'(z) > 0$ .

(c) In general

$$(2.8) \quad \frac{dt}{dz} = \frac{-z}{\nu \| \nabla u \|_2^2} \leq 0$$

with equality only at singular values  $z_T$ , where  $C'(z_T) = 4cz_T^3$ . If  $z > z_T$  (or  $t < T$ ) we have

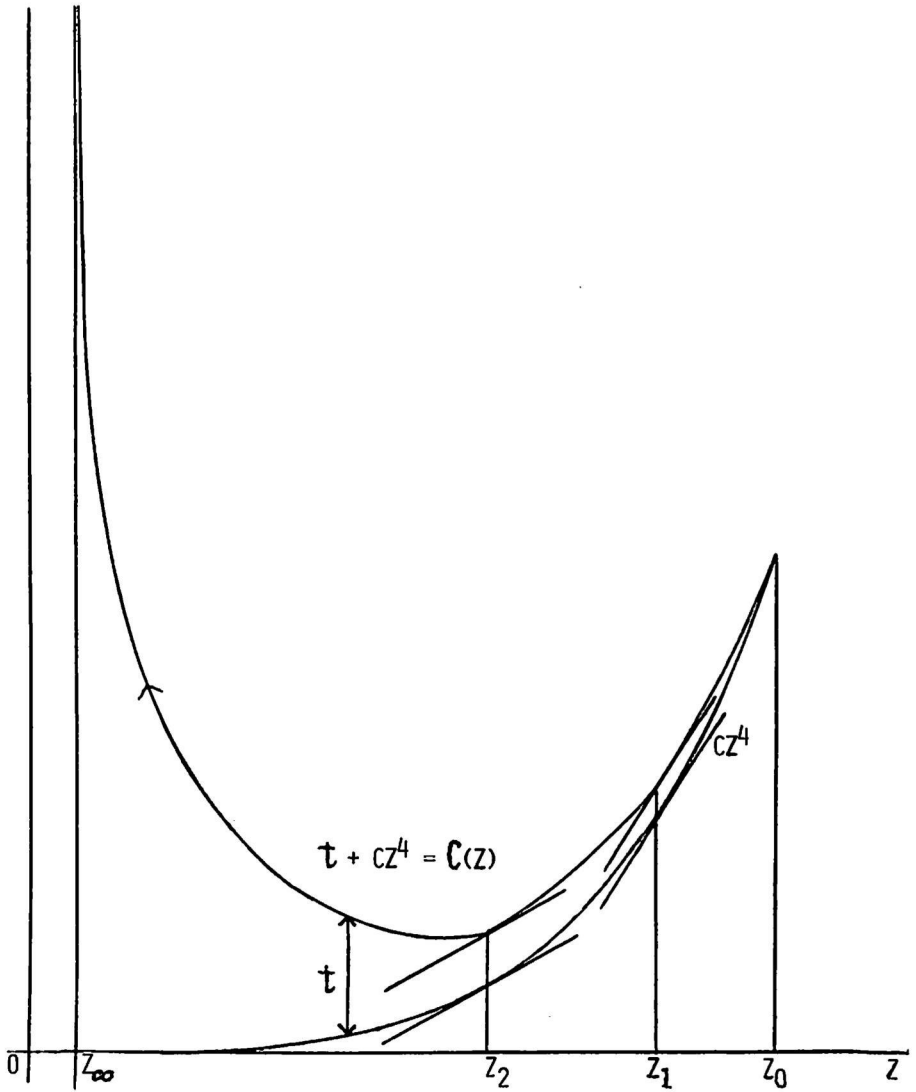
$$(2.9) \quad 0 \geq \frac{dt}{dz} \geq 4c \{ z_T^3 - z^3 \}$$

whence

$$(2.10) \quad \| \nabla u \|_2^2 \geq \frac{z}{4c\nu(z^3 - z_T^3)}, \quad z > z_T$$

If  $\frac{dt}{dz}$  vanishes to order  $(z - z_T)^n$  as  $z \rightarrow z_T + 0$ , then  $T - t$  vanishes to order  $(z - z_T)^{n+1}$  and  $\| \nabla u \|_2^2 = 0 \left[ (T - t)^{-n/n+1} \right]$ ,  $n = 1, 2, \dots$ . Singularities of this order can occur on a set of time dimension at most  $1/n+1$ .

d) If  $T$  is a singular instant and  $t < T$  then



GRAPH OF A CONVEX ENERGY INTEGRAL  
WITH TWO SINGULAR VALUES.

$$(2.11) \quad \frac{1}{\|\nabla u\|_2^2} \leq \int_t^T \left[ K \|\nabla u\|_2^2 - \frac{\nu}{\|u\|_2^2} \right] dt'$$

- e) The limits  $\lim_{t \rightarrow T_1+0} \|\nabla u\|_2^2 = \|\nabla u\|_2^2(T_1+0)$  exist, finite or infinite, and

$$(2.12) \quad \sum_j \frac{1}{\|\nabla u\|_2^2(T_j+0)} \leq \frac{4c\nu z_0^3}{z_{\min}},$$

where  $z_{\min}$  is the minimum point of  $C(z)$ , and the summation is taken over all singular values.

- f) If  $K\|\nabla u\|_2^2 \|u\|_2^2 \leq \nu$  at any time, then  $\|\nabla u\|_2^2$  decreases in  $t$  thereafter, and there can be no later singularities.  
 g) As  $t \rightarrow \infty$ ,  $z \rightarrow z_\infty+0$  monotonically and  $z^2$  is ultimately convex in  $t$ . For  $t_0$  sufficiently large, and  $t > t_0$ ,

$$(2.13) \quad \|\nabla u\|_2^2 \leq \left( \nu \int_{t_0}^t \frac{dt'}{\|u\|_2^2} \right)^{-1} \leq \frac{\|u_0\|_2^2}{\nu(t-t_0)}.$$

### 3. Outline of proof and discussion

Part a) of the theorem is established by the arguments described above, and by two successive integrations with respect to the variable  $z$ . Part b) follows from a) and the fact that at a singular value  $\frac{dt}{dz} \rightarrow 0$  as  $z \rightarrow z_T+0$  so  $C'(z_T+0) = 4Cz_T^3$  is positive. Part c) follows b), with the observations that  $\frac{dt}{dz}$  is negative overall, and  $\frac{dt}{dz} + 4cz^3$  increases with  $z$ . The lower bound for  $\|\nabla u\|_2^2$  and the orders of magnitude then follow easily, and we note that the norms will be analytic functions of  $z$  and  $t$  except at singular values [3, 13;p51]. The dimension of the singular set can be estimated as in [1, 4, p907, 9, 10]. Part d) is a lower bound for  $\|\nabla u\|_2^2$  that follows from (1.11) when  $t$  is replaced by a singular value  $T$  and  $t_0$  is replaced by  $t$ .

Part e) is a consequence of the theorem [5, Theorem 111] that a continuous convex function has right and left hand derivatives at every point. The estimation of the sum follows from the convexity and a bound on the greatest possible slope of  $C(z)$  at  $z=z_0$  where  $t=0$ . Part f) follows from the form (1.10) by direct elementary reasoning. Finally for part g), we refer to (1.11) noting that the integral on the left increases at least linearly with  $t$ , so that  $\|\nabla u\|_2^2$  must tend to zero for large  $t$ . Since the integral on the right is convergent, we can find a  $t_0$  large enough that it is dominated by the first term on the left. The remainder of the estimation is straightforward.

The simplicity of this elementary theory may perhaps help lead to improved understanding of singular solutions which despite many

recent advances [1, 3, 4, 6, 9, 10, 13] are as yet not fully understood and still remain a challenge fifty years after their formulation by Leray [8].

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TENSOR PRODUCTS OF C\*-ALGEBRA FIBRE BUNDLES

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*Presented by G.A. Elliott, F.R.S.C.*

**Abstract.** The C\*-algebra of continuous sections of the tensor product of two C\*-algebra fibre bundles is \*-isomorphic to the tensor product of the C\*-algebras of continuous sections of the two C\*-algebra fibre bundles.

In this paper we use the term C\*-algebra fibre bundle in the sense of [3][4][5]. The C\*-bundles considered by Dupre [2] are more general than those we shall study here. The main result obtained in [2, Section 4] is that there is an isomorphism between sections of the tensor product of C\*-bundles and tensor product of sections of the C\*-bundles. In this note, a special case of this result is stated and proved by different way which is based on the representation theory of C\*-algebra fibre bundles in [4][5]. We recall the definition and results to be found in [3][4][5]: A C\*-algebra fibre bundle  $\Sigma = (E, X, A, p, \nu, \phi_u, G)$  is a fibre bundle in the sense of [7] over a locally compact base space  $X$ , fibre a C\*-algebra  $A$ , fibre-preserving homeomorphisms  $\phi_u : u \times A \longrightarrow p^{-1}(u)$  and an effective topological group  $G$  of \*-automorphisms of  $A$ . For an element  $x$  in the overlap  $u \cap v$ ,  $u, v \in \nu$  and  $a \in A$ , let  $\phi_u^{-1} \phi_v(x, a) = (x, g_{uv}(x)(a))$ ; then  $g_{uv}(x) \in G$  and the map  $g_{uv}$  is continuous from  $u \cap v$  into  $G$ . If  $y \in p^{-1}(x)$ ,  $x \in u \in \nu$ ,

then we can write  $\phi_u^{-1}(y) = (x, t_u(y))$ . Let  $D$  be the set of continuous cross-sections  $\gamma : X \longrightarrow E$  such that  $|\gamma(x)| = \|t_u(\gamma(x))\|_A$  vanishes at infinity (Note that  $|\gamma(x)|$  is independent of the choice of  $u$  in  $U$  containing  $x$ , since  $t_u(\gamma(x)) = g_{uv}(x)(t_v(\gamma(x)))$  and  $g_{uv}(x) \in G$  is an isometry). Under the sup-norm  $\|\gamma\| = \sup\{|\gamma(x)| : x \in X\}$ ,  $D$  becomes a  $C^*$ -algebra with the natural algebraic operations.

Let  $\Sigma_1 = (E_1, A_1, X, p_1, \nu, \phi_u, G_1)$  and  $\Sigma_2 = (E_2, A_2, X, p_2, \nu, \psi_u, G_2)$  be two  $C^*$ -algebra fibre bundles over the same base space  $X$  and with fibre  $C^*$ -algebras  $A_1$  and  $A_2$  respectively. For the definition of tensor products of  $C^*$ -algebras see [6]. Let  $A_1 \otimes_\alpha A_2$  be the  $C^*$ -tensor product of  $A_1$  and  $A_2$  with respect to the least  $C^*$ -norm  $\alpha$ . Then there exists (by [7, 3.2]) a  $C^*$ -algebra fibre bundle  $\Sigma = (E, X, A_1 \otimes_\alpha A_2, p, \nu, \zeta_u, G)$  over the base space  $X$  with fibre  $A_1 \otimes_\alpha A_2$ . We say that  $\Sigma$  is the tensor product of  $\Sigma_1$  and  $\Sigma_2$  and write  $\Sigma = \Sigma_1 \otimes \Sigma_2$ .

Theorem. Let  $\Sigma_1$  and  $\Sigma_2$  be two  $C^*$ -algebra fibre bundles over  $X$  with fibre  $A_1$  and  $A_2$  respectively, and let  $\Sigma = \Sigma_1 \otimes \Sigma_2$ . If  $D$ ,  $D_1$  and  $D_2$  are the  $C^*$ -algebras of continuous sections of  $\Sigma$ ,  $\Sigma_1$  and  $\Sigma_2$  respectively, then  $D$  and  $D_1 \otimes_\alpha D_2$  are  $*$ -isomorphic.

$$C_0(X)$$

(Note that  $D_1$  and  $D_2$  are  $C^*$ -algebras which are modules over the commutative  $C^*$ -algebra  $C_0(X)$ . Define  $D_1 \otimes_\alpha D_2 = (D_1 \otimes_\alpha D_2) / I$ , where  $I$  is the smallest closed two-sided ideal of  $D_1 \otimes_\alpha D_2$  containing all elements of the form  $(f\gamma_1) \otimes \gamma_2 - \gamma_1 \otimes (f\gamma_2)$ ,  $f \in C_0(X)$ ,  $\gamma_1 \in D_1$ ,  $\gamma_2 \in D_2$ ).

**Proof:** If  $\gamma_{1n} \in D_1$ ,  $\gamma_{2n} \in D_2$  and  $f_n \in C_0(X)$ , define  $(T(\sum_{n=1}^m f_n(\gamma_{1n} \otimes \gamma_{2n}))) (x)$  to be the element of  $E$  determined by  $(x, f_n(x) t_u \gamma_{1n}(x) \otimes t_u \gamma_{2n}(x))$  in the construction of  $\Sigma$  from  $\Sigma_1$  and  $\Sigma_2$  (cf. [7, 3.2]), and define

$$|(T(\sum_{n=1}^m \gamma_{1n} \otimes \gamma_{2n}))(x)| = \|\sum_{n=1}^m t_u \gamma_{1n}(x) \otimes t_u \gamma_{2n}(x)\|.$$

(The expression of this equality does not depend on the choice of  $u$ , since  $t_u \gamma_{1n}(x) \otimes t_u \gamma_{2n}(x) = g_{uv}(x) t_v \gamma_{1n}(x) \otimes h_{uv}(x) t_v \gamma_{2n}(x) = (g_{uv}(x) \otimes h_{uv}(x)) (t_v \gamma_{1n}(x) \otimes t_v \gamma_{2n}(x))$ , and  $g_{uv}(x) \otimes h_{uv}(x) \in G$  is an isometry, where  $g_{uv}(x)$  and  $h_{uv}(x)$  are coordinate transformations of  $\Sigma_1$  and  $\Sigma_2$  respectively). Then

$x \longrightarrow (T(\sum_{n=1}^m \gamma_{1n} \otimes \gamma_{2n}))(x)$  is continuous and

$|(T(\sum_{n=1}^m f_n(\gamma_{1n} \otimes \gamma_{2n}))) (x)| = \|\sum_{n=1}^m f_n(x) t_u \gamma_{1n}(x) \otimes t_u \gamma_{2n}(x)\|$  vanishes at infinity on  $X$ , and so  $T(\sum_{n=1}^m f_n(\gamma_{1n} \otimes \gamma_{2n})) \in D$  and  $T$  is a  $*$ -homomorphism of  $D_1 \otimes D_2$  into  $D$ .

If  $\gamma = \sum_{n=1}^m f_n(\gamma_{1n} \otimes \gamma_{2n}) \in D_1 \otimes D_2$  then

$$\begin{aligned} \|\gamma\|_{\alpha} &= \sup\{\|(\pi_1 \otimes \pi_2)(\gamma)\| : \pi_1 \in \hat{D}_1, \pi_2 \in \hat{D}_2\} \\ &= \sup\{\|\sum_{n=1}^m (\pi_1(f_n \gamma_{1n}) \otimes \pi_2(\gamma_{2n}))\| : \pi_1 \in \hat{D}_1, \pi_2 \in \hat{D}_2\} \\ &= \sup\{\|\sum_{n=1}^m \pi_1'(t_u(f_n \gamma_{1n})(x_{\pi_1})) \otimes \pi_2'(t_u(\gamma_{2n}(x_{\pi_2})))\| : \\ &\quad (x_{\pi_1}, u, \pi_1') \in X \times U \times \hat{A}_1, (x_{\pi_2}, u, \pi_2') \in X \times U \times \hat{A}_2\} \\ &\geq \sup\{\|\sum_{n=1}^m \pi_1'(t_u(f_n \gamma_{1n}(x))) \otimes \pi_2'(t_u(\gamma_{2n}(x)))\| : \pi_1 \in \hat{A}_1, \\ &\quad \pi_2 \in \hat{A}_2, x \in X\} \\ &= \sup\{\|\sum_{n=1}^m t_u(f_n \gamma_{1n}(x)) \otimes t_u(\gamma_{2n}(x))\| : x \in X\} \\ &= \sup\{|(T(\sum_{n=1}^m f_n(\gamma_{1n} \otimes \gamma_{2n}))) (x)| : x \in X\} = \|T(\gamma)\|_D, \end{aligned}$$

where  $(x_{\pi_1}, u, \pi_1') \in X \times U \times \hat{A}_1$  corresponds to  $\pi_1 \in \hat{D}_1$  according to [4][5], that is,  $\pi_1'(\gamma_1) = \pi_1(t_u(\gamma_1(x_{\pi_1})))$  for  $\gamma_1 \in \hat{D}_1$ , and the same correspondence holds between  $(x_{\pi_2}, u, \pi_2')$  and  $\pi_2$ .

On the other hand, if  $\gamma_1 \in D_1$  and  $\gamma_2 \in D_2$  then

$$\begin{aligned} \|T(\gamma_1 \otimes \gamma_2)\| &= \sup\{|T(\gamma_1 \otimes \gamma_2)(x)| : x \in X\} \\ &= \sup\{\|t_u \gamma_1(x) \otimes t_u \gamma_2(x)\| : x \in X\} \\ &= \sup\{\|t_u \gamma_1(x)\| \cdot \|t_u \gamma_2(x)\| : x \in X\} \\ &\leq \sup\{\|t_u \gamma_1(x)\| : x \in X\} \sup\{\|t_u \gamma_2(x)\| : x \in X\} \\ &= \|\gamma_1\|_{D_1} \|\gamma_2\|_{D_2}. \end{aligned}$$

Furthermore, we have the equality  $\|T(\sum_{n=1}^m \gamma_{1n} \otimes \gamma_{2n})\|^2 =$

$\|T(\sum_{n=1}^m \gamma_{1n} \otimes \gamma_{2n})\|^2 (\sum_{n=1}^m \gamma_{1n} \otimes \gamma_{2n})$  from the fact that

$|T(\sum_{n=1}^m \gamma_{1n} \otimes \gamma_{2n})(x)|$  vanishes at infinity on  $X$ . Hence by [6, 1.22.7],  $\|\gamma\|_\alpha \leq \|T(\gamma)\|_D$ . Therefore  $\|T(\gamma)\|_D = \|\gamma\|_\alpha$  for  $\gamma \in D_1 \otimes_{C_0(X)} D_2$  and  $T$  can be uniquely extended to an isometric

isomorphism of  $D_1 \otimes_{C_0(X)} D_2$  into  $D$ .

Next we shall show that the extended morphism is surjective and therefore that  $D_1 \otimes_{C_0(X)} D_2$  and  $D$  are \*-isomorphic.

Let  $\gamma \in D$ ,  $\epsilon > 0$ . Set  $K = \{x \in X, |\gamma(x)| = \|t_u \gamma(x)\| \geq \epsilon\}$ , which is compact. If  $x_0 \in K$  and  $x_0 \in u \in v$ , since  $\{\sum_{k=1}^m t_u(\gamma_{1k}(x_0)) \otimes t_u(\gamma_{2k}(x_0)) : \gamma_{1k} \in D_1, \gamma_{2k} \in D_2\}$  is dense in  $A_1 \otimes_{C_0(X)} A_2$ , it follows that there exist  $\gamma_{1k} \in D_1$ ,  $\gamma_{2k} \in D_2$  such that  $|\gamma(x_0) - T(\sum_{k=1}^m (\gamma_{1k} \otimes \gamma_{2k}))(x_0)|$

$$= \|t_u \gamma(x_0) - t_u(T(\sum_{k=1}^m (\gamma_{1k} \otimes \gamma_{2k}))(x_0))\|$$

$$= \|t_u \gamma(x_0) - \sum_{k=1}^m t_u(\gamma_{1k}(x_0) \otimes t_u(\gamma_{2k}(x_0)))\| < \epsilon.$$

By the continuity of  $|\gamma(x)|$ ,  $|\gamma_{1k}(x)|$  and  $|\gamma_{2k}(x)|$  on  $X$ ,  
 $|\gamma(x) - T(\sum_{k=1}^m (\gamma_{1k} \otimes \gamma_{2k}))(x)| < \epsilon$  on a neighbourhood of  $x_0$ .

Therefore there is a finite open covering  $O_1, O_2, \dots, O_n$  of  $K$   
 and for each  $i = 1, \dots, n$  there are  $\gamma_{1ik} \in D_1$  and  $\gamma_{2ik} \in D_2$ ,  
 $i \leq k \leq m_i$  such that

$$|\gamma(x) - T(\sum_{k=1}^{m_i} (\gamma_{1ik} \otimes \gamma_{2ik}))(x)| < \epsilon \text{ for } x \in O_i.$$

Let  $f_1, f_2, \dots, f_n$  constitute a partition of the unity  
 subordinate to  $O_1, O_2, \dots, O_n$ , i.e.,  $f_i \in C_{00}(X)$ ,

$f_i$  vanishes outside  $O_i$  and  $f_1 + f_2 + \dots + f_n = 1$  on  $K$ . Then for  
 each  $i$  and  $k$ ,  $f_i(\gamma_{1ik} \otimes \gamma_{2ik}) \in \underset{C_0(X)}{D_1 \otimes D_2}$ , and

$$|\gamma(x) - T(\sum_{i=1}^n f_i \sum_{k=1}^{m_i} (\gamma_{1ik} \otimes \gamma_{2ik}))(x)| < \epsilon \text{ on } K.$$

Thus  $\|\gamma - T(\sum_{i,k}^{n,m_i} f_i(\gamma_{1ik} \otimes \gamma_{2ik}))\| < \epsilon$  on  $X$ . This shows that  
 $T$  is surjective, and the theorem is proved.

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## MORE NEW IDENTITIES IN MIXED EXTERIOR ALGEBRA

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§0. The purpose of this note is to announce certain identities in mixed exterior algebra and to indicate some of their applications. The notation used throughout is that of [1] and [2], to which the reader is referred for the details of proofs.

§1. Given an  $n$ -dimensional vector space,  $E$ , over a field,  $\Gamma$ , of characteristic 0, we may form its mixed exterior algebra  $\Delta(E^*, E)$ . As well as the product (denoted by a dot) of this algebra, we have a "composition" product and the relations between these structures gives rise to many of the important results of [1]. Reference [2] provides other results, the proofs of which appear in [3], [4].

§2. The dot product makes the diagonal subalgebra,  $\Delta(E) = \bigoplus_{p=1}^n \Delta_p(E)$ , into an associative commutative algebra. The unit tensor  $1 \in \Delta_1(E)$  satisfies  $z \cdot z = z \circ z = z$ ,  $z \in \Delta_1(E)$ , and we write  $u^p$  for the product of  $u$  with itself  $p$  times and divided by  $p!$ . The composition product of  $u$  with itself  $p$  times will be denoted by  $u_p$ . The linear transformations of  $\Delta_1(E)$  given by  $\lambda_z(\hat{z}) = z \circ \hat{z}$  and  $\rho_z(\hat{z}) = \hat{z} \circ z$ ,  $z, \hat{z} \in \Delta_1(E)$ , extend uniquely to derivations of  $\Delta(E)$ , which will be denoted by the same symbols.

§3. The identities referred to in the title of this note are  $\lambda_z(u) = (z \cdot t^{p-1}) \circ u$ ,  $\rho_z(u) = u \circ (z \cdot t^{p-1})$ , where  $z \in \Delta_1(E)$ ,  $u \in \Delta_p(E)$  and  $p=1, 2, \dots, n$ . Their proofs will appear elsewhere.

§4. These identities yield a basis-free proof of a formula given in [5]. It arose in connection with integrability conditions for systems of differential equations. The proof given in [5] involves bases and combinatorial grouping of terms. When translated into the notation of mixed exterior algebra, the formula is

$$\sum_{v=0}^q (-1)^v (z_{q-v} \cdot t^{p-1}) \circ (z^v \cdot t^{p-v}) = (-1)^q (p-q) z^q \cdot t^{p-q},$$

where  $z \in \Delta_1(E)$ .

In view of §3, the summand here may be written as

$$\lambda_{z_{q-v}} (z^v \cdot t^{p-v}),$$

and, since  $\lambda_z$  is a derivation for all  $z \in \Delta_1(E)$ , this, in turn, may be written as  $u_{v-1} + u_v$ , where

$$u_v = z_{q-v} \cdot z^v \cdot t^{p-v-1}.$$

Thus, the alternating sum is simply  $u_{-1} + (-1)^q u_q$ . Since  $u_{-1} = 0$  and  $u_q = t \cdot z^q \cdot t^{p-q-1} = (p-q) z^q \cdot t^{p-q}$ , this completes the proof.

§5. The case when  $p=n$  of the formula in §4 is of interest. From [1] it is known that  $z^v \cdot t^{n-v} = \sigma_v(z)t^n$ , where  $\sigma_v(z) \in \Gamma$  are the coefficients in the characteristic polynomial for  $z \in \Delta_1(E)$ . Thus, when  $p=n$ , the formula of §4 is equivalent to

$$\sum_{v=0}^q (-1)^v \sigma_1(z_{q-v}) \sigma_v(z) = (-1)^q (n-q) \sigma_q(z).$$

If  $z \in \Delta_1(E)$  is regarded as an element of  $L(E)$  and assumed to have  $n$  distinct eigenvalues, then  $\sigma_v(z)$  is the  $v$ -th elementary symmetric function of these eigenvalues and  $\sigma_1(z_v)$  is the sum of their  $v$ -th powers. We therefore have a new proof of the classic relationship between these types of symmetric functions.

§6. The formula of §4 was used in [6] to prove the Cayley-Hamilton theorem, when  $\Gamma$  is the field of complex numbers. We sketch a proof for any field of characteristic 0, based on the formula of §3.

In fact, replacing  $u$  by  $z^n$ ,  $z$  by  $\tilde{z}$  and taking inner products with  $t^n$ , we find

$$\langle \tilde{z}(z^n), t^n \rangle = \langle (\tilde{z} \cdot t^{n-1}) \circ z^n, t^n \rangle.$$

The left hand of this may be shown to be  $\langle \tilde{z}, z \text{adj} z \rangle$ , where  $\text{adj} z$  is the classical adjoint of  $z$ , while the right side is  $\langle \tilde{z}, \text{det} z t \rangle$  (cf. [1]).

Since  $\tilde{z} \in \Delta_1(E)$  is arbitrary, it follows that  $z \text{adj} z = \text{det} z t$ ,  $z \in \Delta_1(E)$ . As in [1], this is equivalent to the Cayley-Hamilton theorem.

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