C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

THE CARTAN FORM FOR VARIATIONAL PROBLEMS

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Presented by G.A. Elliott, F.R.S.C.

Abstract: A new Cartan form θ_L^ℓ for a given Lagrangian L is introduced and is shown to remedy some defects in the classical Cartan form θ_L^1 .

The purpose of this note is to announce a generalization of the classical Cartan form connected with the differential geometric approach to variational problems:

$$\underset{\sim}{L}(\gamma) = \int_N L(x,\gamma(x),\underset{1}{\gamma}(x))\,dx \qquad (1)$$

where $\underset{1}{\gamma} = \{ \partial\gamma^\alpha/\partial x_i \}_{i=1,\ldots,p}^{\alpha=1,\ldots,q}$. While the classical Cartan form is suitable for the needs of mechanics (p=1) or scalar field theories (q=1) it is seen to be inadequate in situations (like electromagnetic field theory, p=q=4) where $\ell \equiv \min(p,q) > 1$. This has important consequences in regard to Noether's theorem on symmetries of variational problems.

The jet bundle approach to this subject [1] starts with a manifold E of dimension p+q which is a fiber bundle over a base space N of dimension p. The cross sections $\gamma:N \to E$ have 1-jets $j^1(\gamma):N \to C^1(E,p)$ which are cross sections of $C^1(E,p)$, the bundle of 1st order contact elements of p-dimensional submanifolds of E. The union of the images of all such 1-jets forms the jet bundle $J^1(E)$ and a Lagrangian is then any smooth

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1980 Mathematics Subject Classification: Primary 49F05, 58A15

real-valued map L on $J^1(E)$. Cartan's idea [2] was to associate with each Lagrangian L a differential p-form θ_L^1 on $J^1(E)$ and to do this in such a way that the symmetries of L are precisely those fiber space automorphisms $g:E \to E$ whose prolongations $g^1:J^1(E) \to J^1(E)$ leave $d\theta_L^1$ invariant: $g^{1*}(d\theta_L^1) = d\theta_L^1$. Regretably this program does not determine all the symmetries of L when the number $\ell = \min(p,q)$ is larger than 1. However the situation can be corrected by introducing an extension θ_L^ℓ of the classical Cartan form as follows.

The definitions are done locally using the natural jet bundle coordinates $(x,u,\underset{1}{u})$ on $J^1(E)$ where $x = (x_1,\ldots,x_p)$, $u = (u^1,\ldots,u^q)$ and $\underset{1}{u} = \{u_i^\alpha\}_{i=1\ldots p}^{\alpha=1\ldots q}$. The contact Pfaffian system consists of the differential 1-forms

$$\phi^\alpha = du^\alpha - \sum_{j=1}^p u_j^\alpha dx_j \tag{2}$$

For $n \le \ell$ and $i_1,\ldots,i_n \in \{1,\ldots,p\}$ let

$$\pi_{i_1\ldots i_n} = \frac{\partial}{\partial x_{i_n}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial x_{i_1}} \lrcorner\, \pi \tag{3}$$

where $\pi = dx_1 dx_2 \ldots dx_p$ and $\lrcorner$ denotes contraction (for simplicity the exterior product symbol $\wedge$ will be omitted: $dx_1 dx_2 = dx_1 \wedge dx_2$) Next introduce the following p-forms on $J^1(E)$:

$$M_L^0 = L\,\pi$$

$$M_L^n = \frac{1}{(n!)^2} \sum_{\substack{i_1\ldots i_n \\ \alpha_1\ \alpha_n}} \frac{\partial^n L}{\partial u_{i_1}^{\alpha_1}\ldots\partial u_{i_n}^{\alpha_n}}\, \phi^{\alpha_1}\ldots\phi^{\alpha_n}\, \pi_{i_1\ldots i_n} \tag{4}$$

where the summation extends over all $i_1,\ldots,i_n \in \{1,\ldots,p\}$ and $\alpha_1,\ldots \alpha_n \in \{1,\ldots,q\}$. The <u>kth Cartan form</u> for L is then defined by ($0 \le k \le \ell$)

$$\theta_L^k = M_L^0 + M_L^1 + \ldots + M_L^k \tag{5}$$

One sees that θ^k is a linear map from the zero forms (the Lagrangians) into the p-forms on $J^1(E)$. θ^1 is the classical Cartan form and the results below offer support for the claim that θ^ℓ is properly called the Cartan form.

THEOREM 1 The following two assertions are equivalent:

(1) $d\theta_L^k = 0$

(2) The Euler-Lagrange Eqs. for L vanish identically, i.e.

$$\frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta} + \frac{\partial^2 L}{\partial u_j^\alpha \partial u_i^\beta} = 0 \qquad (6)$$

$$\frac{\partial L}{\partial u^\alpha} - \sum_i \frac{\partial^2 L}{\partial x_i \partial u_i^\alpha} - \sum_{\beta i} \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} \cdot u_i^\beta = 0 \qquad (7)$$

and L has nullity k, i.e.

$$\frac{\partial^{k+1} L}{\partial u_i^\alpha \partial u_{i_1}^{\alpha_1} \cdots \partial u_{i_k}^{\alpha_k}} = 0 \qquad (8)$$

Sketch of proof: The conditions for a cross section $\gamma: N \to E$ to be an extremal of L can be expressed by the requirement that $j^1(\gamma)^*(X^1 \lrcorner\, d\theta_L^k) = 0$ for each prolongation X^1 of a vector field X on E of the form $X = \sum \xi^i(x)\ \partial/\partial x_i + \sum \eta^\alpha(x,u)\ \partial/\partial u^\alpha$ (cf. [1] p.181). Thus it is clear that if $d\theta_L^k = 0$ then every γ is an extremal of L (which means that L satisfies Eqs. (6)-(7)). The rest of the implication (1) $\Longrightarrow$ (2) together with the reverse implication (which is the non-trivial part of the theorem) requires a computation of the strict components of $d\theta_L^k$. The details of this are presented in [3].

Theorem 1 characterizes the kernal Z_k of the kth Cartan form $L \mapsto d\theta_L^k$ as consisting of those Lagrangians which satisfy Eqs. (6)-(8). One should note that condition (8) is redundant when $k = \ell$. Explicit descriptions of $Z_1 \subset \cdots \subset Z_\ell$ have been given [4,5] and from these one can see that the inclusions $Z_k \subset Z_{k+1}$, $k \in \{1,\ldots,\ell-1\}$ are proper in general. These results are central to the discussion of symmetries.

Because the Pfaffian contact forms (2) pullback to zero under 1-jets of cross sections $\gamma: N \to E$ the action integral (1) for L may be expressed by

$$\underset{\sim}{L}(\gamma) = \int_N j^1(\gamma)^*(\theta_L^k)$$

If (g, g_N) is a bundle morphism of (E,N) let g(L) be the Lagrangian which satisfies $g^{1^*}(\theta_L^0) = \theta_{g(L)}^0$, and for a cross section γ let $g(\gamma)$ be the cross section defined by $g(\gamma) = g \circ \gamma \circ g_N^{-1}$. One says that g is a <u>symmetry</u> of L if $\theta_L^\ell - \theta_{g(L)}^\ell$ is closed. Symmetries (1) map extremals into extremals: γ is an extremal of L iff $g(\gamma)$ is an extremal of L and (2) leave the action integral invariant up to a divergence integral:

$$\underset{\sim}{L}(\gamma) - \underset{\sim}{L}(g(\gamma)) = \int_N dj^1(\gamma)^*\omega$$

For some p-1 form ω on $J^1(E)$ (Poincaré's lemma must be applicable for this to hold). Alternative definitions of symmetries arise from the following:

<u>THEOREM 2</u> <u>The following statements are equivalent</u>:

(a) <u>g is a symmetry of L</u> .

(b) <u>The Euler-Lagrange Eqs. for L-g(L) vanish identically</u>.

(c) $g^{1*}(d\theta_L^\ell) = d\theta_L^\ell$.

<u>Proof</u>: One can show that $g^{1^*}(\theta_L^k) = \theta_{g(L)}^k$ for $k=1,\ldots,\ell$. Then (c) is equivalent to $d\theta_{(L-g(L))}^\ell = 0$ and this is equivalent to (b) by Theorem 1.

Theorems 1 and 2 above are revisions of Theorems 3.1 and 4.1 in [1] which are only true when $\ell = 1$.

<u>COROLLARY</u> <u>The following statements are equivalent</u>:

(a) <u>The vector field</u> $X = \sum_i \xi^i(x)\partial/\partial x_i + \sum_\alpha \eta^\alpha(x,u)\partial/\partial u^\alpha$ <u>on E is an infinitesimal symmetry of L</u>.

(b) $d\theta_{L\#}^\ell = 0$ <u>where</u> $L\# = \mathcal{L}_{X^1}L + \mathrm{div}(\xi)L$

<u>Here</u> $\mathcal{L}_{X^1}$ <u>is the Lie derivative along the prolongation</u> X^1 <u>of X</u>.

Proof: Of course (a) is the requirement that $\mathcal{L}_{X^1}(d\theta_L^{\ell}) = 0$. One can show that $\mathcal{L}_{X^1}(\theta_L^k) = \theta_{L\#}^k$ for any k and so the equivalence of (a) and (b) follows.

When $\ell = \min(p,q)$ is larger than 1, there are physically significant examples of Lagrangians L and vector fields X for which $d\theta_{L\#}^{\ell} = 0$ yet $d\theta_{L\#}^{1} \neq 0$. Thus the classical Cartan form is not adequate for determining all the symmetries.

REFERENCES

1. R. Hermann, Geometry, Physics, and Systems, Dekker, New York, NY, 1973.
2. E. Cartan, Leçons sur les invariants intégraux, Hermann, Paris, 1922.
3. D. Betounes, Extension of the classical Cartan form (to appear)Phys.Rev. D.
4. S. Hojman, Problem of the identical vanishing of the Euler-Lagrange derivatives in field theory, Phys. Rev. D 27 (1983), 451-453.
5. D. Betounes, Infinitesimal symmetries of the kth order (submitted).

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Received January 6, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

REPPLACEABILITY AND μ-UNIQUENESS - A UNIFIED APPROACH

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In this note, we introduce the notion of μ-uniqueness over a subset M of a summability domain. We hence are able to unify the concepts of μ-uniqueness and replaceability and thereby characterize those matrices with $P \neq \overline{\phi}$, which are both μ-unique and replaceable.

Presented by P. Ribenboim F.R.S.C

1. Introduction

We consider infinite complex matrices A with convergent columns, i.e. their summability domain c_A contains the set ϕ of all finite sequences. Such a matrix A is said to be replaceable, if there exists a matrix D with $c_D = c_A$ and $\lim_D x = 0$ for all $x \in \phi$, and A is called μ-unique if for given $\mu \in ¢$, $t \in \ell$ and $s \in c_A^\beta$, the condition

$$\mu \lim_A x + t(Ax) + sx = 0 \text{ for all } x \in c_A$$

implies $\mu = 0$.

Since each $f \in c_A'$ has a representation

$$(*) \qquad f(x) = \mu \lim_A x + t(Ax) + sx \qquad (x \in c_A)$$

with some $\mu \in ¢$, $t \in \ell$ and $s \in c_A^\beta$, μ-uniqueness of A is equivalent to the fact that the nullfunctional does not belong to

$$\mu^{\neq} := \{f \in c_A : \exists\ \mu \neq 0,\ t \in \ell,\ s \in c_A^\beta \text{ s.t. } (*) \text{ holds}\}$$

(Cf. e.g. [1], [3], [5], [6] and [7]). Here we embed the notions of μ-uniqueness and replaceability in a more general setting which not only gives a unified aspect but also enables us to analyse the structure of the distinguished subspace

This research is partially supported by a NSERC grant.

$$P:=\{x \in c_A : (tA)x = t(Ax) \text{ for all } t \in \ell \text{ with } tA \in c_A^\beta\}$$

more closely. It is known (cf. [2] and [5]) that $P = \bar{\phi}$ or $P = \bar{\phi} \oplus \langle u \rangle$ for some $u \in c_A \setminus \bar{\phi}$, where the former is implied by non-replaceability of A, while the latter implies μ-uniqueness. By means of the more general notion it is now possible to fill in the gap and characterize those replaceable μ-unique matrices for which $P = \bar{\phi} \oplus \langle u \rangle$.

2. μ-uniqueness over M

DEFINITION. Let M be a subset of c_A. A is μ-unique over M, if for each $f \in c_A'$ the condition

$$f(x) = 0 \quad \text{for all } x \in M$$

implies $f \notin \mu^{\neq}$.

We observe that μ-uniqueness over a subset of c_A is an invariant property (it depends on c_A rather than on A), since $\mu^{\neq}$ is invariant (see [1], Satz 3). Also, A is μ-unique if and only if A is μ-unique over c_A, and A is non-replaceable if and only if A is μ-unique over ϕ, (see [1], Folgerung 3). Thus the above definition unifies the notions of μ-uniqueness and replaceability.

We give a topological characterization which extends a result due to Wilansky [7], 3.1 and 9.1, who considered μ-uniqueness (Over c_A). In the following theorem,

$$\chi(g) := g(e) - \sum_k g(e^k) \quad \text{for } g \in c',$$

where $e = (1,1,\ldots)$, $e^k = (0,\ldots,0,1,0,\ldots)$ is the unit sequence with "1" in the k-th position and c is the Banach-space of convergent sequences.

THEOREM 1. Let A be a matrix with $\phi \subset c_A$.
Then A is μ-unique over a subset M of c_A if and only if the functional χ lies in the $\sigma(c'',c')$-closure of $D[M] = \{Dx : x \in M\}$ whenever D is a matrix with $c_D = c_A$.

PROOF. Assume that A is non-μ-unique over M. Then, by our definition, there exists a functional $f \in \mu^{\neq}$ with f = 0 on M. By Zeller's theorem (see e.g. [1], Folgerung 2) $f = \lim_D$ for some D with $c_D = c_A$. Thus $D[M] \subset c_0$ (the null-sequences) and χ does not lie in the $\sigma(c'',c')$-closure of D[M].

Conversely, if the condition fails, then there is a matrix D with $c_D = c_A$ and a functional $g \in c'$ such that g = 0 on D[M] and $\chi(g) \neq 0$. Now

$$g(y) = \chi(g) \ \lim y + ty \qquad (y \in c)$$

for some $t \in \ell$. We define $f := g \circ D$; we then have f = 0 on M and

$$f(x) = \chi(g) \ \lim_D x + t(Dx) \quad (x \in c_D).$$

Since $\chi(g) \neq 0$. A is not μ-unique over M. □

3. Application to P.

As we indicated in section 1, the notion of μ-uniqueness over subsets allows us to give a characterization of $P = \overline{\phi} \oplus \langle u \rangle$.

THEOREM 2. Let A be a matrix with $\phi \subset c_A$, and let $u \in c_A \setminus \overline{\phi}$. Then $P = \overline{\phi} \oplus \langle u \rangle$, if and only if A is non-μ-unique over ϕ and is μ-unique over $\phi \cup \{u\}$.

PROOF. If $P = \overline{\phi} \oplus u$, then A is replaceable, hence non-μ-unique over ϕ. Assume that A be non-μ-unique over $\phi \cup \{u\}$. Then there exists $f \in \mu^{\neq}$ (and hence a matrix D with $c_D = c_A$ and $\lim_D = f$) such that $\lim_D = 0$ on $\phi \cup \{u\}$. Since $u \notin \overline{\phi}$ there is a $g \in c_A'$ with $g(u) \neq 0$ and g = 0 on ϕ. From

$$g(x) = \mu \ \lim_D x + t(Dx) + sx \qquad (x \in c_D = c_A)$$

we have (set $x := e^k$)

$$0 = tD + s, \text{ hence } tD \in c_A^{\beta} ,$$

and

$$g(u) = 0 + t(Du) - (tD)u = 0,$$

since $u \in P$. This contradiction proves that the conditions are necessary.

Conversely, if A is non-μ-unique over ϕ, then A is replaceable, and we may assume without loss of generality that $\lim_A e^k = 0$ for all $k \in N$. Then, if A is μ-unique over $\phi \cup \{u\}$, we have $\lim_A u \neq 0$, since otherwise $f := \lim_A$ would be a functional in $\mu^{\neq}$ which vanishes on $\phi \cup \{u\}$. Consequently, $u \notin \overline{\phi} \subset$ kernel of $\lim_A$. Now, let $t \in \ell$ be such that $tA \in c_A^p$. Define $f \in c_A'$ by

$$f(x) := \mu_t \lim_A x + t(Ax) - (tA)x \qquad (x \in c_A),$$

where $\mu_t := [(tA)u - t(Au)] / \lim_A u$. Then $f = 0$ on $\phi \cup \{u\}$. Thus, if A is μ-unique over $\phi \cup \{u\}$, $\mu_t = 0$. This means that A is non-μ-unique. Hence $u \in P$, and consequently $P = \overline{\phi} \oplus \langle u \rangle$. □

We point out that Theorem 2 is particularly interesting for matrices which are μ-unique (over c_A) and replaceable, since there exist such matrices with $P = \overline{\phi}$ (see [4]).

1. W. Beekmann, Uber einige limitierungstheoretische Invarianten, Math. Z., 150 (1976), 195-199.

2. W. Beekmann, J. Boos, K. Zeller, Der Teilraum P im Wirkfeld eines Limitierungsverfahrens ist invariant, Math. Z., 130 (1973), 287-290.

3. W. Beekmann, S.-C. Chang, Some summability invariants, manuscripta math., 31 (1980), 363-378.

4. ___, On the structure of summability fields. Preprint.

5. M. S. MacPhail, A. Wilansky: Linear functionals and summability invariants, Canad. Math. Bull., 17 (1974), 233-242.

6. A. Wilansky, Distinguished Subsets and summability Invariants, J. d'Analyse Math. 12 (1964), 327-350.

7. ___, The μ-property of FK-spaces, Comm. Math. 21 (1978), 371-380.

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Received January 11, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

THE NON-REAL POINT SPECTRUM OF GENERALIZED EIGENVALUE PROBLEMS

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Presented by F.V. Atkinson, F.R.S.C.

Introduction. Let A be self-adjoint, B symmetric (bounded or not) operators on a separable complex Hilbert space $(H, (,))$ each of which is defined in some domain $\mathcal{D}(B)$ respectively, dense in H and such that $\mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}$ is also dense in H. We formulate some results regarding the distribution of those $\lambda \in \mathbb{C}$, $\mathrm{Im}\lambda \neq 0$ for which the equation

$$Ax = \lambda Bx \tag{1.1}$$

has a solution $x \neq 0$ in $\mathcal{D}$. Such a value of λ will be termed an eigenvalue, for short. We will tacitly assume that the totality of eigenvalues of (1.1) forms a proper subset of $\mathbb{C}$. The notation $n(A)$, $(p(A))$, will be used to denote the total number of negative (positive) eigenvalues of A (counting multiplicities). The spectrum and resolvent set of A will be denoted by $\sigma(A), \rho(A)$ respectively. The symbol $N^{\pm}(A;B)$ will denote the total number of eigenvalues of (1.1), counting multiplicities, which lie in $U^{\pm} = \{\lambda \in \mathbb{C} : \mathrm{Im}\,\lambda \gtrless 0\}$

Theorem Let A be self-adjoint, bounded below and have the $n(A) = N$ eigenvalues $\mu_1 \le \mu_2 \le \ldots \le \mu_n < 0$. Let B be symmetric. If

$$0 < \mu_{N+1} = \sup_{x_1,\ldots,x_N} \{ \inf_{\substack{x \in \mathcal{D}(A), \|x\|=1 \\ x \in [x_1,\ldots,x_N]^{\perp}}} (Ax,x)\} \tag{1.2}$$

where $[x_1,\ldots,x_N]^{\perp} = \{x : (x,x_i) = 0,\ i=1,\ldots,N\}$, then

a) The eigenspace corresponding to an eigenvalue $\lambda \in U^{+}$ (resp. U^{-}) is finite-dimensional and of dimension not exceeding N

This research is partially funded by a grant from the Natural Sciences and Engineering Research Council of Canada.

b) There is an at most finite, though possibly empty, set of eigenvalues in U^+ (resp. U^-) and

$$N^{\pm}(A;B) \le N \tag{1.3}$$

Sketch of the proof. Let ψ_i, $1 \le i \le N$, be a set of eigenvectors of corresponding to μ_i and ϕ_i, $1 \le i \le M$, a collection of M independent eigenvectors corresponding to $\lambda \in U^+$. Assume, on the contrary, that $M > N$. Then there exists $C_i \in ¢$, not all zero, such that $\phi = \sum_{i=1} C_i \phi_i$ satisfies $\phi \ne 0$, $(\phi, \psi_i) = 0$, $i = 1,\ldots,N$ and $\phi \in \mathcal{D}(A)$. A simple calculation and use of the result $(A\phi_i, \phi_j) = 0$, $1 \le i,j \le M$, shows that $(A\phi,\phi) = 0$. Since $(\phi,\phi) \ne 0$ use of the operator form of the max-min principle [6, theorem XIII.1] or its min analog yields $\mu_{N+1} \le 0$ which contradicts (1.2). Hence $M \le N$ and this proves (a). In order to prove (b) let $\lambda_1, \lambda_2, \ldots, \lambda_M$ be a collection of distinct eigenvalues in U^+ and $\phi_1, \phi_2, \ldots, \phi_M$ a corresponding set of eigenvectors. A simple induction argument shows that the ϕ_i are linearly independent over ¢. Assume, if possible, that $M > N$. Repeating the argument above without modification, but with due interpretation, we once again obtain the contradiction $\mu_{N+1} \le 0$. Hence $M \le N$ and so the totally $N^+(A;B)$ is finite. Now let $\lambda_1, \lambda_2, \ldots, \lambda_M$ denote any set of eigenvalues of (1.1) in U^+ and $\phi_1, \phi_2, \ldots, \phi_M$ a corresponding set of eigenvectors (with the understanding that if λ_i appears m_i times in the list, there is a set of m_i independent eigenvectors among the ϕ_i corresponding to these). Assuming one final time that $M > N$, we proceed as in (a) to find the stated contradiction. Hence $M \le N$ and so (1.3) follows. The same arguments apply to those eigenvalues $\lambda \in U^-$ and so the corresponding results follow.

The above proof also yields the following result regarding the real point spectrum of (1.1):

Corollary. The total number of eigenvalues $\lambda \in \mathbb{R}^+$ (resp. $\mathbb{R}^-$), counting multiplicities, with the property that corresponding eigenvectors ϕ satisfy

$$(B\phi,\phi) \le 0 \quad (\text{resp. } (B\phi,\phi) \ge 0) \tag{1.4}$$

is at most finite and, in fact, does not exceed N.

For let $\lambda \in \mathbb{R}^+$ and let ϕ_i, $i = 1,2,\ldots,M$ denote a collection of corresponding eigenvectors for which $(B\phi_i,\phi_i) \le 0$. Then $(A\phi_i,\phi_i) \le 0$, $i = 1,2,\ldots,M$ and so the vector ϕ defined above now satisfies $(A\phi,\phi) \le 0$. This, however, also leads to the stated contradiction. The proof of the corresponding result is similar.

Remark. Let $\lambda \in \mathbb{R}^+$ (resp. $\mathbb{R}^-$) and let ϕ_i, $i = 1,2,\ldots,M$, $(M \le \infty)$ denote those eigenvectors for which $(B\phi_i,\phi_i) > 0$, (resp. $(B\phi_i,\phi_i) < 0$). Then on the subspace V spanned by the ϕ_i the form $(B\phi,\phi)$ is positive (resp. negative) and so apart from a finite number of real distinct eigenvalues (whose eigenvectors ϕ satisfy $(B\phi,\phi) = 0$), it may be possible to describe the real eigenvalues of (1.1) variationally as a max-min as is well-known in the case when B is a positive operator.

Applications. Let $A = -\Delta+q$ on $L^2(\mathbb{R}^n)$ where $q: \mathbb{R}^n \to \mathbb{R}$, $q \in C_\infty(\mathbb{R}^n)$. Then A is self-adjoint on $\mathcal{D}(-\Delta)$ by the Kato-Rellich theorem. Assume further that $\sigma(A) \cap (-\infty,0)$ consists only of a finite number of bound-state energies [6, § XIII.2], including degenerate ones, and that (1.2) is satisfied. Finally define B on $L^2(IR^n)$ by $Bf = rf$ where $r: IR^n \to IR$, $r \in L^\infty(IR^n)$, $r \not\equiv 0$. Then the problem

$$-\Delta\psi + q(x)\psi = \lambda r(x)\psi$$

has an at most finite number of pairs of non-real eigenvalues and there holds the estimate (1.3). (See [2], [3] for physical applications of problems of this type i.e., A is an ordinary or partial differential operator and B is a multiplication operator though not necessarily positive). Other applications include the theory of pairs of differential operators, a subject which has received much attention lately, cf., [7] and the references therein and [1]. For the one-dimensional analog of the original application see [5]. We note that in one dimension the bound (1.3) is sharp, at least in the case when A has only one negative eigenvalue, since the problem $-y'' + q(t)y = \lambda r(t)y$, $y(0) = y(2) = 0$, where $q(t) = -9\pi^2/16$; $r(t) = 1$ on $[0,1]$, $r(t) = -1$ on $(1,2]$. It is not difficult to show that this problem admits precisely one pair of pure imaginary eigenvalues with approximate values $\lambda = \pm 4.363\, i$.

Finally let A,B be real symmetric (complex hermitean) $n \times n$ matrices with $0 \notin \sigma(A)$. In this case the bound on the right of (1.3) becomes $\min\{n(A), p(A), n(B), p(B)\}$ and the resulting inequality is sharp in all cases. In fact, the latter estimate may be derived from some very early work of Kronecker - I am grateful to Professor I. Gohberg for pointing this out to me. Since every boundary problem for a difference equation of the form

$$-\Delta(c_n \Delta y_{n-1}) + b_n y_n = \lambda a_n y_n$$

where $\Delta y_n = y_{n+1} - y_n$ is equivalent to a matrix eigenvalue problem of the form (1.1) where A is tridiagonal (Jacobi matrix) and B is diagonal, the results of the theorem and corollary also apply to these.

REFERENCES

1 . F.E. Browder, The Dirichlet and vibration problems for linear elliptic equations of arbitrary order, Proc. Nat. Acad. Sci. U.S.A. 38, (1952) 741-747.

2 . K. J. Brown and S.S. Lin, On the existence of positive eigenfunctions for an eigenvalue problem with an indefinite weight function, J. Math. Anal. Appl. 75, (1980) 112-120.

3 . H.G. Kaper, C.G. Lekkerkerker, A. Zettl, Linear transport theory and an indefinite Sturm-Liouville problem, Lecture Notes in Mathematics, 964, Springer-Verlag, 326 - 361.

4 . T. Kato, Perturbation Theory of Linear Operators, Second Edition, New York, Springer - Verlag, 1980.

5 . A.B. Mingarelli, Indefinite Sturm-Liouville problems, Lecture Notes in Mathematics, 964, Springer - Verlag, 1982, 519-527.

6 . M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. IV, Academic Press, New York, 1978.

7 . A. Schneider, On spectral theory for the linear selfadjoint equation $Fy = \lambda Gy$, Lecture Notes in Mathematics, 846, Springer - Verlag, 306-332.

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Received January 27, 1984

STRUCTURE THEOREMS FOR COMMUTATIVE HJELMSLEV RINGS WITH NILPOTENT RADICALS

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In this note *all rings R are associative and commutative with an identity,* and J is the (Jacobson) radical.

A ring is a *chain ring* if its ideals form a chain. Clearly a chain ring is a *local ring* (= possesses a unique maximal ideal). A chain ring is a *Hjelmslev ring* if every non-unit is a zero divisor. Every finite chain ring is a Hjelmslev ring with nilpotent radical ([4]), and the structure of these rings has been extensively studied in [4] and [5]. There do exist inifinite Hjelmslev rings whose radicals are not nilpotent [cf [12, page 73] and [13, page 45]].

Our purpose here is to use the results of [10] and [12] to obtain structure theorems for infinite commutative Hjelmslev with nilpotent radicals. Such rings are precisely the coordinate rings of Pappian Hjelmslev planes of level n (cf. [2]).

1. Commutative E-rings

A ring R is an *E-ring* ([7]) if and only if R possesses an ideal I so that all ideals of R are of the form I^n. The smallest n so that $I^n = (0)$ is the rank of R. Clearly I is the radical of R.

A ring R is a *valuation ring* if R is a chain ring and an integral domain (cf. [11] or [6]).

* The author gratefully acknowledges the financial support of the National Sciences and Engineering Research Council of Canada.

E-rings and chain rings generate many examples of Hjelmslev rings.

1.1 Proposition (1) Every E-ring with $J \neq (0)$ is a proper Hjelmslev ring (i.e. not a field) with principal nilpotent radical.

(2) If R is a chain ring, then R/Ra is a Hjelmslev ring.

Proof. *(1) Suppose R has rank n, or* $J^n = (0)$. *Now,* $J^2 \subset J$ *or else* $J^n = J = (0)$. *Take* $b \in J \setminus J^2$. *Then,* $b \neq 0$ *and Rb = J. Finally we show that every non-unit is a zero divisor. Let* $a \neq 0$ *be a non-unit. Then,* $a \in J$ *and* $Ra = J^i = Rb^i$. *Hence,* $a = rb^i$, $b^{n-1} \neq 0$ *and* $a \cdot b^{n-i} = r \cdot b^n = 0$. *(2) is from [12, page 70].*

E-rings and valuation rings are in abundance.

1.2 Proposition. Let R be a Noetherian local ring with $J = Ra = \neq (0)$. Then, all non-zero ideals of R have the form Ra^i and R is either an E-ring or a principal valuation ring whose radical is not nilpotent. In both cases, R is a principal ideal ring, and for any non-zero ideal I, R/I is a Hjelmslev ring.

Proof. The first part is just [3, page 379]. The second claim follows from 1.1.

We next collect various characterizations of E-rings.

1.3 Theorem. R is a ring with $J \neq 0$. The following are equivalent.

(1) R is an E-ring.

(2) R is a Hjelmslev ring with nilpotent radical.

(3) R is a Noetherian Hjelmslev ring.

(4) R is an Artinian Hjelmslev ring.

(5) R is a Hjelmslev ring, all zero divisors are nilpotent and $J = Ra$.

(6) R is a Hjelmslev ring with exactly one prime ideal and $J=Ra$.

(7) R is an Artinian local principal ideal ring.

(8) R is an Artinian, Noetherian local principal ideal ring.

Proof. For the equivalence of the first six condition (which holds even in the non-commutative case) use [8, 5.10] and 1.1.

To complete the proof we show that 6 => (7) => (8) => (1)

(6) => (7): Since a chain ring is local, the equivalence of the first six conditions implies that R is an Artinian, Noetherian Hjelmslev ring. Hence, every ideal is finitely generated. But a finitely generated ideal of a Hjelmslev ring is principal ([12, 5.2]).

(7) => (8) : This follows immediately from Hopkin's theorem (cf. [1]).

(8) => (1) : Now, the set of nilpotent elements is the intersection of all prime ideals of R ([9, page 51]). Moreover, since R is Artinian, every prime ideal is maximal [9, page 15]. Hence, J is the set of nilpotent elements. Since $J\neq(0)$, there is $a\neq 0$, $a^n=0$ and so R is not an integral domain. By 1.2, R is an E-ring.

2. STRUCTURE THEOREMS FOR COMMUTATIVE HJELMSLEV RINGS WITH NILPOTENT RADICALS.

By 1.3, a Hjelmslev ring with nilpotent radical is equivalent to an Artinian, Noetherian local principal ideal ring with nilpotent radical. Hence, we may invoke the structure theorems of McLean [10] to obtain the following results.

2.1. Theorem. R is a Hjelmslev ring with $J^n=(0)$ where char(R) = char (R/J) if and only if $R \cong k[x]/(x^n)$ for a field k.

2.2 Theorem R is a Hjelmslev ring with $J^n=(0)$ and char(R/J) $= p \neq 0$ where $R_p = J^m$, $1 \leqslant m \leqslant n$ if and only if $R \cong V[x]/(f(x), x^n)$ where V is a complete Noetherian valuation ring of characteristic zero whose radical is V_p, char$(V/V_p) = p$ with $p^2 \notin V_p$ and $f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1$ (where $a_i \in V_p$ $(1 \leqslant i < m)$ and $a_1 \notin V_{p^2}$) is an Eisenstein polynomial of degree m.

2.3 Theorem For a ring R, the following are equivalent.

(1) R is a Hjelmslev ring with $J^n = (0)$.

(2) $R \cong k[x]/(x^n)$ for a field k or $R \cong V[x]/(f(x),x^n)$ where V is a complete Noetherian valuations ring of characteristic zero whose radical is V_p, for a prime $p \neq 0$, $p^2 \notin V_p$ and f(x) is an Eisenstein polynomial over V.

(3) R is a proper homomorphic image of a discrete valuation ring.

References

[1] I.J. Adamson, Rings, modules and algebras (Oliver and Boyd, Edinburgh 1971).

[2] B. Artmann, Geometric aspects of primary lattices, Pacific J. Math. 43 (1972), 15-25.

[3] N. Bourbaki, Commutative algebra (Addison-Wesley Pub. Co. Reading Mass. 1972).

[4] W. E. Clark and D. A. Clark, Finite chain rings, Abh. Math. Sem. Univ. Hamburg 39(1973), 147-153.

[5] W. E. Clark and J. Liang, Enumeration of finite commutative chain rings, J. Algebra *27* (1973), 445-453.

[6] O. Endler, Valuation theory, Springer-Verlag, New York, Heidelberg, Berlin 1972.

[7] B. Jonsson and G. S. Monk, Representation of primary arguesian Lattices, Pacific J. Math. *30* (1969), 95-139.

[8] J. W. Lorimer, Locally compact Hjelmslev Planes and rings, Can. J. Math. Vol XXXIII No. 4 1981 988-1021.

[9] Hideyuki Matsumura, Commutative algebra, W. A. Benjamin Inc. New York 1970.

[10] K. R. McLean, Commutative Artinian Principal Ideal rings, Proc. London Math. Soc. (3) 26(1973) 249-272.

[11] M. Nagata, Local rings, Interscience, New York and London 1962.

[12] G. Torner, Eine Klassifizierung von Hjelmslev-Ringe und Hjelmslev-Ebenen, Mitt. Math Sem. Giessen 107 (1974).

[13] Hubert Weller, Zusammenhung zwischen verbandstheoretischen Eigenschaften des Verbandes $L(R^3)$ and arithmetischen Eigenschaften des zugehorigen Rings, Mitt. Math. Sem. Giessen 118 (1975).

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Received February 6, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

ON MODAL REPRESENTATIONS OF EXTENSIONS OF PEANO ARITHMETIC

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Presented by L.A. Lorch, F.R.S.C.

Let PA be Peano arithmetic and Pr(x) the Gödel provability formula. The language of propositional modal logic is the language of propositional logic with an additional symbol $\Box$ for a monadic operator. Let $*$ be an assignment of arithmetical statements $\varphi_0, \varphi_1, \ldots$ to propositional variables $p_0, p_1, \ldots$. We define the arithmetical transform P^* of a modal formula P: $(\bot)^* = (0=1)$, $(p_i)^* = \varphi_i$, $*$ commutes with connectives, $(\Box Q)^* = Pr(\ulcorner Q^* \urcorner)$.

According to this interpretation, modal language is a convenient meta-language for schemes of arithmetical formulas which involve the notion of provability in PA . So $(\neg \Box \bot)^* = \neg Pr(\ 0=1\ )$ = Consis PA, $\Box p \to p$ is the modal notation for the so-called local reflection principle etc. What modal formulas are correct in the sense that all their arithmetical transforms are provable in PA? Well known properties of Pr(x) give the correctness of the following schemes of modal formulas: Tautologies, $(\Box P) \wedge \Box(P \to Q)) \to \Box Q$, $\Box(\Box P \to P) \to \Box P$ and rules: modus ponens, $P \vdash \Box P$.

We'll call this modal logic GL . Obviously $GL \vdash P \Rightarrow \forall^* \, PA \vdash P^*$. The remarkable result of Solovay [6] says that $GL \nvdash P \Rightarrow \exists^* \, PA \nvdash P^*$. Solovay's theorem has been uniformized in [4] and in [1,2] where it is shown (by different approaches) that there exists $*$ such that for all modal formulas P , $GL \nvdash P \Rightarrow PA \nvdash P^*$. (Later, this result was found anew

(1) This summarizes the new results presented in lectures at several Canadian Universities in September - November 1983 during the course of a visit under the auspices of the Exchange Agreement between Queen's University and the Steklov Mathematical Institute of the Academy of Sciences of the USSR. The formulations herein benefitted from these discussions, particularly in the Seminar on Logic at Simon Fraser University and the joint Seminar on Logic of York University and the University of Toronto. I thank the participants for their interested and helpful remarks. The full details of this work are expected to appear in Izv. ANSSR. Ser. Mat.

in [3,7] by methods similar to [4].)

Solovay also answered the question about modal formulas for true arithmetical schemes: He proved [6] that $\forall^* P^*$ is true (in the standard model) $\Leftrightarrow$ GL' $\vdash$ P , where GL' is the logic with axioms: theorems of GL , $\Box p \to p$, and the rule of modus ponens.

A Kripke model here is a finite irreflexive tree (K,<) together with a truth assignment $x \Vdash p_i$ or $x \nVdash p_i$ for all $x \in K$ and all propositional variables p_i . We define a forcing relation $x \Vdash P$ for any modal formula P: $x \nVdash \bot$, $x \Vdash P \lambda Q$ (λ is a connective) in the usual "boolean" way, $x \Vdash \Box Q$ iff $\forall y \succ x,\ y \Vdash Q$. We say that a modal formula P is valid in a Kripke model if $x \Vdash P$ for all x of the model. In [5] it is shown that GL $\vdash$ P $\Leftrightarrow$ P is valid in all Kripke models. This implies the decidability of GL .

Let N be an extension of PA with some true formulas as additional axioms. The modal representation g(N) of N was defined in [2] as $\{P \mid \forall^* N \vdash P^*\}$. This set is a modal logic (i.e. closed under modus ponens and substitution) and it lies between GL and GL' . A modal logic was called arithmetically complete [2] if it is g(N) for some N . Obviously $N_1 \subset N_2 \Rightarrow g(N_1) \subset g(N_2)$ and information about the structure of the class $\mathfrak{R}\mathfrak{G}$ of all arithmetically complete modal logics gives certain information about corresponding extensions of PA .

THEOREM 1. There are continuum many logics between GL and GL' which are not arithmetically complete.

Let X be a set of modal formulas. We define GLX as the logic with schemes of axioms: theorems of GL , X , and with modus ponens the only rule. The depth d(x) of a node x in a model (K,<) is a natural number such that for a top node d(x) = 0 and for every other node $d(x) = 1+\max\{d(y) \mid x<y\}$.
The trace t(P) of a modal formula P is, by definition, the subset of the

set ω of all natural numbers such that $n \in t(P) \Leftrightarrow \exists$ a Kripke model containing a node x with $d(x) = n$ where $x \not\Vdash P$.

LEMMA. For any modal formula P, $t(P)$ is either finite or cofinite and there is an algorithm for constructing $t(P)$ if P is given.

Examples: $t(\neg \Box \bot) = \{0\}$, $t(\Box p \to p) = \omega$. For a modal logic H, we define $t(H) = \bigcup t(P)$ for all formulas P such that $H \vdash P$. Let us consider a sequence of modal formulas (atoms) $F_n = \Box^{n+1} \bot \to \Box^n \bot$. Note that $F_0^* =$ Consis PA and for all $n > 0$, $F_n^* =$ Consis $PA^{n-1} \to$ Consis PA^n where $PA^0 = PA$, $PA^{n+1} = PA^n +$ Consis PA^n, ($PA^\omega = \bigcup PA^n$ for all $n \in \omega$). In [2] it was shown that $t(F_n) = \{n\}$ and that for every set X of atoms the logic GLX is arithmetically complete. The class of all logics of this kind was called $\mathfrak{RG}_\omega$. The order $\subset$ on logics from $\mathfrak{RG}_\omega$ coincides with the order $\subset$ on their traces [2] and $g(PA^n) = GL\{F_0, F_1, \dots, F_{n-1}\}$, $g(PA^\omega) = GL\{F_n | n \in \omega\}$ (it is the union of all the logics of $\mathfrak{RG}_\omega$). Let $Y \subset \omega$ and define $\mathfrak{RG}(Y)$ as the class of all arithmetically complete logics with trace Y.

THEOREM 2. (a) The class $\mathfrak{RG}(Y)$ consists of the single logic $GL\{F_n | n \in Y\}$ iff Y is cofinite.

(b) For all $n \in \omega$ the mapping $\varepsilon_n : H \to GL\{P, F_n | H \vdash P\}$ is an isomorphism of $\mathfrak{RG}(Y)$ and $\mathfrak{RG}(Y \cup \{n\})$.

(c) The order $\subset$ on $\mathfrak{RG}$ is the transitive closure of the following orders: the order $H \subset \varepsilon_n(H)$ and orders inside nontrivial classes $\mathfrak{RG}(Y)$.

We see now that all nontrivial classes $\mathfrak{RG}(Y)$ are isomorphic; the least logic in $\mathfrak{RG}(Y)$ is $GL\{F_n | n \in Y\}$ and the largest is $GL\{(\Box P \to P) \vee \neg(\bigwedge_{n \in \omega \setminus Y} F_n)\}$. In terms of the isomorphism ε_n this provides a description of all possible relations between arithmetically complete modal logics with different traces. For further understanding of the structure of $\mathfrak{RG}$ it is sufficient now to study the structure of any nontrivial class $\mathfrak{RG}(Y)$, for example $\mathfrak{RG}(\omega)$. Note that $\mathfrak{RG}(\omega)$ coincides with the class of all arithmetically complete modal logics which lie between GL^{ω} and GL' .

REFERENCES

[1] S.N. ARTEMOV, Extensions of arithmetics and connected with them modal theories, VI Intern. Congress for Logic, Methodology and Philos. of Sci., Hannover, FRG, Abstracts, Sections 1-4, (1979), 15-19.

[2] S.N. ARTEMOV, Arithmetically complete modal theories, (Russian) Semiotics and Information Sciences, Akad. Nauk. USSR., Vsesojuz. Inst. Nauch. i Tehn. Informacii, Moscow, 14 (1980), 115-133.

[3] G. BOOLOS, Extremely undecidable sentences, J. Symbolic Logic, 47 (1982), 191-196.

[4] F. MONTAGNA, On the diagonalizable algebra of Peano arithmetic, Bollet. della Unione Mathematica Italiana, 66-B (1979), 795-812.

[5] K. SEGERBERG, An essay in classical modal logic, Filosofiska Studier, Uppsala, Sweden, 1971.

[6] R.M. SOLOVAY, Provability interpretations of modal logic, Israel Journal of Mathematics, 25 (1976), 287-304.

[7] A. VISSER, Aspects of diagonalization and provability, Dissertation, Utrecht University, 1981.

Received February 13, 1984

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GROUPS OF ELLIPTIC MÖBIUS TRANSFORMATIONS

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Abstract

We report progress in proving that every group of elliptic Möbius transformations is conjugate to a group of orthogonal transformations.

Let $B^{n+1}=\{\vec{x}\in R^{n+1}:\|\vec{x}\|<1\}$ denote the open unit ball in $\mathbb{R}^{n+1}$; $S^n=\partial B^{n+1}$ $=\{\vec{x}\in R^{n+1}:\|\vec{x}\|=1\}$, its bounding n-sphere; $\mathcal{M}_n$, the group of conformal (≡Möbius) transformations of S^n; and $\mathcal{O}_{n+1}$, the group of isometries (≡orthogonal transformations) of S^n. Recall that B^{n+1} equipped with the metric

$$dh = \frac{2ds}{1-\|\vec{x}\|^2}$$

serves as a Poincaré model of hyperbolic (n+1)-space; the isometries of this model are given by (extensions of) the transformations in $\mathcal{M}_n$ provided $n\geq 1$; and the stabilizer of the centre 0 of B^{n+1} is given by $\mathcal{O}_{n+1}\subset\mathcal{M}_n$.

It was shown in [6] (see also [9]) that the familiar classification of two-dimensional Möbius transformations into elliptic, parabolic, and loxodromic types generalizes in a natural way to $\mathcal{M}_n$. An element of $\mathcal{M}_n$ is elliptic iff its action on B^{n+1} has a fixed point or, equivalently, iff it is conjugate to an element of $\mathcal{O}_{n+1}$. We seek to prove that a group $G\subset\mathcal{M}_n$ which consists entirely of elliptic elements has a common fixed point in B^{n+1} and therefore can be conjugated to a subgroup of $\mathcal{O}_{n+1}$.

A finite group $G\subset\mathcal{M}_n$ must consist entirely of elliptic elements and in [8] it was shown that these finite groups are conjugate to groups of orthogonal transformations. Indeed it was shown that if a group $G \subset M_n$ has a bounded orbit in B^{n+1} then it is conjugate to a group of orthogonal transformations.

Recently A.F. Beardon and the author [2] have used the representation of n-dimensional Möbius transformations by (n+2)-dimensional Lorentz transformations (see e.g. [5]) to introduce a norm for n-dimensional Möbius transformations. This norm satisfies

$$\|L\|^2 = n+2+4\sinh^2\rho$$

where ρ is the hyperbolic distance through which the related hyperbolic isometry moves the point 0. The norm formula has many applications and in connection with the present problem it dovetails with the results stated in the last paragraph. A Möbius transformation L is elliptic iff its powers act on B^{n+1} to move 0 in a bounded orbit, hence iff its powers are bounded in norm. On the other hand, a group of Möbius transformations which is uniformly bounded in norm must move 0 in a bounded orbit, and hence must have a common fixed point. The problem of showing that an elliptic group of Möbius transformations is conjugate to an orthogonal group can therefore be stated neatly in terms of the norm: show that a "power bounded" Möbius group is "uniformly bounded", i.e. show that if for any $L \in G$ there exists $K_L > 0$ such that for any integer p, $\|L^p\| < K_L$ then there exists $K > 0$ such that for any $M \in G$, $\|M\| < K$.

Unfortunately this approach has not yet given the desired result. We turn now to analytic and synthetic techniques which have been successful in low dimensions.

A sense-preserving element of M_2 can be written

$$z \to \frac{az+b}{cz+d}, \quad ad-bc = 1,$$

in terms of a complex variable z. The transformation is elliptic iff it is the identity or $-2<a+d<2$; it effects an orthogonal transformation of the Riemann sphere iff $d=\bar{a}$ and $c=-\bar{b}$. Using these criteria Lyndon and Ullman [4] have given a completely analytic proof that an elliptic group of sense-preserving Möbius transformations in dimension n=2 is conjugate to an orthogonal group. In ([1] Theorem 4.3.7) Beardon obtained the same result in a more synthetic way but his proof still depends heavily on complex variables for one crucial lemma.

In the remainder of this paper we show that totally synthetic methods give a fast, elegant proof of our result in dimensions n=0,1,2, i.e. for hyperbolic geometries of dimension m=n+1=1,2,3. Since the emphasis from here onwards is on hyperbolic geometry we shall speak of (hyperbolic) isometries rather than Möbius transformations.

A k-flat in hyperbolic m-space is a subset isometric to hyperbolic k-space: 0-flat≡point; 1-flat≡line; 2-flat≡plane etc. If an elliptic isometry fixes a set S, it fixes the smallest k-flat containing S, hence the fixed sets of elliptic isometries and elliptic groups (assuming the latter are non-void) must be k-flats. In addition to fixed sets in which each point is fixed, we require the notion of invariant sets whose points may be permuted among themselves by the transformation or transformations in question.

Lemma 1. *Let g be an elliptic isometry of hyperbolic m-space H^m and suppose that g leaves invariant a k-flat $H^k \subset H^m$. Then g fixes a point of H^k and hence $g|_{H^k}$ is also an elliptic isometry.*

Proof. Since g leaves H^k invariant it commutes with the projection which maps each point of H^m to the nearest point in H^k. If no point in H^k is fixed, none of the fibers of this projection is fixed and hence no point in H^m is fixed. This contradiction gives the result.

Lemma 2. *In an induction proof of our result it suffices to show that an elliptic isometry group G acting on H^m leaves invariant a k-flat H^k with $k<m$.*

Proof. If each element in G leaves H^k invariant then, according to Lemma 1, each element in G restricts to an elliptic isometry of H^k. Since $k<m$, induction gives a common fixed point to the elliptic group $G|_{H^k}$ and hence to the orginal group G.

Lemma 3. *In an induction proof of our result it suffices to show that an elliptic isometry group G acting on H^m has a normal subgroup $N\neq\{e\}$ which has a non-void fixed set.*

Proof. Since $N\neq\{e\}$ its fixed set must be a k-flat H^k with $k<m$. Since N is normal, G must leave H^k invariant and the result follows from Lemma 2.

Lemma 4. *In an induction proof of our result it suffices to consider elliptic isometry groups consisting of sense-preserving isometries.*

Proof. The result is trivial for groups of order 2. Otherwise the normal subgroup of sense-preserving isometries is different from $\{e\}$ and Lemma 3 applies.

We now proceed to consider dimensions m=1,2,3. When m=1, the catalogue of isometries includes only reflections and translations. Of these, the reflections and the identity are elliptic. The only non-trivial elliptic group is the group of order 2 generated by a single reflection and our result holds.

As an aside we remark that this n=0, m=1 case is anomalous from the point of view of Möbius transformations and the fact that it fits as well as it does with the apparent pattern must be considered fortuitous. Since the 0-sphere consists of just two points $M_0=O_1$ is the group consisting of the identity and the reflection which interchanges these two points. The transformations in M_0 do not extend to give all the isometries of the hyperbolic line. Nevertheless the desired result does hold in the strong sense that any group of elliptic isometries of the line is conjugate to O_1.

When m=2, the catalogue of sense-preserving isometries includes rotations, parallel displacements, and translations but only the rotations are elliptic. If g and h are rotations about different centres then $g^{-1}(h^{-1}gh)$ is the product of a rotation through $-\theta$ about one point and through θ about another. By factoring these rotations into reflections it is easy to see that the product is a translation. Hence the elliptic groups of sense-preserving isometries consist of rotations about a single point and our result holds.

When m=3, the catalogue of sense-preserving isometries includes rotations, parallel displacements, translations and screw displacements and once again the only elliptic isometries are rotations. (See [7] for further details about the catalogue.) The key observation which keeps the m=3 geometry under control is

Lemma 5. *The product of two rotations in hyperbolic 3-space has a fixed point only if the rotation axes are coplanar.*

Proof. If the rotation axes are skew then in particular they are non-intersecting and a fixed point P of the product can only arise because one rotation moves P to $Q\neq P$ and the other restores Q to P. But this puts both axes on the plane equidistant from P and Q: contradiction.

Now it is easy to deduce that in an elliptic group every pair of rotation axes must meet. For if g and h have parallel or ultraparallel axes $g^{-1}(h^{-1}gh)$ is a parallel displacement or a translation.

If every pair of rotation axes meets then either there is a single axis and hence a line of common fixed points or there are two distinct axes a and b with $a \cap b = \{P\}$. In the second case the product of the named rotations has axis c passing through P but not lying in the plane of a and b. This means that other rotation axes can meet a,b, and c only if they pass through P and hence P is a common fixed point for the group.

This completes the proof of our result in dimensions m=1,2,3. We remark that the argument is essentially absolute, i.e. equally valid in Euclidean and hyperbolic geometry. Accordingly it also serves as a contribution towards the solution of the open problem [3].

References

[1] Alan F. Beardon, The Geometry of Discrete Groups, Springer-Verlag, New York, 1983.

[2] A.F. Beardon and J.B. Wilker, The norm of a Mobius transformation, to appear.

[3] D.Ž. Djoković, P306, Canad. Math. Bull. 24 (1981) 252; 25 (1982) 506.

[4] R.C. Lyndon and J.L. Ullman, Groups of elliptic linear fractional transformations, Proc. Amer. Math. Soc. 18 (1967) 1119-1124.

[5] J.B. Wilker, Inversive geometry, The Geometric Vein, Springer-Verlag, New York, 1981, pp. 379-442.

[6] __________, Möbius transformations in dimension n, Period. Math. Hungar. 14 (1983) 93-99.

[7] __________, Topologically equivalent N-dimensional isometries, Topology Appl. 15 (1983) 211-221.

[8] __________, Isometry groups, fixed points, and conformal transformations, Math. Rep. Acad. Sci. Canada IV (1982) 293-297.

[9] __________, Möbius transformations and isoclinal sequences of spheres, Aequationes Math., to appear

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Received February 17, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI (1984) No. 3,
June 1984 juin

SOME RESULTS ON THE DEPTH AND WIDTH OF PARTIAL ORDERS

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A partially ordered set $P = \langle P,<\rangle$ embeds the linear type φ, $P \geq \varphi$, if it has a sub-chain of order type φ. The depth of P is the smallest ordinal number $\gamma^*(P) = \gamma$ such that $P \not\geq \gamma^*$, where, as usual, γ^* denotes the reverse of γ. For example, any well-founded partial order has depth $\leq \omega$. We investigate the question whether the depth of a partial order may be reduced by partitioning into a fixed number of parts. We give a complete solution to this question for linear orders. These results have applications to the theory of partial orderings, to infinite graph theory and to the partition calculus.

For any ordinal $\gamma(\geq 3)$ and cardinal λ, denote by $L(\gamma,\lambda)$ the assertion: if $\langle L,<\rangle$ is a linearly ordered set of depth γ, then there is a partition of L into λ subsets of depth less than γ. It is obvious that $L(\omega,\lambda)$ is false for every λ since, for example, if $\kappa > \lambda$ is an infinite cardinal it cannot be partitioned into λ parts of finite depth. Theorem 4 below shows that $L(\gamma,\lambda)$ is also false for every λ in the case when $\gamma = \nu^+$ is an infinite successor cardinal. However, in all other cases there is some λ such that $L(\gamma,\lambda)$ holds. We have proved the following decomposition theorem. (We write $\alpha^- = \alpha$ if α is a limit ordinal and $\alpha^- = \beta$ if $\alpha = \beta + 1$).

Research supported by NSERC grant #A5198 and NSF grant #8303601

THEOREM 1. *If* $\gamma \geq 3$, $\gamma \neq \omega$, $\gamma \neq \omega_{\alpha+1}$, *then* $L(\gamma,\lambda)$ holds *if* $\lambda \geq \lambda(\gamma)$, *where*

$$\lambda(\gamma) = \begin{cases} 2 \text{ if } \gamma^- \text{ is decomposable} \\ cf(|\gamma|) \text{ if } \gamma^- \text{ is indecomposable and } \gamma < |\gamma|^\omega \\ \omega \text{ if } \gamma^- \text{ is indecomposable and } \gamma \geq |\gamma|^\omega \end{cases}$$

Successive applications of Theorem 1 immediately gives the following useful corollary.

COROLLARY. *If* $<L,<>$ *is a linearly ordered set of depth* γ, *then there is a partition of* L *into* $cf(|\gamma|)$ *parts each having depth at most* $|\gamma|$.

For the case when γ^- is indecomposable and $\gamma \geq |\gamma|^\omega$, we actually prove a stronger statement than Theorem 1. The following result is a generalization of the negative partition relation of [3] that $\xi \not\to (\kappa^n)^1_{n<\omega}$ if $\xi < \kappa^+$.

THEOREM 2. *If the depth of a linearly ordered set* $<L,<>$ *is less than* $\omega_{\alpha+1}$, *then there is an* ω*-partition* $\{L_n : n < \omega\}$ *of* L *such that* L_n *has depth at most* ω_α^n $(n < \omega)$.

It is well-known that any partial order can be extended to a linear order. An order type φ is called *extendable* (see [1]) if whenever $<P,<>$ is a partially ordered set which does not embed φ, then there is a linear extension $<_1 \supseteq <$ such that $<P,<_1> \not\geq \varphi$. For example, η (the order type of the rationals) is extendable (see [1]). F. Galvin and R. McKenzie (unpublished, but see [1]) completely characterized the class of extendable ordinals. It is the smallest class U such that (i) $\omega \in U$, (ii) U is *closed* and (iii) $\alpha\cdot(cf(\alpha))^+ \in U$ *whenever* $\alpha \in U$. In particular, therefore, every infinite cardinal is extendable and so are ordinals like $\omega_{\omega^{\omega_1}}$, $\omega_{\omega^{\omega_2}}$

ω_ω^2, etc. Another immediate corollary of Theorem 1 is the following result for partial orders.

THEOREM 3. *If* γ *is an extendable ordinal*, $\gamma \neq \omega$, $\gamma \neq \omega_{\alpha+1}$, *and if* $\langle P, < \rangle$ *if any partially ordered set of depth* γ, *then there is a partition of* P *into* $cf(|\gamma|)$ *parts each with depth less than* γ.

It is possible that Theorems 1, 2 apply to arbitrary partial orders for all γ, but we have not fully investigated this question.

It is easy to see that if $\kappa \geq 2$ and $\nu \geq \omega$ are cardinal numbers, then ${}^{\nu}\kappa$ has depth ν^+ with the ordinary lexicographic ordering $\prec$ of sequences. We have proved the following result.

THEOREM 4. *If* $\kappa \geq 2$, $\nu \geq \omega$ *are cardinals and* $\lambda < \max\{\nu^+, \kappa\}$, *and if* $\{L_\xi : \xi < \lambda\}$ *is any* λ*-partition of* ${}^{\nu}\kappa$, *then* $\gamma^*(\langle L_\xi, \prec \rangle) = \nu^+$ *for some* $\xi < \lambda$.

Since κ may be chosen arbitrarily large, Theorem 4 shows that $L(\nu^+, \lambda)$ is false for every λ.

In order to prove Theorem 4 we were led to consider certain topological properties of the space ${}^{\nu}\kappa$ endowed with the $\prec$-order topology. These results have an independent interest. For any cardinal μ, a subset of a topological space is of the first (second) μ-category if it is (is not) a union of fewer than μ nowhere dense subsets. Thus, category in the usual sense in the same as ω_1-category. The following two theorems which imply Theorem 4 when $\nu^+ \geq \kappa$, generalize well-known simple facts about real numbers.

THEOREM 5. *If* $\kappa \geq 2$, $\nu \geq \omega$ *are cardinal numbers, then* ${}^{\nu}\kappa$ *is of the second* $(\mathrm{cf}(\nu))^{+}$*-category in the lexicographic order topology.*

THEOREM 6. *If* $\kappa \geq 2$, $\nu \geq \omega$ *are cardinals and* A_{ξ} $(\xi < \nu)$ *are subsets of* ${}^{\nu}\kappa$ *of depth less than* ν^{+}, *then* $A = \bigcup\{A_{\xi} : \xi < \nu\}$ *is of the first* $(\mathrm{cf}(\nu))^{+}$*-category.*

Theorems 1 and 4 provide a rather novel characterization of those ordinal numbers which are infinite successor cardinals.

THEOREM 7. *For any ordinal number* $\gamma (\geq 3)$ *the following statements are equivalent*

(1) $\gamma = \omega$ *or* γ *is an infinite successor cardinal;*

(2) *for every cardinal* λ *there is a linearly ordered set of depth* γ *which is preserved under arbitrary* λ*-partitions;*

(3) *there is a linearly ordered set* $\langle L, < \rangle$ *of depth* γ *and a cardinal* $\lambda \geq 1 + \mathrm{cf}(|\gamma|)$ *such that the depth of* L *is preserved under arbitrary* λ*-partitions.*

Theorem 4 has another application to the theory of partially ordered sets. The *width* of a partial order $\langle P, < \rangle$ is the smallest cardinal number μ such that $|A| < \mu$ holds for every antichain $A \subseteq P$. Say that $\langle P, < \rangle$ has the property $\mathcal{D}(\lambda)$ if and only if there is a partition of P into λ parts $\{P_{\xi} : \xi < \lambda\}$ such that the width of each part is less than the width of P. Dilworth's well-known chain decomposition theorem [2] says that if a partial order has finite width k, it is a union of fewer than k chains. Thus every partially ordered set of finite width $k \geq 3$ has property $\mathcal{D}(2)$. Perles [4] has shown that Dilworth's theorem does not sensibly extend to partial orders of infinite width. In fact, if λ is any cardinal, and $\kappa > \lambda$ is an infinite cardinal then the direct product $\kappa \otimes \kappa$ (in which $(\alpha,\beta) \le (\alpha',\beta')$ if and only if $\alpha \le \alpha'$ and $\beta \le \beta'$) does not contain an infinite antichain (i.e. has width ω), but it is not the union of λ subsets of finite width. Theorem 4 easily implies the following more general result.

THEOREM 8. If $\mu = \omega$ or if $\mu = \nu^+$ is an infinite successor cardinal, then for any cardinal λ there is a partially ordered set of width μ and dimension 2 which does not have property $\mathcal{D}(\lambda)$. (The dimension of partial order $<$ is the smallest cardinal δ such that $< = \bigcap_{\xi<\delta} <_\delta$ is an intersection of δ linear orders.)

PROOF. Since the direct product $\kappa \otimes \kappa$ has dimension 2, we need only consider the case $\mu = \nu^+$, an infinite successor cardinal. Choose $\kappa > \lambda$ and let $\ll$ be any well ordering of ${}^{\nu}\kappa$. Since $\langle {}^{\nu}\kappa, \prec \rangle$ has depth ν^+, it follows that the partial order $\langle {}^{\nu}\kappa, < \rangle$, where $< = \prec \cap \ll$, has width at most ν^+. However, if $\{L_\xi : \xi < \lambda\}$ is any λ-partition of ${}^{\nu}\kappa$, then, by Theorem 4, there is some $\xi < \lambda$ such that $\langle L_\xi, \prec \rangle \ge \nu^*$ (in fact has depth ν^+), and so $\langle L_\xi, < \rangle$ has an antichain of size ν and therefore L_ξ has the same width ν^+, as L. $\square$

The hypothesis $\mu = \omega$ or $\mu = \nu^+$ is needed in Theorem 8 since it fails for uncountable limit cardinals. We have the following result.

THEOREM 9. If μ is an uncountable strong limit cardinal and if $\langle P, < \rangle$ is a partially ordered set of width μ and dimension $\delta < \mathrm{cf}(\mu)$, then $\langle P, \mu \rangle$ has the property $\mathcal{D}\big((\mathrm{cf}(\mu))^{\delta}\big)$.

An interesting open question that remains here is the following: If μ is an uncountable limit cardinal and λ is arbitrary, is there a partially ordered set which does not have the property $\mathcal{D}(\lambda)$? Theorem 9 shows that the dimension of such a partial order must be at least $cf(\mu)$ if $\lambda \geq \mu$.

Theorem 8 implies the following result about graphs of large chromatic number which do not contain large complete subgraphs.

THEOREM 10. *Suppose* $\mu = \omega$ *or* $\mu = \nu^+$ *is an infinite successor cardinal. Then for any cardinal* λ *there is a graph* $G = (V,E)$ *which contains no complete subgraph of size* μ *but, for any partition of the vertex set* V *into* λ *parts, some part contains a complete subgraph of size* μ' *for every* $\mu' < \mu$.

To see this simply consider the complement of the comparability graph of an ordered set $<P,<>$ of width μ and not having property $\mathcal{D}(\lambda)$. We do not know if Theorem 10 is true when μ is an uncountable limit cardinal.

REFERENCES

1. R. Bonnet & M. Pouzet, Linear extensions of ordered sets, in Ordered Sets, (ed. I. Rival), D. Reidel, Holland (1982), pp.125-172.

2. R.P. Dilworth, A. Decomposition theorem for partially ordered sets, Ann. of Math. 51 (1950), 161-166.

3. E.C. Milner, & R. Rado, The pigeon-hole principle for ordinal numbers, Proc. London Math. Soc. (3), 15 (1965), 750-768.

4. M.A. Perles, On Dilworth's theorem in the infinite case, Israel J. Math. 1. (1963), 108-109.

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Received February 21, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

THE ξ-TOPOLOGY ON η_ξ-CLASSES
WITH APPLICATIONS TO REAL ALGEBRAIC GEOMETRY

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Presented by P. Ribenboim F.R.S.C.

Abstract: Any ordered field K can be embedded in an η_ξ-field E, for a suitable admissible ordinal ξ. E has a structure, very much like that of a topology on it, called a ξ-topology, under which every interval is ξ-connected, and many intervals are ξ-compact. These classes behave as expected under ξ-continuous maps.

0. Introduction. Let ξ be the class of all ordinals numbers Ord, or let ξ be in Ord such that it is greater than zero, and such that ω_ξ (the ξ-th infinite cardinal number (9, p.129)) is regular. Then ξ will be called admissible. Let C be a class. If ξ = Ord and if C is a set we will say that the cardinal number of C is less than ω_ξ. Let X be an ordered class and let L and R be subclasses of X. We will write $L < R$ if x^L in L and x^R in R implies $x^L < x^R$. An ordered class E will be said to be an η_ξ-class if given two subclasses L and R of E with $L < R$, and such that L and R are sets of cardinality less than ω_ξ, then there exists e in E for which $L < \{e\} < R$. (For ξ in Ord this idea goes back to Hausdorff (8).)

Let K be an ordered field (that is a set). Unless K is the field of all real numbers, it is not Dedekind-complete, and thus is totally-disconnected, and is not locally compact. Many of the methods used in studying algebraic geometry over the reals seem not to be available over K. One of the usual ways to proceed is to embed K in its real-closure R. Since the Tarski Principal (10) holds, the elementary theory over R is the same as that of the field of all real numbers. On the other hand, much of the standard topological and analytic reasoning one can employ over the field of all real numbers seems not to be available over R.

Let ξ be admissible such that the cardinal number of R is less then or equal to ω_ξ. There exists a real-closed field E that is an η_ξ-class (1,2,5, and 3), and an order-preserving isomorphism f of R into E (6,7). Given a real algebraic-geometric problem, whose coefficients are in R, we can then consider the same problem in which the coefficients are now regarded as being in E.

In order to define and investigate the ξ-topology on E, we need only consider its order structure. In most of the potential applications of this theory, the field structure of E comes into play. Further, applications using the additional structure of the surreal η_ξ-field, ξNo (5,3), are under consideration. An extended version of this paper, with complete proofs, is in preparation (4).

1. <u>The</u> ξ-<u>Topology</u>. Let X be a non-empty ordered class. Let us adjoin to it two new elements, $-\infty$ and $+\infty$, to form a new ordered class X!, ordered in the expected way. For a and b in X! let (a,b) be defined to be {x in X: a < x < b}. Such classes will be called <u>principal</u> <u>open</u> <u>intervals</u> of X. A subclass U will be called <u>open</u> in the <u>interval</u> <u>topology</u> if for all x in U there exist a and b in U such that x is in (a,b) and (a,b) is a subclass of U.

Let ξ be admissible and let β be an ordinal number such that $0 < \beta < \omega_\xi$. Let families $(a_\alpha)_{\alpha < \beta}$ and $(b_\alpha)_{\alpha < \beta}$ of elements be given in X!. Any subclass U of E which can be written in the form

(*) {x in X: there exists $\alpha < \beta$ such that $a_\alpha < x < b_\alpha$}

will be called a ξ-<u>open</u> subclass of X. (Foundational note: Since (a_α,b_α) may be a proper class, we have described U as we did in (*), rather than writing that U is 'the union of $((a_\alpha,b_\alpha))_{\alpha < \beta}$'. The latter expression may not be a meaningful one in the set theory we have chosen to work within (as described e.g., in (9)). We will, however, describe U informally as 'the union of $((a_\alpha,b_\alpha))_{\alpha < \beta}$'. Similar use of 'union' and 'intersection' will be made when expressions like (*) are available.)

Let a subclass V in X be called ξ-<u>closed</u> if X - V is ξ-

open. Then one easily sees that the empty set and X are ξ-open and ξ-closed. The union and intersection of a finite number of ξ-open (resp. ξ-closed) subclasses of X is ξ-open (resp. ξ-closed). Given a 'family' of fewer than ω_ξ ξ-open (resp. ξ-closed) subclasses of X, then the 'union' (resp. 'intersection') of this 'family' of subclasses is again a ξ-open (resp. ξ-closed) subclass of X. Finally, the analogue of the Hausdorff separation condition holds for ξ-open subclasses of X.

Let X* be a subclass of X. A subclass U* of U will be called a _relative_ ξ-_open_ (resp. _closed_) subclass of X* if there exists a ξ-open (resp. ξ-closed) subclass U of X whose intersection with X* is U*. Thus we have an analogy of the relative topology for X*.

X* will be called a _relative_ ξ-_disconnected_ _space_ if it is the union of two non-empty, disjoint, relative ξ-open subclasses of X*. X* will be called a _relative_ ξ-_connected_ _space_ if it is not a relative ξ-disconnected space.

Theorem A. _Let E be an_ η_ξ-_class and let E* be a subclass of E. E* is a relative_ ξ-_connected space if and only if E* is an interval in E_.

X* will be called a _relative_ ξ-_compact_ _space_ if given $\beta < \omega_\xi$ and a 'family $(U^*_\alpha)_{\alpha < \beta}$' of relative ξ-open subclasses of X*, whose 'union' is X^*, then there exists a finite subset I of β such that the union of the U^*_α over I is all of X^*.

Theorem B. Let E be an η_ξ-class and let E^* be an interval in E that either has a greatest (resp. least) element, or has no cofinal (resp. coinitial) subset S of cardinal number less than ω_ξ. Then E^* is a relative ξ-compact space.

2. ξ-Continuous Maps. Let X, Y, and Z be ordered classes, and let f map X to Y and g map Y to Z. Let ξ be admissible. f will be called ξ-continuous if given any ξ-open subclass V of Y, then $f^{-1}(V)$ is a ξ-open subclass in X. It is easy to see that the composition of two ξ-continuous maps is again a ξ-continuous map. It is also clear that if f is a strictly order perserving (resp. reversing) map of X onto Y, then f is ξ-continuous. If f is ξ-continuous, then it is continuous in the interval topology.

Theorem C. Let f be a ξ-continuous map of X into Y. If X is ξ-connected (resp. ξ-compact), then f(X) is a relative ξ-connected (a relative ξ-compact) subspace of Y.

REFERENCES

1. N.L. Alling, A Characterization of Abelian η_α-Groups in terms of their Natural Valuation, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 711-713.

2. -----------, On the Existence of Real-Closed Fields that are η_α-Sets of Power ω_α, Trans. Amer. Math. Soc. 103 (1962), 341-352.

3. -----------, Conway's Field of Surreal Numbers, To appear, Trans. Amer. Math. Soc.

4. -----------, Foundations of Analysis over Surreal Number Fields, in preparation.

5. J.H. Conway, On Numbers and Games, Academic Press, London, 1976.

6. P. Erdős, L. Gillman, and M. Henriksen, An Isomorphism Theorem for Real-Closed Fields, Ann. of Math. 61 (1955), 542-554.

7. L. Gillman, and M. Jerison, Rings of Continuous Functions, van Nostrand, Princeton, 1960.

8. F. Hausdorff, Grundzüge der Mengenlehre, von Veit, Leipzig, 1914.

9. J.D. Monk, Intorduction to Set Theory, Krieger, New York, 1980.

10. A. Tarski, The Completeness of Elementary Algebra and Geometry, Hermann, Paris, 1940.

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Received March 1, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

POSITIVE LINEAR FUNCTIONALS ON VECTOR LATTICES AND ADDITIVE SET FUNCTIONS ON GROUPS

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Presented by E.E. Granirer, F.R.S.C.

Abstract. We generalize a result of Bochner's on integral representations for positive linear functionals defined on vector lattices of bounded functions, closed under products and containing the constant functions. Applications are then made to additive set functions on locally compact abelian groups.

The following result was obtained by Bochner in 1939 (see [Bo, pp. 773-775]).

THEOREM 1.(Bochner) Let V be a real vector lattice, consisting of bounded (real-valued) functions on a set G, closed under products and containing the constant function 1. If L is a positive linear functional on V and J_L is the class of (Jordan) sets E in G (with indicator function I_E) for which

$$\inf\{L(f-g):g \leq I_E \leq f;\ f,g\in V\} = 0$$

then

1) J_L is an algebra of sets,

2) for all $f\in V$, $\{z:f(z)\geq r\}$ lies in J_L for a dense set of reals r, and

3) there exists a (uniquely determined) positive finitely additive set function m_L defined on J_L such that

i) for all $f\in V$,

$$Lf=\int_G f\,dm_L$$

and

ii) m_L is regular in the sense that, for all $E\in J_L$,

$$m_L(E)=\inf\{Lf:I_E \leq f;f\in V\}$$
$$=\sup\{Lg:g\leq I_E,g\in V\}.$$

Here, the integral is taken in the sense of Riemann. It is unfortunate that both the algebra J_L of Jordan sets and the dense set of reals depend on the linear functional L. Our first result, which we now state, eliminates these weaknesses at the expense of uniqueness of the set function representing L. For this theorem and the remainder of the paper, we refer the reader to [D-S,pp. 101-125] for the theory of integration with respect to finitely additive set functions.

THEOREM 2. Let i) V be a real vector lattice consisting of bounded (real-valued) functions on a set G, closed under products and containing the constant function 1, ii) Z and Z^c be the class of zero and cozero sets respectively for all functions in V, and iii) A be the algebra of sets (in G) generated by Z. Then, given a positive linear functional L on V, there exists a positive finitely additive set function μ on A such that

1) for all $f \in V$,

$$Lf=\int_G f d\mu$$

and

2) μ is regular in the sense that, for all $E \in A$,

$$\mu(E)=\inf\{\mu(V):E \text{ in } V,\ V \in Z^c\}$$
$$=\sup\{\mu(Z):Z \text{ in } E,\ Z \in Z\}.$$

REMARK 1. Let V* be the smallest real linear space (with uniform norm) closed under products, containing all functions of the form hf/(f+g) with $f,g,h \in V$ positive, and f+g>0. It is easy to show that the zero set of any given function in V* lies in Z. The use of a positive extension of L from V to V* in the proof of Theorem 2 removes any possibility for uniqueness in general. In view of the following corollary and the fact that V=V* when V consists of all bounded continuous functions on G, we get that our result contains the well known representation theorem of Alexandroff [A]. It also generalizes a theorem of Kirk [K] (see also [Su,p.63]) since it is not assumed that i) V separates points of G, and ii) given Z_1, Z_2 in $\mathbf{Z}$, there exists f in $\mathbf{V}$ such that f=1 on Z_1 and f=0 on Z_2.

COROLLARY. If $\mathbf{V} = \mathbf{V}^*$, then μ is unique.

EXAMPLE 1. Let G be a locally compact abelian group and let $\mathbf{V}$ be the real vector lattice of all real-valued functions in the space AP(G) (with uniform norm) consisting of all continuous complex-valued almost periodic functions defined on G. An application of Theorem 2 to any given positive linear functional L on AP(G) (and so on $\mathbf{V}$) yields the existence of a positive finitely additive set function μ on $\mathbf{A}$ which is inner and outer regular with respect to the zero and cozero sets respectively, and such that

$$Lf = \int_G f \, d\mu$$

for all f in $\mathbf{V}$ (and so in AP(G)). Such a result was first stated in [H] for the case when G is the real line. Unfortunately the proof there not only rests on another result with a known defficiency in its proof (see [Sc] and [D-S, p.379]), but also on the false claim that f/(f+g) lies in $\mathbf{V}$ whenever the functions $f,g \in \mathbf{V}$ are such that f+g>0 (see [H, p.309]). Also, for this latter reason, μ is not necessarily unique; as claimed in [H].

EXAMPLE 2. Let G be a locally compact abelian group with dual group G^ on which is defined a (not necessarily continuous) complex-valued positive definite function p. Recall that such a function is one for which $\Sigma a_i p(z_i\hat{}) \geq 0$ for all positive polynomials $\Sigma a_i z_i\hat{}$ (i.e. finite linear combinations of characters $z_i\hat{}$ in G^ with complex coefficients a_i). Then there exists a positive linear functional L on AP(G) which is uniquely determined by $L(\Sigma a_i z_i\hat{}) = \Sigma a_i p(z_i\hat{})$ for all polynomials $\Sigma a_i z_i\hat{}$. So, by Example 1, there exists a (not necessarily unique) positive finitely additive set function μ on $\mathbf{A}$ for which $Lf = \int_G f \, d\mu$ for all $f \in AP(G)$. Since G^ is contained in AP(G), we get

$$p(z^\wedge) = \int_G z^\wedge \, d\mu$$

for all $z^\wedge \in G^\wedge$. This analogue of Bochner's representation theorem for continuous positive definite functions, generalizes the result stated in [H] for the case when G is the real line.

Given a locally compact abelian group G with dual group $G^\wedge$, let $\langle z, z^\wedge \rangle$ denote the character $z^\wedge \in G^\wedge$ evaluated at $z \in G$. Given m in the space M(G) consisting of all regular complex Borel measures on G, let $m^\wedge : G^\wedge \to \mathbb{C}$ denote its Fourier-Stieltjes transform given by

$$m^\wedge(z^\wedge) = \int_G \langle z, z^\wedge \rangle \, dm(z).$$

Let $M(G)^\wedge$ designate the space $\{m^\wedge : m \in M(G)\}$ (which is closed under products).

Recall that a measure $m \in M(G)$ is said to be continuous if $m(\{z\}) = 0$ for all $z \in G$ and it is said to be discrete if it is concentrated on a countable subset of G. It is shown in [B, pp. 70-71] that the linear functional L on $M(G)^\wedge$ given by $Lm^\wedge = m(\{0\})$ is positive. By means of Theorem 2, we now obtain a measure theoretic approach to Wiener's criteria for the continuity of a Borel measure on G (see [W], [E, pp. 310-312], and [Z, p.108]) which generalizes to a larger set algebra part of the result obtained in [B].

THEOREM 3. *Given a locally compact abelian group G with dual group $G^\wedge$, let A' be the algebra of sets in $G^\wedge$ generated by the class of zero sets for the real-valued functions in $M(G)^\wedge$. Then, there exists a positive finitely additive set function $\mu : A' \to [0,1]$ with the following properties.*

1) $\mu(G^\wedge) = 1$.

2) *μ is inner and outer regular with respect to the zero and cozero sets in A'.*

3) *For all $m \in M(G)$ and all $z \in G$, we have*

$$m(\{z\}) = \int_{G^\wedge} \overline{\langle z, z^\wedge\rangle} m^\wedge(z^\wedge)\, d\mu(z^\wedge)$$

and

$$\Sigma_{z\in G} |m(\{z\})|^2 = \int_{G^\wedge} |m^\wedge(z^\wedge)|^2\, d\mu(z^\wedge)$$

and so m is continuous if and only if

$$\int_{G^\wedge} |m^\wedge(z^\wedge)|^2\, d\mu(z^\wedge) = 0.$$

4) If m_1, $m_2 \in M(G)$ are such that $m_2^\wedge/m_1^\wedge$ is the uniform limit of A'-measurable step functions and m_1 is a discrete measure for which $m_1^\wedge > 0$, then the bounded complex-valued function f on $G^\wedge$ given by

$$f(z) = \int_{G^\wedge} \overline{\langle z, z^\wedge\rangle} (m_2^\wedge(z^\wedge)/m_1^\wedge(z^\wedge))\, d\mu(z^\wedge)$$

is such that

$$m_2(\{z\}) = \int_G f(z-w)\, dm_1(w)$$

for all $z \in G$.

REMARK 2. Note that $m_2^\wedge/m_1^\wedge$ is the uniform limit of A'-measurable step functions whenever $m_1^\wedge$ is bounded away from 0 on $G^\wedge$. When G is the set of all integers (with discrete topology) and $m_1^\wedge > o$ on $G^\wedge$ (i.e. on the unit circle in the complex plane), then $m_1^\wedge$ is bounded away from 0 and Wiener's lemma assures us that f is a discrete measure on G.

ACKNOWLEDGEMENT. This research was supported in part by grants from NSERC (Canada) and FCAC (Québec).

REFERENCES

[A] A.D. Alexandroff, Additive set-functions in abstract spaces, Mat. Sbornik (N.S.) 9 (1941), 543-628.

[B] J.-M. Belley, Construction d'une fonction d'ensembles additive et invariante sur les groupes localement compacts abeliens, Measure Theory and its Applications, Proceedings, Sherbrooke, Canada 1982, Lecture Notes in Mathematics 1033, Springer-Verlag, Heidelberg, 1983.

[Bo] S. Bochner, Additive Set Functions on Groups, Ann. Math. 40 (1939), 769-799.

[D-S] N. Dunford and J.T. Schwartz, Linear Operators. Part I: General Theory, Interscience, New York, 1958.

[E] W.F. Eberlein, A Note on Fourier-Stieltjes Transforms, Proc. Amer. Math. Soc. 6 (1955), 310-312.

[H] E. Hewitt, Linear Functionals on almost periodic functions, Trans. Amer. Math. Soc., 74 (1953), 303-322.

[K] R.B. Kirk, Algebras of bounded real valued functions, I, II, Indag. Math. 31 (1969), 443-463.

[Sc] H.M. Schaerf, Mathematical Reviews, 14 (1953), 545.

[Su] A. Sultan, Measure, Compactification and Representation, Can. J. Math., XXX (1978), 54-65.

[W] N. Wiener, The quadratric variation of a function and its Fourier coefficients, Massachusetts J. Math. 3 (1924), 72-94.

[Z] A. Zygmund, Trigonometric Series I, 2nd ed., Cambridge Univ. Press, 1959.

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Received March 8, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

9_{25} has no period 3

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Presented by K. Murasugi F.R.S.C.

It seems that among the knots with less than ten crossings all periods have been determined save a possible period three of 9_{25} [2]. By using the method of [1] this period can be excluded.

Assume that 9_{25} has period 3. There is a homomorphism ϕ of its knot group G onto a dihedral group D of order 94. It follows from [1] that the factor knot $k = 9_{25}/Z_3$ has the same torsion number $p = 47$ as 9_{25} itself. Let R_2 be the covering space of the complement of 9_{25} with respect to $N_2 = \ker \phi$, and $m_1,\ldots,m_p$ closed curves covering the square m^2 of a meridian m of 9_{25}. Put $M = \langle m_1,\ldots,m_p\rangle \subset H_1(R_2)$. If the factor group $H_1(R_2)/M$ is finite and d is the greatest torsion number of this group the longitude l of 9_{25} has a representation of the form

$$\psi(l) : z \to z + 2/d(y_1\zeta + \ldots + y_{p-1}\zeta^{p-1})$$

with $z \in \mathbb{C}$, $y_i \in \mathbb{Z}$, ζ a primitive p-th root of unity, and 3 is a divisor of each y_i [1]. Calculations on a computer yield $H_1(R_2)/M$ is the trivial group and

$$\psi(l) : z \to z + 2(-3a-3a^2-3a^3-3a^4-5a^5 \ldots)$$

which proves that 9_{25} has not period 3.

[1] U. Lüdicke, Zyklische Knoten. Arch. Math. 32, 588-599 (1979).

[2] K. Murasugi, On Symmetries of Knots, Tsukuba J. Math. 4 No.2, 331-347 (1980).

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Received March 16, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

MEASURES OF INSET INFORMATION ON OPEN DOMAINS - III: WEAKLY REGULAR, SEMISYMMETRIC, β-RECURSIVE ENTROPIES

Bruce Ebanks

Presented by J. Aczel, F.R.S.C.

Abstract. In [7], the general forms of all semisymmetric β-recursive inset entropies on the open domain were found. In this paper we find the forms of those entropies which are also weakly regular.

1. Introduction. This is the third in a series ([7],[6]) of papers on the mixed theory of information on the open domain (i.e., excluding empty sets and zero probabilities). Let B be a ring of subsets of some set Ω (i.e. B contains, with any two sets, their union and difference, hence also their intersection and the empty set 0; see [8].),

$$D_n^o := \{(x_1,\dots,x_n) \mid x_i \in B\setminus\{0\};\ x_i \cap x_j = 0 \text{ for } i\neq j;\ i,j=1,2,\dots,n\},$$

$$\Gamma_n^o := \{(p_1,\dots,p_n) \mid p_i>0;\ i=1,2,\dots,n;\ \sum_{i=1}^n p_i = 1\},$$

for n=2,3,... . The elements of B may be regarded as 'events' and the numbers p_i (i=1,...,n) their probabilities. An inset entropy (on the open domain) is a sequence of maps $I_n: D_n^o\times\Gamma_n^o \to R$ (n=2,3,... ; R the set of real numbers). Originally ([3],[4]), the domains of the maps I_n were allowed to include some boundary points. As we shall see (in the case β=0 below), this restricted the class of inset entropies satisfying the prescribed properties.

2. An inset entropy $I_n: D_n^o\times\Gamma_n^o\to R$ (n=2,3,...) is said to be β-recursive (β∈R) if, for all n>2 and all $(x_1,\dots,x_n;\ p_1,\dots,p_n)\in D_n^o\times\Gamma_n^o$,

$$(1)\quad I_n\begin{pmatrix}x_1,\dots,x_n\\ p_1,\dots,p_n\end{pmatrix} = I_{n-1}\begin{pmatrix}x_1\cup x_2, x_3,\dots,x_n\\ p_1+p_2, p_3,\dots,p_n\end{pmatrix} + (p_1+p_2)^\beta I_2\begin{pmatrix}x_1, x_2\\ r_1, r_2\end{pmatrix},$$

where $r_i = p_i/(p_1+p_2)$ $(i=1,2)$. It is <u>3-semisymmetric</u> if

$$(2)\quad I_3\begin{pmatrix} x_1,x_2,x_3 \\ p_1,p_2,p_3 \end{pmatrix} = I_3\begin{pmatrix} x_1,x_3,x_2 \\ p_1,p_3,p_2 \end{pmatrix},\quad \begin{pmatrix} x_1,x_2,x_3 \\ p_1,p_2,p_3 \end{pmatrix} \in D_3^o\times\Gamma_3^o,$$

and <u>3-symmetric</u> if, for all permutations Π on $\{1,2,3\}$,

$$(3)\quad I_3\begin{pmatrix} x_1,x_2,x_3 \\ p_1,p_2,p_3 \end{pmatrix} = I_3\begin{pmatrix} x_{\pi(1)},x_{\pi(2)},x_{\pi(3)} \\ p_{\pi(1)},p_{\pi(2)},p_{\pi(3)} \end{pmatrix}.$$

Entropies satisfying (1) and (2), or (1) and (3), were studied in [7].

An inset entropy is said to be <u>measurable</u> (or, <u>bounded on a set of positive measure</u>) if the map

$$(4)\quad p\mapsto I_2\begin{pmatrix} x_1,x_2 \\ 1-p,p \end{pmatrix},\quad p\in\,]0,1[,$$

is measurable on $]0,1[$ (or, bounded on a set of positive measure in $]0,1[$) for each fixed $(x_1,x_2)\in D_2^o$. We let Λ be the set of all functions $k\colon\,]0,1[\to\mathbb{R}$ which are bounded on a set of positive measure in $]0,1[$.

Finally, the ring B of sets will be called <u>non-algebraic</u> if D_2^o is nonempty and if, given any $(x,y)\in D_2^o$, there exists a (nonempty) $z\in B$ such that $(x,y,z)\in D_3^o$. An <u>algebra</u> ([8]) of subsets of Ω is a ring containing the universal set Ω. A sufficient (but not necessary) condition for the ring B to be non-algebraic is that B is not an algebra.

<u>Theorem</u>. <u>Suppose B is non-algebraic</u>. <u>The inset entropy</u> $I_n\colon D_n^o\times\Gamma_n^o\to\mathbb{R}$ $(n=2,3,\ldots)$ <u>is (1) β-recursive, (2) 3-semisymmetric, and bounded on a set of positive measure, if and only if there exist maps</u> $g,h\colon D_1^o\to\mathbb{R}$, <u>a constant</u> γ, <u>and an additive set function</u> $a\colon D_1^o\to\mathbb{R}$,

$$(5)\quad a(x\cup y) = a(x)+a(y),\quad (x,y)\in D_2^o,$$

<u>such that, for all</u> $(x_1,\ldots,x_n;\ p_1,\ldots,p_n)\in D_n^o\times\Gamma_n^o$ $(n=2,3,\ldots)$,

$$(6)\quad I_n\begin{pmatrix}x_1,\ldots,x_n\\ p_1,\ldots,p_n\end{pmatrix} = g(x_1)p_1{}^{\beta} + \sum_{i=2}^{n} h(x_i)p_i{}^{\beta} - g(\bigcup_{i=1}^{n} x_i),\ \underline{\text{if}}\ \beta\neq 0,1,$$

$$(7)\quad I_n\begin{pmatrix}x_1,\ldots,x_n\\ p_1,\ldots,p_n\end{pmatrix} = g(x_1)p_1 + \sum_{i=2}^{n} h(x_i)p_i - g(\bigcup_{i=1}^{n} x_i) + \gamma \sum_{i=1}^{n} p_i \log p_i,\quad \underline{\text{if}}\ \beta=1,$$

$$(8)\quad I_n\begin{pmatrix}x_1,\ldots,x_n\\ p_1,\ldots,p_n\end{pmatrix} = g(x_i) + \sum_{i=2}^{n} h(x_i) - g(\bigcup_{i=1}^{n} x_i) + \gamma \log p_1 + \sum_{i=1}^{n} a(x_i)\log p_i,\quad \underline{\text{if}}\ \beta=0.$$

3. <u>Proof of Theorem</u>. It is easy to see that ny inset entropy given by (6), (7), or (8), with arbitrary constant γ, maps $g,h: D_1^o\to R$, and (5) additive set function $a: D_1^o\to R$, is β-recursive, 3-semisymmetric, and bounded on a set of positive measure. Now we prove the converse.

As in [7], we introduce $f: D_2^o\times]0,1[\to R$ by (cf. (4))

$$(9)\quad f(x,y;p) := I_2\begin{pmatrix}x,y\\ 1-p,p\end{pmatrix},\quad (x,y;p)\in D_2^o\times]0,1[.$$

By the 3-semisymmetry, we have

$$I_3\begin{pmatrix}x & ,y,z\\ 1-s-t,s,t\end{pmatrix} = I_3\begin{pmatrix}x & ,z,y\\ 1-t-s,t,s\end{pmatrix}.$$

for all $(x,y,z)\in D_3^o$ and $(s,t)\in\Delta = \{(s,t)\,|\,s,t,s+t\in]0,1[\}$. Expanding both sides of this equation by the β-recursivity for $n=3$, and using (9), we get

$$(10)\qquad f(x\cup y,z;t) + (1-t)^{\beta} f(x,y;s/(1-t)) = f(x\cup z,y;s) + (1-s)^{\beta} f(x,z;t/(1-s)),$$

valid for all $(x,y,z) \in D_3^o$ and $(s,t) \in \Delta$. We now apply a result (Theorem 2; see also Final Remark) from [7], which gives the general solution of (10). There are three cases.

In case $\beta \neq 0,1$, there exist maps $g,h\colon D_1^o \to R$ such that

$$(11)\qquad f(x,y;p) = g(x)(1-p)^{\beta} + h(y)p^{\beta} - g(x\cup y).$$

In case $\beta=1$, f has the form

$$(12)\qquad f(x,y;p) = g(x)(1-p) + h(y)p - g(x\cup y) + S(p) + S(1-p),$$

where $S\colon]0,1[\to R$ is any map satisfying

$$S(pq) = pS(q) + S(p)q, \quad p,q \in]0,1[.$$

An easy calculation shows that the map $T\colon]0,1[\to R$ defined by

$$(13)\qquad T(p) := S(p) + S(1-p), \quad p \in]0,1[,$$

satisfies

$$T(t) + (1-t)T(s/(1-t)) = T(s) + (1-s)T(t/(1-s)), \quad (s,t) \in \Delta,$$

$$T(p) = T(1-p), \quad p \in]0,1[.$$

Also, by (9), (12), and (13), T is in Λ. Hence (cf. [2],[5],[9]) there exists a constant γ such that

$$T(p) = \gamma[p \log p + (1-p)\log(1-p)], \quad p \in]0,1[.$$

Thus, by (12) and (13), f has the form

$$(14)\quad f(x,y;p) = g(x)(1-p) + h(y)p - g(x\cup y) + \gamma[p \log p + (1-p)\log(1-p)].$$

Finally, in case $\beta=0$, f has the form

(15) $f(x,y;p) = g(x) + h(y) - g(x\cup y) + \ell(x,1-p) + \ell(y,p) + L(1-p)$,

where $\ell\colon D_1^o\times]0,1[\to R$, $L\colon]0,1[\to R$ respectively satisfy

(16) (a) $\ell(x\cup y,p) = \ell(x,p)+\ell(y,p)$, (b) $\ell(x,pq) = \ell(x,p)+\ell(x,q)$,

(17) $L(pq) = L(p) + L(q)$.

By hypothesis, we thus have in Λ for each fixed $(x,y)\in D_2^o$,

(18) $p\mapsto \ell(x,1-p) + \ell(y,p) + L(1-p)$.

Replacing y by $y\cup z$ (so that $(x,y,z)\in D_3^o$) in (18), using (16a), and comparing the result with (18), we have

$$p\mapsto \ell(z,p)$$

in Λ for each $z\in D_1^o$. By (16b), this means that

(19) $\ell(x,p) = a(x)\log p$, $x\in D_1^o$, $p\in]0,1[$,

for some map $a\colon D_1^o\to R$, which must be (5) additive because of (16a). By (18) and (19), L is in Λ. Hence, by (17),

$$L(p) = \gamma \log p, \quad p\in]0,1[,$$

for some constant γ, and by (15) and (19) f is given by

(20) $f(x,y;p) = g(x)+h(y)-g(x\cup y)+[a(x)+\gamma]\log(1-p)+a(y)\log p$.

Now, (9) with (11),(14), or (20) gives the form of I_2, and the forms (6),(7),(8) for $I_n (n=3,4,\ldots)$ follow by the (1) β-recursivity.

4. Remarks. We have also proved that (11),(14), and (20) are the only solutions of (10) for which $p\mapsto f(x,y;p)$ is in Λ for each $(x,y)\in D_2^o$.

Moreover, as a corollary, if we strengthen the (2) 3-semisymmetry to the (3) 3-symmetry, then we obtain (6),(7), and (8) with g replaced by h, and also $\gamma=0$ in case $\beta=0$.

The difference between these results and those of [3],[4],[1] (in which the domain of I_n contains boundary points) occurs in the case $\beta=0$. Here we find 'extra' solutions $\gamma \log p_1 + \Sigma a(x_i) \log p_i$ for additive $a: D_1^o \to R$.

References

1. Aczél, J., A mixed theory of information. VII: Inset information functions of all degrees, C.R. Math. Rep. Acad. Sci. Canada 2 (1980), 125-129.

2. Aczél, J.; Daróczy, Z., On Measures of Information and Their Characterizations. Academic, New York, San Francisco, London, 1975, §3.4.

3. Aczél, J; Daróczy, Z., A mixed theory of information. I: Symmetric recursive and measurable entropies of randomized systems of events, R.A.I.R.O. Informatique théorique 12 (1978), 149-155.

4. Aczél, J; Kannappan, Pl., A mixed theory of information. III: Inset entropies of degree β, Inform. and Control 39 (1978), 315-322.

5. Didderich, G.T., Boundedness on a set of positive measure and the fundamental equation of information, Publ. Math. Debrecen, to appear.

6. Ebanks, B.R., Measures of inset information on open domains - II: Additive inset entropies with measurable sum property, submitted.

7. Ebanks, B.R.; Maksa, Gy., Measures of inset information on open domains - I: Inset entropies and information functions of all degrees, submitted.

8. Halmos, P.R., Measure Theory. Van Nostrand, Toronto, New York, London, 1950.

9. Maksa, Gy., Bounded symmetric information functions, C.R. Math. Rep. Acad. Sci. Canada 2 (1980), 247-252.

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Received April 12, 1984

C.R. Math. Rep. Acad. Sci. Canada - Vol. VI, No. 3, June 1984 juin

A PROPERTY OF ARCS OF ORDER n IN R_n

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Introduction. Let Γ denote an arc in a real affine n-space R_n which satisfies certain differentiability conditions. Obviously, there are (n-1)-flats through n distinct points of Γ. We require that Γ be of (real) order n; that is, no (n-1)-flat meets Γ in more than n points. Such an arc then lies on the boundary of its convex hull $H(\Gamma)$; cf.[1].

The differentiability conditions yield that osculating k-flats exist at each point of Γ for $k = 1,2,\dots,n$. Non-parallel osculating (n-1)-flats of two distinct points of an arc Γ of order n intersect in an (n-2)-flat. We show that this (n-2)-flat does not meet $H(\Gamma)$.

1. Generalities

1.1. If $s_1, s_2 \in R_1$ and $s_1 < s_2$, we write $[s_1,s_2] = \{t \in R_1 | s_1 \le t \le s_2\}$ and $(s_1, s_2] = [s_1, s_2]\backslash\{s_1\}$. An arc in R_n is a continuous map

$$\Gamma: [s_1,s_2] \to R_n.$$

It is convenient to identify Γ with $\Gamma([s_1,s_2])$. The topology of $[s_1,s_2]$ defines (open) neighbourhoods on Γ.

The arc shall be differentiable in the following sense. For every $s \in [s_1,s_2]$, let $\Gamma_{-1}(s) = \emptyset$ and $\Gamma_0(s) = \Gamma(s)$. If $\Gamma_{k-1}(s)$ is already defined and its existence postulated then we require:

i) If $t \neq s$ is sufficiently close to s in $[s_1,s_2]$, then the flat $\langle\Gamma_{k-1}(s),\Gamma(t)\rangle$ spanned by $\Gamma_{k-1}(s)$ and $\Gamma(t)$ has dimension k and

ii) it converges as t tends to s. Its limit is the osculating k-flat $\Gamma_k(s)$; $k = 1,2,\dots,n$.

Henceforth, Γ will denote a differentiable arc of order n in R_n.

Let $L \subset R_n$ be a flat such that

$$L \cap \Gamma_k(s) = \Gamma_{k-1}(s).$$

Then $\Gamma(s)$ is counted as a k-fold intersection point of L and Γ. The flat L is said to meet Γ m times if the sum of the multiplicities of all intersection points of L and Γ is equal to m. Since Γ is of order n, it is known that any (n-1)-flat meets $\Gamma|_{(s_1,s_2]}$ at most n times and

(1) $\Gamma_k(s) \cap \Gamma_{n-k-1}(t) = \emptyset$ for $s \neq t$ in $(s_1,s_2]$; $k = 0,1,\dots,n-1$.

A necessary and sufficient condition for every (n-1)-flat to meet Γ itself at most n times is the Sauter Condition (cf. [3]).

(2) $\Gamma_k(s_1) \cap \Gamma_{n-k-1}(s_2) = \emptyset$ for $k = 0,1,\dots,n-1$.

Finally we note, the differentiability of Γ implies that Γ is strongly differentiable in the following sense. Let L_k^λ be a sequence of k-flats converging to L_k^λ such that all the points of $L_k^\lambda \cap \Gamma$ converge to $\Gamma(s)$. If each L_k^λ meets Γ (k+1)-times then $L_k^\lambda = \Gamma_k(s)$. In particular, the $\Gamma_k(s)$ depend continuously on s; cf. [3].

1.2 Let P_n denote a real projective n-space. A curve in P_n is a continuous map $\Delta: P_1 \to P_n$. We identify again Δ with $\Delta(P_1)$ and require that Δ satisfy the differentiability conditions in 1.1 with $[s_1,s_2]$ and R_n replaced by P_1 and P_n respectively. An arc Δ in P_n is of course a continuous map from a proper segment of P_1 to P_n.

Let Δ be a differentiable arc [curve] of order n in P_n. The dual arc [dual curve] Δ^* in the dual space P^*_n of P_n has the osculating spaces $\Delta^*_{-1}(s) = \emptyset$, $\Delta^*_n(s) = P^*_n$ and

(3) $\Delta^*_k(s) = \Delta_{n-k-1}(s)$; $k = 0,1,\ldots,n-1$.

Thus $\Delta^*(s) = \Delta^*_0(s) = \Delta_{n-1}(s)$. From [2], we note that Δ^* is a differentiable arc [curve] of order n in P^*_n.

Imbedding R_n in a real projective n-space P_n;

$$\Gamma:[s_1,s_2] \to R_n \subset P_n$$

is a differentiable arc of order n in P_n. It is known that Γ can be extended to a differentiable curve Δ of order n in P_n if and only if Γ satisfies (2); cf.[3]. In that case, the dual curve Δ^* of Δ contains the dual arc Γ^* of Γ. Since Δ^* is of order n and Δ^* has no end-points, it follows that Δ^* then satisfies (2). Thus if Γ satisfies the Sauter Condition then so does Γ^*.

Finally, we note the following evident fact about the convex hull of a set in R_n.

1.3 LEMMA. Let Σ be a bounded and path-wise connected set in R_n and let L_{n-2} be an (n-2)-flat. Then

(4) $L_{n-2} \cap H(\Sigma) = \emptyset$

if and only if some (n-1)-flat through L_{n-2} does not meet Σ.

2. An application of the Sauter Condition.

2.1 Let $\Gamma:[s_1,s_2] \to R_n \subset P_n$ denote a differentiable arc of order n which satisfies (2). Let $s_1 \le t_1 < t_2 \le s_2$. Then

$$(5) \qquad L_{n-2} = \Gamma_{n-1}(t_1) \cap \Gamma_{n-2}(t_2)$$

is an (n-2)-flat disjoint from Γ and

$$(6) \qquad L_{n-1}(s) = \langle L_{n-2}, \Gamma(s)\rangle$$

is an (n-1)-flat for every $s \in [s_1,s_2]$. Obviously,

$$(7) \qquad L_{n-1}(s) = \Gamma_{n-1}(t_\lambda) \Leftrightarrow s = t_\lambda; \quad \lambda = 1,2 .$$

Let s range from s_1 to s_2 in $[s_1,s_2]$. Then $L_{n-1}(s)$ rotates continuously about L_{n-2}.

2.2 LEMMA. Under the hypotheses of 2.1;

1. $L_{n-1}(s)$ rotates monotonically about L_{n-2},
2. there is an (n-1) flat through L_{n-2} which does not meet Γ and
3. $L_{n-2} \cap H(\Gamma) = \emptyset$.

PROOF.1. Suppose that the rotation of $L_{n-1}(s)$ changes directions at s_0. As Γ is strongly-differentiable, this yields.

$$\Gamma_1(s_0) \subset L_{n-1}(s_0) = \langle \Gamma_{n-1}(t_1) \cap \Gamma_{n-1}(t_2), \Gamma(s_0)\rangle.$$

Hence $\Gamma_1(s_0) \cap \Gamma_{n-1}(t_1) \cap \Gamma_{n-1}(t_2)$ is a point and dually by (3),

$$L^*_{n-1} = \langle \Gamma^*_{n-2}(s_0), \Gamma^*_0(t_1), \Gamma^*_0(t_2)\rangle$$

is an (n-1)-flat in P^*_n. It meets Γ^* at least (n+1)-times but as Γ^*, the dual arc of Γ, is of order n and satisfies (2); this is a contradiction.

2. We complete Γ to a differentiable curve Δ of order n in P_n. We may assume that $\Gamma = \Delta|_{[s_1,s_2]}$ and we redefine

$$L_{n-1}(s) = \langle L_{n-2}, \Delta(s)\rangle, \quad s \in P_1.$$

As s ranges through P_1, $L_{n-1}(s)$ rotates monotonically about L_{n-2}. Obviously (7) remains valid and thus the equation

$$\overline{L}_{n-1} = L_{n-1}(s)$$

has exactly one solution $s \in P_1$ for every (n-1)-flat $\overline{L}_{n-1}$ through L_{n-2}. Hence $L_{n-1}(\overline{s}) \cap \Gamma = \emptyset$ for $\overline{s} \in P_1\setminus[s_1,s_2]$.

3. We apply 2.2.2 and 1.3. □

REMARK. Dualizing 2.2.2, we obtain that there is a point on $\langle\Gamma(t_1), \Gamma(t_2)\rangle$ which does not lie on any osculating (n-1)-flat of Γ.

2.3 THEOREM. Let $\Gamma:[s_1,s_2] \to R_n$ be a differentiable arc of order n. Then

(8) $\Gamma_{n-1}(t_1) \cap \Gamma_{n-1}(t_2) \cap \operatorname{int} H(\Gamma) = \emptyset$ for $t_1 < t_2$ in $[s_1,s_2]$.

PROOF. If $t_1 = s_1$ then $\Gamma_{n-1}(s_1) \cap \Gamma = \{\Gamma(s_1)\}$ yields that $\Gamma_{n-1}(s_1)$ supports $H(\Gamma)$ and thus (8). Let $s_1 < t_1$.

For $s_1 < s < t_1$, $\Gamma|_{[s,s_2]}$ satisfies the Sauter Condition by (1) and thus by 2.2.3,

$$\Gamma_{n-1}(t_1) \cap \Gamma_{n-1}(t_2) \cap H(\Gamma|_{[s,s_2]}) = \emptyset.$$

Letting s tend to s_1, we now obtain (8). □

REFERENCES

[1] D. Derry, Convex hulls of simple space curves. Can. J. Math., 8 (1956), 383-388.

[2] R. Park, Topics in direct differential geometry. Can. J. Math., 24 (1972), 98-148.

[3] P. Scherk, Über differenzierbare Kurven and Bögen II Elementarbogen and Kurve n-ter Ordnung im R_n. Časopis pěst. mat. a fys. 66 (1937), 172-191.

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Received April 20, 1984

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