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UN THEOREME DE PRESERVATION POUR LES FACTORISATIONS

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Presented by P. Ribenboim

Résumé

A l'aide du théorème d'isomorphisme de S. Shelah et de théorèmes de préservation classiques, nous établissons un théorème de préservation pour les factorisations d'homomorphismes. Nous donnons également une généralisation d'un théorème de S. Burris qui servira d'exemple d'intervention naturelle de la notion de factorisation en théorie des modèles.

Les notations sont celles de [2].

On dit qu'un ensemble d'énoncés T (dans un langage du premier ordre) est <u>préservé sous les factorisations</u> ou, de façon équivalente, que la classe de ses modèles M(T) est <u>fermée pour les factorisations</u>, si pour tout homomorphisme $f:A \longrightarrow B$ avec $A \models T$ et $B \models T$, on a $Im(f) \models T$ (où Im(f) est l'image homomorphe de f).

Le résultat suivant nous sera utile: sa démonstration est implicite dans celle du lemme 3.2.1 de [2] :

<u>Lemme</u>. <u>Soit A une structure, $\triangle$ un ensemble d'énoncés fermé pour la disjonction finie et soit T une théorie consistante. Alors il existe un modèle B de T tel que ($\sigma \in \triangle$ et $B \models \sigma$) $\Rightarrow A \models \sigma$ si et seulement si ($\sigma \in \triangle$ et $T \models \sigma$) $\Rightarrow A \models \sigma$.</u>

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<u>Théorème</u>. <u>Un ensemble d'énoncés T est préservé sous les factorisations si et seulement s'il est équivalent à un ensemble d'énoncés positifs et d'énoncés universels.</u>

<u>Démonstration</u>. La direction ($\Leftarrow$) découle aisément du fait que Im(f) est sous-structure de $\mathcal{B}$.

Pour la direction opposée, on définit les ensembles d'énoncés T_1, T_2, Δ_1 et Δ_2, respectivement les conséquences universelles de T, les conséquences positives de T, les énoncés universels et les énoncés positifs. On doit montrer que $(T_1 \cup T_2) \models T$.

Soit donc $A \models (T_1 \cup T_2)$.

a) Soit $\delta_1 \in \Delta_1$. Si $T \models \delta_1$, alors $\delta_1 \in T_1$ et donc $A \models \delta_1$. Par le lemme, il existe donc $B_1 \in M(T)$ tel que $B_1 \models \delta \Rightarrow A \models \delta$ pour tout $\delta \in \Delta_1$. Par la proposition duale de 5.2.2 de [2], il existe donc B' et A' tels que $B_1 \prec B$, $A \cong A'$ et $A' \subset B'$ (i.e. B_1 est sous-structure élémentaire de B, A est isomorphe à A' et A' est sous-structure de B').

b) Comme en a), on déduit qu'il existe $B_2 \in M(T)$ tel que $B_2 \models \delta_2$ implique $A \models \delta_2$ pour tout $\delta_2 \in \Delta_2$. Par 5.2.12 de [2], il existe A'', B'' et un homomorphisme surjectif $h : B'' \twoheadrightarrow A''$ tels que $A \prec A''$ et $B_2 \prec B''$.

On a donc le schéma

$$B_2 \prec B'' \overset{h}{\twoheadrightarrow} A'' \succ A \cong A' \hookrightarrow B' \succ B_1$$

où B'' et B' sont dans M(T) puisque B_1 et B_2 y sont.

Puisque A'' et A' sont élémentairement équivalentes, elles ont des ultrapuissances (sur un même filtre D) $\prod_D A''$ et $\prod_D A'$ isomorphes (voir [4] ou le théorème 6.1.15 de [2]). Or $\prod_D$ peut être vu comme un endofoncteur de la catégorie M(T) qui préserve les inclusions, où $\prod_D f$, pour un homomorphisme f, est défini ponctuellement (voir [3] p. 115). Il est clair que $\prod_D f$ est surjectif lorsque f l'est, d'où on a le diagramme

$$\prod_D B'' \xrightarrow{\prod_D h} \!\!\!\!\rightarrow \prod_D A'' \xrightarrow{\simeq} \prod_D A' \hookrightarrow \prod_D B'$$

avec $\prod_D B''$ et $\prod_D B'$ dans M(T), puisque toute structure est élémentairement équivalente à ses ultrapuissances.

L'hypothèse sur T implique donc que $\prod_D A' \models T$, et donc $A \models T$, ce qui démontre le théorème.

Les factorisations interviennent naturellement par exemple dans le contexte suivant.

<u>Définitions</u>. 1) Le <u>skolemisé</u> Ψ^* d'un énoncé Ψ sous forme prenex $\forall x_1 \exists y_1 \cdots \forall x_n \exists y_n \phi(\vec{x},\vec{y})$ dans un langage L est l'énoncé $\forall x_1 \forall x_2 \cdots \forall x_n \phi(\vec{x}, F_1(x_1), \cdots, F_n(x_1 \cdots x_n))$ dans l'expansion $L_\Psi{}^*$ obtenue de L en lui ajoutant les symboles de fonctions $F_1, \cdots, F_n$. Si T est une théorie dans L, on note T^* l'ensemble $\{\Psi^* | \Psi \in T\}$ d'énoncés dans $L^* = \bigcup_{\Psi \in T} L_\Psi{}^*$.

2) Si Ψ est comme en 1), on note $\Psi!$ l'énoncé

$$\Psi \wedge \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n \forall z_1 \cdots \forall z_n ((\phi(\vec{x},\vec{y}) \wedge \phi(\vec{x},\vec{z})) \longrightarrow \bigwedge_{i=1}^{n} (y_i = z_i)).$$

Si T est une théorie, on note T! l'ensemble $\{\Psi! | \Psi \in T\}$.

Si M(T) désigne la catégorie dont les morphismes sont les homomorphismes entre les modèles de T, on a un foncteur oubli $\mathcal{O}_T{}^*$: $M(T^*) \longrightarrow M(T)$ évident qui fait commuter

$$\begin{tikzcd} M(T^*) \arrow[rr, "\mathcal{O}_T^*"] \arrow[dr, "G^*"'] & & M(T) \arrow[dl, "G"] \\ & \mathrm{Ens} & \end{tikzcd}$$

où G^* et G sont les foncteurs oubli dans la catégorie des ensembles. La proposition suivante généralise le théorème 8 de [1] et se démontre sans grande difficulté.

<u>Proposition</u>. <u>Soit T un ensemble d'énoncés négatifs, d'énoncés positifs et d'énoncés universels. Alors $\mathcal{O}_T^*$ est un isomorphisme si et seulement si $T \equiv T^!$ et T est préservé sous les factorisations</u>.

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A NON COMPACT DIFFERENTIABLE SEMIGROUP ARISING FROM AN ABSTRACT DELAY EQUATION

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Presented by D.V. Atkinson

0. Introduction

In this paper we want to summarize recent results ([5]) on retarded equations in Banach spaces and prove some theorems concerning the associated solution semigroup. The same problem has been studied in Hilbert spaces in [1] and has applications to parabolic partial differential equations with delays in the highest order derivatives.

1. Notation

Let E be a Banach space with norm $\|\cdot\|$ and $A:D_A \subset E \to E$ the generator of a bounded analytic semigroup $T(t)$ in E. D_A will be endowed with the graph norm. For each $\alpha \in]0,1[$ let us define the following intermediate spaces between D_A and E (see e.g. [4]):

$$D_A(\alpha,\infty) = \{x \in E,\ \|x\|_\alpha = \sup_{t>0} \|t^{1-\alpha}AT(t)x\| < \infty\}$$

$$D_A(\alpha) = \{x \in E,\ \lim_{t\to 0} t^{1-\alpha}AT(t)x = 0\}$$

with norm $\|x\|+\|x\|_\alpha$. We have $D_A \subset D_A(\alpha) \subset D_A(\alpha,\infty)$. If $a,b \in \mathbb{R}$ and X is a Banach space then $B(a,b;X)$, $C(a,b;X)$ and $C^\alpha(a,b;X)$ are the usual Banach spaces of functions from $[a,b]$ to X which are bounded, continuous and α-hölder continuous respectively. We will use also the little-hölder continuous functions' space $h^\alpha(a,b;X) = \{u:[a,b] \to X;\ \lim_{\varepsilon\to 0} \sup_{|t-s|<\varepsilon} |t-s|^{-\alpha}\|u(t)-u(s)\| = 0\}$ with the norm of $C^\alpha(a,b;X)$ (see [3]). If $n \in \mathbb{N}$ the spaces $C^{n+\alpha}(a,b;X)$ and

$h^{n+\alpha}(a,b;X)$ have an obvious definition. Finally $W^{1,1}(a,b)$ is the usual Sobolev space of functions u from $[a,b]$ to C.

2. Abstract delay equation

Let $A_1, A_2 \in \mathcal{L}(D_A, E)$, $a \in L^1(-r,0)$ and let f and y be defined in $[0,T]$ and $[-r,0]$ respectively. For the delay equation:

$$(2.1) \quad \begin{cases} u'(t) = Au(t)+A_1u(t-r)+ \int_{-r}^{0} a(\theta)A_2u(t+\theta)d\theta+f(t) \;, & 0 \le t \le T \\ u\,(t) = y(t) & , -r \le t \le 0 \end{cases}$$

the following global existence, uniqueness and regularity results can be proved:

Theorem 2.1 *Given* $f \in C^{\alpha}(0,T;E)$ *and* $y \in C^{\alpha}(-r,0;D_A)$ *satisfying:*

$$(2.2) \quad y^* \triangleq Ay(0)+A_1y(-r)+ \int_{-r}^{0} a(\theta)A_2y(\theta)d\theta+f(0) \in D_A(\alpha,\infty)$$

there exists a unique solution u of (2.1); moreover $u \in C^{\alpha}(-r,T;D_A) \cap C^{1+\alpha}(0,T;E)$ *and* $u' \in B(0,T;D_A(\alpha,\infty))$. *If in addition* $f \in h^{\alpha}(0,T;E)$, $y \in h^{\alpha}(-r,0;D_A)$ *and* $y^* \in D_A(\alpha)$ *then we have also* $u \in h^{\alpha}(-r,T;D_A) \cap h^{1+\alpha}(0,T;E)$ *and* $u' \in C(0,T;D_A(\alpha))$. *Finally there exists* $C>0$ *depending on* $\alpha, a(\cdot)$, *A and* A_2 *such that*

$$\|u\|_{C^{\alpha}(0,T;D_A)\cap C^{1+\alpha}(0,T;E)} + \|u'\|_{B(0,T;D_A(\alpha,\infty))} \le$$

$$\le C\,\{\|f\|_{C^{\alpha}(0,T;E)}+\|y\|_{C^{\alpha}(-r,0;D_A)}+\|y^*\|_{D_A(\alpha,\infty)}\} \;.$$

Theorem 2.2 *If* $f \in C^{1+\alpha}(0,T;E)$ *and* $y \in C^{1+\alpha}(-r,0,D_A)$ *satisfy the conditions*

$$y'(0) = Ay(0)+A_1y(-r)+ \int_{-r}^{0} a(\theta)A_2y(\theta)d\theta+f(0)$$

$$y^{**} \triangleq Ay'(0)+A_1y'(-r)+ \int_{-r}^{0} a(\theta)A_2y'(\theta)d\theta+f'(0) \in D_A(\alpha,\infty)$$

then the solution u of (2.1) is such that $u \in C^{1+\alpha}(-r,T;D_A) \cap C^{2+\alpha}(0,T;E)$, $u'' \in B(0,T;D_A(\alpha,\infty))$ *and for each* $t \in [0,T]$

$$u''(t) = Au'(t)+A_1u'(t-r)+ \int_{-r}^{0} a(\theta)A_2u'(t+\theta)d\theta+f'(t) \;.$$

If in addition $f\in h^{1+\alpha}(0,T;E)$, $y\in h^{1+\alpha}(-r,0;D_A)$ and $y^{**}\in D_A(\alpha)$ then we have also $u\in h^{1+\alpha}(-r,T;D_A)\cap h^{2+\alpha}(0,T;E)$ and $u''\in C(0,T;D_A(\alpha))$.

The proofs of these theorems are based on maximal regularity results for the problem $u'(t)=Au(t)+f(t)$, $0\le t\le T$; $u(0)=u_0$ when $f\in C^\alpha(0,T;E)$ (see [4], [5]).

3. The solution operator semigroup

Let us introduce the Banach space

$$Z = \{y\in h^\alpha(-r,0;D_A)\ ;\ y^* = Ay(0)+A_1y(-r)+\int_{-r}^{0} a(\theta)A_2y(\theta)d\theta\in D_A(\alpha)\}$$

with the norm

$$\|y\|_Z = \|y\|_{h^\alpha(-r,0;D_A)}+\|y^*\|_{D_A(\alpha)}$$

As from theorem 2.1 we deduce the existence in $[-r,+\infty[$ of a unique solution u of (2.1) with $f=0$, the solution operator semigroup $S(t): Z\to Z$ can be defined for each $t\ge 0$ as follows

$$(3.1)\qquad (S(t)y)(\theta) = u(t+\theta)\ ,\quad -r\le\theta\le 0$$

<u>Theorem 3.1</u> <u>$S(.)$ is a strongly continuous semigroup in Z and its generator Λ is given by $\Lambda y=y'$ for</u> $y\in D_\Lambda=\{y\in h^{1+\alpha}(-r,0;D_A);$ $y'(0)=Ay(0)+A_1y(-r)+\int_{-r}^{0}a(\theta)A_2y(\theta)d\theta$; $y^{**}\hat{=}Ay'(0)+A_1y'(-r)+\int_{-r}^{0}a(\theta)\cdot$ $\cdot A_2y'(\theta)d\theta\in D_A(\alpha)\}$.

The use of the little-hölder spaces is necessary to prove the continuity of $S(\cdot)$. Moreover to characterize D_Λ we can apply the regularity result stated in theorem 2.2.

It is known ([6]) that the study of the spectrum $\sigma(\Lambda)$ is useful to study the asymptotic behavior of $S(\cdot)y$ and hence of the solutions of (2.1) if $\sup\{\mathbb{R}z; z\in\sigma(\Lambda)\}$ coincides with the type of $S(\cdot)$; this is true e.g. when $S(\cdot)$ is compact or differentiable. But the next theorems show that in general this is not the case.

<u>Theorem 3.2</u> <u>If $A \notin \mathcal{L}(E)$, $A_1=\gamma A$ with $\gamma \in \mathbb{R}-\{0\}$ and $A_2=0$ then $S(\cdot)$ is not compact nor differentiable.</u>

<u>Proof</u> Let us observe that if $\lambda \in \mathbb{C}$, $y \in D_\Lambda$ and $\bar{y} \in Z$ satisfy $\lambda y-\Lambda y = \bar{y}$ then necessarily $x=y(0) \in D_A$ and

(3.2) $y(\theta)=e^{\lambda\theta}x + \int_\theta^0 e^{\lambda(\theta-s)}\bar{y}(s)ds$, $-r \leq \theta \leq 0$

(3.3) $\lambda x-(1+\gamma e^{-\lambda r})Ax = \bar{y}(0)+\gamma \int_{-r}^0 e^{-\lambda(r+s)}A\bar{y}(s)ds$

Now the set Γ of the solutions λ of $1+\gamma e^{-\lambda r}=0$ is given by $\lambda=\lambda^*+ i\frac{2\pi k}{r}$, $k \in Z$ where $\lambda^*=\frac{\lg|\gamma|}{r}$ if $\gamma<0$ and $\lambda^*=\frac{\lg|\gamma|}{r} + i\frac{\pi}{r}$ if $\gamma>0$. Let us choose now $\lambda \in \Gamma$ and $\bar{y}(t)=e^{\lambda t}(\lambda_0-A)^{-1}w$ $(-r \leq t \leq 0)$ with λ_0 in the resolvent of A and $w \notin D_A$: it is easy to verify that equation (3.3) cannot have a solution $x \in D_A$ therefore λ belongs to $\sigma(\Lambda)$. In addition if $\lambda \in \Gamma$ and $\lambda \neq 0$ then, setting $\bar{y}=0$ in (3.2)-(3.3) we get $x=0$: so that $y \equiv 0$ by virtue of (3.2). In conclusion there exist elements of $\sigma(\Lambda)$ which are not eigenvalues: hence $S(\cdot)$ is not compact (see section 2.2 of [2]). Moreover the location of Γ in the complex plane proves the nondifferentiability of $S(\cdot)$ by virtue of Pazy's theorem ([2], pag. 54).

<u>Theorem 3.3</u> <u>If $A \notin \mathcal{L}(E)$, $A_1=0$, $A_2=A$ and $a(\theta)=a\neq 0$ then $S(\cdot)$ is noncompact.</u>

<u>Proof</u> If we proceed as above we must replace (3.3) with

(3.4) $\lambda x -(1+a \int_{-r}^0 e^{\lambda\theta}d\theta)Ax = \bar{y}(0)+a\int_{-r}^0 e^{\lambda\theta}d\theta \int_\theta^0 e^{-\lambda s}A\bar{y}(s)ds$

It can be proved that there exist $\lambda \neq 0$ such that $1+a\int_{-r}^0 e^{\lambda\theta}d\theta=0$: moreover by choosing the same $\bar{y}$ as above the proof can be completed.

We want to show now that if we have only a distributed delay term with a more regular density $a(\cdot)$ then the semigroup $S(\cdot)$ is differentiable:

<u>Theorem 3.4</u> <u>If $A_1=0$ and $a(\cdot) \in W^{1,1}(-r,0)$ then $S(t)$ is differ-</u>

entiable for $t > r$.

<u>Proof</u> Our result will be demonstrated if we prove that $S(r+\epsilon)y \in D_A$ for each $\epsilon > 0$ and $y \in Z$. This means that the solution u of

$$(3.4) \qquad \begin{cases} u'(t)=Au(t)+\int_{-r}^{0} a(\theta)A_2u(t+\theta)d\theta \quad , & t \geq 0 \\ u(t)=y(t) & -r \leq t \leq 0 \end{cases}$$

must verify the following conditions

$$(3.5) \qquad u \in h^{1+\alpha}(\epsilon,\epsilon+r;D_A)$$

$$(3.6) \qquad u'(r+\epsilon) = A\ddot{u}(r+\epsilon) + \int_{-r}^{0} a(\theta)A_2u(r+\epsilon+\theta)d\theta$$

$$(3.7) \qquad Au'(r+\epsilon) + \int_{-r}^{0} a(\theta)\, A_2u'(r+\epsilon+\theta)d\theta \in D_A(\alpha).$$

Setting $g(t) = \int_{-r}^{0} a(\theta)A_2u(t+\theta)d\theta$, $t \geq 0$ it can be proved that $g \in h^{1+\alpha}(0,T;E)$ for each $T>0$: as (3.4) implies that $u(t)=e^{At}y(0)+ + \int_0^t e^{A(t-s)}g(s)ds$, $t \geq 0$, by using theorems 4.1, 4.2 and 4.3 of [4] we get (3.5) and that $u''(t) \in D_A(\alpha)$ for $t > 0$: but from (3.4) we have $u''(t)=Au'(t)+ \int_{-r}^{0} a(\theta)A_2u'(t+\theta)d\theta$ and so (3.7) follows. Finally (3.6) is a consequence of (3.4).

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ON PROJECTIONS OF REAL ALGEBRAIC VARIETIES

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Presented by P. Ribenboim

Abstract. It is shown that any closed semialgebraic set whose Zariski-closure is irreducible is the projection under a finite map of an irreducible real algebraic set (theorem 1.4). Some applications are given.

§1. The main result.

Let R be an arbitrary real closed field. Given an irreducible real algebraic set $V \subset R^n$, we denote by p its ideal in $R[X_1,\dots,X_n]$ and by V_c its set of central points -i.e. the closure of the regular points of V. We work always with the order topology of R^n. Finally, let π stand for the projection into the n first coordinates.

1.1. Lemma. Let $f_1,\dots,f_p$ be polynomials in $R[X_1,\dots,X_n]$. Then for any $\lambda_1,\lambda_2 \in R$, $0 < \lambda_2 < \lambda_1$, the set of zeros $\tilde{W}$ of the polynomial $F(T,\underline{X}) \in R[X_1,\dots,X_n,T]$,

$$F(T,X) = Qf_p(T^2-\lambda_2 f_1)-QT^2(T^2-\lambda_1 f_1)-(T^2-\lambda_2 f_1)\sum_{i=2}^{p-1} T^2(T^2-2f_i)Q_i$$

satisfies $\pi\colon \tilde{W} \to R^n$ is finite and

$$\pi(\tilde{W}) = \{f_1 \geq 0\} \cup \dots \cup \{f_p \geq 0\},$$

where

$$Q(T,\underline{X}) = \prod_{i=2}^{p-1} (T^2-f_i) \quad \text{and}$$

$$Q_i(T,\underline{X}) = Q(T,\underline{X})/(T^2-f_i) \qquad (i=2,\dots,p-1)$$

Proof. See [A-G]

1.2. Proposition. Let $f_1,\dots,f_p$ be polynomials in $R[X_1,\dots,X_n]$. Assume that the semialgebraic set $S = V \cap (\{f_1 \geqq 0\} \cup \dots \cup \{f_p \geqq 0\})$ is Zariski-dense in V. Then there exists an irreducible algebraic set $W \subset R^{n+1}$ such that $\pi: W \to V$ is finite and $\pi(W) = S$.

Proof. By using Bertini's theorem (cf. [H], p.275) we prove that λ_1, λ_2 in 1.1 can be chosen such that $W = \mathfrak{W} \cap (V \times R)$ is irreducible.

1.3. Definition. A semialgebraic set $S \subset R^n$ is called irreducible if the smallest algebraic set which contains it is irreducible.

Let $S \subset R^n$ be an irreducible semialgebraic set. S can be written in the form, see [A-G],

$$S = (S_1 \cap \dots \cap S_m) \cap V$$

where V is the Zariski closure of S and

$$S_i = \{f_{1i} \geqq 0\} \cup \dots \cup \{f_{pi} \geqq 0\}, \quad f_{ki} \in R[X_1,\dots,X_n],$$

for all $(k,i) \in \{1,\dots,p\} \times \{1,\dots,m\}$.

Then by a repeated use of Bertini's theorem and induction on m we manage to prove:

1.4. Theorem. Let $S \subset R^n$ be an irreducible semialgebraic set. Let $V \subset R^n$ be the Zariski-closure of S. Then there exist a positive integer m and an irreducible algebraic set $W \subset R^{n+m}$ such that:

(1) $\pi: W \to V$ is finite

(2) $\pi(W) = S$.

1.5. Remark. Notice that the condition of the irreducibility of S in theorem 1.4. is necessary. Indeed, it is immediate to check that no irreducible real algebraic set can project onto $S = \{(x,0): x \in R\} \cup \{(0,y): y \in R\} \subset R^2$.

1.6. Remark. For applications to the space of orders of function fields it is convenient to observe that in case $S = \overline{\overset{\circ}{S}}$ -where both closure and interior are taken in V- we can conclude that $\pi(W_c) = S$. This follows from the finiteness of π by using [B], p. 170.

§2. Applications.

Let K be a function field over R. We denote by X(K) its space of orders endowed with Harrison's topology.

Given any formally real extension E of K, we consider the restriction morphism ε defined by

$$\varepsilon: X(E) \to X(K): P \to P \cap K.$$

A subset y of X(K) is called clopen if it is open and closed. Our first application answers a question proposed in [E-L-W]:

2.1. Theorem. Let Y be any clopen subset of $X(K)$. Then there exists a finite extension E of K such that $Y = \operatorname{im} \varepsilon$.

Proof. Take a non singular model V of K and consider the semi-algebraic subset $\overline{Y}^{\wedge}$ of V associated to Y according to [D-R]. Then 2.1. follows from 1.6. and theorem 2.10 in [D-R].

Next we state theorem 1.4 in terms of the real spectrum of real finitely generated R-algebras (see [C-R]).

2.2. Proposition. Let A be a real domain, finitely generated as R-algebra. Let C be a constructible set of $\operatorname{Spec}_R A$ with non-empty interior. Then there exists a real domain B, integral over A, such that $C = j^*(\operatorname{Spec}_R B)$.

Proof. The result follows from 1.4. from the one - to-one correspondence (proved in [C-R]) between constructible sets of $\operatorname{Spec}_R R[V]$ and semialgebraic subsets of V, for a model V of A.

We end by showing how theorem 1.4. can be used to produce "exotic" real irreducible algebraic sets.

2.3. Proposition. For any natural numbers n,p, there exists an irreducible non-singular real hypersurface in $\mathbb{P}^{n+1}(\mathbb{R})$ with p equidimensional connected components.

Proof. Take p disjoint closed balls $B_1,\dots,B_p$ of $\mathbb{R}^n$. After a careful choice of the centers and radius, it can be checked that the hypersurface W of 1.2. which projects onto $S = B_1 \cup \dots \cup B_p$

verifies the requested conditions.

2.4. Proposition. For any finite increasing sequence of non-negative integers $s_o < s_1 < \ldots < s_r$ there exists an irreducible real hypersurface $W \subset \mathbb{R}^{s_r+1}$ with pieces of dimensions $s_o, s_1, \ldots, s_r$ (and no pieces of other dimensions). Moreover, if $s_o > 0$, W can be chosen connected.

Proof. We take the closed ball B in $\mathbb{R}^{s_r}$ of radius 1 centered at $b = (0,\ldots,0,1)$ and, linear subspaces $L_o,\ldots,L_{r-1}$ of dimensions $s_o,\ldots,s_{r-1}$ respectively and transversal each other.

If $\ell_{j,1},\ldots,\ell_{j,s_r-s_j}$ are the linear forms which define L_j, we set $f_j = -(\ell_{j,1}^2 + \ldots + \ell_{j,s_r-s_j}^2)$ and $f_r = 1 - x_1^2 - \ldots - x_{s_r-1}^2 - (x_{s_r}-1)^2$.

Then:

$$S = B \cup L_o \cup \ldots \cup L_{r-1} = \{f_o \geq 0\} \cup \ldots \cup \{f_r \geq 0\}.$$

So the hypersurface $W \subset \mathbb{R}^{s_r+1}$ in proposition 1.2 projecting onto S satisfies the requested conditions.

For the moreover part it is enough to take into account that $f_o(0) = \ldots = f_r(0) = 0$. Then from the very definition of F in 1.1., the unique point of W which projects onto $\underline{0} \in \mathbb{R}^{s_r}$ is $(\underline{0},0)$.

2.5. Remark. Notice that 2.3 and 2.4 remain valid for any real closed field R if we define a semialgebraic set $S \subset R^n$ to be connected if it is not union of two disjoint and non empty open semialgebraic sets (see [B]).

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ON t-CLOSURES

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Presented by K. Murasugi

Let R be a Noetherian integral domain with field of quotients K, and assume that the integral closure $\bar{R}$ in K, is finite over R. Let A be a subring between R and $\bar{R}$. In [5], Traverso defined the seminormalization ${}^{+}_{A}R$ of R in A as follows; ${}^{+}_{A}R = \{ a \in A : a_p \in R_p + J(A_p)$ for all $p \in \operatorname{Spec} R \}$, where $J(A_p)$ is the Jacobson radical of A_p. Then he proved that the ring ${}^{+}_{A}R$ is characterized as the greatest subring B between R and A which satisfies the following two properties: (1) The contraction map $\operatorname{Spec} B \to \operatorname{Spec} R$ is an injection. (2) For all $P \in \operatorname{Spec} B$, $k(P \cap R) \cong k(P)$. Then it follows that R is seminormal in A (i.e., $R = {}^{+}_{A}R$) if and only if R is (2,3)-closed in A (i.e., whenever $a \in A$ satisfies that $a^2, a^3 \in R$, then $a \in R$) [2].

In [3], local quasinormality is defined, and it is proved that the following statements are equivalent: (i) R is locally quasinormal in A. (ii) R is seminormal and locally u-closed in A. (iii) R is t-closed in A. ([3, Corollary 2.13] and [4, Theorem 1]).

The aim of this note is to define t-closure, and to characterize it just by condition (2) in the characterization of seminormalization.

We recall some definitions and notation: Let R, A, $\bar{R}$ and ${}^{+}_{A}R$ be as above. For $a \in A$, we say that the element a satisfies the condition (t;R) if both

a^2-ra and $a^3-ra^2 \in R$ for some $r \in R$. We say that R is t-*closed* in A if whenever $a \in A$ satisfies the condition $(t;R)$, then $a \in R$. We say that R is u-*closed in* A if whenever $a \in A$ satisfies both a^2-a and $a^3-a^2 \in R$, then $a \in R$. And R is called *locally* u-*closed in* A if R_p is u-closed in A_p for all $p \in \operatorname{Spec} R$. Let $\operatorname{MPic}(R)$ denote the *cokernel* of the map $\operatorname{Pic}(R) \to \operatorname{Pic}(R[X,1/X])$, where X is an indeterminate over R. Then we say that R is *quasinormal in* A if $\operatorname{MPic}(R) \to \operatorname{MPic}(A)$ is injective. And R is said to be *locally quasinormal in* A if R_p is quasinormal in A_p for all $p \in \operatorname{Spec} R$. In this notation, if $A = \bar{R}$, we simply say, for example, that R is t-*closed* instead of saying that R is t-closed *in* $\bar{R}$. Here it should be noted that if R is normal, then $\operatorname{MPic}(R) = 0$ [1, Corollary 5.10], therefore an integral domain R is quasinormal if and only if $\operatorname{Pic}(R) \cong \operatorname{Pic}(R[X,1/X])$. Let ${}^t_AR$ denote the intersection of those subrings B between R and A, such that B is t-closed in A. Then clearly, ${}^t_AR$ is the smallest subring which is t-closed in A, so we call this ring the t-*closure of* R *in* A.

Now we define subrings R_i of A, $i = 0,1,\dots$, as follows: Put $R_0 = R$. If $a_1 \in A - R_0$ satisfies the condition $(t;R_0)$, then we put $R_1 = R_0[a_1]$. If $a_2 \in A - R_1$ satisfies the condition $(t;R_1)$, then we put $R_2 = R_1[a_2]$. Inductively, if $a_i \in A - R_{i-1}$ satisfies the condition $(t;R_{i-1})$, then we put $R_i = R_{i-1}[a_i]$. Then this procedure will stop at some R_n. Put $R_* = R_n$. It immediately follows that the ring R_* is the smallest t-closed subring between R and A. Thus $R_* = {}^t_AR$.

REMARK. The argument mentioned above shows that the t-closure ${}^t_AR$ can be obtained by adjoining a finite number of elements $a_1,\dots,a_n \in A$ to R, as above. And $R_* = {}^t_AR$ is independent of the choices of such elements a_i.

For $a \in A$ and $P \in \mathrm{Spec}\, A$, we denote by $a(P)$, as usual, the image of a in the residue field $k(P)$ of P. Put $C = \{ a \in A : a(P) \in k(P \cap R)$ for all $P \in \mathrm{Spec}\, A \}$. Then we have the following theorem, which gives the characterization of the t-closure stated in the first paragraph.

<u>THEOREM 1.</u> <u>Let the notation be as above. Then the rings R_* and C coincide with ${}^t_AR$. Therefore the t-closure of R in A is the greatest subring B between R and A such that $k(P) \cong k(P \cap R)$ for all $P \in \mathrm{Spec}\, B$.</u>

Proof. We have already seen that $R_* = {}^t_AR$. So it remains to prove that $C = {}^t_AR$. We first show that C is t-closed in A. Let $a \in A$ satisfy the condition $(t;C)$: a^2-ca, $a^3-ca^2 \in C$ for some $c \in C$. Now suppose, for contrary, that $a(P) \notin k(P \cap R)$. Then, in particular, $a(P) \neq 0$. It can be seen that $a(P)\cdot(a^2-ca)(P)$, $(a^2-ca)(P) \in k(P \cap R)$, and $a(P)\cdot(a-c)(P) = 0$. Therefore $a(P) = c(P)$, and so $a(P) \in k(P \cap R)$, the desired contradiction. Thus C is t-closed in A, and hence $C \supset {}^t_AR$. Secondly we show the reverse containment. For this purpose, we recall a theorem of quasinormality [3, Theorem 2.12], which asserts that there exists a sequence of rings $A = F_0 \supset F_1 \supset \dots \supset F_d = {}^t_AR$ such that for each i, $1 \le i \le d$, F_i is locally quasinormal in F_{i-1}, and each $q \in \mathrm{Ass}_{F_i}(F_{i-1}/F_i)$ has a unique lying-over prime ideal Q in F_{i-1}. From this one can see that $F_i = \{ f \in F_{i-1} : f(Q) \in k(q)$ for all $Q \cap F_i = q \in \mathrm{Ass}_{F_i}(F_{i-1}/F_i)\}$ (see, [6, Theorem 2.10]). It then follows that $F_i \supset C$ for all i, $1 \le i \le d$, since for $c \in C$ and $Q \in \mathrm{Spec}\, F_{i-1}$ with $Q \cap F_{i-1} = q \in \mathrm{Ass}_{F_i}(F_{i-1}/F_i)$, as Q being unique, we have that for some $P \in \mathrm{Spec}\, A$ with $P \cap F_{i-1} = Q$, $c(P) \in k(P \cap R) = k(q \cap R)$, and so $c(Q) \in k(q)$. Thus we have that $C \subset {}^t_AR$. The final assertion is now obvious. The theorem is completely proved.

As noted in the second paragraph, for a normal domain R, MPic(R) = 0 [1, Corollary 5.10]. Hence we obtain the following.

THEOREM 2. For a Noetherian integral domain R with $\bar{R}$ finite over R, the following statements are equivalent.

(1) R is locally quasinormal.

(2) R is t-closed.

(3) R is seminormal and locally u-closed.

(4) There is no subring B between R and $\bar{R}$ other than R itself for which the folllowing property is satisfied: for every $P \in \operatorname{Spec} B$, $k(P) \cong k(P \cap R)$.

We complete this note by giving an example of a sequence of rings such that $R = {}^{+}R \subsetneq {}^{t}R \subsetneq \bar{R}$.

EXAMPLE. Let $k[X,Y]$ be a polynomial ring over a field k in two indeterminates X, Y. Let p be the prime ideal in $k[X,Y]$ generated by the element Y^2-X. Put $C = k[X] + p$ and $R = k[X^2-X, X^3-X^2] + p$. Then $\bar{R} = k[X,Y]$, and $C = \{ a \in \bar{R} : a(p) \in k(X) \}$, therefore C is t-closed. Further, the conductor $c(C/R)$ from C to R, is the ideal $(X^2-X, X^3-X^2)C + p$. Then passing to the residue rings C and R modulo $c(C/R)$, we see that R is seminormal in C. Hence R is seminormal. Thus $R = {}^{+}R \subsetneq C = {}^{t}R \subsetneq \bar{R}$.

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POLYHEDRA WITH TRANSITIVITY PROPERTIES *

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Presented by H.S.M. Coxeter

ABSTRACT

We investigate polyhedra in 3-dimensional Euclidean space which are homeomorphic to embedded manifolds and have groups of symmetries that act transitively on their faces, edges, or vertices. The most unexpected result is that the vertex-transitive polyhedra can have positive genus.

Polyhedra in three-dimensional space have been studied for thousands of years and particular attention has been paid to those polyhedra which are highly symmetric. One way in which the symmetry properties can be described is by means of the transitivity classes of elements such as vertices, edges or faces. For example, the *uniform polyhedra* can be defined as those polyhedra whose faces are regular polygons and for which the symmetry group is transitive on the vertices; the *regular polyhedra* or Platonic solids can be defined as those polyhedra for which the symmetry group is transitive on the vertices, on the edges and also on the faces. It should be noted here that we are using the word *polygon* in the sense of a simple closed curve which is the union of a finite number of line segments; the closed bounded plane set having such a curve as its boundary will be called a *polygonal region*. If the line segments are disjoint except for their endpoints and no two consecutive ones are collinear then they are called the *edges* and their intersections are called the *vertices* of the polygon and of the polygonal region. A *polyhedron* is the union of a finite number of polygonal regions which form a manifold without boundary and the three-dimensional set bounded by such a manifold is called a *polyhedral solid*. If the polygonal regions are disjoint except for their edges, and no two adjacent regions are coplanar, then they are called the *faces* of the polyhedron. Thus we are excluding from consideration the Kepler-Poinsot polyhedra and other stellated polyhedra in which either the polygons defining the faces or the polyhedron itself have self-intersections. For simplicity we shall

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use descriptions such as convex, star-shaped etc. for polyhedra and polygons though strictly speaking they refer to the corresponding polyhedral solids or polygonal regions. In either case, of course, the condition of being star-shaped does not preclude convexity.

The *isogonal* (=vertex-transitive) polyhedra and the *isohedral* (=face-transitive) polyhedra have been studied by many authors, mathematicians and crystallographers (see, for example, Hess [1883], Fedorov [1885], Brückner [1900], Buerger [1963], Robertson & Carter [1970], Grünbaum & Shephard [1981], [1982]). However, in all these cases the treatment was needlessly restrictive due to either implicit or explicit assumptions such as that of convexity, or the mistaken belief that it is enough to determine the polyhedra of one of these kinds for those of the other kind to be automatically obtainable by "duality".

Our results include the following:

Theorem 1. Each isohedral polyhedron is star-shaped and has star-shaped faces.

Examples of isohedral non-convex polyhedra are shown in Figure 1.

Theorem 2. Each isotoxal (=edge-transitive) polyhedron is either isohedral or isogonal, and is convex.

Theorem 3. Each isogonal polyhedron has all its vertices on a sphere and all its faces are convex.

The theorems imply that the isohedral and isotoxal polyhedra must be of genus 0, but there is no such implication for isogonal polyhedra. In fact, the following is true.

Theorem 4. Non-convex isogonal polyhedra of genera 0, 1, 3, 5, 7, 11 and 19 exist.

These seem not to have been described previously although isolated examples have occasionally been considered in other contexts; see, for example, Wunderlich [1965], Jessen [1967], Goldberg [1978].

The nonconvex isogonal polyhedra of genus 0 have the same combinatorial structure as the convex ones, and can be obtained from these by suitably "twisting" certain sets of faces while the other faces may have to be stretched or otherwise modified so as to conform. In Figure 2 we show examples of such polyhedra.

To construct a family of isogonal polyhedra which are toroidal (that is, of genus 1) we start from a non-convex n-gonal isogonal antiprism with $n \geq 7$. This is obtained from the usual convex antiprism by twisting one of the n-gons relative to the other; it must be such that the only edges on the boundary of its convex hull are the 2n edges of the two n-gons. We delete these n-gons and complete the polyhedron by combining the remaining mantle of triangular faces with the mantle of its convex hull; depending upon the twist that was imparted to the n-gons the outer mantle may be prismatic (with n rectangles) or antiprismatic (with 2n triangles). Other toroidal isogonal polyhedra can be obtained by taking as the outer mantle that of a suitable non-convex antiprism as well. The resulting polyhedra have either six triangles, or three triangles and two quadrangles, meeting at each vertex; examples are shown in Figure 3.

A similar construction yields the isogonal polyhedra of higher genus. For example, for that of genus 5 we start from a non-convex isogonal polyhedron P of the same type as the snub cube (see Figure 2), with the property that the only edges on the boundary of its convex hull are those of the six square faces. We then delete these square faces and complete the polyhedron by adjoining the triangular faces of the convex hull of P. See Figure 4 for the resulting polyhedron.

The other isogonal polyhedra of higher genus can be obtained from the snub tetrahedron (genus 3), the snub octahedron (genus 7), the snub dodecahedron (genus 11) and the snub icosahedron (genus 19), by an analogous construction.

A more complete description of the various polyhedra with transitivity properties requires an appropriate definition of "type". The most sensible one seems to be based on the convexity behaviour at the edges and vertices. Details will be presented in a later publication.

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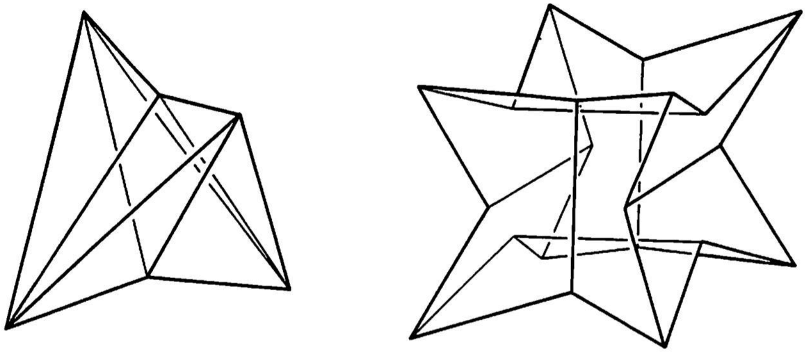

Figure 1.

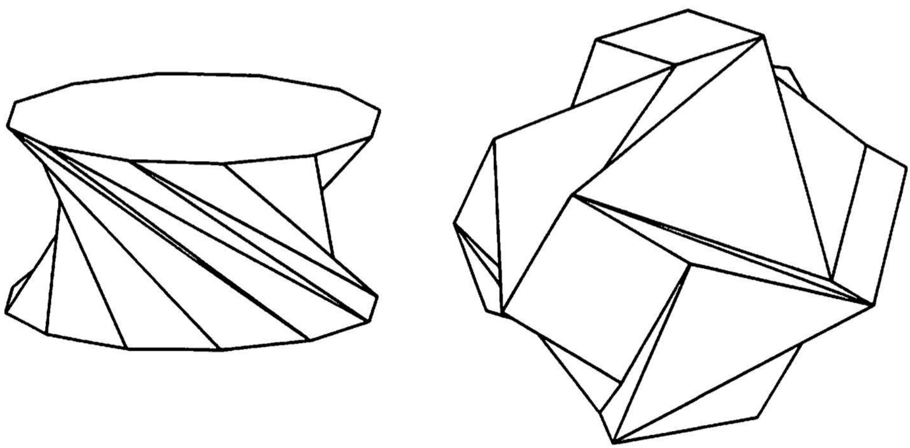

Figure 2.

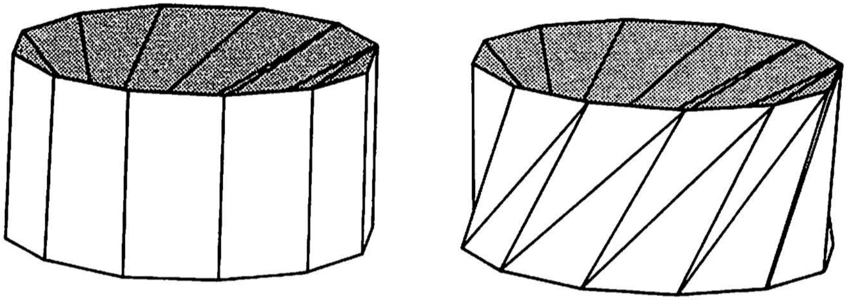

Figure 3.

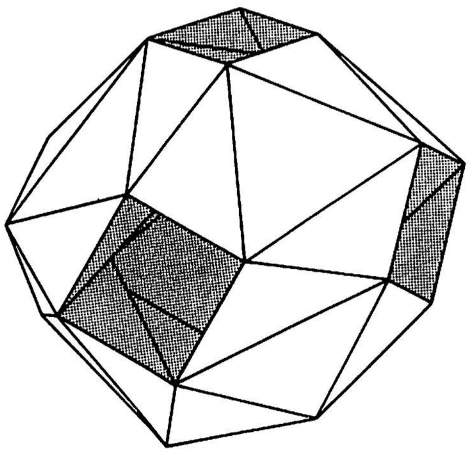

Figure 4.

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A HURWITZ TYPE FORMULA FOR SINGULAR CURVES

Nadia CHIARLI

Presented by P. Ribenboim

Summary. The classical Hurwitz formula for finite, separable morphisms of non-singular curves, is extended to the singular case. Applications are given to rank computation.

The aim of this note is to extend the classical Hurwitz formula for finite, separable morphisms $\pi : X \to Y$ of non-singular curves, to the case when X is singular.

The main result is the following: if $p_a(X)$ is the arithmetic genus of X, $g(Y)$ the genus of Y and d the degree of π, then

$$2p_a(X) - 2 = d(2g(Y) - 2) + \deg \mathcal{R}$$

where $\mathcal{R}$ is the discriminant divisor associated with π (th. 2). If X is non-singular, this gives the classical Hurwitz formula.

As an application we determine the rank of a curve and an upper-bound for the number of singular points it can have (under suitable assumptions).

NOTATIONS AND ASSUMPTIONS

Let X, Y be two complete, irreducible curves over an algebraically closed field k, with Y non-singular; let $\pi : X \to Y$ be a separable morphism of degree d and denote by $\nu : \bar{X} \to X$ the normalization.

If $\sigma = \pi \circ \nu$, let $\mathcal{D}^{(\pi)}$ (resp. $\mathcal{D}^{(\sigma)}$) be the discriminant sheaf ideal associated with π (resp. σ) and let R be the divisor on Y associated with $\mathcal{D}^{(\pi)}$.

(For definition and details on the discriminant sheaf ideal associated with a morphism see [1]).

Finally, if Q is a closed point of Y and v_Q is the valuation associated with the D.V.R. $\mathcal{O}_{Y,Q}$, denote $v_Q(\sigma) = v_Q(\mathcal{D}_Q^{(\sigma)})$ (resp. $v_Q(\pi) = v_Q(\mathcal{D}_Q^{(\pi)})$) ; hence $\deg R = \sum_{Q\in Y} v_Q(\pi)$.

MAIN RESULT

LEMMA 1. <u>Let Q be a (closed) point of Y and let</u> B_Q (resp. B'_Q) <u>be the semilocal ring corresponding to</u> $\sigma^{-1}(Q)$ (resp. $\pi^{-1}(Q)$).

<u>Then</u>:

i) $p_a(X) - g(\overline{X}) = \sum_{Q\in Y}$ <u>length</u> (B_Q/B'_Q)

ii) $\sum_{Q\in Y} v_Q(\sigma) = \sum_{\substack{\overline{P}\in\overline{X}\\ \sigma(\overline{P})=Q}} (e_{\overline{P}} - 1)$, <u>where</u> $e_{\overline{P}}$ <u>is the ramification index of</u> $\mathcal{O}_{\overline{X},\overline{P}}$ <u>with respect to</u> σ.

PROOF. i) follows from [8], p. 72.

ii) It is $\sum_{Q\in Y} v_Q(\sigma) = \sum_{\substack{P\in X\\ \sigma(\overline{P})=Q}} f_{\overline{P}}(e_{\overline{P}} - 1)$ (see [7] prop. 10, p. 26), but $f_{\overline{P}} = [k(m_{\overline{P}}) : k] = 1$ for all $\overline{P}\in\overline{X}$ (where $m_{\overline{P}}$ is the maximal ideal of B_Q corresponding to $\overline{P}$), and our claim follows.

THEOREM 2. $2p_a(X) - 2 = d[2g(Y) - 2] + \sum_{Q\varepsilon Y} v_Q(\pi)$.

PROOF. $\sigma : \overline{X} \to Y$ is a separable morphism of degree d ; hence, by the Hurwitz classical formula and lemma 1 ii) it follows

$$2g(\overline{X}) - 2 = d[2g(Y) - 2] + \sum_{Q\varepsilon Y} v_Q(\sigma)$$

Now, by [3] 1.1, $v_Q(\pi) = 2$ length $(B_Q/B'_Q) + v_Q(\sigma)$ and the claim follows from lemma 1 i).

SOME APPLICATIONS.

Through this section we assume char $k = 0$.

Recall that, if X $\mathbb{P}^r$ is a complete, irreducible curve, the <u>rank</u> of X is the non-negative integer

rk X = # tangent hyperplanes to X through a generic $\mathbb{P}^{r-2}$

= # tangent lines to X meeting a generic $\mathbb{P}^{r-2}$

A non difficult computation of constants shows that we may choose a $\mathbb{P}^{r-2}$, say L , such that $L \cap X = \emptyset$ and L is not contained in any of the osculating hyperplanes to X , in any of the multitangent hyperplanes to X , in any of the tangent hyperplanes to X passing through a singular point of X .

From now on, we will assume that L is as above and Y is a line such that $Y \cap L = Y \cap X = \emptyset$; we will assume also that $\pi : X \to Y$ is the projection from L . Thus π is a separable morphism of degree d , where d is the degree of X .

PROPOSITION 3. rk $X = 2p_a(X) - 2 + 2d - \sum_{Q\varepsilon\pi(X_{sing})} v_Q(\pi)$.

PROOF. By our choice of L it follows that $\sum_{Q\in Y} v_Q(\pi) =$ $= \sum_{Q\in S} v_Q(\pi) + \sum_{Q\in T} v_Q(\pi)$ where $S = \{Q\in Y \mid (L\ , Q) \cap X_{sing} \neq \emptyset\}$ $T = \{Q\in Y \mid (L\ , Q) \cap X_{sing} = \emptyset$ and $(L\ , Q)$ is tangent to $X\}$ ($(L\ , Q)$ denotes the hyperplane determined by L and Q).

Since $\sum_{Q\in T} v_Q(\pi) = \mathrm{rk}\ X$, our claim follows from th.2.

In some cases it is easy to compute $\sum_{Q\in S} v_Q(\pi)$, hence, from prop. 3, $\mathrm{rk}\ X$, as shown by the following.

PROPOSITION 5. Let $P\in X_{sing}$ be an s-fold point, $Q = \pi(P)$ and put $A = \mathcal{O}_{Y,Q}$, $R = \mathcal{O}_{X,P}$. Then:

i) if $\mathrm{endim}\ R = 2$, then $v_Q(\pi) = \sum_i s_i(s_i - 1) + \sum_\alpha (\nu_\alpha - 1)$, where the first sum is extended to P and to all its infinitely near points, while the second sum is extended to all the branches of X at P, the ν_α's being the corresponding orders.

ii) if $gr(R)$ is the affine coordinate ring of s lines in $\mathbb{A}^{r+1}$ in generic $(s-1)$ and s position, then $v_Q(\pi) = 2sd_0 - 2\binom{d_0 + r}{r+1}$, where d_0 is the least integer such that $s \leq \binom{d_0 + r}{r}$. (For definition of $(s-1)$ and s position see [4] p. 119).

PROOF. By [3], 1.1. $v_Q(\pi) = 2$ length $(\overline{R}/R) + v_Q(\sigma)$ (where $\overline{}$ denotes the integral closure).

i) Since R is Gorenstein, then 2 length $(\overline{R}/R) =$ length $(\overline{R}/\mathfrak{b})$ where $\mathfrak{b}$ is the conductor; now, it is well known that length $(\overline{R}/\mathfrak{b}) = \sum_i s_i(s_i - 1)$ (e.g. apply [5], cor. 3.12). Moreover we have $v_Q(\sigma) = \sum_\alpha(\nu_\alpha - 1)$ (see proof of lemma in [2]).

ii) From [6], th. 4.2 and th. 4.4 it follows that $gr(R)$ is reduced and the conductor of R in $\overline{R}$ is m^{d_0}, where m is the maximal ideal of R ; moreover ([6], p. 92) $gr(\overline{R}) = \overline{gr(R)}$. An easy computation shows that length $(\overline{R}/R) =$ $=$ length $(gr(\overline{R})/gr(R))$, and our claim follows by [4], prop. 6.

REMARK 6. When $r = 2$, prop. 4 and prop. 5 give the classical generalized Plücker's class formula (see [2]).

EXAMPLE 7. Let $r = 3$ and let $X_{sing} = \{t$ triple seminormal points$\}$. What is an upper bound for t ? Since condition ii) of prop. 5 is satisfied, then, for ever $Q \in R$,

$v_Q(\pi) = 6d_0 - 2\binom{d_0+3}{4}$, with $d_0 = 1$; so $v_Q(\pi) = 4$, and, since $\mathrm{rk}\, X \geq 0$, we get, from prop. 3, $t \leq \frac{1}{2}[p_a(X) - 1 + d]$.

REMARK 8. If F is a surface of degree n in $\mathbb{P}^3$, with δ ordinary double points as its only singularities, and G is the first polar of a generic point of $\mathbb{P}^3$ with respec to F, then the complete intersection of F and G is a curve X having just δ nodes as its singularities ([9]). Therefore, in this case, we have

$$\mathrm{rk}\, X = \frac{1}{2}n(n-1)(2n-5) - 2 + 2n(n-1) - 2\delta$$

Viewed another way, if we know a lower-bound for $\mathrm{rk}\, X$, the formula allows us to determine an upper-bound for δ ; unfortunately, the trivial inequality $\mathrm{rk}\, X \geq 0$ gives a loose upper-bound for δ (see [9]), but it is challenging to search for a non-trivial lower-bound for rk X , depending on n .

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ON VARIETIES OF 3-ALGEBRAS

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Presented by P. Ribenboim

1. Introduction

Let V be a variety of algebras over a field F satisfying the condition that there exists an integer $n \geq 2$ such that if R belongs to V and I is an ideal of R then I^n is an ideal of R. Such varieties have been called n-varieties by Zwier [11]. All algebras belonging to an n-variety are called n-algebras. Many authors have studied n-varieties for n=2([1], [2], [3], [4], [7], [8], [9] and [11]). Examples of 2-varieties are associative, alternative and Lie algebras. A variety V of algebras is a 2-variety (see [2]) if and only if all algebras of V satisfy the following two identities:

$$xy \cdot z = \alpha_1 zx \cdot y + \alpha_2 xz \cdot y + \alpha_3 y \cdot zx + \alpha_4 y \cdot xz + \alpha_5 zy \cdot x + \alpha_6 yz \cdot x + \alpha_7 x \cdot zy + \alpha_8 x \cdot yz$$

$$z \cdot xy = \beta_1 zx \cdot y + \beta_2 xz \cdot y + \beta_3 y \cdot zx + \beta_4 y \cdot xz + \beta_5 zy \cdot x + \beta_6 yz \cdot x + \beta_7 x \cdot zy + \beta_8 x \cdot yz$$

where the α's and the β's are assumed to be in F. 2-varieties have been completely classified in [7].

In this paper I study n-varieties for n=3. Examples are Jordan and standard algebras. Such varieties (see [8]) are defined by four degree four identities each of which depends on 72 constants. The only thought of this huge number of parameters would discourage anybody from undertaking the task of studying these identities. But the matrix technique ([5] and [6])that we have been using in several occasions and in particular in the classification of 2-varieties allows one to keep computations under control, since it allows to find a standard way for representing identities and a straightforward way to compare and classify them. Moreover, this technique allows to handle the problem by

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making use of the computer, because, as we will see shortly, it enables to translate the problem of recognizing a 3-algebra to a very simple problem of linear algebra. Full details and proofs are forthcoming in [10].

2. Matrix representation of a 3-variety

Let R be a nonassociative algebra defined over a field of characteristic not 2 or 3. The fact that R is a 3-algebra amounts to saying that there exist 288 constants $\alpha_1,\dots,\alpha_{72},\beta_1,\dots,\beta_{72},\gamma_1,\dots,\gamma_{72},\delta_1,\dots,\delta_{72}$ such that the following set of four degree four identities hold. We shall write the four defining identities in Table I in the next page. Table I should read in the following manner. There are five different types of association in a degree four identity; $(x_1x_2)(x_3y)$, $(x_1x_2\cdot x_3)y$, $(x_1\cdot x_2x_3)y$, $x_1(x_2x_3\cdot y)$ and $x_1(x_2\cdot x_3y)$. Each term of the identity appearing in one of these forms is set under the corresponding heading, which is respectively: RR.RR, (RR.R)R, (R.RR)R, R(RR.R) and R(R.RR). The left column of the table indicates what permutation of the four elements x_1, x_2, x_3 and y is being considered. The four columns α_i, β_i, γ_i and δ_i under each heading give rise to the four different identities.

Of course, Table I is only a different way of writing the four identities stated in [8]. Writing the identities in the form of Table I allows one to use the group representation technique stated in [5] and [6]. If char F $\neq$ 2,3, the group algebra FS_4 is isomorphic to the direct sum of matrix algebras over F,

$FS_4 \cong F \oplus F \oplus F_{2\times2} \oplus F_{3\times3} \oplus F_{3\times3}$.

The isomorphism is given for instance in [6], page 55. Every degree four identity may be written therefore in the form

$$(*)\quad \begin{array}{ccccccccc} RR.RR & \oplus & (RR.R)R & \oplus & (R.RR)R & \oplus & R(RR.R) & \oplus & R(R.RR) \\ g_1 & & g_2 & & g_3 & & g_4 & & g_5 \end{array}$$

where each g_i $(i=1,\dots,5)$ belongs to FS_4 and so will be of the kind:

$$g_i = (h^i) \oplus (k^i) \oplus \begin{pmatrix} p_1^i & p_2^i \\ p_3^i & p_4^i \end{pmatrix} \oplus \begin{pmatrix} a_1^i & a_2^i & a_3^i \\ a_4^i & a_5^i & a_6^i \\ a_7^i & a_8^i & a_9^i \end{pmatrix} \oplus \begin{pmatrix} a_1'^i & a_2'^i & a_3'^i \\ a_4'^i & a_5'^i & a_6'^i \\ a_7'^i & a_8'^i & a_9'^i \end{pmatrix}$$

TABLE I

	RR.RR	(RR.R)R	(R.RR)R	R(RR.R)	R(R.RR)
I	$\alpha_1\ \beta_1\ \gamma_1\ \delta_1$	-1	-1		$\alpha_{61}\ \beta_{61}\ \gamma_{61}\ \delta_{61}$
(123)	$\alpha_2\ \beta_2\ \gamma_2\ \delta_2$				$\alpha_{63}\ \beta_{63}\ \gamma_{63}\ \delta_{63}$
(124)	$\alpha_3\ \beta_3\ \gamma_3\ \delta_3$	$\alpha_{28}\ \beta_{28}\ \gamma_{28}\ \delta_{28}$			
(132)	$\alpha_4\ \beta_4\ \gamma_4\ \delta_4$				$\alpha_{62}\ \beta_{62}\ \gamma_{62}\ \delta_{62}$
(134)	$\alpha_5\ \beta_5\ \gamma_5\ \delta_5$	$\alpha_{30}\ \beta_{30}\ \gamma_{30}\ \delta_{30}$			
(142)	$\alpha_6\ \beta_6\ \gamma_6\ \delta_6$	$\alpha_{26}\ \beta_{26}\ \gamma_{26}\ \delta_{26}$	$\alpha_{41}\ \beta_{41}\ \gamma_{41}\ \delta_{41}$	$\alpha_{53}\ \beta_{53}\ \gamma_{53}\ \delta_{53}$	
(143)	$\alpha_7\ \beta_7\ \gamma_7\ \delta_7$		$\alpha_{38}\ \beta_{38}\ \gamma_{38}\ \delta_{38}$	$\alpha_{50}\ \beta_{50}\ \gamma_{50}\ \delta_{50}$	$\alpha_{65}\ \beta_{65}\ \gamma_{65}\ \delta_{65}$
(234)	$\alpha_8\ \beta_8\ \gamma_8\ \delta_8$	$\alpha_{25}\ \beta_{25}\ \gamma_{25}\ \delta_{25}$	$\alpha_{40}\ \beta_{40}\ \gamma_{40}\ \delta_{40}$	$\alpha_{52}\ \beta_{52}\ \gamma_{52}\ \delta_{52}$	
(243)	$\alpha_9\ \beta_9\ \gamma_9\ \delta_9$		$\alpha_{37}\ \beta_{37}\ \gamma_{37}\ \delta_{37}$	$\alpha_{49}\ \beta_{49}\ \gamma_{49}\ \delta_{49}$	$\alpha_{64}\ \beta_{64}\ \gamma_{64}\ \delta_{64}$
(12)(34)	$\alpha_{10}\ \beta_{10}\ \gamma_{10}\ \delta_{10}$		$\alpha_{39}\ \beta_{39}\ \gamma_{39}\ \delta_{39}$	$\alpha_{51}\ \beta_{51}\ \gamma_{51}\ \delta_{51}$	$\alpha_{66}\ \beta_{66}\ \gamma_{66}\ \delta_{66}$
(13)(24)	$\alpha_{11}\ \beta_{11}\ \gamma_{11}\ \delta_{11}$	$\alpha_{27}\ \beta_{27}\ \gamma_{27}\ \delta_{27}$	$\alpha_{42}\ \beta_{42}\ \gamma_{42}\ \delta_{42}$	$\alpha_{54}\ \beta_{54}\ \gamma_{54}\ \delta_{54}$	
(14)(23)	$\alpha_{12}\ \beta_{12}\ \gamma_{12}\ \delta_{12}$	$\alpha_{29}\ \beta_{29}\ \gamma_{29}\ \delta_{29}$			
(12)	$\alpha_{13}\ \beta_{13}\ \gamma_{13}\ \delta_{13}$				$\alpha_{69}\ \beta_{69}\ \gamma_{69}\ \delta_{69}$
(13)	$\alpha_{14}\ \beta_{14}\ \gamma_{14}\ \delta_{14}$				$\alpha_{68}\ \beta_{68}\ \gamma_{68}\ \delta_{68}$
(14)	$\alpha_{15}\ \beta_{15}\ \gamma_{15}\ \delta_{15}$	$\alpha_{35}\ \beta_{35}\ \gamma_{35}\ \delta_{35}$			
(23)	$\alpha_{16}\ \beta_{16}\ \gamma_{16}\ \delta_{16}$				$\alpha_{67}\ \beta_{67}\ \gamma_{67}\ \delta_{67}$
(24)	$\alpha_{17}\ \beta_{17}\ \gamma_{17}\ \delta_{17}$	$\alpha_{31}\ \beta_{31}\ \gamma_{31}\ \delta_{31}$	$\alpha_{46}\ \beta_{46}\ \gamma_{46}\ \delta_{46}$	$\alpha_{58}\ \beta_{58}\ \gamma_{58}\ \delta_{58}$	
(34)	$\alpha_{18}\ \beta_{18}\ \gamma_{18}\ \delta_{18}$		$\alpha_{43}\ \beta_{43}\ \gamma_{43}\ \delta_{43}$	$\alpha_{55}\ \beta_{55}\ \gamma_{55}\ \delta_{55}$	$\alpha_{70}\ \beta_{70}\ \gamma_{70}\ \delta_{70}$
(1234)	$\alpha_{19}\ \beta_{19}\ \gamma_{19}\ \delta_{19}$	$\alpha_{34}\ \beta_{34}\ \gamma_{34}\ \delta_{34}$		-1	-1
(1243)	$\alpha_{20}\ \beta_{20}\ \gamma_{20}\ \delta_{20}$		$\alpha_{45}\ \beta_{45}\ \gamma_{45}\ \delta_{45}$	$\alpha_{57}\ \beta_{57}\ \gamma_{57}\ \delta_{57}$	$\alpha_{72}\ \beta_{72}\ \gamma_{72}\ \delta_{72}$
(1324)	$\alpha_{21}\ \beta_{21}\ \gamma_{21}\ \delta_{21}$	$\alpha_{36}\ \beta_{36}\ \gamma_{36}\ \delta_{36}$			
(1342)	$\alpha_{22}\ \beta_{22}\ \gamma_{22}\ \delta_{22}$	$\alpha_{33}\ \beta_{33}\ \gamma_{33}\ \delta_{33}$	$\alpha_{48}\ \beta_{48}\ \gamma_{48}\ \delta_{48}$	$\alpha_{60}\ \beta_{60}\ \gamma_{60}\ \delta_{60}$	
(1423)	$\alpha_{23}\ \beta_{23}\ \gamma_{23}\ \delta_{23}$	$\alpha_{32}\ \beta_{32}\ \gamma_{32}\ \delta_{32}$	$\alpha_{47}\ \beta_{47}\ \gamma_{47}\ \delta_{47}$	$\alpha_{59}\ \beta_{59}\ \gamma_{59}\ \delta_{59}$	
(1432)	$\alpha_{24}\ \beta_{24}\ \gamma_{24}\ \delta_{24}$		$\alpha_{44}\ \beta_{44}\ \gamma_{44}\ \delta_{44}$	$\alpha_{56}\ \beta_{56}\ \gamma_{56}\ \delta_{56}$	$\alpha_{71}\ \beta_{71}\ \gamma_{71}\ \delta_{71}$

Let us observe that when we fill all array (*) for i=1,2,3,4 we will write the above matrices vertically instead of horizontally.
This matrix representation determines a mapping from F^{288} to F^{480} (480=24x5x4 being the entries of the matrices of the above array). Not all fills of these matrices represent the defining identities of a 3-algebra. The fills of the matrices which correspond to a 3-algebra lie in the range of the map we have been describing. One sees that the fill of the first three representations (i.e. the two 1x1 and the 2x2) and of part of representations #4 and #5 are arbitrary; the remaining elements of representations #4 and #5 are determined by these. Knowledge of the range of this mapping is vital for checking when some identities are the defining identities of a 3-algebra.

3. Recognizing a 3-algebra

Let R be an algebra defined by a collection of identities of degree four (or reducible to degree four). The question we are going to solve is the following: is there a quick and standard way which allows one to see if the identities are the defining identities of a 3-algebra, without having to undertake tedious and difficult manipulations of the identities? We shall use the same method we have used in [7] for the classification of 2-varieties. The difficulty in the present case is the huge amount of parameters. Anyhow the translation of the problem to a matter of linear algebra allows to handle part of the computations with the help of a computer.

Remember that a fill T of matrices representing some identities is weaker than a fill T^1 if representation by representation the row space of T is contained in the row space of T^1. Two fills T and T^1 are equivalent if the row space of T is contained in the row space of T^1 and viceversa. Now, one easily sees that, disregarding the terms of the form RR.RR (which in fact do not affect a set of identities being or not being the defining identities of a 3-algebra), any collection of degree four identities is equivalent to a finite set of at most four identities of degree four. If R is any algebra defined by any number of degree four (or reducible to degree four) identities these are going to be the defining identities of a 3-algebra if and only if there is an equivalent or weaker fill T of the matrices representing the identities which is in the range. Now, since an equivalent or weaker fill of T is obtainable multiplying each representation of T respectively by two 1x1, one 2x2 and two 3x3 matrices, this new fill is going to be in the range if and only if certain relations between its elements are true. This happens if and only if a certain system of linear equations of 192 equations in 384 unknowns is compatible. Since the matrix of the system is of the kind

$$A = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

where A is a 48x96 matrix, the task before us is not so impossible as one might at first sight expect, since one needs only to care about a smaller size matrix.

We have thus proved the following theorem.

Theorem: Let R be an algebra over a field F of characteristic $\neq 2$, $\neq 3$, defined by one or more degree four identities. Then R is a 3-algebra if and only if a certain system of 192 equations in 384 unknowns is compatible.

4. Conclusions

We will summarize here the steps one needs to undertake by this method:

1) By use of the matrix technique, give a matrix representation of the identities.
2) Reduce the matrices to row canonical form in each representation: by this way we obtain a standard form of the identities, which enables to compare them very easily and to classify them. Most of all, it allows to see if some identities are redundant.
3) Build the matrix A, which is a 48x96 matrix whose entries depend on the elements of the matrix representation of the identities.
4) Using the matrix A build a certain system of linear equations that we are not going to describe here (for full details see [10]). The algebra R is

a 3-algebra if and only if the system is compatible.

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SOME REMARKS ON THE ORDER OF AN ENTIRE FUNCTION ASSOCIATED WITH A SECOND-ORDER DIFFERENTIAL EQUATION, II.

Angelo B. Mingarelli

Presented by D.V. Atkinson

Let $G : [a,b] \times \mathbf{C} \to \mathbf{C}$ where $-\infty < a < b < +\infty$. Assume that $G(\cdot,\lambda) \in L(a,b)$ for each fixed $\lambda \in \mathbf{C}$. Then solutions of an initial value problem associated with

$$-y'' + G(t,\lambda)y = 0, \qquad t \in [a,b], \tag{1}$$

exist and are unique [3].

The question of interest here is the analytic dependence of a non-trivial solution $y(t,\lambda)$ of (1) upon $\lambda \in \mathbf{C}$ for fixed t. The following result is obtained.

Theorem 1. (a) Let $G(\cdot,\lambda) \in L(a,b)$ and $\partial G/\partial\lambda(\cdot,\lambda) \in L(a,b)$ for each fixed $\lambda \in \mathbf{C}$. For each fixed $t \in [a,b]$ let $G(t,\lambda)$ be an entire function of λ.

Then $y(t,\lambda)$ is an entire function of λ for each $t \in [a,b]$.

(b) If, in addition to the hypotheses in (a), G satisfies for some $p \geq 0$,

$$\int_a^t \left| 1 + \frac{G(s,\lambda)}{|\lambda|^p} \right| ds = O(1), \tag{2}$$

as $|\lambda| \to +\infty$, uniformly for $t \in [a,b]$, then $y(t,\lambda)$ is of order not exceeding $p/2$.

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Proof. The technique described in [1, §11.5] may be adapted, with minor changes, to (1) in order to prove (a).

The proof of (b) is based upon an idea which appears to have originated in [1, §8.2]. We need only sketch the argument. Fix $\lambda \in \mathbb{C}$ and let $u(t) \equiv |\lambda|^p |y(t,\lambda)|^2 + |y'(t,\lambda)|^2$. Then

$$u'(t) = 2|\lambda|^p \operatorname{Re}\left\{[1 + \frac{G(t,\lambda)}{|\lambda|^p}]\, y(t,\lambda)\bar{y}'(t,\lambda)\right\},$$

and consequently,

$$|u'(t)| \le 2|y|\,|y'|\,|\lambda|^p \left|1 + \frac{G(t,\lambda)}{|\lambda|^p}\right| .$$

Use of the relation $(|\lambda|^{p/2}|y| - |y'|)^2 \ge 0$ now yields

$$|u'(t)| \le |\lambda|^{p/2} u(t) \left|1 + \frac{G(t,\lambda)}{|\lambda|^p}\right| .$$

Thus

$$\left|\frac{d}{dt} \log u(t)\right| \le |\lambda|^{p/2} \left|1 + \frac{G(t,\lambda)}{|\lambda|^p}\right|$$

and a quadrature followed by an exponentiation gives the result

$$u(t) \le c \exp\left\{|\lambda|^{p/2} \int_a^t \left|1 + \frac{G(s,\lambda)}{|\lambda|^p}\right| ds\right\}$$

$$= O\left\{\exp[(\text{const})|\lambda|^{p/2}\right\},$$

and (b) follows.

This theorem appears to have interesting applications. First, we note that the Sturm-Liouville equation, (1) with $G(t,\lambda) = q(t) - \lambda r(t)$, $q,r \in L(a,b)$ is included in theorem 1. Here q,r are not necessarily real-valued. Theorem 1(b) gives the bound 1/2 for $y(t,\lambda)$, a result which is classical and which, in fact, is sharp.

Another consequence is that the solution $y(t,\lambda)$ of the equation

$$-y'' + (P(t)\lambda^2 + Q(t)\lambda + R(t))y = 0, \tag{3}$$

under the assumptions $P,Q,R \in L(a,b)$ is an entire function of order not exceeding 1 in the event that $\int_a^b |P(s)|ds > 0$ and of order not exceeding 1/2 otherwise. The former result was also obtained by Langer [6] using techniques which dealt particularly with equations of the form (3). Equations of the form (3) also arise in mathematical physics, cf. [4], [5].

Now we note that the generalized eigenvalue problem

$$-y'' + q(t)y = \lambda\left\{i(\rho y)' + i\rho y' + wy\right\} \tag{4}$$

is a particular case of a problem considered by Everitt [2]. The assumptions therein were $\rho : [a,b] \to \mathbb{R}$, $\rho \in AC_{loc}(a,b)$; $w : [a,b] \to \mathbb{R}$, $w \in L_{loc}(a,b)$; $q : [a,b] \to \mathbb{R}$, $q \in L_{loc}(a,b)$. One open problem in [2] regarded the order of $y(t,\lambda)$ as a function of λ. Sharp estimates were derived in [7]. Therein we showed that whenever $\int_a^b |\rho(s)|ds > 0$ the order of $y(t,\cdot)$ is at most 1 and at most 1/2 otherwise.

However, under the stated assumptions, the change of dependent variable

$$y(t) = z(t)\exp\left\{-i\lambda \int_a^t \rho(s)ds\right\} \tag{5}$$

carries (4) into:

$$z'' + (\rho^2(t)\lambda^2 + w(t)\lambda - q(t))z = 0 \tag{6}$$

which is an equation of the form (3). The sharp estimates 1, 1/2 now follow from the analogous results for (3). Note that the transformation (5) may be used to obtain information about the spectrum of (4) by reducing the problem to the study of the spectrum of (6). For example, the spectral function of (6) was studied by Langer [6] and so more information can be gathered about the spectrum of (4) than is already known, cf., [2].

Let $G(t,\lambda) \equiv a_0(t)\lambda^n + a_1(t)\lambda^{n-1} + \ldots + a_{n-1}(t)\lambda + a_n(t)$ where $a_i(t) \in L(a,b)$, $i = 0,1,\ldots,n$. Inserting this into (2) and solving the inequality one finds $p = n$. Consequently the solutions $y(t,\lambda)$ of the corresponding equation (1) are of order not exceeding $n/2$, as long as $\int_a^b |a_0(s)|ds > 0$. It seems very likely that the order is precisely $n/2$, (here $n \geq 1$ is integral).

Finally we note that if $G(t,\lambda) \equiv \sin \lambda$, then for each $p \geq 0$ there exists a sequence $\lambda_n \in \mathbf{C}$ with the property that (2) fails as $|\lambda_n| \to \infty$. In this case it is readily verified that $y(t,\lambda)$ is of infinite order. This shows that (2) is in some sense sharp.

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LEAST SQUARES, TRANSMUTATION, AND THE MARČENKO EQUATION

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Presented by D.V. Atkinson

Abstract. It is shown how the general Marčenko (M) equation of [2;4] arises as a minimizing criterion in characterizing a certain transmutation.

1. Introduction. Given two operators P and Q of the form $Qu = (\Delta_Q u')'/\Delta_Q - \hat{q}(x)u$ on say $(0,\infty)$ we consider transmutations B satisfying $BP = QB$ (acting on suitable objects - cf. [1]) and we report here on some new directions. Thus one sees in [7;8] the splitting of spectral measures related to a type of autocorrelation of readout data (cf. [11] for the classical interplay of time series analysis, spectral theory, Krein strings, etc.). In what appears to be a similar spirit the results of [5;6;9] characterize various transmutations via least squares minimizing procedures (appropriate Gelfand-Levitan (G-L) equations arise as Euler equations). This development was motivated by some work in [12] (cf. also [10]) and in fact the patterns are strongly related to various work in linear estimation, prediction, and filtering theory (cf. [14;15]). The present note involves a new result showing that the general M equation of [2;4] also arises in a similar manner. The technique also provides some new insight into the structure of this general M equation and the relation between general G-L and M equations (cf. [1;3]).

2. Background. We take here $Pu = u'' - q(x)u$ to be an operator of Fourier type (on $(-\infty,\infty)$) as in [2;4] (thus q is real, even, positive, continuous, and say $q(x)\exp 2Hx \in L^1(0,\infty)$); this is stronger than necessary but convenient. Take $Qu = (Au')'/A - \hat{q}(x)u$ (on $(0,\infty)$) with say $A \in C^2$, $0 < a \leq A \leq \tilde{A} < \infty$, $A \to A_\infty$ "rapidly" as $x \to \infty$, $A(0) = 1$ (which is no restriction), and $\hat{q}$ satisfying suitable conditions involving e.g. $\int_0^\infty (1+x)|\hat{q}(x)|dx < \infty$ (more general Q are possible - cf.

[1;2;4] - but this is again convenient). For such operators one takes φ_λ^Q to be the solution of $(*)_Q$: $Qu = -\lambda^2 u$ with $\varphi_\lambda^Q(0) = 1$ and $D_x\varphi_\lambda^Q(0) = 0$ while $\Phi_{\pm\lambda}^Q(x)$ denotes the Jost solutions of $(*)_Q$ satisfying $\Phi_{\pm\lambda}^Q(x) \sim A_\infty^{-\frac{1}{2}}\exp(\pm i\lambda x)$ as $x \to \infty$. One has $\varphi_\lambda^Q = c_Q(\lambda)\Phi_\lambda^Q + c_Q(-\lambda)\Phi_{-\lambda}^Q$ and (assuming no bound states) the spectral measure for Q (and φ_λ^Q) is $d\omega = \hat{\omega}d\lambda = (1/2\pi|c_Q|^2)d\lambda$; we set also $\Psi_\lambda^Q = \Phi_\lambda^Q/c_Q(-\lambda)$. On the other hand for P one defines in addition χ_λ^P as the solution of $(*)_P$ satisfying $\chi_\lambda^P(0) = 0$ and $D_x\chi_\lambda^P(0) = -1$. One thinks of φ_λ^P and χ_λ^P on $(-\infty,\infty)$ and sets $M_1(\lambda) = c_P(-\lambda)$ with $M(\lambda) = 1/2i\lambda F(\lambda)$ where $\chi_\lambda^P = (1/2i\lambda)\{F(\lambda)\Phi_{-\lambda}^P - F(-\lambda)\Phi_\lambda^P\}$. Then $\Phi_\lambda^P = 2i\lambda\{M(\lambda)\varphi_\lambda^P - M_1(\lambda)\chi_\lambda^P\}$ and one defines $\Sigma_\lambda^P = 2i\lambda\{M(\lambda)\varphi_\lambda^P + M_1(\lambda)\chi_\lambda^P\}$ (so $\Sigma_\lambda^P(-x) = \Phi_\lambda^P(x)$). Setting further $\rho(\lambda) = 1/4i\lambda MM_1$ and $A(\lambda) = -\{MM_1^- + M^-M_1\}/2MM_1$ ($M^- = M(-\lambda)$) one has $\Phi_{-\lambda}^P = \rho\Sigma_\lambda^P + A\Phi_\lambda^P$ (we write also $d\nu = \hat{\nu}d\lambda$ with $\hat{\nu} = 1/2\pi|c_P|^2$.

The theory of the M equation revolves around three transmutations $P \to Q$ (cf. [1;2;4;13]), namely, B: $\varphi_\lambda^P \to \varphi_\lambda^Q$, $\tilde{B}$: $\Psi_\lambda^P \to \Psi_\lambda^Q$, and $\check{B}$: $\Phi_\lambda^P \to \Phi_\lambda^Q$. These have kernels given respectively by the spectral pairings $\beta(y,x) = \langle\varphi_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_\nu$, $\tilde{\beta}(y,x) = \langle\varphi_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_\omega$, and $\check{\beta}(y,x) = \langle\Phi_\lambda^Q(y),\varphi_\lambda^P(x)/c_P(-\lambda)\rangle_\lambda/2\pi$ with triangularity $\beta(y,x) = 0$ for $x > y$ while $\tilde{\beta}(y,x) = 0 = \check{\beta}(y,x)$ for $x < y$. The extended G-L equation of [1] can be written $B = \tilde{B}\tilde{W}$ where $\tilde{W}(x,y) = \langle\varphi_\lambda^P(x),\varphi_\lambda^P(y)\rangle_\mu$ with $d\mu = (\hat{\nu}^2/\hat{\omega})d\lambda$ and one has from [2;4] $\tilde{B} = \check{B}\mathcal{H}$ where $\mathcal{H}$ has kernel $\mathcal{H}(x,s) = (1/2\pi)\int_{-\infty}^{\infty}(M_1(\lambda)/c_Q(-\lambda)) \Sigma_\lambda^P(s)\Phi_\lambda^P(x)\rho(\lambda)d\lambda$ (for action $s \to x$ and $\mathcal{H}(x,s) = 0$ for $x > s$, which we refer to as anticausality). The general extended M equation of [2;4] (generalizing [1;3; 13]) is now $B\tilde{\mathcal{H}} = \tilde{B}\tilde{W}\tilde{\mathcal{H}} = \check{B}(\mathcal{H}\tilde{W}\tilde{\mathcal{H}}) = BK$ where $\tilde{\mathcal{H}} \sim \mathcal{H}^*$. Here $\mathcal{H}$ is anticausal as indicated with $\tilde{\mathcal{H}}$ causal (by adjointness) so $B\tilde{\mathcal{H}}$ is causal and for $x > y$ the M equation becomes $\ker(\check{B}K)(y,x) = 0$ which will be written out below. Finally we recall that $\tilde{\beta}(y,x) = A^{-\frac{1}{2}}(y)\delta(x-y) + \tilde{K}(y,x)$ (cf. [7;8;9]) where $\tilde{K}$ is a function with $\tilde{K}(y,x) = 0$ for $x < y$. Similarly $\beta(y,x) = A^{-\frac{1}{2}}(y)\delta(x-y) + \hat{K}(y,x)$ and the G-L equation $B = \tilde{B}\tilde{W}$ becomes in kernel form (■) $A^{-\frac{1}{2}}(y)\tilde{\Omega}(x,y) + \tilde{K}(y,x) + \int_y^\infty \tilde{K}(y,\xi)\tilde{\Omega}(\xi,x)d\xi = 0$ (writing $\tilde{W}(x,y) = \delta(x-y) + \tilde{\Omega}(x,y)$). One notes here that it is an integrated form of such G-L equations (where the explicit $A^{-\frac{1}{2}}$ disappears) which is used in solving inverse problems (cf. [7;8;9]).

3. Minimization. In the present circumstances one can write $\check{\beta}(y,x) = A^{-\frac{1}{2}}(y)\delta(x-y) + \check{K}(y,x)$, $H(x,s) = \delta(x-s) + h(x,s)$, and $K(t,x) = \delta(t-x) + K(t,x)$ where $K(t,x)$ is symmetric, $h(x,s) = 0$ for $s < x$, and $\check{K}(y,x) = 0$ for $x < y$. Then for $x > y$ the M equation becomes(♦) $A^{-\frac{1}{2}}(y)K(y,x) + \check{K}(y,x) + \int_y^\infty \check{K}(y,s)K(s,x)ds = 0$. We note also that (*) $\tilde{K}(y,x) = \check{K}(y,x) + A^{-\frac{1}{2}}(y)h(y,x) + \int_y^x \check{K}(y,s)h(s,x)ds$. Now $\tilde{B}\varphi_\lambda^P = W\varphi_\lambda^Q$ where $W = \hat{\omega}/\hat{\nu}$ and it was shown in [6;9] that $\tilde{K}$ is characterized as the kernel which minimizes $\tilde{\Xi} = \int_0^T \int_0^\infty \{A^{-\frac{1}{2}}(y)\varphi_\lambda^P(y) - W(\lambda)\varphi_\lambda^Q(y) + (\tilde{K}\varphi_\lambda^P)(y)\}^2 d\mu dy$ where $\tilde{K}$ is allowed to range over a suitable class of anticausal kernels. In fact the minimizing kernel $\tilde{K}$ for $\tilde{\Xi}$ can be uniquely characterized as the solution of the G-L equation (▪). In this direction one writes now $\hat{\Xi} = \int_0^T \int_0^\infty (W\varphi_\lambda^Q - A^{-\frac{1}{2}}\varphi_\lambda^P)^2 d\mu dy$ (which will make sense here) and then $\tilde{\Xi}$ can be written as $\tilde{\Xi} = \hat{\Xi} + \mathrm{Tr}\,\tilde{K}(1+\tilde{\Omega})\tilde{K}^* + \mathrm{Tr}\,A^{-\frac{1}{2}}(\tilde{\Omega}\tilde{K}^* + \tilde{K}\tilde{\Omega})$. Let us think of our trial operators $\tilde{K}$ as arising from a construction (*) with ker $\check{B} = A^{-\frac{1}{2}}(y)\delta(x-y) + \check{K}(y,x)$ and ker $\tilde{B}$ = ker $(\check{B}H) = A^{-\frac{1}{2}}(y)\delta(x-y) + \tilde{K}(y,x)$. Then $\mathrm{Tr}\,\check{B}K\check{B}^*$ for example enters naturally into the equations (since $\tilde{\Xi} = \hat{\Xi} + \mathrm{Tr}\,\tilde{K}\tilde{K}^* - \mathrm{Tr}\,A^{-1}\tilde{\Omega} + \mathrm{Tr}\,\check{B}(H\tilde{\Omega}\tilde{H})\check{B}^*$) and after some calculation one obtains (note (♦) ≡ the M equation of [2;4] but is written differently)

Theorem. Writing $H\tilde{W}\tilde{H} = K = \delta + K$ the minimization problem for $\tilde{\Xi}$ reduces to minimizing $\tilde{\Xi} = \dot{\Xi} + \mathrm{Tr}\,\check{K}(\delta + K)\check{K}^* + 2\mathrm{Tr}\,A^{-\frac{1}{2}}K\check{K}^*$ over a suitable class of anticausal kernels $\check{K}$ ($\dot{\Xi}$ does not depend on $\check{K}$). The solution $\check{K}_0$ to this problem is then characterized as the unique solution $\check{K}$ of the M equation (♦).

Remark. The derivation of general M equations in [1;2;3;4] was motivated by [13] and the passage from $B = \check{B}H\tilde{W}$ to $B\tilde{H} = \check{B}K$ was in a certain way an ad hoc procedure. We see from our least squares minimization technique however that $K = H\tilde{W}\tilde{H}$ arises naturally by adjointness and the M equation $B\tilde{H} = \check{B}K$ is intrinsically connected to the G-L equation $B = \check{B}H\tilde{W}$.

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SYZYGIES AND VECTOR BUNDLES

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Abstract: In this note we discuss the application of our theorem on the ranks of syzygies to establishing constraints on the extension of vector bundles.

In working out the details for a solution to the syzygy problem [3], we were led quite naturally to the circumstance in which the syzygy in question represented a vector bundle on the punctured spectrum of a regular local ring. Having obtained an affirmative answer to the syzygy problem we were able to establish results on the vanishing (or non vanishing) of the cohomology of vector bundles of "small" rank (see [3; theorem 2.4]). After completion of this paper we realized that these results had consequences in the context of extending vector bundles on projective space $\mathbb{P}^n$ to $\mathbb{P}^{n+m}$. Barth and Van de Ven [1] in the case of rank 2 and Sato [7] and Tyurin [9] in general showed that the only infinitely extendable bundles were direct terms of line bundles. Horrocks [6] asked the corresponding question in regard to punctured spectra of regular local rings. In this article we summarize our results [4], [5] in this context which were an outgrowth of [3]. For our main result in [3] to apply we must assume throughout that all rings contain a field.

1. Cohomological constraints on lifting vector bundles. Throughout this section we assume that R is a regular local ring of dimension $n+1$, with $n>2$, and that S denotes a regular local ring having parameter t such that $S/tS \cong R$. We use script letters to

denote vector bundles on the punctured spectrum of a local ring (i.e. the space one obtains after deleting the maximal ideal) and the corresponding latin capital to denote its module of sections, e.g. E is the module of sections $\mathcal{E}$. If $\mathcal{F}$ is a vector bundle on the punctured spectrum of S, then $\mathcal{F}$ lifts the vector bundle $\mathcal{E}$ on the punctured spectrum of R provided the restriction of $\mathcal{F}$ to the punctured spectrum of R is isomorphic to $\mathcal{E}$. We remark that in terms of the modules of sections this does not say that there is an exact sequence $0 \to F \xrightarrow{t} F \to E \to 0$ but rather that there is an exact sequence $0 \to F \xrightarrow{t} F \to E \to L \to 0$ where L has finite length. The cohomology $H^i(\mathcal{E})$ is the usual sheaf cohomology on the punctured spectrum.

Our first result gives a flavor of constraints that occur in the lifting process in case certain sheaf cohomology vanishes. The proof is a rather straightforward induction.

Thoerem 1.1: Let $\mathcal{E}$ be a vector bundle on the punctured spectrum of R and suppose p, d, and q are integers such that $1 \leq p < d \leq q \leq n-1$. If $H^p(\mathcal{E})$ and $H^q(\mathcal{E})$ are zero while $H^d(\mathcal{E})$ is nonzero, then $\mathcal{E}$ can be lifted at most $q - p+2$ successive times. In particular if $q = p+2$, then $\mathcal{E}$ cannot be lifted at all.

To illustrate the above result, let $n \geq 5$ and let E be a k^{th} syzygy of a nonzero module of finite length where $3 \leq k \leq n-2$. Then the bundle $\mathcal{E}$ associated to E has exactly one nonzero cohomology, namely $H^{n+1-k}(\mathcal{E})$. Thus $\mathcal{E}$ has no liftings to higher dimensions.

What initially aroused our interest in questions concerning the lifting of bundles is the next result.

Theorem 1.2: Let $\mathcal{E}$ be a vector bundle on the punctured spectrum of R and assume that $H^1(\mathcal{E}) = \cdots = H^{k-2}(\mathcal{E}) = 0$ with $k \geq 3$. If the bundle $\mathcal{F}$ lifts $\mathcal{E}$ to the punctured spectrum of S, then $H^1(\mathcal{F}) = \cdots = H^{k-1}(\mathcal{F}) = 0$ and the sequence of sections $0 \to F \xrightarrow{t} F \to E \to 0$ is exact.

In module theoretic language, Theorem 1.2 says that, if E is a k^{th} syzygy over R with $k \geq 3$, then F is a $(k+1)^{st}$ syzygy over S. Our main result of this section now follows.

Theorem 1.3: Let $\mathcal{E}$ be a vector bundle of rank r such that E is not a free R-module and such that $H^1(\mathcal{E}) = \cdots = H^{k-2}(\mathcal{E}) = 0$ with $k \geq 3$. Then $\mathcal{E}$ cannot be lifted more than r-k successive times.

As an application of Theorem 1.3, consider projective space $\mathbb{P}^n$ with $n \geq 2$. Then the tangent bundle, which represents an n^{th} syzygy of rank n, cannot be lifted to $\mathbb{P}^{n+1}$, a similar statement holds for the Tango bundle [8] which is of rank n-1.

We end this section with a result which together with Theorem 1.3 is used to establish Theorem 2.1 of the next section.

Theorem 1.4: Let E be a vector bundle of rank r on the punctured spectrum of R and assume that E is not a direct sum of line bundles. Assume the dimension of R is at least 4 and that $H^2(E) = 0$. If $H^1(E)$ is generated by ℓ elements, then E can be lifted at most $r+\ell-4$ times.

2. The lifting of stably equivalent bundles.

We continue the notation of the previous sections with the addition that $\mathcal{O}_R$ and $\mathcal{O}_S$ are the structure sheaves for the punctured spectra of R and S, respectively. We also assume that R is of dimension n+1 with $n \geq 2$.

If the dimension of R is less than or equal to 2, then any bundle on the punctured spectrum of R is trivial since its module of sections is reflexive and hence free. On the other hand there are nontrivial bundles on the punctured spectrum of R as soon as its dimension is at least 3. These bundles will restrict to trivial bundles, that is, a direct sum of line bundles, in dimension 2. Thus the trivial bundle of the restriction lifts to a nontrivial bundle. However, if the dimension of R is at least 3 and if E is a trivial bundle on the punctured spectrum of R, say $E \cong \mathcal{O}_R^{\ell}$, then the only lifting of E to the punctured spectrum of S is $\mathcal{O}_S^{\ell}$. To see this, we suppose that E is a vector bundle on the punctured spectrum of R and that F is a nontrivial bundle on the punctured spectrum of S which lifts E. Then some $H^i(F)$ is nonzero for $0 < i < n+1$. The long exact sequence on cohomology yields that $H^i(E)$ and $H^{i-1}(E)$ are both nonzero. Since $n \geq 2$ it must be the case that $H^j(E)$ is nonzero for some $0 < j < n$. Thus E is nontrivial.

Let E be a vector bundle on the punctured spectrum of R. Then bundles of the form $E \oplus \mathcal{O}_R^{\ell}$ are said to be stably equivalent to E

(see [6]). The remarks above show that bundles of the form $\mathcal{E}\oplus\mathcal{O}_R^{\ell}$ have the property that each of their lifts is again infinitely liftable in case $\mathcal{E}$ is a trivial bundle. In this section we record the converse. That is, if $\mathcal{E}$ is nontrivial then there is an ℓ determined by cohomological data and a series of liftings of $\mathcal{E}\oplus\mathcal{O}_R^{\ell}$ which terminate in a bundle which cannot be lifted. The proof of our next result uses both Theorem 1.3 and Theorem 1.4 as well as induction.

Theorem 2.1: Let $\mathcal{E}$ be a nontrivial bundle on the punctured spectrum of R. Let $\mathcal{E}$ have rank r and let h be the length of $H^1(\mathcal{E}^*)$. Then there is a sequence of liftings of $\mathcal{E}\oplus\mathcal{O}_R^{\ell}$, with $\ell<h$, which continues at most $r + d + 2h - 2$ times, where $d = h(h+1)/2$.

In order to settle this problem of infinite liftability over punctured spectra, it remains to show that all lifting "paths" for a given bundle $\mathcal{E}$ are finite (perhaps with a uniform bound) in the situation where the module of sections E is a second but not third syzygy. Perhaps to indicate the more delicate nature of the lifting problem in regard to second syzygies, we record the following phenomena. Let L be a nonzero module of finite length over R and let E be a second syzygy for L. Considering L as a module over S, let F be a second syzygy for L over S, that is, we have an exact sequence

$$0 \to F \to S^a \longrightarrow S^b \to L \to 0.$$

Reducing this exact sequence modulo t gives a sequence

$$0 \to (F/tF) \to R^a \longrightarrow R^b \to L \to 0$$

which is exact except for R^a where there is homology isomorphic to L. Replacing F/tF by its double dual (F/tF)** with respect to R, however, repairs the inexactness. Hence, we obtain an exact sequence

$$0 \to (F/tF)^{**} \to R^a \longrightarrow R^b \to L \to 0$$

over R from which it follows that $(F/tF)^{**} = E\oplus R^{\ell}$, for some $\ell \geq 0$.

Thus it follows that F lifts $E\oplus\mathcal{O}_R^{\ell}$. Furthermore, if we allow ℓ to increase without bound, we may lift $E\oplus\mathcal{O}_R^{\ell}$ arbitrarily many times; in particular, $E\oplus\mathcal{O}_R^{\infty}$ lifts forever. Finally, we observe that Theorem 1.4 guarantees that ℓ must increase as more stable lifts are desired. The infinite liftability of $E\oplus\mathcal{O}_R^{\infty}$ is analagous to the "Eilenberg trick" that if P is a projective R module then $P\oplus R^{\infty}$ is a free R module.

3. Concerning the structure of k^{th} syzygies of rank k.

A test case for the lifting problem in regard to second syzygies is that of rank two bundles. As a consequence we have begun to consider the module theoretic properties of such syzygies in more depth. In particular we have begun a study of syzygies of minimal rank. Such syzygies are not rare. Indeed Bruns [2] has shown that a k^{th} syzygy M of rank larger than k must contain a free submodule F so that M/F is a k^{th} syzygy of rank exactly k. On the other hand we have shown [3] that any nonfree k^{th} syzygy must have rank at least k. We have the following result which appears in [5].

Theorem 3.1: Let R be a regular local ring which contains a field and let M be a k^{th} syzygy of rank k, with $k \geq 2$. Then M has a presentation

$$0 \to K \to B \to M \to 0$$

in which B is a second syzygy for $\mathrm{Ext}^{k-1}(M^*,R)$ and K is a $(k+2)^{nd}$ syzygy.

In addition we establish a uniqueness result. The grade of $\mathrm{Ext}^{1}(M^*,R)$ is one more than expected. In case M is a bundle on the punctured spectrum one obtains a uniqueness statement as in Schanuel's Lemma. Lastly, if F is a rank 2 vector bundle over the punctured spectrum of S which lifts E over the punctured spectrum of R one obtains a commulative diagram

$$\begin{array}{ccccccccc} 0 & \to & K'/tK' & \to & B'/tB' & \to & F'/tF' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K & \longrightarrow & B & \dashrightarrow & E & \to & 0 \end{array}$$

in which K', K and B', B are as in Theorem 3.1 for S and R, respectively. However, we have little information on the kernels and images.

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THE LOCAL TOTAL COHOMOLOGY OF NON LINEAR EVOLUTION EQUATIONS

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INTRODUCTION

Recent research in evolution equations which are derived from a variational problem has indicated the importance of conserved quantities [2]. These conserved quantities are functionals which have their domain in the spacelike sections of the base manifold. The base manifold for the considered equations is the direct product $M \times \mathbb{R}$, where M is an n-dimensional manifold. It is therefore natural to look for n-forms, which live on the prolonged evolution equation. This is itself a submanifold P of an appropriate jet bundle over $M \times \mathbb{R}$. The n-forms one looks for are horizontal and closed over the solutions of the equations, which are the integrable sections of P.

Associated to the integrable sections of P there is a de Rham type differential operator, which sends horizontal p-forms into horizontal (p+1)-forms on P and which over the solutions is nothing else but the de Rham differential operator. This operator allows one to construct a differential complex, the total complex, which in contrast with the usual de Rham complex on any manifold is locally non trivial. It is then evident that the local cohomology group, $_{::}H^n$, represents classes of local conserved quantities.

The main theorem of this paper states that all local cohomology groups of order smaller than n are zero on every evolution equation.

1. THE JET BUNDLE AND THE TOTAL COMPLEX

All manifolds are real, paracompact and C^∞. Let M be a manifold, then C(M) denotes the ring of C^∞-real functions on M and $J^k \equiv J^k(M)$ is the k-jet bundle of germs of elements of $C(M)$. Each J^k is a vector bundle for the source map $\alpha : J^k \longrightarrow M$. We indicate the target map by β.

The tangent bundle $T(J)$ contains a special subbundle, the total vector bundle $\mathcal{H}(J)$; this is the subbundle of $T(J)$ which is tangent to the integrable sections of J. Sections of this bundle are total vector fields on J and the set of C^∞ total vector fields will be denoted by $\mathcal{T}$.

As a consequence of the isomorphism $Jf_\star : T_xM \longrightarrow \mathcal{H}_p(J)$, with $Jf(x) = p$, any vector field $X \in \mathfrak{X}(M)$ has a unique total lift $X^\#$, section of $\mathcal{H}(J)$. The normalizer of $\mathcal{T}$ in $\mathfrak{X}(J)$ is called the set of contact vector fields on J. The set of contact vector fields contains a subset of those fields X with $\alpha_\star X = 0$. This subset, denoted by $\mathcal{L}$, will be called the set of special contact vector fields on J, [1][4].

<u>PROPOSITION 1.1.</u> [1],[4]. <u>Each $X \in \mathcal{L}$ is determined by $X \lrcorner d\beta$.</u> ∎

As a consequence of this property there is a bijection, $\natural$, between $C(J)$ and $\mathcal{L}$. If $\phi \in C(J)$ then $\phi^\natural \in \mathcal{L}$ is such that $\phi^\natural \lrcorner d\beta = \phi$.

The module of differential forms $\wedge(J)$ contains the submodule $\alpha^\star \wedge (M)$ of horizontal forms over the ring $C(J)$.

Associated to the distribution $\mathcal{H}(J)$, one has the total differential D, defined by $D : C(J) \longrightarrow \alpha^\star \wedge^1(M)$ such that for any $\phi \in C(J)$ and $X \in \mathcal{T}$, $D\phi(X) = X(\phi)$ [1],[4],[8].

<u>PROPOSITION 1.2.</u> (1) <u>D extends uniquely to $\alpha^\star \wedge (M)$</u>,

(2) $D \circ D = 0$. ∎

The complex $(\alpha^\star \wedge (M), D)$ is the total complex on J and will be denoted by $E = \oplus_k E^k$, where $E^k = \alpha^\star \wedge^k(M)$ and $E^0 = C(J)$.

<u>PROPOSITION 1.3.</u> [5]. <u>Let M be a manifold of dimension n, with vanishing de Rham cohomology, then the following sequence is exact</u> :

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} E^0 \xrightarrow{D} E^1 \to \dots \to E^{n-1} \xrightarrow{D} E^n \ . \ \blacksquare$$

2. THE TOTAL COMPLEX ON P.D.E.

Let $\Sigma \subset J^k$ be a formally integrable P.D.E. The ℓ^{th} order prolongation $\Sigma^{(\ell)} \subset J^{k+\ell}$ is a vector bundle over Σ [3]. We denote the inverse limit by $\Sigma^{\#} \subset J$. The total vector bundle on J, restricted to $\Sigma^{\#}$ will be in $T\Sigma^{\#} = \varprojlim \pi_\ell^{\star} T\Sigma^{(\ell)}$, where the map π_k are the restricted projections on $\Sigma^{\#}$. We will denote this restricted bundle by ${}_{\star}\mathcal{H}$. The bundle ${}_{\star}\mathcal{H}$ is involutive as a consequence of the formal integrability of Σ. Hence the pull-back of the differential operator D to $\Sigma^{\#}$, which is denoted by ${}_{\star}D$, will satisfy ${}_{\star}D \circ {}_{\star}D = 0$.

The ring of real functions $C(\Sigma^{\#})$, is the ring $C(J)$ modulo those elements which vanish on $\Sigma^{\#}$. The module $\alpha^{\star} \wedge (M)$, on $\Sigma^{\#}$, is considered as a ring over $C(\Sigma^{\#})$. This enables us to define the total complex ${}_{\star}E = (\alpha^{\star} \wedge (M), {}_{\star}D)$ on $\Sigma^{\#}$.

The sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} {}_{\star}E^0 \xrightarrow{{}_{\star}D} {}_{\star}E^1 \to \dots \to {}_{\star}E^{n-1} \xrightarrow{{}_{\star}D} {}_{\star}E^n$$

will in general not be locally exact. We denote the cohomology groups by ${}_{\star}H^k$.

Because all our consideration are purely local, we will from now on suppose $M = \mathbb{R}^n$. We will make use of the Euclidean structure of $\mathbb{R}^n$ and the standard volume form η (which is translation invariant) on $\mathbb{R}^n$. The results generalize to Riemannian manifolds.

The volume form η, which is identified with $\alpha^{\star}\eta$ on J, defines a bijection $\tau : {}_{\star}E^{n-1} \longrightarrow {}_{\star}\mathcal{C}$, where ${}_{\star}\mathcal{C}$ is the set of total vector fields on $\Sigma^{\#}$, given by : if $\lambda \in {}_{\star}E^{n-1}$, then $\tau(\lambda) \in {}_{\star}\mathcal{C}$ such that $\tau(\lambda) \lrcorner \eta = \lambda$. One has $\tau^{-1}(\tau(\lambda)) = \lambda$ and we will denote τ for τ^{-1}.

DEFINITION 2.1. The total divergence of a $X \in {}_{\star}\mathcal{C}$ is defined by $(\mathrm{div}_T\, X)\eta = {}_{\star}D(X \lrcorner \eta)$. $\blacksquare$

PROPOSITION 2.2. $\lambda \in {}_{\star}B^{n-1}$ iff $\mathrm{div}_T\ \tau(\lambda) = 0$. ∎

The correspondance τ allows us to introduce a Lie algebra structure on ${}_{\star}E^{n-1}$.

DEFINITION 2.3. Let $\lambda_1, \lambda_2 \in {}_{\star}E^{n-1}$; then $\{\lambda_1, \lambda_2\} = \tau([(\lambda_1), \tau(\lambda_2)])$. ∎

PROPOSITION 2.4. $\nu \in {}_{\star}Z^{n-1}$ iff there exists an antisymmetric matrix A^{ij}, with coefficients in $C(\sum^{\#})$ such that $\nu = \sum_{i,j} \partial_i^{\#} A^{ij} \eta^j$, with $\eta^i = \partial_i \lrcorner\ \eta$.

This allows us to write $\{\tau(X), \tau(Y)\} = {}_{\star}D(X^i Y^j - X^j Y^i)\eta^{ij} - (\mathrm{div}_T\ X) \cdot \tau(Y) + (\mathrm{div}_T\ Y) \cdot \tau(X)$, with $\eta^{ij} = \partial_j \lrcorner\ \partial_i \lrcorner\ \eta$.

PROPOSITION 2.5. Let $\lambda_1, \lambda_2 \in {}_{\star}B^{n-1}$, then $\{\lambda_1, \lambda_2\} \in {}_{\star}Z^{n-1}$. ∎

Hence ${}_{\star}B^{n-1}$, equipped with the bracket $\{.,.\}$, is a Lie algebra.

The bracket $\{.,.\}$ induces a bracket on ${}_{\star}H^{n-1}$, which becomes an abelian Lie algebra.

3. EVOLUTION EQUATIONS

We will from now on adopt the following notations : $J(M) = \check{J}$, $M = \mathbf{R}^n$, $J(M \times \mathbf{R}) = J$. The index $\vee$ will indicate that the quantity is considered on J. We define an evolution equation as a section $\psi : \check{J}^k \times \mathbf{R} \longrightarrow J^1$, and will, in addition, always require ψ to be independent of $\mathbf{R}$. The parameter t on $\mathbf{R}$ is the time coordinate. In terms of natural coordinates on $\check{J} \times \mathbf{R}$, this equation is $u_t = \psi(x^i, u, u_{(1)}, \ldots, u_{(k)})$, where $u_{(\ell)}$ stands for the ℓ^{th} order partial derivatives of u.

Because any evolution equation is by construction formally integrable, one has the canonical lift $\psi^{\#} : J \times \mathbf{R} \longrightarrow J$.

The evolution equation $u_t - \psi = 0$ defines a submanifold $\sum \subset J^k$ and $\mathrm{Im}\ \psi^{\#} = \sum^{\#}$.

In the following we will identify $\check{J} \times \mathbb{R}$ with $\sum^{\#}$ via the map $\psi^{\#}$ without mentioning it. On $\sum^{\#}$ one has the fundamental 1-form ${}_{\star}\rho = \psi^{\star} D\beta$, with β the target map.

Because $C(\check{J})$ is included in $C(\check{J} \times \mathbb{R})$, we may extend the operator $\natural$, in a trivial way to $C(\check{J} \times \mathbb{R})$. The image of $\natural$ will be denoted by $\mathcal{L}(\check{J} \times \mathbb{R})$. As a consequence of these constructions one obtains :

<u>PROPOSITION 3.1.</u> <u>The vector bundle</u> ${}_{\star}\mathcal{H}$ <u>is spanned by</u> $\mathcal{H}$ <u>and</u> ${}_{\star}\partial_t^{\#} = \partial_t + \psi^{\natural}$.

It is the vector field $\psi^{\natural}$, which determines completely the evolution equation. Because the function ψ is independent of t, this vector field can be considered on J, which is an alternative way of defining the same evolution equation.

<u>PROPOSITION 3.2.</u> <u>Let</u> ${}_{\star}\mathcal{L}$ <u>be the special contact vector fields in</u> $\mathcal{L}(\check{J} \times \mathbb{R})$ <u>which are sections of</u> $T\sum^{\#}$. <u>Then</u> $X \in {}_{\star}\mathcal{L}$ <u>iff</u> $X \in \mathcal{L}(\check{J} \times \mathbb{R})$ <u>and</u> $[X, {}_{\star}\partial_t^{\#}] = 0$.

Because any $X \in \mathcal{L}(\check{J} \times \mathbb{R})$ is defined by an element of $C(\check{J} \times \mathbb{R})$, it follows that for $A \in C(\check{J} \times \mathbb{R})$, $A^{\natural}$ will be in ${}_{\star}\mathcal{L}$ iff $A_{,t} = \psi^{\natural}(A) - A^{\natural}(\psi)$. The time independent elements of ${}_{\star}\mathcal{L}$ commute with $\psi^{\natural}$.

4. THE LOCAL TOTAL COHOMOLOGY OF EVOLUTION EQUATIONS

As a consequence of the direct product $M \times \mathbb{R}$, any ${}_{\star}E^k$ has a decomposition into the direct sum ${}_{\star}E^{k,0} \oplus {}_{\star}E^{k-1,1}$. This allows us to consider the double complex $\oplus_{k,\ell\,\star}E^{k,\ell}$ $(k = 0,1,\ldots,n\ ,\ \ell = 0,1)$, with differential operator ${}_{\star}D = \check{D} + D_t$ satisfying $\check{D} \circ D_t + D_t \circ \check{D} = 0$, $\check{D} \circ \check{D} = 0$ and $D_t \circ D_t = 0$.

Consequently, the horizontal rows of the following diagram are exact :

$$\begin{tikzcd}
C(\mathbb{R}) \arrow[r] & {}_{\star}E^{0,0} \arrow[r, "\check{D}"] \arrow[d, "D_t"] & {}_{\star}E^{1,0} \arrow[r] \arrow[d, "D_t"] & \cdots \arrow[r, "\check{D}"] & {}_{\star}E^{n,0} \arrow[d, "D_t"] \\
 & {}_{\star}E^{0,1} \arrow[r, "\check{D}"] & {}_{\star}E^{1,1} \arrow[r] & \cdots \arrow[r] & {}_{\star}E^{n,1}
\end{tikzcd}$$

We will now state our main theorem.

<u>THEOREM 4.1.</u> $_{\star}H^{\ell} = 0 \quad , \quad \forall \ell \quad , \quad 0 < \ell < n.$ ∎

The bundle J is a vector bundle over $\check{J}$, which allows us to consider C(J) as a subset of $C(\check{J})$. Let $\check{\delta}$ be the variational derivative on $C(\check{J})$.

<u>PROPOSITION 4.2.</u> [4][6][8]. <u>Let</u> $\varphi \in C(\check{J})$, <u>then</u> $\partial_t \varphi = \mathrm{div}_T(X)$, <u>for some</u> $X \in \check{\mathcal{C}}$, <u>iff</u> $\check{\partial}\varphi = 0.$ ∎

Let $\nu \in {_{\star}E^n}$, then ν may be written as $\phi \cdot \eta + \sigma$, with $\sigma \in {_{\star}E^{n-1,1}}$. We define the map $j : {_{\star}E^n} \to C(\check{J} \times \mathbb{R}) : \nu \mapsto \phi$, and define $\mathcal{B} = j({_{\star}B^n})$, $\mathcal{A} = \mathrm{Ker}\ \check{\delta}$ in $C(J \times \mathbb{R})$ and $_{\star}K = \mathcal{B}/\mathcal{A}$.

<u>LEMMA 4.3.</u> <u>If</u> $\lambda_1, \lambda_2 \in {_{\star}B^N}$ <u>such that</u> $\lambda_1 - \lambda_2 \in {_{\star}E^{n-1,1}}$, <u>then</u> $\lambda_1 - \lambda_2 \in {_{\star}Z^n}$. ∎

<u>THEOREM 4.4.</u> <u>The map j defines an isomorphism j</u> : $_{\star}H^n \longrightarrow {_{\star}K}$.

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A REMARK ON SMITH'S RESULT ON A DIVISOR PROBLEM IN ARITHMETIC PROGRESSIONS

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Presented by J.H.H. Chalk

Let $d_r(n)$ be the number of the factorizations of n into r positive integers. If k and h are arbitrary relatively prime integers with $k \geq 1$, then

$$\sum_{\substack{n \leq x \\ n \equiv h \pmod k}} d_r(n) = x\, P_r(\log(x)) + \Delta_r,$$

where P_r is a polynomial of degree $r-1$, and Δ_r is the error term.

In [1], Professor R.A. Smith proved the following result. If $x \geq k^{\frac{1}{2}(r+1)}$, then for any positive ε,

$$\Delta_r = D_r + O(x^{(r-1)/(r+1)+\varepsilon} . k^{\varepsilon}) \tag{1}$$

where D_r is an explicit (but rather complicated) term, which is independent of x.

The estimation of the term D_r is a remaining problem. In the 'Note added in proof' of [1], Professor R.A. Smith announced the result $D_r = O(k^{\frac{1}{2}r+\varepsilon})$, and said, "it seems likely that this upper bound can be improved to $k^{\frac{1}{2}(r-1)+\varepsilon}$, for any positive ε."

Recently, I proved the following estimate

$$\Delta_r = O(x^{r/(r+2)+\varepsilon} \cdot k^{-1+(r-1)/(r+2)+\varepsilon}) \tag{2}$$

if $x \geq k^{\frac{1}{2}(r+1)}$. In particular, if we put $x = k^{\frac{1}{2}(r+1)}$, then $\Delta_r = O(k^{\frac{1}{2}(r-1)+\varepsilon})$. On the other hand, if we substitute the value $x = k^{\frac{1}{2}(r+1)}$ in (1), then $\Delta_r = D_r + O(k^{\frac{1}{2}(r-1)+\varepsilon})$. On comparing these two results, we have

$$D_r = O(k^{\frac{1}{2}(r-1)+\varepsilon}),$$

for $x = k^{\frac{1}{2}(r+1)}$. But, as D_r is independent of x, this estimate always holds. This proves the statement which was conjectured and suggested by Professor R.A. Smith.

My method for proving (2) is somewhat similar to that of [1], but depends on an idea of Heath-Brown [2], instead of the method of Hooley, which was used by Professor R.A. Smith.

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