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ON PRIME RIGHT ALTERNATIVE ALGEBRAS AND ALTERNATORS

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Presented by P. Ribenboim, F.R.S.C.

Abstract. We study subvarieties of the variety of right alternative algebras over a field of characteristic $\neq 2, \neq 3$ such that the defining identities of the variety force the span of the alternators to be an ideal and do not force an algebra with identity element to be alternative. We call a member of such a variety a right alternative alternator ideal algebra. We characterize the algebras of this subvariety by finding an identity which holds if and only if an algebra belongs to the subvariety. We use this identity to prove that if R is a prime, right alternative alternator ideal algebra with an idempotent $e \neq 0, \neq 1$ such that $(e,e,R)=0$, then either R is alternative or R belongs to one of four exceptional varieties.

Introduction. Let R be an algebra over a field of characteristic $\neq 2, \neq 3$. We need not suppose R finite dimensional over F, but we do need F has enough elements so that certain linearizations can be done, the size of F depending on the degree of the identity to be linearized. An associator (a,b,c) is defined as $(a,b,c)=(ab)c-a(bc)$, the commutator $[a,b]$ as $[a,b]=ab-ba$ and an alternator is an associator of the form $(a,a,b),(a,b,a)$

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or (b,a,a). Let M be the subspace spanned by all alternators. We are interested in studying right alternative algebras in which M is assumed to be a two-sided ideal. In this paper a right alternative alternator ideal algebra will mean slightly more than a right alternative algebra where M is an ideal. It will mean an identity holds which forces M to be an ideal and that this identity does not imply alternativity in the presence of an identity element. Several examples of algebras in the literature have turned out to be right alternative alternator ideal algebras. The best known are (-1,1) algebras [6] but other identities are studied in [1],[2], and [4]. Work in this area has been done in [3],[4] and [8]. In [4]using a group representation technique to study identities it is shown that for right alternative algebras the assumption that M is an ideal is rather weak. In that same paper a simple process is given to test if additional identities (in addition to right alternativity) imply or do not imply that M is an ideal. In this paper we sharpen this result by determining when the identity is reducible or irreducible.

Characterization of right alternative alternator ideal algebras.

We prove the following theorem:

Theorem 1. <u>Let R be a right alternative algebra with identity element, and suppose that R satisfies some identity which forces</u>

the span of the alternators to be an ideal. Then, for some fixed $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ in F, R satisfies the following identity , where $M(a,b,c)=(a,b,c)+(b,a,c)$:

$$3[a,(b,b,d)]-[b,M(a,b,d)]+[b,M(a,d,b)]=$$

$$\beta_1 M([a,b],b,d) + \beta_2 M([a,b],d,b) + \beta_3 M([a,d],b,b)+ \tag{1}$$

$$\beta_4 M([b,d],a,b)+ \beta_5 M([b,d],b,a).$$

Conversely, if R is a right alternative algebra satisfying (1) for some choice of $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in F$, then R is a right alternative alternator ideal algebra.

Specific algebras satisfying (1) are referred to by listing their coefficients as $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ and by saying that the algebra is of the type $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$.

We also compare different types of identities in right alternative algebras that imply that the span of the alternators is an ideal, i.e. we compare different characterizations of right alternative alternator ideal algebras.

Presence of an idempotent. Let R be a right alternative alternator ideal algebra with identity element 1. Suppose it contains an idempotent $e\neq 0, \neq 1$ such that $(e,e,R)=0$. We can thus write R as the additive direct sum of the four summands R_{11}, R_{10}, R_{01} and R_{00}, where $x\in R_{ij} \Leftrightarrow ex=ix$ and $xe=jx$. By making use of Theorem 1 and therefore of identity

(1) we prove that $R_1R_{01}+R_0R_{10}$ is a trivial ideal of R, i.e. an ideal I such that $I^2=0$. The only possible exceptions are for algebras of the types (-3/2,1,1/2,-2,3/2) and (0,1,-1,-2,3). If we now let R be semiprime, we prove that, with the only possible exceptions of algebras of types (-3/2,1,1/2,-2,3/2), (0,1,-1,-2,3), (-2,1,1,-2,1) and (3,-4,1,2,-3), all alternators lie in R_1+R_0. Furthermore, M(R,R,R) and $R_{10}R_{01}+R_{10}+R_{01}+R_{01}R_{10}$ are ideals which annihilate each other.

We are thus in condition to prove the final theorem, and, by using the same argument of page 108 in [3],we see that the assumption of the existence of an identity element is unnecessary:

Theorem 2. Let R be a prime, right alternative alternator ideal algebra with an idempotent $e\neq 0,\neq 1$ such that $(e,e,R)=0$. Then R is alternative. The only possible exceptions are algebras of the four types mentioned above.

We do not know whether or not these exceptional cases are truly exceptions to the theorems. They arise as exceptions to our proofs. Part of nonassociative algebra is deciding which identities are worth studying. We believe these four exceptional algebras are defined by very reasonable identities.

Having survived as exceptions to the techniques of this paper, they are certainly of interest for further study.

Full details of these and other results are forthcoming in [9] .

Acknowledgements

The computer analysis in this paper was done using a general program which was written chiefly by Leslie Hogben. This program was designed by the second-named author and Dr. Hogben. It analyzes degree four identities of nonassociative algebras over fields of characteristic $\neq 2, \neq 3$.

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QUASI-VALUATION OF RINGS WITH CORE ELEMENTS

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We consider a class of rings which are integral domains but are not fields. We study these rings by investigating a property of certain elements, called the core property, by means of suitably defined valuations of such rings, called quasi-valuations. We use a quasi-valuation to characterize the core elements, and also showing that the existence of a strongest quasi-valuation is a characteristic of rings with core elements.

All rings R are commutative without identities.

DEFINITION. A non-zero element c in a ring R is a core element if, and only if, every non-zero element x of R divides some power of c. (If $x = c$, then c is said to divide itself trivially.)

A mapping v of a ring R into the set of non-negative reals together with $+\infty$ is a quasi-valuation on R if, and only if, (1) $v(x) = \infty$ for $x = 0$, (2) $v(xy) \geq v(x) + v(y)$, (3) $v(x^2) = 2v(x)$, (4) $v(x+y) \geq \min(v(x),v(y))$, (5) $v(x) > 0$ for some non-zero x in R.

Let u and v be quasi-valuations on a ring R such that $u \neq v$. Then u is _smaller_ than v if, for all x in R, $u(x) \leq v(x)$, and is _stronger_ than or equal to v if, for any real number N there is a real number M such that, for all x in R, $u(x) > M$ implies that $v(x) > N$.

THEOREM 1. _Let c be a core element of an integral domain C. Then x in C, $x \neq 0$, is a core element if, and only if,_

$$\lim_{n\to\infty} \frac{h_c(x,n)}{n} > 0$$

where $h_c(x,n)$ denotes the greatest upper bound of $H = \{ k : k = 0,1,2,\ldots,c^k | x^n \}$ $(n=1,2,\ldots)$.

THEOREM 2. _If a ring R has a strongest quasi-valuation, then R has core elements_.

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A NOTE ON DUBREIL'S THEOREMS ON THE NUMBER OF GENERATORS OF PERFECT IDEALS OF CODIMENSION 2

Peter Schenzel

Presented by P. Ribenboim

Introduction

In his article [3] P.Dubreil proved two estimates for the number of generators of unmixed ideals of codimension two in $k[X_1,X_2,X_3]$, k an infinite field. For a modern approach and a generalization to $A = k[X_1,\dots,X_r]$, $r \geq 2$, see A.V.Geramita's paper [4] and the preprint [2]. The purpose of this note is to give a new and different argument for proving these theorems.

Put $R = A/I$ for a homogeneous perfect ideal I of codimension 2. For a finitely generated graded A-module M define $a(M) = \inf\{n \in \mathbb{Z} \mid M_n \neq 0\}$, where M_n denotes the n-th graded piece of M.

Dubreil's 1^{st} Theorem. $\mu(I) \leq a(I) + 1.$

Here μ denotes the minimal number of generators. Because k is infinite there are two forms F,G taking part of a minimal generating set of I such that $\underline{F} = (F,G)A$ is a complete intersection and $\deg F = a(I)$. Set $f = \deg F$ and $g = \deg G$. Let $h(I) = \max\{n \in \mathbb{Z} \mid (R_+)^n \nsubseteq \underline{x} R\}$, where $\underline{x} = \{x_1,\dots,x_{r-2}\}$ is an R-regular sequence consisting of forms of degree one. As shown in the Proposition below h(I) does not depend on $\underline{x}$.

Dubreil's 2^{nd} Theorem. $\mu(I) \leq f + g - h(I).$

Using the Hilbert-Burch theorem there is a straightforward proof of the first theorem. The second result is a sharpening of the first. The new point of view is to show that the first result yields the second as a corollary.

One might ask for a corresponding result for perfect ideals of codimension 3. If I is a Gorenstein ideal, i.e., R is a Gorenstein ring, we have

Theorem. $\mu(I) \leq 2\, a(I) + 1$.

An auxiliary statement

In order to show the results let us introduce some numerical invariants of R, see [6]. Let $H(n,R)$ denote the Hilbert function of R. Then it becomes a polynomial $h(n,R)$ for n large. $F_R(T)$ denotes the Hilbert series, i.e.,

$$F_R(T) = \sum_{n \geq 0} H(n,R)\, T^n = g_R(T)/(1-T)^{\dim R},$$

where $g_R(T)$ is a polynomial. Set $h(R) = \deg g_R(T)$. Note that

$$h(R) = \max\{n \in \mathbb{Z} \mid \Delta^{\dim R} H(n,R) \neq 0\}.$$

By the Hilbert syzygy theorem R admits a finite free minimal resolution $F_\bullet$ over A. Define

$$t(R) = \max\{s \in \mathbb{Z} \mid A(-s) \text{ occurs as a summand in } F_\bullet\}.$$

Proposition ([6], Theorem C). *For a d-dimensional Cohen-Macaulay ring R the following integers are equal:*

(i) $\max\{n \in \mathbb{Z} \mid H(n,R) \neq h(n,R)\}$,

(ii) $\max\{n \in \mathbb{Z} \mid (R_+)^n \not\subseteq \underline{x} R\} - \sum_{i=1}^{d} t_i$, *where* $\underline{x} = \{x_1, \ldots, x_d\}$ *is an R-sequence with* $t_i = \deg x_i$, $i = 1, \ldots, d$,

(iii) $h(R) - \dim R$, *and*

(iv) $t(R) - \dim_k R_1$.

Proofs

If I is a perfect ideal of codimension 2 in A the Hilbert-Burch theorem, see e.g. [1], provides a minimal free resolution

$$0 \longrightarrow \bigoplus_{i=1}^{t} A(-r_i) \xrightarrow{\mathfrak{a}} \bigoplus_{j=1}^{t+1} A(-s_j) \longrightarrow A \longrightarrow A/I \longrightarrow 0 \qquad (x)$$

where $\mathfrak{a}$ is a $t\times(t+1)$-matrix with entries 0 or forms of positive degree. Moreover, I is generated by the maximal minors of $\mathfrak{a}$. Hence, any generating form has a degree $\geqq t$. Because $t + 1 = \mu(I)$ it follows $\mu(I) \leqq a(I) + 1$.

In order to show the second claim set $J = \underline{F} : I$. Note that I and J are linked over $\underline{F}$ in the sense of C.Peskine and L.Szpiro [5]. Accordingly [5], Proposition 2.5 and Proposition 3.6, it follows that J is a perfect ideal of codimension 2 with $\mu(J) = \mu(I) - 1$. The first result yields $\mu(I) \leqq a(J) + 2$. Hence, it is enough to show that $a(J) = f+g-h(R)-2$. In order to do this consider the short exact sequence

$$0 \longrightarrow \underline{F} \longrightarrow J \longrightarrow \underline{F} : I/\underline{F} \longrightarrow 0.$$

Because $\underline{F} : I/\underline{F} \cong \mathrm{Ext}_A^2(A/I,A)(-f-g)$ it follows from (x) by dualizing that $a(\underline{F} : I/\underline{F}) = -t(R)+f+g$. Therefore, $a(\underline{F} : I/\underline{F}) < f = a(\underline{F})$ and $a(J) = -h(R)-2+f+g$ accordingly the Proposition. Thus, $\mu(I) \leqq f+g-h(R)$ as required. Because $a(J) < a(I)$ the second bound is sharper than the first.

Using the structure theorem of Gorenstein ideals of codimension 3 given by D.A.Buchsbaum and D.Eisenbud [1] the proof of the third statement follows the same line of reasoning as in the proof of the first result. Hence we omit it.

For some applications of Dubreil's Theorems and some comments see the articles [4] and [2].

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OPTIMIZATION AND REARRANGEMENTS OF THE COEFFICIENT IN THE DIFFERENTIAL EQUATIONS(S) $y'' \pm qy = 0$

M. Essén[1)]

Presented by L. Lorch, F.R.S.C.

Abstract. For coefficients q in certain classes of integrable functions, we determine $\sup y(T)$ and $\inf y(T)$ when q varies in the class and y is a solution of the differential equation(s) $y'' \pm qy = 0$, $y(0) = 1$, $y'(0) = \alpha$.

For $f \in L^1(0,T)$, we define the distribution function $m(t,f) = |\{s \in (0,T): f(s) > t\}|$, where T is a fixed positive number and $|\cdot|$ denotes Lebesgue measure. Let $\Phi : [0,T] \to [0,1]$ be a nonincreasing, right continuous function and let $\mathcal{P}(\Phi)$ be those functions p^2 in $L^\infty(0,T)$ which satisfy $||p||_\infty \leq 1$, $p \geq 0$ and $m(\cdot,p^2) = m(\cdot,\Phi)$. For $p^2 \in \mathcal{P}(\Phi)$, we consider the differential equation

(1) $$y'' - p(t)^2 y = 0, \quad y(0) = 1, \; y'(0) = 0, \quad t \in (0,T).$$

Problem 1. Determine $\inf y(T)$, $p^2 \in \mathcal{P}(\Phi)$.

Problem 2. Determine $\sup y(T)$, $p^2 \in \mathcal{P}(\Phi)$.

Problem 1 was solved by Essén (cf. [1], Theorem 5.2, pp.43-44 and 56-64 and Theorem A in [2]: the infimum is assumed when p is nondecreasing. A discrete version can be found in [3]).

To solve the supremum problem, we use ideas of L.-E. Zachrisson and imbed $\mathcal{P}(\Phi)$ in the weak* closure $\Omega(\Phi)$ of the set of finite sums of the form

1) Part of this work was done while the author was visiting McMaster University with support from NSERC Grant # A7322.

$\Sigma c_j \Phi_j$ where $\{\Phi_j\}$ is a sequence in $\mathcal{G}(\Phi)$, and $\{c_j\}$ is a sequence of nonnegative numbers with sum 1. If y is a solution of (1), we have two new problems:

Problem 3. Determine $\inf y(T)$, $p^2 \in \Omega(\Phi)$.

Problem 4. Determine $\sup y(T)$, $p^2 \in \Omega(\Phi)$.

It is easy to see that there exists at least one function p_0^2 and an associated solution y_0 of (1) such that $y_0(T)$ is maximal or minimal when p^2 varies in $\Omega(\Phi)$. To y_0, we associate the function

$$Q(t) = y_0(t)^2 \int_t^T y_0(s)^{-2} ds, \quad t \in (0,T).$$

Theorem 1. Let (p_0, y_0) be an extremal couple for Problem 4. Then p_0 is nonincreasing. In the open set O where

$$\int_0^t p_0(s)^2 ds < \int_0^t \Phi(s) ds, \quad t \in (0,T), \tag{2}$$

we have $Q'(t) = 0$.

Remark. In intervals where $Q'(t) = 0$, $p_0(t)$ will be constant.

Example. Let $B \in (0,T)$ be given and consider $\Phi = \Phi_B$ given by $\Phi_B(t) = 1$, $0 \le t < B$; $\Phi_B(t) = 0$, $B \le t \le 1$. It follows from Theorem 1 that an extremal coefficient is of the form $p_0(t) = 1$, $0 \le t < a$; $p_0(t) = c < 1$, $a \le t < b$; $p_0(t) = 0$, $b \le t \le T$. It remains to determine a, b and c. After some computations, we find that

a) if $T \le B + \coth B$, the open set O is empty and $p_0^2 = \Phi_B$;

b) if $T > B + \coth B$, a is the unique solution of the equation

$$T = a + \coth a + (B - a)(\coth a)^2$$

$b = T - \coth a$ and we have determined p_0 which is unique in this special case.

The situation is more complicated if we consider Problem 4 for solutions of the equation

(3) $\quad y'' - p(t)^2 y = 0,\ y(0) = 1,\ y'(0) = \alpha \geq 0,\quad t \in (0,T).$

Using our results on the solution of the supremum problem for equation (3), we can solve the following problem of E. Trubowitz:

Let $B > 0$ be given and let $E = E_B$ be the class of functions $q \in L^1(0,T)$ which are such that $\int_0^T |q| = B$. Consider the equation

(4) $\quad y'' - qy = 0,\ y(0) = 1,\ y'(0) = \alpha \geq 0,\quad t \in (0,T).$

The problem is to determine $\sup y(T)$, $q \in E_B$, when y is a solution of (4).

Theorem 2. If $0 < T \leq (B + \alpha)/\alpha^2$, *the solution of the problem of Trubowitz is as follows:*

$$\sup_{q \in E_B} y(T) = \begin{cases} 1 + (B + \alpha)T, & T(B + \alpha) \leq 1, \\ 2 \exp(\sqrt{T(B + \alpha)} - 1), & T(B + \alpha) > 1. \end{cases}$$

The extremal coefficient is

$$q_0 = \begin{cases} B\delta\ , & T(B + \alpha) \leq 1, \\ \gamma\delta + ((B + \alpha)/T)\chi\ , & T(B + \alpha) > 1, \end{cases}$$

where δ *is the Dirac functional,* χ *is the characteristic function of the interval* $(0,\ T - \sqrt{T/(B + \alpha)})$ *and* $\gamma = \sqrt{(B + \alpha)/T} - \alpha$.

If $T > (B + \alpha)/\alpha^2$, *the results are as follows:*

$$\sup_{q\in E_B} y(T) = 2\alpha B^{-1}(\sqrt{1 + B(T + \alpha^{-1})} - 1)\exp(\sqrt{1 + B(T + \alpha^{-1})} - 1),$$

and the extremal coefficient is $q_0 = \alpha(1 + \alpha d)^{-1}\chi$, *where* χ *is the characteristic function of the interval* $(d,\ d + (d + \alpha^{-1})^2 B)$ *and*

$$d = B^{-1}\sqrt{1 + B(T + \alpha^{-1})} - \alpha^{-1} - B^{-1}.$$

Remark. The case $\alpha = 0$ has been solved by J. E. Wilkins [10].

Theorem 3. Let (p_0, y_0) be an extremal couple for Problem 3 (the infimum problem. Then p_0^2 is the increasing rearrangement Φ^{**} of Φ. The infimum over the larger class $\Omega(\Phi)$ coincides with the infimum over the smaller class $\mathcal{G}(\Phi)$.

A detailed discussion of Theorems 1 - 3 is given in Essén [4].

In the proof of Theorem 1, we put $u = y'/y$ and obtain

$$u' + u^2 = p^2, \qquad u(0) = 0, \qquad t \in (0,T).$$

It is easy to see that Problem 4 is equivalent to finding sup $\int_0^T u$, $p^2 \in \Omega$. We know that there exists an extremal coefficient p_0. A standard argument from the calculus of variations leads us to a situation where we need a partial order introduced by Hardy, Littlewood and Pólya (cf. [7], 10.2 and 10.13). The suggestion to use this partial order is due to L.-E. Zachrisson.

To $f \in L^1(0,T)$, we associate the decreasing function f^* and the increasing function f^{**} which are such that $m(\cdot,f) = m(\cdot,f^*) = m(\cdot, f^{**})$. If f and g are in $L^1(0,T)$, we say that f majorizes g, written $g \prec f$, if

$$\int_0^t g^* \le \int_0^t f^*, \quad t \in (0,T); \qquad \int_0^T g^* = \int_0^T f^* .$$

A crucial property of these rearrangements is that is f and g are non-negative on $(0,T)$ with $f \in L^1(0,T)$ and $g \in L^\infty(0,T)$, then

$$\int_0^T f^* g^{**} \le \int_0^T fg \le \int_0^T f^* g^* .$$

We also need the following characterization of $\Omega(\Phi)$ (cf. Ryff [9]):

$$\Omega(\Phi) = \{g \in L^1(0,T): \ g \prec \Phi\} .$$

In the proof of Theorem 3, we use the techniques described above and the fact that the extremal coefficient is nondecreasing (cf. Problem 1).

The optimization technique which we use in the proofs of Theorems 1 and 3 can also be used to study the equation

(5) $\quad y'' + qy = 0, \quad y(0) = 1, \quad y'(0) = \alpha, \quad t \in (0,T),$

where $q \in \Omega(\Phi)$ (cf. Essén [5]). For solutions y of (5), we determine

$\sup y(T)$, $q \in \Omega$ (we assume that there exist solutions which are positive on $(0,T)$, when q varies in Ω),

$\inf y(T)$, $q \in \Omega$ (we assume that all solutions of (5) are positive on $(0,T)$ when q varies in Ω).

We can also determine $\sup y(T)$ and $\inf y(T)$ when q varies in E_B and deduce the following analogue of the classical Lyapunov criterion: let $q \in L^1(0,T)$, let $q_+ = \max(q,0) \in E_B$ and assume that the solution of equation (5) has a zero in $(0,T]$. Then

(6) $\quad B \geq \alpha + T^{-1}, \quad \alpha T \leq 1$

(7) $\quad B \geq 4/(T + \alpha^{-1}), \quad \alpha T > 1.$

Our methods can be used to study oscillation problems for equation (5) in cases where we know something about $m(\cdot,q)$.

Our methods are constructive. As an example, we mention that the expressions in Theorem 2 or in (6) and (7) are found by computation. It is not necessary to guess.

Finally, we consider the eigenvalue problem

(8) $\quad -y'' + qy = \lambda y, \quad y(0) = y(1) = 0, \quad t \in (0,1),$

where $q \in E_B$. Let $\lambda_1(q)$ be the first positive eigenvalue. Find $\Lambda(B) = \sup \lambda_1(q)$ when q varies in E_B! This is a question of A. Ramm [8]. We prove that $\Lambda(B) = (\pi + \sqrt{\pi^2 + 4B})^2/4$.

Remark. A. G. Ramm has received several solutions of his problem. What is special in our case is that our solution follows in a natural way from the theory developed in [4] and [5].

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DIMENSION DE HAUSDORFF D'UNE COURBE CIRCULAIRE SIMPLE DE VON KOCH

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Presented by H. Schwerdtfeger, F.R.S.C.

Soit X la trace dans le plan d'une courbe circulaire simple de Von Koch selon la terminologie introduite dans [1], c'est-à-dire la trace d'une courbe continue simple décomposable en d'arcs successifs, bout à bout, géométriquement semblables à X dans les rapports $r_1,\ldots,r_d$ respectivement. Soit α le paramètre d'agitation de cette courbe (toujours selon la terminologie de [1]), c'est-à-dire la solution de l'équation $r_1^\alpha+\ldots+r_d^\alpha = 1$. Montrons que α est la dimension de Hausdorff de X conformément à la conjecture de S. Dubuc (c f. [1], p. 108).

Par construction, X est constitué d arcs de courbe $X_1,\ldots,X_d$ bout à bout et géométriquement semblables à X dans les rapports respectifs $r_1,\ldots,r_d$. Nous ferons la convention suivante: pour un certain paramétrage de la courbe, lorsque le paramètre t augmente de 0 à 1, les arcs sont parcourus dans l'ordre $X_1,\ldots,X_d$.

On observe que pour tout entier $m \geq 1$, X se décompose en une chaîne de d^m arcs (que nous appelerons <u>arcs fondamentaux d'ordre m</u>) notés $X_{j_1,\ldots,j_m}$ semblables à X dans les rapports respectifs $r_{j_1}r_{j_2}\ldots r_{j_m}$ $(1 \leq j_\ell \leq d$ pour $\ell = 1,\ldots,m)$. Si deux arcs fondamentaux sont consécutifs, ils ont un seul point en commun; s'ils sont non consécutifs, ils sont disjoints car la courbe est simple. Le "dernier" arc fondamental d'ordre m dans la chaîne (i.e. celui qui se termine par l'image de $t = 1$) est $X_{d,\ldots,d}$. Il est semblable à X dans le rapport r_d^m .

Désignons par $X'_{d,\ldots d}$ l'arc "pointé" obtenu en enlevant de l'arc fondamental $X_{d,\ldots,d}$ d'ordre m son origine. Soit K_m le compact $\subset X$ défini par $K_m = X \setminus X'_{d,\ldots,d}$. Soient $\psi_1,\psi_2,\ldots,\psi_{d^m-1}$ d^m-1 similitudes de $\mathbb{R}^2$ qui transforment respectivement X en ses d^m-1 premiers arcs fondamentaux d'ordre m. Posons

$$E_m = \bigcap_{n=1}^{\infty} \bigcup_{a_1,\ldots,a_n=1}^{d^m-1} \psi_{a_1}\circ\ldots\circ\psi_{a_n}(K_m) \quad .$$

L'ensemble parfait totalement discontinu $E_m \subset X$ se décompose alors en d^m-1 portions disjointes $\psi_i(E_m)$ $(i = 1,\ldots,d^m-1)$ semblables à E_m dans les rapports respectifs $r_{j_1}r_{j_2}\ldots r_{j_m}$ avec $(j_1,\ldots,j_m)\in\Gamma_m$, où

$$\Gamma_m = \{(j_1,\ldots,j_m) \neq (d,\ldots,d) \ : \ 1 \leq j_\ell \leq d \text{ et } \ell = 1,\ldots,m\} \quad .$$

Selon la terminologie de [2], E_m est un parfait isotypique au sens strict de type $(d^m-1 \ , \ \xi_1,\ldots,\xi_{d^m-1})$ où

$\{\xi_1,\ldots,\xi_{d^m-1}\} = \{r_{j_1}\ldots r_{j_m} \ : \ (j_1,\ldots,j_m)\in\Gamma_m\}$. D'après [2] et [3] la dimension de Hausdorff α_m de E_m est donnée par

$$\sum_{(j_1,\ldots,j_m)\in\Gamma_m} (r_{j_1}\ldots r_{j_m})^{\alpha_m} = 1 \ .$$

Les r_j étant < 1 et α vérifiant

$$\sum_{j_1,\ldots,j_m=1}^{d} (r_{j_1}\ldots r_{j_m})^{\alpha} = (r_1^\alpha+\ldots+r_d^\alpha)^m = 1 \quad ,$$

on a immédiatement $\alpha_m < \alpha$. D'autre part,

$$\sum_{(j_1,\dots,j_m,j_{m+1})\in\Gamma_{m+1}} (r_{j_1}\cdots r_{j_m} r_{j_{m+1}})^{\alpha_m} = \sum_{j=1}^{d} r_j^{\alpha_m} \sum_{(j_1,\dots,j_m)\in\Gamma_m} (r_{j_1}\cdots r_{j_m})^{\alpha_m}$$

$$+ r_d^{m\alpha_m} \sum_{j=1}^{d-1} r_j^{\alpha_m}$$

$$= \sum_{j=1}^{d} r_j^{\alpha_m} + r_d^{m\alpha_m} \sum_{j=1}^{d-1} r_j^{\alpha_m} > 1$$

car $\alpha_m < \alpha$ et $\sum_{j=1}^{d} r_j^{\alpha} = 1$. Compte tenu de la définition de α_{m+1}, on a donc

$\alpha_m < \alpha_{m+1} < \dots < \alpha$. Posons $\alpha' = \sup_{m\geq 1} \alpha_m$. Si $\alpha' < \alpha$ alors

$$1 = \sum_{(j_1,\dots,j_m)\in\Gamma_m} (r_{j_1}\cdots r_{j_m})^{\alpha_m} > \sum_{(j_1,\dots,j_m)\in\Gamma_m} (r_{j_1}\cdots r_{j_m})^{\alpha'}$$

$$= (r_1^{\alpha'}+\dots+r_d^{\alpha'})^m - r_d^{m\alpha'} \to \infty$$

quand $m \to \infty$ car $r_1^{\alpha'}+\dots+r_d^{\alpha'} > 1$. Cette contradiction montre que $\alpha' = \alpha$.

Notons dim S la dimension de Hausdorff d'un ensemble $S \subset \mathbb{R}^2$.

Puisque $E_m \subset X$ pour tout $m \geq 1$ avec

$$\dim E_m = \alpha_m \to \alpha \qquad (\text{lorsque } m \to \infty),$$

on obtient

$$\dim X \geq \alpha \ .$$

L'inégalité réciproque

$$\dim X \leq \alpha$$

est triviale (il suffit de considérer les recouvrements de X par ses arcs fondamentaux d'ordre m, et laisser $m \to \infty$). D'où

$$\dim X = \alpha \ .$$

Remarque - Il semble même plausible que la mesure de X en dimension α soit non-nulle (on sait qu'elle est finie). Toutefois la démonstration de cette conjecture devrait nécessiter une méthode différente de celle que nous venons d'employer pour démontrer un résultat plus faible.

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CONJUGACY SEPARABILITY OF CERTAIN CLASSES OF GROUPS

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Presented by D.M. Solitar, F.R.S.C.

Abstract

We report in this note that groups of the forms (1) $\langle a,b;(ab)^n = a^{\ell n} = b^{mn} = 1\rangle$; (2) $\langle b,t;t^{-1}b^m tb^{\ell}\rangle$, are conjugacy separable.

1. Introduction. A group A is said to be conjugacy separable (c.s.) if for every pair of elements x,y such that x,y are not conjugate in A there exists a finite homomorphic image $\bar{A}$ of A such that the images of x,y in $\bar{A}$ are not conjugate in $\bar{A}$. A.W. Mostowski [8] showed that finitely presented c.s. groups have solvable conjugacy problem. This raises the question which groups are c.s. Partial results were obtained by P. Stebe for free products (g.f.p.) of free groups with cyclic amalgamations [10] and Fuchsian groups [11]. By repeated applications of Solitar's theorem ([7], p.212), J.L. Dyer [6] proved that g.f.p. of c.s. groups with finite amalgamations are c.s. Using this result Dyer showed that g.f.p. of finitely generated torsion-free nilpotent groups or free groups with cyclcic amalgamations are c.s. In the same paper Dyer also showed that 1-relator groups with non-trivial centres are c.s. (cf. S.M. Armstrong [2]). Since

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1-relator groups with torsion behave in a nice way and in view of G. Baumslag's conjecture [4] that they are residually finite ($_RF$), we ask whether 1-relator groups with torsion are c.s. The answer to this question is expected to be very hard since a proof of Baumslag's conjecture is still unknown. However 1-relator groups of the form $\langle b,t;(t^{-1}b^{m}tb^{\ell})^{s}\rangle$, $s > 1$, behave particularly nice. They have been proved to be $_RF$ by B. Baumslag and F. Levin [3] and also by Allenby, Moser and Tang [1]. In this note we give a brief sketch of the proofs of the following results:

Theorem A. Let $P = \langle a,b;(ab)^{n} = a^{\ell n} = b^{mn} = 1\rangle$. Then P is c.s.

Note that P is a degenerate Fuchsian group and belongs to the class of groups of the form $\langle a,b;(ab)^{n} = a^{\ell} = b^{m} = 1\rangle$, which are called polyhedral groups by Coxeter and Moser [5]. Thus it would be interesting to know whether all polyhedral groups are c.s.

Theorem B. Let $G = \langle b,t;(t^{-1}b^{m}tb^{\ell})^{s}\rangle$, $s > 1$. Then G is c.s.

The detail proofs of these results are rather long and tedious and will appear elsewhere.

2. Sketch of proofs. Let x,y be elements of a group A. We use $x \sim_A y$ to denote x and y are conjugate in A and $x \not\sim_A y$ for x,y not conjugate in A. To prove theorem A we make use of Stebe's result [11] that every element of infinite

order in P is conjugacy distinguishable (c.d.), where an element $x \in P$ is said to be c.d. if $y \in P$ and $x \not\sim_P y$ then there exists a finite homomorphic image $\bar{P}$ of P such that $\bar{x} \not\sim_{\bar{P}} \bar{y}$. Thus if we can show that every element of finite order in P is c.d. then P is c.s. This is accomplished by applying theorem 1.5 of C.H. Sah [9].

The basis of our proof for theorem B rests on the following results:

Theorem 1. (Theorem 4, Dyer [6]). If the groups X, Y are c.s. and U is finite, then $X *_U Y$ is c.s.

Theorem 2. (Collins, Theorem 13 [6]). Let A be c.s. and L, M be finite isomorphic subgroups. Then the HNN extension of A by the infinite cycle $\langle t \rangle$ with L and M as the associated subgroups is c.s.

Theorem 3 (Theorem 3.5, Tang [12]). $B = \langle h,k;(h^m k^\ell)^s \rangle$ is c.s.

To prove B we first note that G is an HNN extension of the base group $B' = \langle b_1, b_0; (b_1{}^m b_0{}^\ell)^s \rangle$, where $b_i = t^{-i} b t^i$. By theorem 3, B' is c.s. Our task is to show that given $x,y \in G$ such that $x \not\sim_G y$ there exists an integer w such that $\bar{x} \not\sim_{G_w} \bar{y}$, where $G_w = \langle h,k,t;(h^m k^\ell)^s = h^{\ell m w} = k^{\ell m w} = 1,$ $t^{-1}ht = k \rangle$ and $\bar{x}, \bar{y}$ are the images of x,y in G_w respectively. (Note that we identify b_1, b_0 with h, k respectively.) Now G_w is an HNN extension of $B_w = \langle h,k;(h^m k^\ell)^s = h^{\ell m w} = k^{\ell m w} = 1 \rangle$, $s > 1$, with finite associate subgroups $\langle h; h^{\ell m w} \rangle$ and $\langle k; k^{\ell m w} \rangle$.

If B_w is c.s., then, by theorem 2, G_w is c.s. It follows immediately that G is c.s.

Now B_{ws} is c.s. for all $w \geq 1$. This follows from the fact that

$$B_{ws} = (H' \underset{h^m = u}{*} D) \underset{v = k^\ell}{*} K',$$

where $H' = \langle h; h^{\ell mws} \rangle$, $K' = \langle k; k^{\ell mws} \rangle$ and $D = \langle u,v;\ (uv)^s = u^{\ell ws} = v^{mws} = 1 \rangle$. By theorem A, D is c.s. Thus, by two applications of theorem 1, B_{ws} is c.s.

To show that there exists a suitable integer w (or rather ws) such that $\bar{x} \not\sim_{G_w} \bar{y}$, we establish the following lemma.

Lemma. (i) *Let* $x,y \in B$ *such that* $x = h^i y k^j$. *Then, for all integers* $w > 0$ *if* $\bar{x} = \bar{h}^\alpha \bar{y} \bar{k}^\beta$ *in* B_w, *then* $\bar{h}^i = \bar{k}^\alpha$ *and* $\bar{k}^j = \bar{k}^\beta$.

(ii) *Let* $x,y \in B$ *such that not both* $x,y \in H$ *(or* K*) and such that* $x = h^i y h^j$ $(k^i y k^j)$. *Then, for all sufficiently large* w *if* $\bar{x} = \bar{h}^\alpha \bar{y} \bar{h}^\beta (\bar{k}^\alpha \bar{y} \bar{k}^\beta)$ *in* B_w, *then* $\bar{h}^i = \bar{h}^\alpha$ *and* $\bar{h}^j = \bar{h}^\beta$ $(\bar{k}^i = \bar{k}^\alpha$ *and* $\bar{k}^j = \bar{k}^\beta)$.

This lemma enables us to show that by choosing insufficiently large $\bar{x} \not\sim_{G_w} \bar{y}$ whenever $x \not\sim_G y$. Since G_w is c.s., it follows that there exists a finite image $\tilde{G}$ of G_w, whence G, such that $\tilde{x} \not\sim_G \tilde{y}$ if $x \not\sim_{\tilde{G}} y$. This proves that G is c.s.

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AN APPLICATION OF FALTINGS' RESULTS TO FERMAT'S LAST THEOREM

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Abstract: We show, as a consequence of Faltings' results, that Fermat's Last Theorem holds for an infinite sequence of relatively prime exponents.

In this note we consider the problem of finding integer solutions to

$$(*) \qquad x^n + y^n = z^n$$

where n is an integer ≥ 3. We say that FLT (Fermat's Last Theorem) holds for n if (*) has no non-zero integer solutions. Also, we say the first case of FLT holds for n if (*) has no solutions in non-zero integers x, y, and z with g.c.d.$(xyz,n_0) = 1$ where n_0 is the odd part of n (ie. the largest odd divisor of n); and we say the second case of FLT holds for n if (*) has no solutions in non-zero integers with g.c.d.$(xyz,n_0) \neq 1$. Maillet [2] proved that for any prime p and for any positive integer r sufficiently large (depending on p) the first case of FLT holds for p^r. Using Faltings' results [1], we prove

Theorem. Given any prime p, there is a number R (depending on p) such that for all integers $r \geq R$, FLT holds for p^r.

For p = 2, Fermat proved this theorem with R = 2. For odd primes p the theorem is a direct consequence of the following lemma by setting n = p and taking R so that $p^{R-1} > M_0(p)$.

Lemma. For any integer $n \geq 3$, there is an $M_0 = M_0(n)$ such that for any integer $m > M_0$, FLT holds for mn.

Proof. We have from Faltings' results that for $n \geq 3$ (*) has finitely many solutions in non-zero relatively prime integers x, y, and z. Call these solutions $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_k, y_k, z_k)$. Let M_0 be the maximal positive integer such that some z_i $(1 \leq i \leq k)$ is an M_0^{th} power (of an integer). If (*) has no non-zero relatively prime solutions, $M_0 = 0$ suffices. Now, we have for all integers $m > M_0$

$$(x^m)^n + (y^m)^n = (z^m)^n$$

has no solutions in non-zero relatively prime integers x, y, and z. Hence, FLT holds for the exponent mn.

The above theorem clearly implies the existence of an infinite sequence of positive relatively prime integers n_i, $i = 1, 2, 3, \ldots$, such that FLT holds for each n_i, $i = 1, 2, 3, \ldots$. We note that the same result for the first case of FLT follows from Maillet's Theorem. Moreover, Maillet's proof is constructive. Finally, we note that a direct and simple proof of this corollary of Maillet's Theorem has been given by Powell [3].

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