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ZERO - FREE REGIONS FOR BERNOULLI POLYNOMIALS

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Presented by P. Ribenboim, F.R.S.C.

Abstract: It is shown that there exists a double parabolic region which is free of zeros of Bernoulli polynomials.

1. Introduction. While the real zeros of Bernoulli polynomials have been studied quite extensively (among others, by D.H. Lehmer and by K. Inkeri), very little is known about the distribution of the complex zeros. To my knowledge, the only published work on this question is by R. Spira and L. Carlitz [6], where it is shown that the critical strip $\{z = \sigma + it \mid 0 \leq \sigma \leq 1,\ t \neq 0\}$ contains no zeros of any Bernoulli polynomials. In a preceding paper [1], the author has announced his result that the zeros of the Bernoulli polynomials $B_n(z)$ lie in the disk $|z - 1/2| \leq (n-2)/2\pi$. This is used to prove

Theorem 1: The double parabolic region defined by

$$x^2 < c|y|\ , \quad z = x + iy, \quad c = 2/33,$$

contains no zeros of any polynomial $B_n(z+1/2)$, for $n \geq 129$.

2. Zero-free strips. It is possible to make a slight improvement on the result of Spira and Carlitz.

Theorem 2: For $n \geq 1$, $B_n(z)$ has no non-real zero in the strip defined by $1 - \alpha \leq \mathrm{Re}(z) \leq \alpha$, where $\alpha = 1.1577$.

Sketch of proof: We follow the idea in [6]. Using a Taylor expansion, we get

$$B_n(a+ib) = i^n \sum_{2j\le n} \binom{n}{2j}(-1)^j B_{2j}(a) b^{n-2j} + \sum_{2j+1\le n} \binom{n}{2j+1} B_{2j+1}(a)(ib)^{n-2j-1}.$$

We denote the left-hand sum by S. In view of Spira and Carlitz's result, and of the functional equation $B_n(1-z) = (-1)^n B_n(z)$, it suffices to regard $B_n(a+ib)$ with $a > 1$, $b > 0$. It can be seen by induction that

$$(-1)^j B_{2j}(a) < 0 \quad \text{for } 1 \le a \le \alpha,\ j \ge 2,$$

where α is the largest (real) zero of $B_4(a)$. Hence for $1 \le a \le \alpha$

$$S < \binom{n}{0} B_0(a) b^n - \binom{n}{2} B_2(a) b^{n-2}$$

$$= b^{n-2}\left[b^2 - \frac{n(n-1)}{2}\left(a^2 - a + \frac{1}{6}\right)\right] \le b^{n-2}\left[b^2 - \frac{n(n-1)}{12}\right].$$

We apply the Kakeya - Eneström theorem [3] to $B_n(z+1/2)$, and find that all zeros of $B_n(z+1/2)$ lie in the disk $|z| \le (n(n-1)/24)^{1/2}$. Hence we may restrict our attention to $b \le (n(n-1)/24)^{1/2}$. So in this case $S < 0$ for $b > 0$, and therefore $B_n(a+ib) \ne 0$.

3. Connections to the sine and cosine functions. Let

$$S_n(z) = \sum_{j=0}^{n} \frac{z^j}{j!}, \quad T_{2k}(z) = \sum_{j=0}^{k} (-1)^j \frac{z^{2j}}{(2j)!},$$

and similarly $T_{2k+1}(z)$ be the sections of the series of the exponential, sine, and cosine functions, respectively.

The Bernoulli polynomials have a Fourier expansion (see e.g.[4])

$$B_{2k}(t) = (-1)^{k-1} \frac{2(2k)!}{(2\pi)^{2k}} \sum_{m=1}^{\infty} \frac{\cos(2\pi m t)}{m^{2k}}, \quad 0 \le t \le 1,\ k \ge 1,$$

with a similar expansion for odd index polynomials. This is used to prove the following

<u>Lemma 1</u>: <u>For all</u> $z \in \mathbb{C}$, $n \geq 2$, <u>we have</u> (with $k = [n/2]$)

$$\left|\frac{(2\pi)^n}{2n!} B_n(z+1/2) - (-1)^k T_n(2\pi z)\right| < 2^{-n-1}[S_n(4\pi|z|) + e^{-4\pi|z|}].$$

If z is bounded then the right-hand side of this tends to 0 as $n \to \infty$. Since the polynomials $T_n(2\pi z)$ converge uniformly on a compact subset of $\mathbb{C}$ to $\sin(2\pi z)$, or to $\cos(2\pi z)$ (according to their parity), Lemma 1 implies the following result which I did not find in the literature.

<u>Theorem 3</u>: <u>The following sequences are uniformly convergent on every compact subset of</u> $\mathbb{C}$.

(a) $$(-1)^k \frac{(2\pi)^{2k+1}}{2(2k+1)!} B_{2k+1}(z+\tfrac{1}{2}) \longrightarrow \sin(2\pi z),$$

(b) $$(-1)^k \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}(z+\tfrac{1}{2}) \longrightarrow \cos(2\pi z).$$

This suggests to study

4. <u>Zero-free regions for the</u> $T_n(z)$. D.J. Newman and T.J. Rivlin [2] have proved that there is a parabolic region defined by $y^2 \leq cx$ ($z = x+iy$, $c = 0.74$) which is free of zeros of the $S_n(z)$. E.B. Saff and R.S. Varga [5] have improved and generalized this result. While their generalization does not cover the sections of the sine and cosine functions, Newman and Rivlin's method can be adapted to these. Using the ideas in [2], we can prove

Lemma 2: For all $z = x + iy$ we have

$$n!\,|T_n(z)| \geq n!\,S_n(y)\frac{1-e^{-2y}}{2} - \frac{1+e^{-2y}}{2}\,x|z|^n - \frac{1}{2}\left(\frac{1}{n+y} + \frac{e^{-2y}}{n-y}\right)|z|^{n+1}. \qquad (1)$$

Furthermore, we need

$$S_n(y) \geq \frac{1}{2}e^y \qquad \text{if} \quad 0 \leq y \leq n, \qquad (2)$$

which was proved in [2]. We can apply (2) since as a consequence of the Kakeya - Eneström theorem [3] we may restrict our attention to $|z| \leq n - 1/2$. Now we get

Theorem 4: The sections $T_n(z)$ of the sine and cosine functions have no zeros in the double parabolic region defined by

$$x^2 < c|y| \quad (z = x + iy), \text{ with } c = 1/2.$$

Sketch of proof: We may restrict our attention to $x \geq 0$, $y \geq 0$. For those numbers z satisfying the inequality in Th.4, we have

$$|z| \leq (y^2 + cy)^{1/2}, \quad x \leq (cy)^{1/2}.$$

We substitute this and (2) into (1), and after some arithmetic, using Stirling's formula for n! and the inequality

$$\left(\frac{y + c/2}{n}\right)^n \leq e^{y+c/2-n},$$

we eventually arrive at

$$n!\,|T_n(z)| \geq e^y\left(\frac{n}{e}\right)^n n^{1/2}\left[\frac{399}{1600}(2\pi)^{1/2} - \left(\frac{401}{800}c^{1/2} + \frac{13}{100}\right)e^{c/2}\right].$$

For $c = 1/2$, the square bracket expression is positive, so $|T_n(z)| > 0$.

Remark: The constant c can be improved if we assume $n \geq n_0$ (sufficiently large). However, c cannot exceed the (unique positive) solution of $\pi = 2ce^c$ (c.f. [2]), which is approximately $c \cong 0.7454$.

5. Proof of Theorem 1 (sketch). Under the hypothesis of Th.1 we have $|z| \leq y + c/2$. Using a Taylor expansion, we get

$$S_n(4\pi|z|) \leq \sum_{j=0}^{n} S_{n-j}(4\pi y)\,\frac{(2\pi c)^j}{j!}. \tag{3}$$

Also

$$2^{-m}S_m(4\pi y) = S_m(2\pi y) - \sum_{\lambda=1}^{m} 2^{-\lambda}S_{m-\lambda}(2\pi y) \quad \text{for } 0 \leq m \leq n. \tag{4}$$

Inequality (2) can be generalized to the case $y \geq n$; with a slight modification in the proof in [2] we find that for $a \geq n$ and $0 \leq y \leq a$ we have

$$S_n(y) \geq (e^{2(a-n)}+1)^{-1}\,e^y. \tag{5}$$

By the main result in [1], we may restrict our attention to $2\pi y \leq n-2$. Hence, using (3) and (4) with (2) and (5), we get

$$2^{-n-1}S_n(4\pi|z|) < \tfrac{1}{2}S_n(2\pi y) - A(c)\,e^{2\pi y},$$

where

$$A(c) = 0.195 - c\pi(\tfrac{1}{8} + \tfrac{\pi c}{8}).$$

With Lemma 1 we now get, if we denote

$$B = \frac{(2\pi)^n}{2n!}\,B_n(z+\tfrac{1}{2}),$$

$$|B| > |T_n(2\pi z)| - [\tfrac{1}{2}S_n(2\pi y) - A(c)\,e^{2\pi y} + 2^{-n-1}\,e^{-4\pi|z|}].$$

Now we apply Lemma 2; the term $\frac{1}{2}S_n(2\pi y)$ disappears, and four very small terms can be incorporated into A(c). Hence

$$|B| > A(c)\,e^{2\pi y} - \frac{\pi x}{n!}(2\pi|z|)^n - \frac{1}{2n!}\,\frac{(2\pi|z|)^{n+1}}{n+2\pi y}.$$

We proceed essentially as in the proof of Theorem 4, taking into account that by Theorem 2 we may restrict our attention to $2\pi y > 26.5$. Finally,

$$|B|n! > e^{2\pi y} n^{1/2} (\frac{n}{e})^n [\, A(c)(2\pi)^{1/2} - ((\frac{\pi c}{2})^{1/2} + \frac{(n-1)^{1/2}}{2n + 53}]\, e^{\pi c} \;.$$

For $c = 2/33$ and $n \geq 129$, the right-hand side is positive.

This completes the proof of Theorem 1.

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SPECIALISATIONS DE POLYNOMES

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RESUME. Ce travail, inspiré par le théorème d'irréductibilité de Hilbert, étudie la structure arithmétique des polynômes spécialisés P(x,Y) où P est un polynôme irréductible à deux indéterminées X,Y.

INTRODUCTION. Soit P un polynôme irréductible dans Q [X,Y] ; d'après le théorème d'irréductibilité de Hilbert (1892) [4] l'ensemble des nombres rationnels x tels que P(x,Y) est irréductible dans Q [Y] est un ensemble infini.
En 1929, dans son mémoire sur les E-fonctions [6], C.L. Siegel donne également, sans démonstration, plusieurs résultats sur les valeurs de certaines G-fonctions. Bien qu'ils excluent le cas particulier des fonctions algébriques, qui est celui qui nous intéresse ici, ces résultats, en ouvrant une nouvelle voie d'approche du problème, ont eu une influence importante sur le développement du théorème de Hilbert. Cinquante ans plus tard, P. Bundschuh [3], généralisant des travaux de T. Schneider [5], et V.G. Sprindzuk [7],[8],[9] aborderont le cas exclu par Siegel en utilisant des méthodes analogues aux siennes. E. Bombieri, peu après [2], élargira le cadre de ces derniers résultats : il démontre en effet en 1980, un énoncé général sur les valeurs de G-fonctions, comme Siegel l'avait fait pour les E-fonctions.
Les théorèmes que nous montrons ici contiennent tous les résultats précédents relatifs aux valeurs de fonctions algébriques ; leur démonstration repose sur un développement de la méthode de Gel'fond ; ils généralisent le théorème d'irréductibilité de Hilbert en précisant à quelles conditions (effectives), P(x,Y) est divisible par un polynôme de degré donné.

NOTATIONS.
Les valeurs absolues v d'un corps de nombres F sont normalisées de telle façon que :

si v/p (c'est-à-dire si v prolonge la métrique p-adique sur Q, p étant un nombre premier)

$$|p|_v = p^{-1}$$

si v/∞ (c'est-à-dire si v est archimédienne)

$$|x|_v = |x| \quad \text{pour tout nombre rationnel } x$$

(| | désignant la valeur absolue usuelle sur $\mathbb{Q}$).

Si v est une place de F, nous noterons F_v le complété de F pour la métrique v et d_v^F le degré local de la place v par rapport à $\mathbb{Q}$ défini par :

$$d_v^F = [F_v : \mathbb{Q}_v] \ .$$

Si x est un nombre algébrique non nul, h(x) désigne la hauteur logarithmique absolue de x définie de la manière suivante : si F est un corps de nombres auquel x appartient

$$h(x) = \frac{1}{[F : \mathbb{Q}]} \sum_v d_v^F \operatorname{Log} \max (1, |x|_v) \quad ,$$

la sommation étant étendue à toutes les places de F.

§ 1 - UN PREMIER RESULTAT AVEC UNE SEULE PLACE

Soient k un corps de nombres, ξ_o un élément de k, et P un polynôme irréductible dans $k[X,Y]$. On fait sur P l'hypothèse notée (H)

(H) - Il existe une série formelle $\mathbf{Y} = \sum_{m \geq o} \eta_m (X-\xi_o)^m$ à coefficients $\eta_m, m \geq o$ algébriques vérifiant :

$$P(X,\mathbf{Y}) = 0 \ -$$

On note K le corps $k((\eta_m)_{m \geq o})$; il est facile de voir que K est un corps de nombres. Sous ces hypothèses, on démontre le résultat suivant :

THEOREME 1 - Il existe deux constantes a,b (effectives), strictement positives et ne dépendant que de P, ξ_o telles que :
si ξ est un nombre algébrique différent de ξ_o, d un entier positif et v une place du corps $K(\xi)$ et si :

$$|\xi-\xi_o|_v^{d_v^{K(\xi)}/[K(\xi):\mathbb{Q}]} < a \exp\left\{- \frac{d\,[k(\xi):k]}{\deg_Y P} h(\xi-\xi_o) - b\sqrt{h(\xi-\xi_o)}\right\} \quad ,$$

alors $P(\xi,Y)$ est divisible dans $k(\xi)[Y]$ par un polynôme irréductible dans $k(\xi)[Y]$ de degré strictement supérieur à d.

A cause de l'inégalité de Liouville, l'hypothèse de l'énoncé ne peut être réalisée que si d<q. Le cas d=q-1 est intéressant puisque la conclusion du théorème est dans ce cas : $P(\xi,Y)$ est irréductible dans $k(\xi)[Y]$.

Le théorème 1 contient simultanément deux résultats de P.Bundschuh ([3] THEOREME 1) et de V.G. Sprindzuk ([7] THEOREME 1). Dans l'énoncé de P. Bundschuh v est archimédienne et k=Q ; l'énoncé de V.G. Sprindzuk correspond au cas particulier du théorème 1 où k=K=Q, P est absolument irréductible (c'est-à-dire irréductible sur la clôture algébrique de Q), ξ est un nombre rationnel, v une place finie de Q et d=q-1. En outre, dans ces deux énoncés, on fait sur P l'hypothèse notée (H'):

(H') - Le polynôme $P(\xi_o,Y)$ admet une racine simple η_o -

Et d'après un lemme classique, si P vérifie (H'), alors P vérifie (H) avec $K=k(\eta_o)$. Le théorème 1 est un corollaire du théorème 2, qui est l'objet du paragraphe 2. Son énoncé fait intervenir simultanément plusieurs places, tenant compte ainsi à la fois des points de vue archimédiens et p-adiques.

§ 2 - LE THEOREME PRINCIPAL

Soient k, ξ_o, P, $\mathbf{Y}$, K comme dans le paragraphe 1. Pour toute place v de K, soit R_v le rayon de convergence de $\mathbf{Y}$ pour la métrique v ; d'après des théorèmes classiques d'analyse, R_v est strictement positif. La série formelle $\mathbf{Y}$ induit donc sur la boule ouverte $B(\xi_o,R_v)=\{x\in K_v / |x-\xi_o|_v<R_v\}$ une fonction Y_v, strictement analytique [1] sur toute boule fermée de $B(\xi_o,R_v)$ vérifiant :

$$\text{pour tout } x \text{ dans } B(\xi_o,R_v) \quad , \quad P(x, Y_v(x)) = 0 \ .$$

Le théorème suivant est le principal résultat de cette note.

<u>THEOREME 2</u> - <u>Il existe deux constantes (effectives)</u> A,B <u>strictement positives, ne dépendant que de</u> P <u>et de</u> ξ_o <u>ayant la propriété suivante</u> :
<u>si</u> ξ <u>est un élément de</u> k, <u>différent de</u> ξ_o, Q <u>un polynôme dans</u> k [Y] <u>divisant</u> $P(\xi,Y)$ <u>dans</u> k [Y] <u>et</u> $S(\xi_o,\xi,Q)$ <u>l'ensemble des places de</u> K <u>vérifiant</u> :

$$|\xi-\xi_o|_v < R_v \quad \text{et} \quad Q(Y_v(\xi)) = 0 \ ,$$

<u>alors</u>

$$\left| \frac{1}{[K:Q]} \sum_{v\in S(\xi_o,\xi,Q)} d_v^K \ \mathrm{Log}\min(1,|\xi-\xi_o|_v) + \frac{\deg Q}{\deg_Y P}\, h(\xi-\xi_o) \right| < A+B\sqrt{h(\xi-\xi_o)} \ .$$

V.G. Sprindzuk a démontré ce résultat en 1982 [9] dans le cas où P est absolument irréductible et vérifie l'hypothèse (H') avec $\eta_o\in k$ (et donc K=k)(En fait, l'hypothèse d'irréductibilité absolue est ici superflue puisqu'elle résulte de (H) si K=k). D'autre part, les constantes A et B dépendent dans son énoncé également du degré de k sur Q. Dans son article sur les G-fonctions [2], E. Bombieri montre que les relations k-linéairement indépendantes liant les valeurs en un point

ξ de k, de G-fonctions à coefficients dans k, linéairement indépendantes sur k(X) ne sont pas trop nombreuses. Or si P vérifie l'hypothèse (H) avec de plus k=K, les séries $1, Y, \ldots, Y^{(\deg_Y P-1)}$ sont des G-fonctions à coefficients dans k linéairement indépendantes sur k(X). L'application du théorème de Bombieri à ce cas particulier permet d'obtenir le théorème 1 dans le cas k=K.

§ 3 - APPLICATIONS

Dans ce paragraphe, nous supposons, outre les hypothèses du théorème 2, que $\xi_o = 0$. Le corollaire suivant, montre, sous certaines conditions, l'irréductibilité des polynômes spécialisés $P(\xi^m, Y)$ où m est un entier suffisamment grand.

COROLLAIRE - Soit ξ un élément de k non nul, de hauteur $h(\xi)$ non nulle (i.e ξ n'est pas une racine de l'unité) et vérifiant la propriété suivante :
pour tout ensemble S non vide et strictement inclus dans l'ensemble des places v de K telles que $|\xi|_v < 1$, la quantité $(\sum_{v\in S} d_v^K \operatorname{Log}|\xi|_v)/h(\xi)$ n'appartient pas à $\mathbb{Q}$.
Alors il existe un entier (effectif) m_o ne dépendant que de P,k et de ξ tel que :
pour tout entier m, si $m \geq m_o$ alors $P(\xi^m, Y)$ est irréductible dans $k[Y]$.

L'hypothèse faite sur ξ dans le corollaire est satisfaite s'il existe une place v_o telle que le nombre réel $|\xi|_{v_o}$ soit strictement inférieur à 1 et n'appartienne pas au groupe multiplicatif engendré par les $|\xi|_v$ tels que $|\xi|_v < 1$ et $v \neq v_o$.

En particulier le corollaire s'applique dans les cas suivants :

a) La famille des $|\xi|_v$, où v décrit l'ensemble des places de K telles que $|\xi|_v < 1$ est non vide et multiplicativement libre ([9] pour le cas K = k)

b) L'ensemble des places v de K telles que $|\xi|_v < 1$ a exactement un élément (par exemple si $\xi = p^s$ où p est un nombre premier, s un entier plus grand que 1 et K = $\mathbb{Q}$ ou encore si $\xi = \frac{1}{s}$ où s est un entier non nul et K est inclus dans un corps quadratique imaginaire). Dans ce cas, on a un résultat plus précis : il existe une constante h_o ne dépendant que de P telle que : si $h(\xi) \geq h_o$ alors $P(\xi,Y)$ est irréductible sur k.

c) $k = \mathbb{Q}$ et il existe un nombre premier p tel que $|\xi|_p < 1$ et que l'idéal engendré par p dans l'anneau des entiers de K est une puissance d'un idéal premier.

d) $k = \mathbb{Q}$, K est inclus dans un corps quadratique imaginaire et $|\xi| < 1$.

e) $k = K = \mathbb{Q}$ (Conséquence de c) et d), voir aussi [8]).

L'exemple du polynôme $P = Y^2 - X$ montre que l'hypothèse (H) n'est pas superflue dans l'énoncé du corollaire et donc dans celui du théorème 2.

§ 4 COMPLEMENT AU THEOREME 2

Nous donnons ici un énoncé légèrement plus faible que celui du théorème 2, mais d'où ont disparu les fonctions algébriques.

Soient k un corps de nombres, ξ_o un élément de k, P un polynôme irréductible dans $k[X,Y]$. On suppose dans ce paragraphe qu'il existe un nombre algébrique η_o tel que

$$P(\xi_o,\eta_o) = 0 \qquad P'_Y(\xi_o,\eta_o) \neq 0 .$$

On note K le corps $k(\eta_o)$ et P_{ξ_o,η_o} le polynôme de $K[X,Y]$ défini par

$$P_{\xi_o,\eta_o}(X-\xi_o,\, Y-\eta_o) = P(X,Y) .$$

On a alors :

$$P_{\xi_o,\eta_o} = \sum_{\substack{o \leq i \leq \deg_X P \\ o \leq j \leq \deg_Y P}} C_{ij}\, X^i\, Y^j \qquad \text{avec} \quad C_{00} = 0 \quad \text{et} \quad C_{01} \neq 0 .$$

Soit ε un nombre réel strictement compris entre 0 et le minimum des $|C_{01}|_v$ où v décrit l'ensemble des places archimédiennes de K ; pour toute place v de K, on note $\beta_v(\varepsilon)$ la quantité définie par :

$$\begin{cases} \beta_v(\varepsilon) = \min\left(1, \dfrac{|C_{01}|_v}{\operatorname{Max}_{2\leq j\leq \deg_Y} |C_{oj}|_v}\right) & \text{si v est finie} \\[2ex] \beta_v(\varepsilon) = \min\left(1, \dfrac{|C_{01}|_v - \varepsilon}{\deg_Y P(\deg_Y P-1)\operatorname{Max}_{2\leq j\leq \deg_Y P} |C_{oj}|_v}\right) & \text{si v est infinie.} \end{cases}$$

Enfin nous supposons toute place v de K prolongée à la clôture algébrique de Q. Alors sous ces hypothèses et notations on a le résultat suivant :

THEOREME 3 - Il existe deux constantes (effectives) A', B' strictement positives, ne dépendant respectivement que de ξ_o, P,ε et ξ_o,P ayant la propriété suivante. Soient ξ un élément de k différent de ξ_o,η un nombre algébrique tels que $P(\xi,\eta) = 0$; si $T(\xi_o,\xi,\eta)$ est l'ensemble des places v de K vérifiant :

il existe un conjugué η_v de η sur k tel que :

$$|\eta_v - \eta_o|_v < \beta_v(\varepsilon) ,$$

alors

$$\left|\frac{1}{[K:Q]}\sum_{v\in T(\xi_o,\xi,\eta)} d_v^K \operatorname{Log}\min(1,|\xi-\xi_o|_v)+\frac{[k(\eta):k]}{\deg_Y P}h(\xi-\xi_o)\right| < A'+B'\sqrt{h(\xi-\xi_o)} .$$

On a $\beta_v(\varepsilon) = 1$ sauf pour un nombre fini de places de K ; mais en général le résultat devient faux si l'on remplace tous les $\beta_v(\varepsilon)$ par 1 dans l'énoncé du théorème 3 (Considérer le polynôme $P=X^2+Y^2+2Y$ et la famille de points $P_h = (\xi_h,\eta_h)$ où $h>0$ et $\xi_h = \frac{2^{h+1}3^h}{2^{2h}+3^{2h}}$ $\eta_h = -\frac{2\, 3^{2h}}{3^{2h}+2^{2h}}$)

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THE HILBERT FUNCTION OF SOME UNIONS OF LINES

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Presented by P. Ribenboim, F.R.S.C.

In our study of seminormality of unions of planes [1], B. Dayton and I, without explicitly realizing it, were working with the Hilbert function of the co-ordinate ring A of a union of planes passing through the origin in $\mathbb{A}_k^r$ (which is the same as the homogeneous co-ordinate ring of a union X of lines in $\mathbb{P}_k^{r-1}$). The computation of the Hilbert funtion of such a ring is in general very difficult, as is shown by the case of skew lines [3]. This note shows that when there are many intersection points the computation can be much simpler. The groundfield k is arbitrary, and all lines and points are k-rational.

First let me describe the algebraic situation in more detail. Suppose there are s lines. Then the normalization of A is $\bar{A} = \Pi_{i=1}^{s} k[t_i,u_i]$. The rings A and $\bar{A}$ are graded, and the inclusion $A \to \bar{A}$ preserves degrees. The induced maps $A \to k[t_i,u_i]$ are surjections, with kernel I_i. If $\ell_i \cap \ell_j = \phi$ then $A/(I_i + I_j) \cong k$. If $\ell_i \cap \ell_j \neq \phi$ then $A/(I_i+I_j) \cong k[t]$. (ℓ_i the line corresponding to I_i.) Let $C = \Pi_{i=1}^{s} C_i$ be the conductor of A in $\bar{A}$. The ideals $\sqrt{C_i}$ (the radical in $k[t_i,u_i]$) can be read directly from the geometry - namely $\sqrt{C_i}$ is the homogeneous ideal in $k[t_i,u_i]$ of $\ell_i \cap (\bigcup_{j\neq i} \ell_j)$. Explicitly $\sqrt{C_i} = (t_i,u_i)$ if $\ell_i \cap (\bigcup_{j\neq i} \ell_j) = \phi$. If $\ell_i \cap (\bigcup_{j\neq i} \ell_j)$ contains d points ($d > 0$) then $\sqrt{C_i} = (\Pi_{j=1}^{d}(b_j t_i - a_j u_i))$ where (a_j,b_j) are the homogeneous co-ordinates in ℓ_i of these d points. Hence (if $d > 0$) C_i is radical if and only if there exists a form of degree d vanishing on all lines except ℓ_i. It is often easy to give such a form explicitly. The proof of Theorem 5 below, and some of the examples in [1], show ways of doing this. (Lemma 2.5 of [1] gives an elementary proof of these facts.)

The Hilbert function of $A = \oplus_{i\geq 0} A_i$ is $H_A(i) = \dim_k A_i$ $i \geq 0$. The entire sequence will be abbreviated $H(A)$. Sometimes $H(A)$ will be referred to as the Hilbert function of $X = \mathrm{Proj}(A)$. The main computational tool is:

Theorem 1: Let $A^{(s+1)}$ be the homogeneous co-ordinate ring of $s+1$ lines in $\mathbb{P}^{r-1}$, and let $A^{(s)}$ be the homogeneous co-ordinate ring of some s of these lines. Let $I = \ker(A^{(s+1)} \twoheadrightarrow A^{(s)})$. Then I is isomorphic to C_{s+1}, the $(s+1)^{st}$ component of the conductor of $A^{(s+1)}$ in $\overline{A^{(s+1)}} = \Pi_{i=1}^{s+1} k[t_i,u_i]$.

Proof: By [2], Lemma 3.2, $C_{s+1} = \{\alpha \in k[t_{s+1},u_{s+1}] \mid (0,0,\ldots,0,\alpha) \in A^{(s+1)}\}$. But the latter is the definition of I. This completes the proof of Theorem 1.

Thus $H(A^{(s+1)}) = H(A^{s}) + H(I)$. This Theorem is most useful for computing when C_{s+1} is radical in $k[t_{s+1},u_{s+1}]$. Then $H(I)$ is the sequence $0\ 2\ 3\ 4\ 5 \rightarrow$ if ℓ_{s+1} intersects no other line, or the sequence $1\ 2\ 3\ 4 \rightarrow$ shifted d units to the right if ℓ_{s+1} intersects d other lines $(d > 0)$.

Theorem 2: For any set of n skew lines in $\mathbb{P}^{r-1}$ $(r \geq 4)$ with homogeneous co-ordinate ring A, we have

(a) $H_A(i) \geq (i+1)^2$ if $i \leq n-1$

and (b) $H_A(i) = (i+1)n$ if $i \geq n-1$.

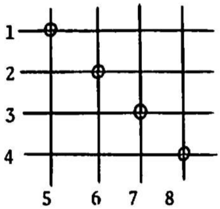
Equality occurs in (a) if and only if we have n lines from one ruling system of a non-singular quadric surface in $\mathbb{P}^3$.

Proof: Parts (a) and (b) are a restatement of Lemma 5 of [1]. Suppose the lines lie in a quadric surface in $\mathbb{P}^3$. Let B be the homogeneous co-ordinate ring of this quadric surface. Then we have a surjection $B \to A$ so $H_A(i) \leq H_B(i) = (i+1)^2$ which proves the reverse inequality in (a). If equality holds in (a) then $H_A(1) = 4$ and $H_A(2) = 9$ so the lines lie in a quadric in $\mathbb{P}^3_k$. It follows from the computation in Theorem 3 below that the lines are skew.

Theorem 3: *Let* $H_{n,m}(i)$ *be the Hilbert function of* n *lines from one ruling system in a non-singular quadric surface* Q *in* $\mathbb{P}^3$, *and* m *lines from the other system* $(0 \leq m \leq n)$. *Then* $H_{n,m}(i) = (i+1)^2$ *if* $i < n$ *and* $H_{n,m}(i) = (i+1)n + m(i-n+1)$ *if* $i \geq n$.

Proof: The Hilbert function of the n skew lines is given by Theorem 2. Now apply m times the method of computation described after Theorem 1 (the appropriate component of the conductor is radical). This completes the proof.

A double 4 is a set of lines with the following configuration:



(where the o denotes non-intersection). If the double 4 lies in $\mathbb{P}^3_k$, let Q_{ijk} be a form defining the unique quadric containing ℓ_i, ℓ_j, ℓ_k, and for $\ell_i \cap \ell_j \neq \phi$, H_{ij} will be the equation of the hyperplane spanned by $\ell_i \cup \ell_j$.

Theorem 4: Let X consist of all lines except ℓ_4 of a double 4 in $\mathbb{P}^3$. Then X has Hilbert function 1 4 10 19 26 → (continuing with differences 7). In particular X lies in a unique cubic hypersurface.

Proof: The skew lines 5 6 7 8 cannot lie in a quadric (otherwise lines 1 and 5 would intersect). By Theorem 2 they must have Hilbert function 1 4 10 16 20 → (differences 4). Now add lines 1, 2, 3 in order. $Q_{567}H_8$ is a form of degree 3 vanishing on lines 5, 6, 7, 8 but not 1. (H_8 is the equation of any hyperplane containing ℓ_8, but not ℓ_1), $Q_{567}H_{18}$ is a form of degree 3 vanishing on lines 5, 6, 7, 8 and 1 but not 2. $Q_{678}H_{25}$ is a form of degree 3 vanishing on lines 5, 6, 7, 8, 1 and 2 but not 3. Thus the method explained after Theorem 1 can be used (with radical conductors), yielding the desired Hilbert function.

Theorem 5: A double 4 configuration in $\mathbb{P}^3$ has Hilbert function 1 4 10 20 28 → (continuing with differences 8) if it does not lie in a cubic, and 1 4 10 19 28 36 → (continuing with differences 8) if it does. Given all lines except ℓ_4, there is at most one way to choose ℓ_4 so that the configuration lies in a cubic.

Proof: If the unique cubic $F = 0$ containing lines 1 2 3 5 6 7 8 (Theorem 4) does not contain ℓ_4 then F is a form of degree 3 vanishing on all but ℓ_4. Theorems 1 and 4 now yield the first Hilbert function. Now assume that ℓ_4 lies in the cubic $F = 0$. Then $f_1 = Q_{568}H_{17}H_{27}$ and $f_2 = Q_{678}H_{25}H_{35}$ are two forms of degree 4 vanishing on all lines except ℓ_4. When restricted to ℓ_4 they are linearly independent since f_1 has a double root at $\ell_4 \cap \ell_7$ and f_2 has a double root at $\ell_4 \cap \ell_5$. But $\sqrt{C_4}$ has Hilbert function 0 0 0 1 2 3 ... , so C_4 has Hilbert function 0 0 0 0 2 3 ... . Theorems 1 and 4 now yield the second Hilbert function.

Now let $F = 0$ be the unique cubic containing all lines except ℓ_4. This surface intersects Q_{567} in a curve of bidegree 3,3. The intersection contains 3 lines (namely ℓ_5, ℓ_6, ℓ_7) from one ruling system. Hence it must contain 3 lines (possibly repeated) from the other. Two of these will intersect ℓ_8. Thus there is at most one line lying in the cubic that completes the double 4 configuration.

Theorem 6: If a double 5 lies in a cubic it has Hilbert function 1 4 10 19 30 40 → (differences 10). If a double 5 does not lie in a cubic it has Hilbert function 1 4 10 20 30 → (differences 10).

Proof: (For a picture of the double 5 configuration refer to [1] p.110). The double 5 lies in a cubic if and only if any sub-double 4 lies in a cubic. Add the remaining 2 lines. The appropriate conductor components are easily seen to be radical so Theorems 1 and 5 yield the desired Hilbert function.

Remark: In an earlier version of this work [4, §6] I had to assume that the cubic occuring in Theorems 5 and 6 was non-singular. This assumption is no longer necessary.

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DIAMONDS ARE NOT THE CAUCHY EXTENSIONISTS' BEST FRIEND

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Abstract. Arguments are presented concerning the advantages of hexagons/triangles $H=\{(x,y)\,|\,x,y,x+y\in I\}$ (I an interval) over diamonds $D=\{(x,y)\,|\,|x|+|y|<r\}$ as base domains from which the Cauchy equation is to be extended. The cases where 0 is in the interior, on the boundary or in the exterior of I are separately considered. Examples are given where (in the last case) H is empty while I is nonempty (of positive length) or H is nonempty but there exists no extension from H.

1. In [5] the process of extension (later of quasi-extension) of the Cauchy equation began with the diamond

$$D = \{(x,y) \mid |x|+|y| < r\}, \tag{1}$$

that is, it is proved that, if the Cauchy equation

$$f(x+y) = f(x)+f(y)$$

is satisfied by $f\colon]-r,r[\to \mathbb{R}$ for all $(x,y)\in D$, then there exists a (unique) $g\colon \mathbb{R}\to\mathbb{R}$ such that

$$g(x) = f(x) \textit{ for all } x\in]-r,r[,$$

but

$$g(x+y) = g(x)+g(y) \textit{ for all } (x,y)\in \mathbb{R}^2.$$

In [5] the next step was to prove a similar theorem starting with the circular disk

$$C = \{(x,y) \mid x^2+y^2<r^2\}, \tag{2}$$

because the authors seem to have thought this necessary in order to advance to arbitrary open (and/or connected) sets, by using (2) as neighbourhood. But, of course, (1) can just as well serve as neighbourhood (of 0) as (2), so later the diamonds (1) were considered to be the extensionists' best friend.

The (small) point I wish to make is that the hexagon

$$H' = \{(x,y) \mid x,y,x+y\in]-r,r[\} \tag{3}$$

or, more generally,

$$H = \{(x,y) \mid x,y,x+y\in I\},\ I \text{ a proper real interval,} \tag{4}$$

in particular, if

$$0 \text{ is } \textit{in the interior} \text{ of } I, \tag{5}$$

and/or

$$I \text{ open,} \tag{6}$$

are more appropriate for the purpose.

2. My first argument is that (3) [or (4) with (5) and (6)] are also neighbourhoods of 0. The second is that an extension theorem for (4) [even without (6) and with a weakened condition (5)] is just as easy (or easier) to prove (cf. [1]).

THEOREM. If $f: I \to \mathbb{R}$ satisfies the Cauchy equation

$$f(x+y) = f(x)+f(y) \text{ for all } (x,y)\in H, \tag{7}$$

where H is given by (4) and I satisfies (5), then there exists a (unique) $g: \mathbb{R}\to\mathbb{R}$ such that

$$g(x) = f(x) \text{ for } x\in I \tag{8}$$

but

$$g(u+v) = g(u)+g(v) \text{ for all } (u,v)\in \mathbb{R}^2. \tag{9}$$

Proof. From (7) and (4) we have $f(nx)=nf(x)$, if $nx\in I$, that is,

$$f\left(\frac{y}{n}\right) = \frac{f(y)}{n} \text{ if } y\in I. \tag{10}$$

On the other hand, for every $u\in\mathbb{R}$ there exists an $n\in\mathbb{N}$ such that $x=u/n\in I$. We *define*

$$g(u) = nf(x) = nf\left(\frac{u}{n}\right) \qquad \left(x = \frac{u}{n} \in I\right). \tag{11}$$

We first show that this definition is *unambiguous*. Indeed, if $u=nx=mz$ $(x,z\in I;\ m,n\in\mathbb{N})$ then $x/m=z/n$ and, by (10),

$$\frac{f(x)}{m} = f\left(\frac{x}{m}\right) = f\left(\frac{z}{n}\right) = \frac{f(z)}{n},$$

that is, $nf(x)=mf(z)$, as asserted. - The definition (11) gives (8) (for $n=1$).

We prove (9) by choosing, for given $u,v\in\mathbb{R}$, the positive integer n so large that u/n, v/n and $(u+v)/n$ are all in I. By (11) and (7),

$$g(u+v) = nf\left(\frac{u}{n}+\frac{v}{n}\right) = nf\left(\frac{u}{n}\right)+nf\left(\frac{v}{n}\right) = g(u)+g(v).$$

So *the extension* indeed *exists*, as asserted. - Finally, we show that the extension g is *unique*. Indeed, if both g and h satisfy (8) and (9), then (9) implies

$$g(nt) = ng(t), \quad h(nt) = nh(t) \quad \text{for all } t\in\mathbf{R}, n\in\mathbf{N}. \tag{12}$$

Choosing, for any given $u\in\mathbf{R}$, again an n such that $u/n\in I$, (12) and (8) (valid also for h) give

$$g(u) = ng(\frac{u}{n}) = nf(\frac{u}{n}) = nh(\frac{u}{n}) = h(u). \qquad \square$$

It was not supposed in the theorem or proof that the interval is finite (if it is not, the hexagon degenerates into an infinite triangular region or into $\mathbf{R}^2$). - The above proof shows that the Theorem can be generalized in obvious ways to algebraic and topological structures.

3. The third argument concerns a generalization of (5), which allows 0 to be *on the boundary* of I (whether 0 belongs to I or not). The simplest way to solve (7) in this situation is to reduce it to the above Theorem, where 0 is in the interior of a new interval

$$\bar{I} = \{x-y \mid x,y\in I\}. \tag{13}$$

This can be done in the following way (cf. [2] for a similar, but not identical, situation). First, one may define

$$\bar{f}(z) = \begin{cases} f(z) & \text{if } z\in I \\ -f(-z) & \text{if } z\in -I. \\ 0 & \text{if } z=0 \end{cases} \tag{14}$$

If $0\in I$, then (7) with $y=0$ gives $f(0)=0$ anyway, so this definition is unambiguous. We see that

$$\bar{f}(x) = f(x) \text{ if } x\in I \tag{15}$$

and

$$\bar{f}(-z) = -\bar{f}(z) \text{ for all } z\in\bar{I}. \tag{16}$$

We will prove that

$$\bar{f}(s+t) = \bar{f}(s)+\bar{f}(t) \text{ on } \bar{H} = \{(s,t) \mid s,t,s+t\in\bar{I}\}. \tag{17}$$

By (7) and (15) this is true if $s,t,s+t\in I$ and, by (16), for $t=-s$. It trivially holds if $s=0$ or $t=0$. By (16), if (17) is true for (s,t), it holds also for $(-s,-t)$ (notice that, by the definitions (13) and (17) of $\bar{I}$ and $\bar{H}$, $(s,t)\in\bar{H} \Rightarrow (-s,-t)\in\bar{H}$). Because of the symmetry of (17) in s and t, just one case of (17) remains to be proved: $s\in I$, $t\in -I$ and $z=s+t\in I$. Then $-t\in I$ and $(z,-t)\in H$ $(z+(-t)=s\in I)$. So, by (7) and (14),

$$\bar{f}(s) = f[z+(-t)] = f(z)+f(-t) = \bar{f}(s+t)-\bar{f}(t)$$

that is, as asserted,

$$\bar{f}(s+t) = \bar{f}(s)+\bar{f}(t).$$

Also, this $\bar{f}$ is unique. Indeed, if there would also be an f' satisfying (15) and (17), then we had $f'(0)=0$, $f'(-z) = -f'(z)$, so

$$f'(z) = \left\{\begin{matrix} f(z) & (z\in I) \\ -f(-z) & (z\in -I) \\ 0 & (z=0) \end{matrix}\right\} = \bar{f}(z).$$

We can now apply the Theorem to $\bar{I}$, $\bar{H}$ and $\bar{f}$ in order to get the following (cf. also [1]).

COROLLARY. If I is a proper real interval which has 0 *on its boundary, if H is defined by (4), and if f satisfies the Cauchy equation on H, then there exists a unique $g:\mathbf{R}\to\mathbf{R}$ which satisfies (8) and (9).* □

In this case H degenerates to a *triangle*. Maybe this is what gave the idea of considering diamonds in the case where $\bar{I}=]-r,r[$. But we have just seen that the natural first extension $\bar{f}$ from this triangle goes to a hexagon.

4. As mentioned above, also the hexagon (4) with (5) and (6) can be considered a neighbourhood of 0 and so the further arguments in [5] concerning extensions and quasi-extensions from open and/or connected sets can be made just as well for hexagons. A *quasi-extension* from a set $S\subset\mathbf{R}^2$ of the Cauchy equation

$$f(x+y) = f(x)+f(y) \quad (x,y)\in S$$

to $\mathbf{R}^2$ is a function g satisfying (9) and

$$\left.\begin{matrix} g(x)-g(x_0)=f(x)-f(x_0) & \text{for all } x\in S_1=\{x\,|\,\exists y:\ (x,y)\in S\} \\ g(y)-g(y_0)=f(y)-f(y_0) & \text{for all } y\in S_2=\{y\,|\,\exists x:\ (x,y)\in S\} \\ g(z)-g(x_0+y_0)=f(z)-f(x_0+y_0) & \text{for all } z\in S_3=\{x+y\ |\ (x,y)\in S\} \end{matrix}\right\} \quad (18)$$

for one (and therefore every) point $(x_0,y_0)\in S$. The triple (18) of equations is clearly a generalization of (8). This generalization is necessary since, even for simple sets S, no extension may exist. This can happen to hexagons (actually triangles) H, as defined in (4) *if* $0<\alpha<\beta<3\alpha$ *or* $0>\alpha>\beta>3\alpha$, *where* α *and* β *are the endpoints of* I. Actually, H has an empty interior if $0<\alpha<\beta\leq 2\alpha$ or $0>\alpha>\beta\geq 2\alpha$, but its interior is nonempty if $2\alpha<\beta<3\alpha$ (for $0<\alpha<\beta$) or $2\alpha>\beta>3\alpha$ (for $0>\alpha>\beta$) and still there are solutions of (7) which have no extensions. Take, for instance, $I=]2,5[$, thus $S=H=\{x,y\ |\ x,y,x+y\in]2,5[\}$, $S_1=S_2=]2,3[$, $S_3=]4,5[$. The function f: $(]2,3[\cup]4,5[)\to\mathbf{R}^2$ given by

$$f(x) = \begin{cases} x+1 & (x\in]2,3[\,) \\ x+2 & (x\in]4,5[\,) \end{cases} \tag{19}$$

is a solution of (7) on H which has *no* extension $g: \mathbf{R}\to\mathbf{R}$ ($g(x)=f(x)$ if $x\in]2,3[\cup]4,5[$) satisfy (9). Indeed, this f is bounded on $]2,3[$, so g would have to be bounded too. But (see e.g. [4]) all solutions of (9), bounded on an interval, are of the form $g(t) = ct$ (c a constant) and none of these is an extension of (19). But the identity function ($g(x)=x$ for all $x\in\mathbf{R}$) is a quasi-extension.

Generally, it has been shown in [5] that there exists a unique extension of the Cauchy equation from a proper region to $\mathbf{R}^2$. A proper region is defined here as a union of a nonempty open connected set with a (not necessarily proper) subset of its boundary points. Actually, in [5] no boundary points were considered, but they can be easily added. So, *whenever H, as defined by (4), has a nonempty interior, there exists a unique quasi-extension of the Cauchy equation from H to* $\mathbf{R}^2$. *If* $\alpha\le 0\le\beta$ *($\infty\ge\beta>\alpha\ge-\infty$) or* $0<3\alpha<\beta$ *or* $0>3\alpha>\beta$ ($\beta = \pm\infty$ possible), *then this quasi-extension is an extension (a unique extension).*

A final argument in favour of hexagons is that they are affectionately known and well applied in the *theory of webs* (see e.g. [3]).

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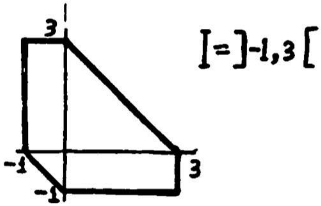

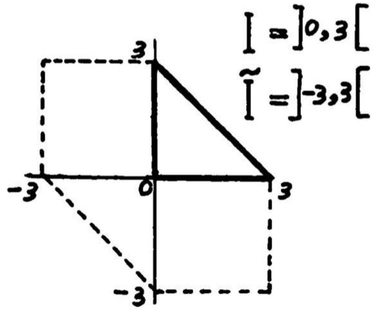

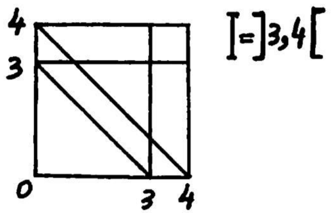

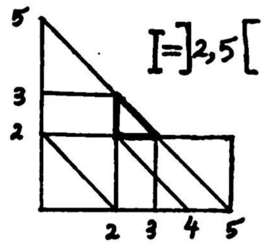

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ON INTEGRABLE SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

Karol Baron

Presented by J. Aczél, F.R.S.C.

The following problem arose in a research of A.Boyarsky (a personal communication). For which r from the interval $[3,4]$ does there exist an integrable function $f: [0,1] \to [0,+\infty)$ such that

$$f\left(\tfrac{1}{2}+x\right) + f\left(\tfrac{1}{2}-x\right) = 2rxf\left(r\left(\tfrac{1}{2}+x\right)\left(\tfrac{1}{2}-x\right)\right) \quad \left(\forall x \in [0,\tfrac{1}{2}]\right), \tag{1}$$

$$\int_0^1 f(x)dx = 1 \tag{2}$$

and

$$\operatorname{supp} f \subset \left[\frac{r^2}{4}\left(1-\frac{r}{4}\right), \frac{r}{4}\right]. \tag{3}$$

We are going to show that, if $r \in [3,4]$ fulfils the inequality

$$\frac{r^2}{4}\left(1-\frac{r}{4}\right) \geqslant \frac{1}{2}, \tag{4}$$

then such a function does not exist. For the other case the problem remains open.

Suppose for the indirect proof that we are given an integrable function $f: [0,1] \to [0,+\infty)$ and an $r \in [3,4]$ satisfying conditions (1)-(4). Then, in view of properties (3), (4) and (1), we have

$$f\left(\tfrac{1}{2}+x\right) = 2rxf\left(r\left(\tfrac{1}{2}+x\right)\left(\tfrac{1}{2}-x\right)\right) \quad \left(\forall x \in [0,\tfrac{1}{2}]\right).$$

Consequently, after integration over the interval $[0,x]$ and suitable changes of variables,

$$\int_{\frac{1}{2}}^{\frac{1}{2}+x} f(s)ds = \int_{r\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)}^{\frac{r}{4}} f(s)ds \quad \left(\forall x \in [0,\tfrac{1}{2}]\right).$$

Extending the intervals of integration by conditions (3) and (4) to 0 and 1, respectively, and making use of equality (2), we get hence

$$\int_0^{\frac{1}{2}+x} f(s)ds = 1 - \int_0^{r\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)} f(s)ds \quad \left(\forall x \in [0,\tfrac{1}{2}]\right).$$

Introducing the function

(5) $$F(x) = \int_0^x f(s)ds, \quad x \in [0,1],$$

we may rewrite this property as

$$F\left(\frac{1}{2}+x\right) + F\left(r\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)\right) = 1 \quad \left(\forall x \in [0,\tfrac{1}{2}]\right)$$

or, replacing x by $\frac{1}{2}-x$ in the above equality, as

$$F(1-x) + F(r(1-x)x) = 1 \quad \left(\forall x \in [0,\tfrac{1}{2}]\right).$$

Now, writing here $1-x$ in place of x we obtain

$$F(x) + F(rx(1-x)) = 1 \quad \left(\forall x \in [\tfrac{1}{2},1]\right).$$

Finally, with the function $\varphi: [\frac{1}{2},1] \to [0,\frac{r}{4}]$ given by the formula

(6) $$\varphi(x) = rx(1-x),$$

we have

(7) $$F(x) + F[\varphi(x)] = 1 \quad \left(\forall x \in [\tfrac{1}{2},1]\right).$$

Let us observe that (since $r \geq 3$) $\frac{1}{2} < 1-\frac{1}{r} < \frac{r}{4}$ and it follows from the definition (6) of the function φ and from the inequality (4) that

(8) $$\varphi\left([\tfrac{1}{2},1-\tfrac{1}{r}]\right) = [1-\tfrac{1}{r},\tfrac{r}{4}], \quad \varphi\left([1-\tfrac{1}{r},\tfrac{r}{4}]\right) \subset [\tfrac{1}{2},1-\tfrac{1}{r}].$$

In particular, if $x \in [\frac{1}{2},1-\frac{1}{r}]$, then $\varphi(x) \in [\frac{1}{2},1]$ and, recalling condition (7), we see that

$$F[\varphi(x)] + F(\varphi[\varphi(x)]) = 1.$$

This together with property (7) yields

(9) $$F(x) = F[\Phi(x)] \quad \left(\forall x \in [\tfrac{1}{2},1-\tfrac{1}{r}]\right),$$

where the function $\Phi: [\frac{1}{2},1-\frac{1}{r}] \to [\frac{1}{2},1-\frac{1}{r}]$ is defined by the formula (cf. properties (8) of the function φ)

(10) $$\Phi(x) = \varphi[\varphi(x)].$$

Note that Φ is a continuous strictly increasing and convex function which therefore may have in the interval $(\frac{1}{2},1-\frac{1}{r})$ at most one fixed point (or $\Phi(x)=x$ on an interval which, however, is not the case here— cf. formulas (10) and (6)).

Let us distinguish two cases.

If the function Φ has no fixed point in the interval $(\frac{1}{2}, 1-\frac{1}{r})$, then either $\Phi(x) < x$ in the whole interval $(\frac{1}{2}, 1-\frac{1}{r})$ or $x < \Phi(x)$ in the whole interval $(\frac{1}{2}, 1-\frac{1}{r})$. Consequently (cf. [1, Theorem 2.11]), the function $F|_{[\frac{1}{2}, 1-\frac{1}{r}]}$, being a continuous automorphic function (cf. (9)), has to be a constant function. It follows from the definition (5) of the function F and from the properties (3) and (4) that $F(\frac{1}{2}) = 0$ and so $F(x) = 0$ in each point x of the interval $[\frac{1}{2}, 1-\frac{1}{r}]$. In particular $F(1-\frac{1}{r}) = 0$ which, however, contradicts property (7) since $1-\frac{1}{r}$ is a fixed point of the function φ.

If the function Φ has (exactly one) fixed point, say c, in the interval $(\frac{1}{2}, 1-\frac{1}{r})$, then we argue as above considering functions

$$\Phi|_{[\frac{1}{2}, c]} \quad \text{and} \quad \Phi|_{[c, 1-\frac{1}{r}]}$$

instead of Φ.

The proof is finished.

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THE DIRAC OPERATOR AND THE CHANGE OF THE METRIC

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Presented by P. Ribenboim, F.R.S.C.

Abstract: Given any two isotopic immersions i,j of an oriented manifold M into an Euclidean space $(\mathbb{R}^k,<,>)$, their differentials di and dj are related by $dj = g\cdot di\cdot f$, where $g \in C^\infty(M,SO(k))$ and f is a bundle isomorphism of TM symmetric with respect to $i^*<,>$. Then the Dirac and the Laplace-Beltrami operator of $j^*<,>$ are expressed in terms of the respective operators of $i^*<,>$ and the factors g and f.

1) The Dirac Operator

Let (M,G) be an oriented Riemannian manifold of dimension n which allows a spin structure $P(M)$. Fix a negative definite inner product on $\mathbb{R}^n$ and call its Clifford algebra $C_n(-)$. Let λ be a representation of $C_n(-)$ in some vector space V. It reduces to a representation λ of the spin group $Spin(n)$ in V. Then the spinor bundle $V(M)$ is the associated vector bundle to $P(M)$ via λ, $V(M) = P(M) \times_{Spin(n)} V$. Since the tangent bundle TM of M is associated to $P(M)$, too, the representation λ gives rise to a Clifford multiplication μ of tangent vector fields and spinor fields. μ extends to smooth sections of the tensor algebra $\otimes TM$ and (using the metric isomorphism) also of $\otimes T^*M$. Every linear connection in TM induces an 'associated' connection in the spinor bundle V(M). Let ∇_V be such a connection in V(M) interpreted as an operator $\nabla_V : \Gamma(V(M)) \longrightarrow \Gamma(T^*M \otimes V(M))$. The Dirac operator $D : \Gamma(V(M)) \longrightarrow \Gamma(V(M))$ is then defined by

$$D = \mu \circ \nabla_V .$$

The Laplace-Beltrami operator Δ acting on smooth sections of V(M) is given by

$$\Delta = - \operatorname{tr} \nabla_V{}^2 .$$

Since the Clifford multiplication is covariantly constant with respect to any connection ∇_V, the square of the Dirac operator turns into $D^2 = \mu \circ \nabla_V{}^2$ and finally $D^2 = \Delta + \mu \circ R_V$ where

$$R_V : \Gamma(V(M)) \longrightarrow \Gamma(T^*M \otimes T^*M \otimes V(M))$$

denotes the curvature tensor of the connection ∇_V. For details, see e.g. [1] and [5]. Now we change the connection ∇_V to $\nabla_V(\tau) = \nabla_V + \tau$, where $\tau \in A^1(M, L(V(M)))$ is a bundle valued one form on M considered as an operator

$$\tau : \Gamma(V(M)) \longrightarrow \Gamma(T^*M \otimes V(M)) .$$

Then for the Dirac operator $D(\tau)$ of the connection $\nabla_V(\tau)$ we find $D(\tau) = D + \mu \circ \tau$.

2) The Change of the Metric

To any Riemannian metric $\bar{G}$ on M there is a unique smooth bundle isomorphism f of TM , positive definite and symmetric with respect to G , such that $\bar{G}(X,Y) = G(fX,fY)$ for all smooth vector fields X and Y on M (cf.[3]) .

Choose a spin structure $\bar{P}(M)$ for $\bar{G}$ such that $P(M)$ and $\bar{P}(M)$ are isomorphic as principal bundles. Denote the associated spinor bundle by $\bar{V}(M)$ and the corresponding Clifford multiplication by $\bar{\mu}$. The isomorphism between the principal bundles induces an isomorphism $f_V : \bar{V}(M) \longrightarrow V(M)$ between the spinor bundles. We consider the linear connection ∇_V in $V(M)$ and the connection $\nabla_V(f)$ in $\bar{V}(M)$ defined by $\nabla_V(f)=f_V^{-1}\cdot\nabla_V\cdot f_V$ where the dot means pointwise formed composition. To express the Dirac operator $D(f)$ of $\nabla(f)$ in terms of f_V and the Dirac operator D of ∇_V we need the following two operators:

$$\hat{\nabla}_V : \Gamma(V(M)) \longrightarrow \Gamma(T^*M \otimes V(M))$$

and

$$\hat{D} : \Gamma(V(M)) \longrightarrow \Gamma(V(M))$$

defined by

$$(\hat{\nabla}_V)_X\psi = (\nabla_V)_{fX}\psi$$

and

$$\hat{D}\psi = (\mu \circ \hat{\nabla}_v)\psi ,$$

respectively, for all smooth vector fields X and spinor fields ψ on M. Then:

Theorem 1 The Dirac operator $D(f)$ corresponding to the connection $\nabla_v(f)$ satisfies

$$D(f) = f_v^{-1} \cdot \hat{D} \cdot f_v .$$

Proof: By construction, $\bar{\mu} = f_v^{-1} \cdot \mu \cdot (f^b \otimes f_v)$, where f^b is obtained from f via the metric isomorphism. Hence, $D(f) = f_v^{-1} \cdot \mu \cdot (f^b \otimes f_v) \circ f_v^{-1} \cdot \nabla_v \cdot f_v = f_v^{-1} \cdot \hat{D} \cdot f_v$.

Corollary 1 The square of the Dirac operator $D(f)$ is given by $D^2(f) = f_v^{-1} \cdot [- \operatorname{tr} \hat{\nabla}_v^{\,2} + \mu \circ \hat{R}_v] \cdot f_v$, where R_v is the curvature tensor of ∇_v and $\hat{R}_v(X,Y)\,\psi = R_v(fX,fY)\,\psi$ for all smooth vector fields X,Y and spinor fields ψ on M.

The curvature $R_v(f)$ of $\nabla_v(f)$ satisfies $\bar{\mu} \circ R_v(f) = f_v^{-1} \cdot (\mu \circ \hat{R}_v) \cdot f_v$ which in turn yields:

Corollary 2 The Laplace-Beltrami operator $\Delta(f)$ is expressed by $\Delta(f) = - f_v^{-1} \cdot (\operatorname{tr} \hat{\nabla}_v^{\,2}) \cdot f_v$.

Consider now two smooth isotopic immersions i,j of M into an Euclidean space $(\mathbb{R}^k, \langle , \rangle)$. For n big enough, any two immersions are isotopic (cf.[4]).

Then $dj = g \cdot di \cdot f$ where $g \in C^{\infty}(M,SO(k))$ and f is as above a smooth bundle isomorphism of TM, positive definite with respect to $G = i^*\langle,\rangle$ (cf.[3]). Let $\nabla(i)$ and $\nabla(j)$ be the corresponding Levi-Cività connections of $G = i^*\langle,\rangle$ and $\bar{G} = j^*\langle,\rangle$. The form $g \cdot di$ defines a connection $\nabla(g)$ by

$$di\ \nabla(g)_X Y = di\ \nabla(i)_X Y + (g^{-1} dg(X) di(Y))^{T_i}$$

for all smooth vector fields X,Y on M. T_i means the orthogonal projection onto $TiTM$. Clearly,

$$\nabla(j) = f^{-1} \cdot \nabla(g) \cdot f \quad .$$

Denote the induced connections in the corresponding spinor bundles $V(M)$ and $\bar{V}(M)$ by $\nabla_v(i)$, $\nabla_v(g)$ and $\nabla_v(j)$. We still have the equation

$$\nabla(j) = f_v^{-1} \cdot \nabla_v(g) \cdot f_v \quad .$$

<u>Theorem 2</u> <u>The Dirac operator</u> $D(j)$ <u>corresponding to the Levi-Cività connection</u> $\nabla(j)$ <u>in TM satisfies</u>

$$D(j) = f_v^{-1} \cdot \hat{D}(g) \cdot f_v \quad ,$$

<u>where</u> $D(g)$ <u>is the Dirac operator of</u> $\nabla_v(g)$. <u>The Laplace-Beltrami operator</u> $\Delta(j)$ <u>is expressed by</u>

$$\Delta(j) = - f_v^{-1} \cdot (\mathrm{tr}\ \hat{\nabla}_v^{2}(g)) \cdot f_v \quad .$$

Remark: In the case of conformal deformation of the original metric, this expression coincides with the result of H.Baum [2] for the spin representation.

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ALL TRAPEZOID FUNCTIONS ARE CONJUGATE

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Presented by J. Aczél, F.R.S.C.

For each number e in the open interval (0,1/2), define the <u>trapezoid</u> <u>function</u> t_e on the closed unit interval [0,1] by

$$(1) \qquad t_e(x) = \begin{cases} x/e, & x \text{ in } [0,e], \\ 1, & x \text{ in } [e,1-e], \\ (1-x)/e, & x \text{ in } [1-e,1], \end{cases}$$

(see Figure 1).

At the 21st International Symposium on Functional Equations (Konolfingen, Switzerland, August 6-13, 1983), W. A. Beyer asked whether any two different trapezoid functions are <u>conjugate</u>, i.e., whether for $e_1 \neq e_2$ there exists a one-one function f mapping [0,1] onto itself such that

$$(2) \qquad t_{e_1} = f^{-1} \circ t_{e_2} \circ f.$$

In this note we answer Beyer's question affirmatively by appealing to the following:

Conjugacy Theorem. Two functions are conjugate if and only if they have isomorphic orbit structures.

The notion of orbit needed here is that introduced by Kuratowski in a note at the end of [Tambs Lyche 1924]: Given a function g, denote its $n^{\underline{th}}$ iterate by g^n, and define $g^0(x) = x$ for all x in the union U of the domain and range of g. Then x and y in U <u>are</u> <u>in</u> <u>the</u> <u>same</u> g-<u>orbit</u> if there exist non-negative integers m,n such that

(3) $g^m(x) = g^n(y)$.

Thus g and h have isomorphic orbit structures if and only if there is a one-one function f mapping U onto V such that, for any x,y in U and any non-negative integers m,n,

(4) $g^m(x) = g^n(y)$ implies $h^m(f(x)) = h^n(f(y))$.

Orbits have natural representations as directed graphs: see Figure 2 and [Rice, Schweizer, and Sklar 1980]; the latter reference also contains a proof of the Conjugacy Theorem.

If a function g maps a set into itself, then each orbit of g is either <u>cyclic</u> or <u>acyclic</u>. An n-<u>cyclic</u> g-<u>orbit</u> is one that contains an n-<u>cycle</u>, i.e., a set of n distinct points $x_1,\ldots,x_n$ such that

(5) $g(x_m) = x_{m+1}$ for $m = 1,\ldots,n-1$, and $g(x_n) = x_1$.

Thus a 1-cycle is a fixed point of g, and more generally, each point in an n-cycle of g is a fixed-point of g^n. An <u>acyclic</u> g-orbit is one that contains no n-cycle for any $n \geq 1$.

Turning to the trapezoid functions, we observe that the preimage under t_e of 0 is the pair $\{0,1\}$; the preimage of 1 is the closed interval $[e,1-e]$, which we decompose into the union of the pair $\{e,1-e\}$ and the open interval $(e,1-e)$; the preimage of any y in the half-open interval $[0,1)$ is the pair $\{ey,1-ey\}$; and the preimage of any open subinterval (a,b) of $(0,1)$ is the pair of open intervals $\{(ea,eb),(1-eb,1-ea)\}$. Thus the t_e-orbit containing 0 is a 1-cyclic orbit, which looks like a binary tree with a 1-cycle at its root and which may conveniently be depicted as in Figure 3. All other cyclic orbits have the appearance illustrated in Figure 2; and the acyclic orbits look, locally, like the "tails" of the orbit in Figure 2.

Since these facts are independent of the value of e, it follows immediately that for any e_1, e_2 in $(0,1/2)$:

(a) The 1-cyclic t_{e_1}-orbit containing 0 is isomorphic to the 1-cyclic t_{e_2}-orbit containing 0; the 1-cyclic t_{e_1}-orbit not containing 0 is isomorphic to the 1-cyclic t_{e_2}-orbit not containing 0;

(b) for $n \geq 2$, any n-cyclic t_{e_1}-orbit is isomorphic to any n-cyclic t_{e_2}-orbit;

(c) any acyclic t_{e_1}-orbit is isomorphic to any acyclic t_{e_2}-orbit.

Thus to prove t_{e_1} and t_{e_2} conjugate, it remains to prove that:

(i) For each $n \geq 2$, the number of n-cyclic t_{e_1}-orbits is equal to the number of n-cyclic t_{e_2}-orbits;

(ii) the number of acyclic t_{e_1}-orbits is equal to the number of acyclic t_{e_2}-orbits.

To prove (i), observe that any trapezoid function t_e is piecewise linear, with a graph consisting of a flat piece and 2 "slopes". Since the range of each slope is the entire interval [0,1], under iteration each slope will generate a compressed replica of the whole graph of t_e. Thus the graph t_e^2 (see Figure 1) has 2 compressed replicas of the graph of t_e and therefore contains 4 slopes. By induction, the graph of t_e^n has 2^n slopes. Each slope intersects the graph of the identity function in exactly 1 point, whence t_e^n has precisely 2^n fixed-points. Since this is true for every n, a standard combinatorial argument [Sklar 1975] (see also [Rice 1979]) shows that the number of n-cyclic t_e-orbits is given by

$$\frac{1}{n} \sum_{d|n} \mu(d) 2^{n/d},$$

where μ is the Möbius function and the summation is over the divisors of n. This expression is independent of e, and this proves (i).

To prove (ii), we begin by observing that for any e, the open intervals of the t_e-orbit containing 0 (see Figure 3) together form the complement of a Cantor set. The remaining points in the t_e-orbit containing 0, plus the points in all other cyclic t_e-orbits, together form a denumerable set. Thus, when all the points in all the cyclic orbits of t_e are removed from the unit interval, what remains is a Cantor set minus a denumerable set. Since any Cantor set has the cardinality of the continuum, this remaining set, which is evidently the set of all points in the acyclic orbits of t_e, has the cardinality of the continuum. Now any individual acyclic orbit is denumerable; hence, independently of e, the cardinality of the set of acyclic orbits of t_e is the cardinality of the continuum, and this proves (ii).

Finally, we note that it is not difficult to construct a continuous function f satisfying (2). Indeed, f can be taken to be strictly increasing, and linear on the intervals of the t_{e_1}-orbit containing 0. The details of the construction will be presented elsewhere.

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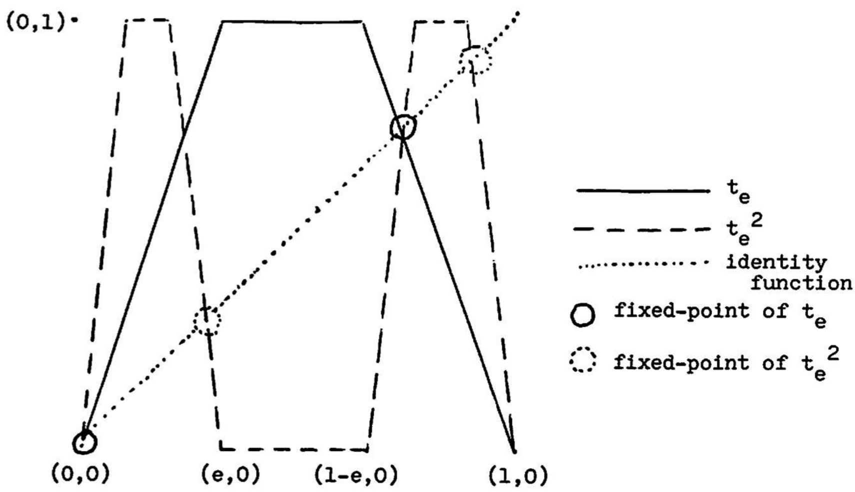

Figure 1
Graphs of t_e and t_e^2.

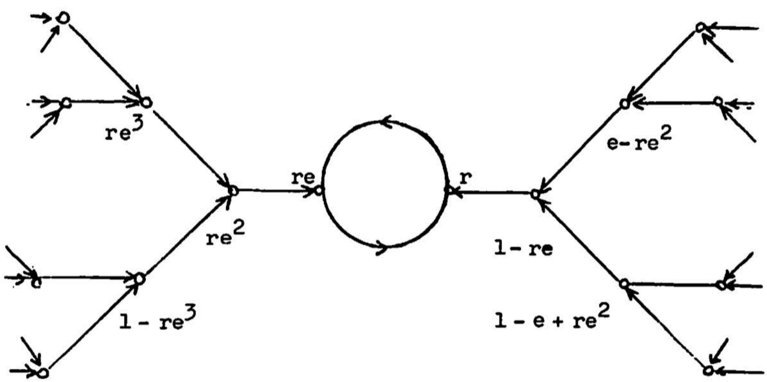

Figure 2. 2-cyclic orbit of t_e $(r = (1+e^2)^{-1})$.

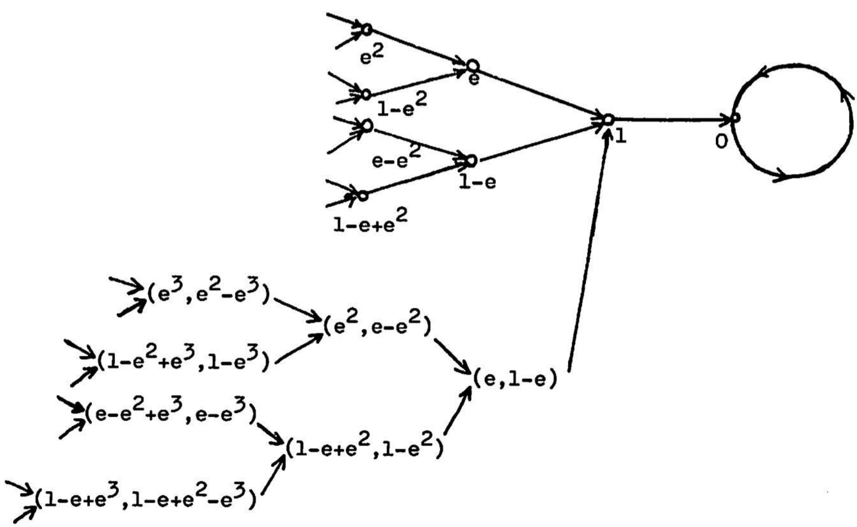

Figure 3. Orbit of t_e containing 0.

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EIGENFUNCTION EXPANSION OF GENERALIZED FUNCTIONS

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Presented by J.G. Arthur, F.R.S.C.

ABSTRACT

Expansion of a large class of generalized functions in terms of normalized eigenfunctions of a general Sturm-Liouville problem is given. Expansions of certain generalized functions of Gelfand-Shilov are noted as special cases.

1. Introduction. Eigenfunction expansion of a class of generalized functions was given by us [3]. The spaces of testing functions were not precisely defined there. Indeed the testing function space $N(I)$, $I = (a,b)$, $-\infty < a < b < \infty$, was assumed to be such that it contained the normalized eigenfunctions $\{\Psi_n(x)\}_{n=1}^{\infty}$ of the Sturm-Liouville problem:

$$\begin{aligned} &L_x \phi = \lambda \phi, \qquad L_x = q(x) - d^2/dx^2 \\ &\phi(a) \cos\alpha + \phi'(a) \sin\alpha = 0 \qquad\qquad (1.1) \\ &\phi(b) \cos\beta + \phi'(b) \sin\beta = 0. \end{aligned}$$

As the asympototic behaviours of the eigenfunctions of (1.1) are precisely known depending upon the nature of $q(x)$, here I precisely define the testing function space and hence the generalized functions which can be represented in series of eigenfunctions of (1.1). Among many other generalized functions the elements of $(S_\alpha^\beta)'$ are also expanded in terms of eigenfunctions of (1.1). The finite interval case was successfully treated in [3]. In this paper I confine myself to the infinite interval case.

2. The testing function space $N_w(I)$.

Let I denote the interval $(0,\infty)$ or $(-\infty,\infty)$. We consider the eigenfunctions $\psi_n(x)$ in the case where the interval is $(0,\infty)$, with given boundary condition at $x = 0$, and $q(x)$ is continuous and increases steadily to ∞ as $x \to \infty$, or where the interval is $(-\infty,\infty)$, $q(x)$ decreases steadily in $(-\infty,0)$. For convenience we assume that $q(0) = 0$. Then for any positive number λ_n, there is a positive number X_n such that $q(X_n) = \lambda_n$; and all zeros of $\psi_n(x)$ lie in the interval $(0,X_n)$.

According to Titchmarsh [4, p. 166] for fixed x,

$$|\psi_n(x)| < 2/3 \ \lambda_n \ X_n^{3/2} .$$

If we assume further, that $q(x)$ is three times differentiable (in the sequel we shall assume that $q(x) \in C^\infty$), and as $x \to \infty$,

$$\frac{q'(x)}{q(x)} = O(1/x) , \quad \frac{q''(x)}{q'(x)} = O(1/x) , \quad \frac{q'''(x)}{q''(x)} = O(1/x) ,$$

then from Titchmarsh [4, p. 168],

$$|\psi_n(x)| \leq K (q(x) - \lambda_n)^{-1/4} \exp\left[- \int_{X_n}^{x} (q(u) - \lambda_n)^{\frac{1}{2}} du\right] ,$$

where K is independent of x. Since $q(u)$ is convex downwards, $q(u) \geq 2 \lambda_n$ if $u \geq 2X_n$. Hence if $x \geq 4X_n$,

$$\int_{X_n}^{x} (q(u) - \lambda_n)^{\frac{1}{2}} du \geq \int_{2X_n}^{x} (\tfrac{1}{2} q(u))^{\frac{1}{2}} du.$$

Thus

$$|\psi_n(x)| \leq K (q(x)- \lambda_n)^{-1/4} \exp\left[- \int_{2X_n}^{x} (\tfrac{1}{2} q(u))^{\frac{1}{2}} du\right]$$

for $x \geq 4X_n$.

Now, we define the weight function w by

$$w(x) = (q(x))^{1/4} \exp\left[- \int_{0}^{x} (\tfrac{1}{2} q(u))^{\frac{1}{2}} du\right] .$$

Then the testing function space $N_w(I)$ consists of all the complex valued infinitely differentiable functions $\phi(x)$ defined over I such that

$$\gamma_k(\phi) = \sup_{x \in I} \left| w(x)\, L_x^{(k)} \phi(x) \right| < \infty ,$$

for all $k = 0,1,2,\ldots,$ q(x) being assumed to be infinitely differentiable over I. Then the topology over $N_w(I)$ is generated by $\{\gamma_k\}_{k=0}^{\infty}$. The space $N_w(I)$ is locally convex, sequentially complete, Hausdorff topological vector space. The dual of $N_w(I)$ will be denoted by $N_w'(I)$. The space D(I) of testing functions of compact supports is a subspace of $N_w(I)$ and the restriction of $f \in N_w'(I)$ to D(I) is in D'(I).

The following theorem improves [3, Theorem 2, p.5].

THEOREM 1. Let f be an arbitrary element of $N_w'(I)$ and let $\psi_n(x)$ be the normalized eigenfunctions of the Sturm-Liouville problem (1.1). Define the finite Sturm-Liouville transform of f by

$$F(n) := \langle f(x) , \psi_n(x) \rangle , \qquad n = 1,2,3,\ldots,$$

then for each $\phi(t) \in D(I)$,

$$\lim_{N \to \infty} \left\langle \sum_{n=1}^{N} \psi_n(t)\, F(n) , \phi(t) \right\rangle = \langle f(t) , \phi(t) \rangle .$$

Proof. The proof is similar to that of Theorem 2 in [3].

3. The testing function space $N_q(I)$.

In this section we assume that q is rapidly increasing in I and satisfies the conditions 1 and 2 of Giertz [2, pp. 53, 57], and hence the eigenfunctions of the problem :

$$L_x \phi = \lambda \phi , \qquad \int_{-\infty}^{\infty} |\phi(x)|^2 \, dx = 1$$

satisfy the estimate

$$(q(x))^m \, |\psi_n(x)| \leq K_o(m)\, \lambda_n^{m+1} , \quad m = 0,1,2,\ldots,$$

for all $x \in I$.

The testing function space $N_q(I)$ is defined to be the collection of all complex valued infinitely differentiable functions ϕ defined on I which satisfy

$$\beta_{m,k}(\phi) = \sup_{x \in I} |(q(x))^m D^{(k)} \phi(x)| < \infty \quad ,$$

for each $m,k = 0,1,2,\ldots$. The topology over $N_q(I)$ is generated by $\{\beta_{m,k}\}_{m,k=0}^{\infty}$. The following theorem, being analogous to Theorem 1, provides expansion of generalized functions of rapid growth belonging to $N_q'(I)$.

THEOREM 2. *Let* $f \in N_q'(I)$ *and* $\psi_n(x)$ *be the normalized eigenfunctions of the problem* (1.1). *Define* $F(n)$ *by*

$$F(n) = \langle f(x) \, , \, \psi_n(x) \rangle \quad , \quad n = 1,2,3,\ldots \, .$$

Then for each $\phi(t) \in D(I)$,

$$\lim_{N\to\infty} \langle \sum_{n=1}^{N} \psi_n(t) \; F(n) \, , \, \phi(t) \rangle = \langle f \, , \, \phi \rangle \, .$$

REMARK 1. Since $q(x) = x^4$, $\exp(x^4)$, $\exp(\exp x^4)$,... satisfy conditions 1 and 2 of Giertz referred earlier, the space $N_q(I)$ coincides with the W_M - space of Gelfand and Shilov [1] when $M(x) = \ln q(x)$ and $q(x) = \exp(x^4)$, $\exp(\exp x^4)$,...

4. *The special case* $q(x) = x^{2k}$.

The case $q(x) = x^{2k}$ is of special interest because it gives rise to an expansion of generalized functions belonging to certain $S_\alpha^{\beta\prime}$ - space.

In this case from [4] we find that the eigenvalues

$$\lambda_n = O(n^{2k/(2k+1)}) \, , \quad n \to \infty \, .$$

The normalized eigenfunctions $\psi_n(x)$ possess the estimates :

$$|\psi_n(x)| \leq 2/3 \quad \lambda_n^{1+3/(4k)} \exp\{-(4(k+1))^{-1} \, |x|^{k+1}\} \quad ,$$

for $|x| > 2\lambda_n^{1/(2k)}$.

Therefore, for any small positive number $a < (2(k+1))^{-1}$, we have

$$|\psi_n(x)| \leq K_n \exp(-a|x|^{k+1}), \tag{4.1}$$

where

$$K_n = 16/3 \ \lambda_n^{1+3/(4k)} \exp(a\, 4^{k+1} \lambda_n^{(k+1)/(2k)}).$$

Furthermore, following the method suggested by Titchmarsh [4, p.172] the eigenfunctions $\psi_n(x)$ can be extended to entire functions $\psi_n(z)$ satisfying the estimate

$$|\psi_n(z)| \leq 8/3\ (1+2k)^{\frac{1}{2}} \ \lambda_n^{1+3/(4k)} \exp(4d\ \lambda_n^{(k+1)/(2k)} + (1+d^{-k})|z|^{k+1}) \tag{4.2}$$

for arbitrary $d > 0$.

Consequently, in view of the criterion of Gelfand and Shilov [1, p.220], $\psi_n(x) \in S_\alpha^\beta$, with $\alpha = 1/(k+1)$ and $\beta = k/(k+1)$.

The following theorem provides expansion of $f \in S_{1/(k+1)}^{k/(k+1)}$ in terms the normalized eigenfunctions of the operator $-d^2/dx^2 + x^{2k}$.

THEOREM 3. If $a_n = O(\exp(-\delta\lambda_n^{(k+1)/(2k)}))$, $\delta > 0$ and

$$f(x) = \sum_{n=0}^{\infty} a_n\ \psi_n(x), \tag{4.3}$$

then $f(x) \in S_\alpha^\beta$ with $\alpha = 1/(k+1)$ and $\beta = k/(k+1)$.

Proof. In the estimate (4.1) we can take $a < \delta/4^{k+1}$. So that

$$|f(x)| \leq A \sum_{n=0}^{\infty} \exp(-\delta\ \lambda_n^{(k+1)/(2k)})\ 16/3\ \lambda_n^{1+3/(4k)} \exp(a\ 4^{k+1}\ \lambda_n^{(k+1)/(2k)}) \cdot \exp(-a|x|^{k+1})$$

$$\leq A' \exp(-a|x|^{k+1}), \tag{4.4}$$

because the series is convergent. Also, on replacing x by z in (4.3), using (4.2) and choosing $d < \delta/4$, we have

$$|f(z)| \leq \sum_{n=o}^{\infty} |a_n| \; |\psi_n(z)|$$

$$\leq 4/3 \; \exp((1+d^{-k}) \, |z|^{k+1}) \sum_{n=o}^{\infty} \lambda_n^{1+3/(4k)} \exp(-(\delta-4d) \, \lambda_n^{(k+1)/(2k)})$$

$$\leq 4/3 \; A'' \exp((1+d^{-k}) \, |z|^{k+1}) \; .$$

Therefore by the same criterion of Gelfand and Shilov, $f(x) \in S_\alpha^\beta$.

The following theorem provides a condition on $d_n=(T, \psi_n)$ so that $T \in S_\alpha^{\beta\,\prime}$ can be represented by

$$T = \sum_{n=o}^{\infty} d_n \, \psi_n \; . \qquad (4.5)$$

THEOREM 4. Every $T \in S_\alpha^{\beta\,\prime}$, $\alpha = 1/(k+1)$, $\beta = k/(k+1)$, possesses expansion (4.5) if and only if $d_n = o(\exp(\varepsilon \, \lambda_n^{(k+1)/(2k)}))$, ε is arbitrary small positive number.

For $k = 1$ the above theorem gives the well-known expansion of distributions $T \in S_{\frac{1}{2}}^{\frac{1}{2}\,\prime}$ in series of Hermite functions.

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