C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No. 5, October 1983 octobre

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C.R. Math. Rep. Acad. Sci. Canada - Vol.V, No.5, October 1983 octobre

# THE MAXIMUM MODULUS OF ZEROS OF POLYNOMIALS

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*Presented by P. Ribenboim, F.R.S.C.*

Abstract. The theorem of Kakeya and Eneström on the maximum modulus of zeros of polynomials with positive coefficients is extended to certain classes of polynomials with unimodal coefficients. Upper bounds for the moduli of the zeros of Bernoulli polynomials are found.

1. The following theorem, due to Kakeya and Eneström, is well-known in the theory of the distribution of zeros of polynomials.

Theorem A (see e.g. [1]): Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a polynomial with positive coefficients such that

$$0 < a_0 \leq a_1 \leq \ldots \leq a_n . \tag{1}$$

Then the zeros of $p(z)$ lie in the closed unit disk $|z| \leq 1$.

If we change hypothesis (1) to

$$0 < a_0 \leq a_1 \leq \ldots \leq a_{n-j} > a_{n-j+1} \geq \ldots \geq a_n > 0 , \tag{2}$$

with some $j \geq 1$, then Theorem A will no longer hold in general. We shall study the question: under which additional conditions on the coefficients do the zeros still lie in the disk $|z| \leq 1$ ?

2. The first result deals with the case $j = 1$. It is independent of the later considerations on Bernoulli polynomials.

Theorem 1: Let $p(z) = a_0 + a_1 z + \ldots + a_{2k} z^{2k}$ be a polynomial such that

$$0 < a_0 \leq a_1 \leq \ldots \leq a_{2k-1} > a_{2k} > 0.$$

Denote

$$b_j = (a_{2j} - a_{2j-1})/(a_{2j+1} - a_{2j}) \quad \text{for} \quad 0 \leq j \leq k-1 \quad (a_{-1} := 0).$$

If

$$a_{2k}/(a_{2k-1} - a_{2k}) \geq \max\{b_j, 1/b_j\} \quad \text{for all} \quad 0 \leq j \leq k-1,$$

then all the zeros of $p(z)$ lie in the closed unit disk.

Example: Let $p(z) = 1 + \frac{3}{2} z + 2z^2 + 3z^3 + 2z^4$.
We have $b_0 = 2$ and $b_1 = 1/2$, while $a_4/(a_3 - a_4) = 2$.
Hence Theorem 1 applies; the zeros lie in $|z| \leq 1$. (The Kakeya-Eneström theorem gives a maximum modulus of 3/2; computation: the moduli of the zeros are approximately 0.946 and 0.747.)

Sketch of proof: We write

$$p(z)(1-z) = f(z) - g(z),$$

where

$$f(z) = a_0 + (a_1 - a_0)z + \ldots + (a_{2k-1} - a_{2k-2})z^{2k-1}$$

$$g(z) = (a_{2k-1} - a_{2k})z^{2k} + a_{2k} z^{2k+1}.$$

For $r > 1$ and arbitrarily close to 1, we show that

$$|f(z)| < |g(z)| \quad \text{for} \quad |z| = r;$$

this is done by grouping the coefficients of $f(z)$ in pairs. We get the result by applying Rouché's theorem, using the fact that $g(z)$ has all its zeros in $|z| \leq 1$.

Remark: Theorem 1 can be generalized by grouping some or all coefficients of $f(z)$ in triplets.

3. Bernoulli polynomials. We define the infinite series

$$S_{2j} := \sum_{m=1}^{\infty} (-1)^{m-1} m^{-2j}, \qquad j \geq 1,$$

and a sequence of polynomials $g_n(z)$ of degree $k=[n/2]$, $n \geq 3$, by

$$g_n(z) = a_0^{(n)} z^k + a_1^{(n)} z^{k-1} + \ldots + a_{k-1}^{(n)} z + a_k^{(n)},$$

where

$$a_0^{(n)} = 1/2, \quad a_j^{(n)} = \frac{n(n-1)\ldots(n-2j+1)}{(n-2)^{2j}} S_{2j} \quad \text{for } 1 \leq j \leq k.$$

The $g_n(z)$ are closely related to the Bernoulli polynomials $B_n(z)$ (For their definition and properties, see e.g. [2]).

Lemma 1: Let $k = [n/2]$ and $\delta = n - 2k$. Then for $n \geq 3$,

$$B_n(z+\tfrac{1}{2}) = z^{\delta}(-1)^k \frac{2(n-2)^{2k}}{(2\pi)^{2k}} g_n\left(-\left(\frac{2\pi}{n-2}z\right)^2\right).$$

The proof requires Euler's formula

$$B_{2j} = (-1)^{j-1} \frac{2(2j)!}{(2\pi)^{2j}} \zeta(2j),$$

the equality $(1-2^{1-2j})\cdot\zeta(2j) = S_{2j}$ for $j \geq 1$, and some standard properties of the Bernoulli polynomials.

The main result on the $g_n(z)$ is

Theorem 2: All the zeros of the polynomials $g_n(z)$, for $n \geq 129$, lie in the closed unit disk $|z| \leq 1$.

With Lemma 1 we immediately obtain the following

Corollary 1: For all $n \geq 129$, the zeros of the Bernoulli polynomials $B_n(z)$ lie in the disk

$$|z - 1/2| \leq (n-2)/2\pi. \qquad (3)$$

4. Proof of Theorem 2. First we shall see that the coefficients $a_j^{(n)}$ of $g_n(z)$ are of type (2).

Lemma 2: For $n \geq 3$ there is an index $\lambda$ such that

$$a_k^{(n)} \leq a_{k-1}^{(n)} \leq \dots \leq a_\lambda^{(n)} > a_{\lambda-1}^{(n)} > \dots > a_1^{(n)} > a_0^{(n)},$$

and

$$\lambda = 3 \quad \text{for} \quad 129 \leq n \leq 839, \qquad \lambda < \frac{\log((n-2)/3)}{2\log 2} \quad \text{for} \quad n \geq 840.$$

Our aim is to apply Rouché's theorem. We have

$$Q_n(z) := (1-z)g_n(z) = p_n(z) - q_n(z),$$

where

$$p_n(z) = a_k^{(n)} + (a_{k-1}^{(n)} - a_k^{(n)})z + \dots + (a_\lambda^{(n)} - a_{\lambda+1}^{(n)})z^{k-\lambda},$$

$$q_n(z) = (a_\lambda^{(n)} - a_{\lambda-1}^{(n)})z^{k-\lambda+1} + \dots + (a_1^{(n)} - a_0^{(n)})z^k + a_0^{(n)} z^{k+1}.$$

By Lemma 2, both $p_n(z)$ and $q_n(z)$ have nonnegative coefficients. We need more information on the sizes of the coefficients of $p_n(z), q_n(z)$.

Lemma 3: (a) The sequence $\{a_j^{(n)} - a_{j+1}^{(n)}\}_j$ for $1 \leq j \leq k-1$ is unimodal; for $n \geq 129$ it has a maximum at some $j_0$ in the interval

$$\frac{1}{2}\left(\sqrt{n-2} + \frac{1}{2} + \frac{1/3}{\sqrt{n-2}}\right) < j_0 < \frac{1}{2}\left(\sqrt{n-2} + \frac{1}{2} + \frac{5/4}{\sqrt{n-2}}\right) + 1.$$

(b) In this case we have, for $n \geq 129$

$$a_{j_0}^{(n)} - a_{j_0+1}^{(n)} < \frac{1}{\sqrt{n-2}}\left( 2 - \frac{1}{\sqrt{n-2}} + \frac{2}{n-2} \right) \exp\left( -\frac{1}{2} + \frac{5/6}{\sqrt{n-2}} + \frac{2}{n-2} \right).$$

Comparing Lemmas 2 and 3(a), we find that the coefficients of $q_n(z)$ are increasing. Therefore, by Theorem A, all k+1 zeros of $q_n(z)$ lie in $|z| \leq 1$. Using Rouché's theorem and a theorem of Hurwitz, we conclude that $Q_n(z)$ has k+1 zeros in $|z| \leq 1$ provided that

$$|p_n(z)| < |q_n(z)| \quad \text{for } |z| = 1, \quad z \neq 1. \tag{4}$$

Hence we are done if we can verify (4).

To do this, we apply geometrical arguments, using the fact that (by Lemma 2) $q_n(z)$ has very few, namely $\lambda+1$ nonzero coefficients, compared to the $k-\lambda+1$ coefficients of $p_n(z)$. Furthermore, by Lemma 3(b), the maximum of the coefficients of $p_n(z)$ tends to zero as $n \to \infty$, while $q_n(z)$ has the coefficient $a_0^{(n)} = 1/2$ for all n. Note that $p_n(1) = q_n(1) = a_\lambda^{(n)}$, so the sums of the coefficients of both polynomials are equal.

To verify (4) for large arguments of z (i.e. for $\pi/2\lambda \leq |\arg z| \leq \pi$), we use the following

<u>Lemma 4</u>: <u>Let</u> $f(\alpha) = a_0 + a_1 e^{i\alpha} + a_2 e^{i2\alpha} + \ldots + a_n e^{in\alpha}$,

<u>such that</u> $0 \leq a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n \geq 0$.

<u>Then</u>
$$|f(\alpha)| \leq 2\,\frac{\cos(\alpha/2)}{\sin\alpha}\, a_k .$$

This is proved with an argument similar to that of M. Tomić in his alternative proof [3] of the Kakeya - Eneström theorem.

On the other hand, we have

<u>Lemma 5</u>: <u>For all</u> $n \geq 129$ <u>and</u> $|z| = 1$, <u>we have</u> $|q_n(z)| > \frac{22}{95}$.

If we compare Lemmas 4 and 5, taking into account Lemma 3(b), we can verify (4) for large arguments $\alpha$ of $z$. Similar, though more complicated geometrical arguments are used to verify (4) for small arguments $\alpha$ (i.e. $0 < \alpha \leq \pi/2\lambda$).

5. Professor U. Fixman remarked that in Corollary 1 we actually have a strict inequality.

Indeed, since $B_n(z)$ has rational coefficients (the Bernoulli numbers $B_n$ are rationals), the left-hand side of (3) is an algebraic number for any zero $z$ of $B_n(z)$. The right-hand side, however, is certainly transcendental. Hence there can be no equality in (3).

<u>Acknowledgements</u>. This paper is part of my Ph.D. thesis, written at Queen's University. I would like to thank my supervisor, Professor P. Ribenboim, for his advise and encouragement. I also thank Professor U. Fixman for suggesting several improvements.

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Received July 13, 1983

C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No. 5, October 1983 octobre

# AN APPLICATION OF A THEOREM BY HJELMSLEV

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Introduction. Let $\Gamma$ denote a (closed) curve in a real projective n-space which satisfies certain smoothness conditions. Obviously, there are (n-1)-flats meeting $\Gamma$ in n or more points. The curve $\Gamma$ is said to be of (real) order n if it is met by no (n-1)-flat in more than n points.

Very few characterizations of curves of order n seem to be known; cf. [1, 5.2]. A famous one by Hjelmslev [2] states that $\Gamma$ is of order n if no (n-2)-flat meets $\Gamma$ in n points and every point of $\Gamma$ has a neighbourhood of order n on $\Gamma$. Using this result, we present two new characterizations.

## 1. Definitions.

Let $P^n$ denote a real projective n-space with the standard topology; $n \geq 1$. A k-flat in $P^n$ will also be denoted by $P^k$; $-1 \leq k \leq n$.

A curve in $P^n$ is a continuous map $\Gamma: P^1 = \{s,t,\ldots\} \to P^n$. For convenience, we identify this map with the point set $\Gamma(P^1)$. The topology on $P^1$ defines (open) neighbourhoods on $\Gamma$. A neighbourhood of a point $\Gamma(s)$ is decomposed by $\Gamma(s)$ into two (open) one-sided neighbourhoods.

The curve $\Gamma$ shall be differentiable in the following sense. Let $\Gamma_{-1}(s) = \emptyset$ and $\Gamma_0(s) = \Gamma(s)$. If $\Gamma_{k-1}(s)$ is already defined and its existence postulated then we require:

(i) If $t \neq s$ is sufficiently close to s, then the flat $\langle\Gamma_{k-1}(s), \Gamma(t)\rangle$ spanned by $\Gamma_{k-1}(s)$ and $\Gamma(t)$ has the dimension k, and

(ii) it converges as t tends to s. Its limit is the osculating k-flat $\Gamma_k(s)$; $k = 1,2,\ldots,n$ and $s \in P^1$.

Thus $\Gamma_n(s) = P^n$ and $\Gamma_{n-1}(s)$ does not meet some neighbourhood of $\Gamma(s)$ on $\Gamma$, outside $\Gamma(s)$.

The curve $\Gamma$ is elementary if every point $\Gamma(s)$ has two disjoint one-sided neighbourhoods each of which is of order n. Obviously, an elementary curve meets every (n-1)-flat only in a finite number of points. Thus given $s \in P^1$ and any $P^{n-1} \subset P^n$, there is a neighbourhood $N(s)$ of s in $P^1$ such that $P^{n-1} \cap \Gamma(N(s)) \subseteq \{\Gamma(s)\}$. Then $P^{n-1}$ supports [cuts] $\Gamma$ at s if the two one-sided neighbourhoods of $\Gamma(s)$ in $\Gamma(N(s))$ lie locally on the same side [on different sides] of $P^{n-1}$.

Let $P_k(s) = \{P^{n-1} \subset P^n | P^{n-1} \cap \Gamma_{n-1}(s) = \Gamma_k(s)\}$; $k = 0,1,\ldots,n-1$. By [3, p.102], either every flat of $P_k(s)$ supports $\Gamma$ at s or every such flat cuts $\Gamma$ at s. We call $\Gamma$ regular at s if $P^{n-1} \in P_k(s)$ supports $\Gamma$ at s if and only if k is even. Then $\Gamma$ is regular if it is regular at each $s \in P^1$. Henceforth we assume that $\Gamma$ is elementary and regular. By [3, p.113], each point of $\Gamma$ has a neighbourhood of order n on $\Gamma$.

Let $\Gamma(s) \in P^k \subset P^n$. If $P^k \cap \Gamma_{n-1}(s) = \Gamma_{m-1}(s)$, we say that $P^k$ meets $\Gamma$ at s with multiplicity m (or "m times"). Suppose $\Gamma$ is of order n. Then it is well known that the sum of the multiplicities of all the intersections of $\Gamma$ with any (n-1)-flat is at most n. Thus such a $\Gamma$ satisfies

$$\Gamma_k(s) \cap \Gamma_{n-k-1}(t) = \emptyset \tag{1}$$

for every pair $s \neq t$ and $k = 0,1,\ldots,n-1$.

## 2. Projection.

Given a point $P^0$ and any $P^{n-1}$ not through $P^0$, define

$$\tilde{\Gamma}_k(s) = \begin{cases} \langle P^0, \Gamma_k(s)\rangle \cap P^{n-1} & \text{if } P^0 \notin \Gamma_k(s) \\ \Gamma_{k+1}(s) \cap P^{n-1} & \text{if } P^0 \in \Gamma_k(s);\ k = 0,1,\ldots,n-1. \end{cases}$$

Then the projection $\tilde{\Gamma}: \begin{cases} P^1 \to P^{n-1} \\ s \to \tilde{\Gamma}_0(s) \end{cases}$ of $\Gamma$ from $P^0$ into $P^{n-1}$

is a differentiable curve in $P^{n-1}$ with the osculating k-flats $\tilde{\Gamma}_k(s)$, $k = 0,1,\ldots,n-1$; cf [3, p.101]. We note that if $P^0 = \Gamma(t)$ and $\Gamma(t) \notin \Gamma_{n-1}(s)$ for $s \neq t$ then $\tilde{\Gamma}$ is regular and elementary with $\Gamma$.

We generalize the preceding definition. Suppose the flats $P^i$ and $P^{n-i-1}$ in $P^n$ are disjoint; $0 < i \le n-2$. Let

$$\Gamma^i_k(s) = \langle P^i, \Gamma_{k+m+1}(s)\rangle \cap P^{n-i-1}$$

where m is chosen such that $\Gamma^i_k(s)$ is a k-flat. Then the <u>projection</u>

$$\Gamma^i : \begin{cases} P^1 \to P^{n-i-1} \\ s \to \Gamma^i_0(s) \end{cases}$$ <u>of</u> $\Gamma$ <u>from</u> $P^i$ <u>into</u> $P^{n-i-1}$ is a

differentiable curve in $P^{n-i-1}$ with the osculating k-flats $\Gamma^i_k(s)$; $k = 0,1,\dots,n-i-1$.

Finally, we note the well known relationship between projection from points and from flats. Let $P^{i-1} \subset P^i$ and $P^{n-i-1} \subset P^{n-i}$; $P^{i-1} \cap P^{n-i} = P^i \cap P^{n-i-1} = \emptyset$. Let $\Gamma^{i-1}[\Gamma^i]$ denote the projection of $\Gamma$ from $P^{i-1}$ into $P^{n-i}$ [from $P^i$ into $P^{n-i-1}$]. The projection of $P^i$ from $P^{i-1}$ into $P^{n-i}$ is a point Q. Then $P^i = \langle P^{i-1},Q\rangle$ and $\Gamma^i$ is identical with the projection of $\Gamma^{i-1}$ from Q into $P^{n-i-1}$.

## 3. Two characterizations.

THEOREM 1. <u>Let</u> $\Gamma:P^1 \to P^n$ <u>be a regular and elementary curve</u>; $n \ge 1$. <u>Then the following are equivalent</u>:

1. $\Gamma$ <u>is of order</u> n.
2. <u>There is a</u> t <u>such that</u>

   a) $\Gamma(t) \notin \Gamma_{n-1}(s)$ <u>for all</u> $s \ne t$ <u>and</u>

   b) <u>no</u> (n-2)-<u>flat through</u> $\Gamma(t)$ <u>meets</u> $\Gamma$ <u>in more than</u> n-1 <u>points counting multiplicities</u>.
3. <u>There is a</u> t <u>such that</u>

[n,k] $\quad \Gamma_k(s) \cap \Gamma_{n-k-1}(t) = \emptyset$ <u>for all</u> $s \ne t$; $k = 0,1,\dots,n-1$.

PROOF. Clearly, 1. implies 2. and 3.; cf. (1).
Given 2., we may assume $n > 1$. By a), the projection $\tilde{\Gamma}$ of $\Gamma$ from $\Gamma(t)$

is regular and elementary in $P^{n-1}$; cf. **2**. By b), every (n-3)-flat of $P^{n-1}$ meets $\tilde{\Gamma}$ in at most n-2 points. Hence $\tilde{\Gamma}$ is of order n-1 by Hjelmslev's theorem. In particular, $\tilde{\Gamma}$ satisfies [n-1,k] for k = 0,1,...,n-2. The definitions of the osculating flats of $\tilde{\Gamma}$ then yield

$$<\Gamma_k(s),\ \Gamma(t)> \cap\ \Gamma_{n-k-1}(t) = \{\Gamma(t)\}$$

and thus $\Gamma$ satisfies [n,k] for k = 0,1,...,n-2. By a), [n,n-1] is also true and hence 2. implies 3.

Given 3., we may again assume n > 1. We prove 1. by induction. The projection $\tilde{\Gamma}$ of $\Gamma$ from $\Gamma(t)$ is again regular and elementary. By [n,k] and **2**., $\tilde{\Gamma}$ satisfies [n-1,k] for k = 0,1,...,n-2. Hence $\tilde{\Gamma}$ is of order n-1 by the induction hypothesis and every (n-2)-flat in $P^{n-1}$ meets $\tilde{\Gamma}$ at most n-1 times. Thus every (n-1)-flat through $\Gamma(t)$ meets $\Gamma$ at most n times. But then every (n-2)-flat meets $\Gamma$ in at most n-1 points and 1. follows by Hjelmslev's theorem. □

The preceding proof also yields the following:

COROLLARY. *Let* $\Gamma$ *be a regular and elementary curve in* $P^n$, $n \geq 1$. *Suppose there is a* t *satisfying* 2. a) *and the projection of* $\Gamma$ *from* $\Gamma(t)$ *is of order* n-1. *Then* $\Gamma$ *is of order* n.

THEOREM 2. *Let* $\Gamma: P^1 \to P^n$ *be a regular and elementary curve*; $n \geq 2$. *Let* $t_1,\dots,t_{k+1}$ *be mutually distinct parameters*; $0 \leq k \leq n-2$. *Put*

$$L^{-1} = \emptyset,\ L^i = <\Gamma(t_1),\dots,\Gamma(t_{i+1})>$$

*and let* $\Gamma^{i+1}$ *denote the projection of* $\Gamma$ *from* $L^i$; $0 \leq i \leq k$. *If*

a)$_k$ $L^k$ *is a* k-*flat*,

b)$_k$ $L^i \not\subset <L^{i-1}, \Gamma_{n-i-1}(s)>$ *for all* $s \neq t_{i+1}$ *and for* i = 0,1,...,k *and*

c)$_k$ $\Gamma^{k+1}$ *is of order* n-k-1,

*then* $\Gamma$ *is of order* n.

PROOF. The case $k = 0$ is identical with the Corollary. Suppose the theorem has been proved up to $k-1$ and assume $a)_k - c)_k$. We apply the following projections:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\quad L^k \quad} & \Gamma^{k+1} \\ {\scriptstyle L^{k-1}}\downarrow & \nearrow_{\;\Gamma^k(t_{k+1})} & \\ \Gamma^k & & \end{array} \qquad ; \text{ cf. } \underset{\sim}{2}.$$

Since $a)_k$ and $b)_k$ imply $a)_{k-1}$ and $b)_{k-1}$, $\Gamma^k$ is readily seen to be regular. By [3,p.113], the projection of an elementary curve from any point is elementary and thus $\Gamma^k$ is elementary. Also by $b)_k$,

$$<L^{k-1}, \Gamma(t_{k+1})> = L^k \not\subset <L^{k-1}, \Gamma_{n-k-1}(s)>$$

and hence by $\underset{\sim}{2}$., $\Gamma^k(t_{k+1}) \notin \Gamma^k_{n-k-1}(s)$ if $s \neq t_{k+1}$. Therefore the Corollary and $c)_k$ imply $c)_{k-1}$. Our assertion now follows from the induction hypothesis. □

REMARK. We note the case $k = n-2$. The condition $c)_{n-2}$ can be reformulated as follows: each $(n-1)$-flat through

$$L^{n-2} = <\Gamma(t_1),\ldots,\Gamma(t_{n-1})>$$

meets $\Gamma$ in at most one additional point (and hence in exactly one additional point).

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Received July 14, 1983

C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No.5, October 1983 octobre

# THE SIMPLE GROUPS PSL(2, 7) AND PSL(2, 11)

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Klein and Fricke [7, pp. 712-715] discussed a plane quartic curve of genus 3 whose group of self-collineations is PSL(2, 7). This group has two conjugate sets of 7 octahedral subgroups $S_4$. Each such subgroup permutes 4 bitangents whose 8 points of contact lie on a conic, and each set of 7 quadruples of bitangents comprises all the 28. For these bitangents, the notation of Coxeter [3, p. 134, first column of Table 2] agrees with that of Edge [5, p. 184, on the right of his two tables] if we identify Coxeter's 0 and $\infty$ with Edge's 7 and 0. The coordinates used by Coxeter arise from Ciani's equation for the quartic curve [1, pp. 364-365], namely

$$x^4 + y^4 + z^4 + 3\bar{c}(y^2z^2 + z^2x^2 + x^2y^2) = 0 ,$$

where $c = (-1 + i\surd 7)/2$, so that $c^2 + c + 2 = 0$. It is convenient to define also $b = c - \bar{c} = i\surd 7$.

The two sets of 7 conics may be expressed as

$$u_\nu = 0, \qquad v_\nu = 0,$$

where $\nu$ runs over GF[7]. Cyclic permutation of x, y, z will be seen to have the effect of multiplying $\nu$ by 4 : an operation of period 3 over GF[7].

$$u_0 = -2c(x^2 + y^2 + z^2), \qquad v_0 = x^2 + y^2 + z^2 - b\bar{c}(yz + zx + xy),$$

$$u_2 = -c^3x^2 + y^2 + z^2 - 2byz, \qquad v_1 = x^2 + y^2 + z^2 + b\bar{c}(-yz + zx + xy),$$

$$
\begin{aligned}
u_1 &= x^2 - c^3y^2 + z^2 - 2bzx, & v_4 &= x^2 + y^2 + z^2 + b\bar{c}(yz - zx + xy),\\
u_4 &= x^2 + y^2 - c^3z^2 - 2bxy, & v_2 &= x^2 + y^2 + z^2 + b\bar{c}(yz + zx - xy),\\
u_6 &= -c^3x^2 + y^2 + z^2 + 2byz, & v_5 &= -\bar{c}^2(cx^2 + \bar{c}y^2 + \bar{c}z^2),\\
u_3 &= x^2 - c^3y^2 + z^2 + 2bzx, & v_6 &= -\bar{c}^2(\bar{c}x^2 + cy^2 + \bar{c}z^2),\\
u_5 &= x^2 + y^2 - c^3z^2 + 2bxy, & v_3 &= -\bar{c}^2(\bar{c}x^2 + \bar{c}y^2 + cz^2).
\end{aligned}
$$

These 14 quadratic forms are nearly the same as those in the last column of Coxeter's Table 2 [3, p. 134], but the coefficients $-2c$ and $-\bar{c}^2$ have been inserted so as to exhibit certain linear dependencies among them:

$$\Sigma u_\nu = 0, \qquad \Sigma v_\nu = 0,$$

$$cv_\nu = u_{\nu+1} + u_{\nu+2} + u_{\nu+4},$$

$$\bar{c}u_\nu = v_{\nu+3} + v_{\nu+6} + v_{\nu+5}$$

[7, p. 759]. (Notice that the squares 1, 2, 4 in GF[7] are the powers of 2, while the non-squares 3, 6, 5 are the same multiplied by 3.) These expressions for $cv_\nu$ and $\bar{c}u_\nu$ are epitomized in Figure 1, where the point $v_0$ is joined to $u_1$, $u_2$, $u_4$, and so on. This is the Levi graph for the finite projective plane (Fano plane) PG(2, 2) [2, p. 118]. In other words, the two sets of 7 conics are represented in a natural manner by the points and lines of that finite plane.

Figure 1 is 'complementary' to Coxeter's Fig. 7 [3, p. 135] in two senses. We can derive Figure 1 by deleting the 7 edges $u_\nu v_\nu$ and interchanging the letters u and v; or we can place the 14 vertices in new positions and join all pairs of vertices

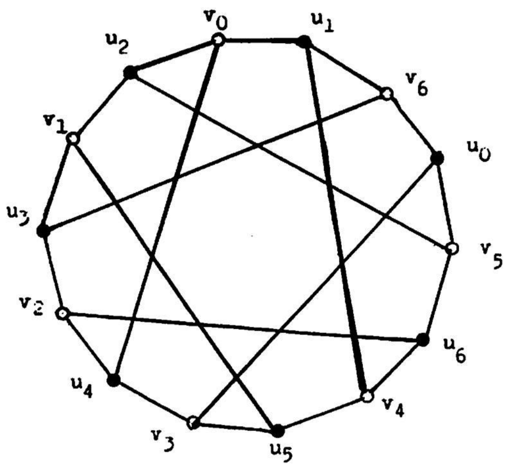


FIGURE 1

that were not joined before. Notice that, in both these figures, all incidences are preserved when the suffixes are doubled.

There is a striking analogy between Klein's plane quartic of genus 3 in the complex projective plane and a certain curve of order 20 and genus 26 in complex projective 4-space [6, p. 648], whose group of self-collineations is PSL(2, 11). This group has two conjugate sets of 11 icosahedral subgroups $A_5$. Each such subgroup permutes the 40 points of intersection of the curve with a certain quadric.

For the two sets of 11 quadrics, Klein and Fricke [8, p. 428] obtained equations

$$x_\nu = 0 \quad \text{and} \quad y_\nu = 0$$

where $\nu$ runs over GF[11]; $x_\nu$ and $y_\nu$ are abbreviations for certain very complicated quadratic forms in the coordinates $z_0$, $z_1$, $z_2$, $z_3$, $z_4$, $z_5$. By writing

$$C = (-1 + i\surd 11)/2\ ,$$

so that $C^2 + C + 3 = 0$, we can express the linear dependencies among these 22 quadratic forms as follows:

$$\Sigma x_\nu = 0, \qquad \Sigma y_\nu = 0,$$

$$Cy_\nu = x_{\nu+1} + x_{\nu+3} + x_{\nu+9} + x_{\nu+5} + x_{\nu+4}\ ,$$

$$\bar{C}x_\nu = y_{\nu+2} + y_{\nu+6} + y_{\nu+7} + y_{\nu+10} + y_{\nu+8}$$

[8, p. 429]. (Notice that the squares 1, 3, 9, 5, 4 in GF[11] are the powers of 3, while the non-squares 2, 6, 7, 10, 8 are the same multiplied by 2.) These expressions for $Cy_\nu$ and $\bar{C}x_\nu$ are epitomized in Figure 2, where the point $y_0$ is joined to $x_1$, $x_3$, $x_9$, $x_5$, $x_4$, and so on. Figure 3 is its 'complement', epitomizing the consequent relations

$$-Cy_\nu = x_\nu + x_{\nu+2} + x_{\nu+6} + x_{\nu+7} + x_{\nu+10} + x_{\nu+8}\ ,$$

$$-\bar{C}x_\nu = y_\nu + y_{\nu+1} + y_{\nu+3} + y_{\nu+9} + y_{\nu+5} + y_{\nu+4}\ .$$

Notice that, in both these figures, all incidences are preserved when the suffixes are trebled.

Figure 3 is the Levi graph for the regular skew polytope $_5\{3, 5, 3\}_5$ [4, Figure 1]. In other words, the two sets of 11 quadrics are represented in a natural manner by the 11 vertices

$$0 = x_0\ ,\quad 1 = x_1\ ,\quad 2 = x_2\ , \ldots,\quad t = x_{10}$$

and 11 hemi-icosahedral facets

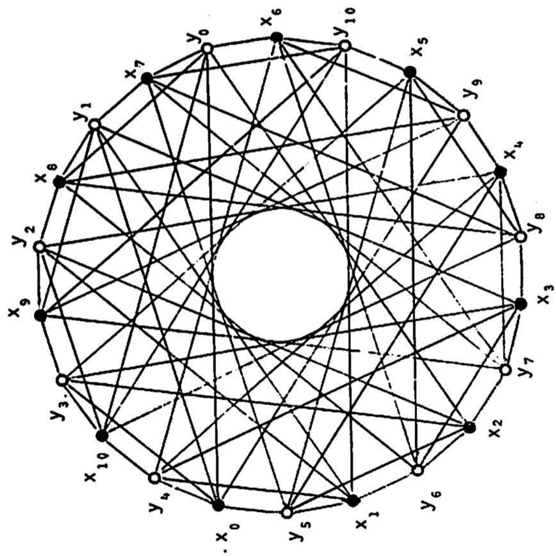


FIGURE 3

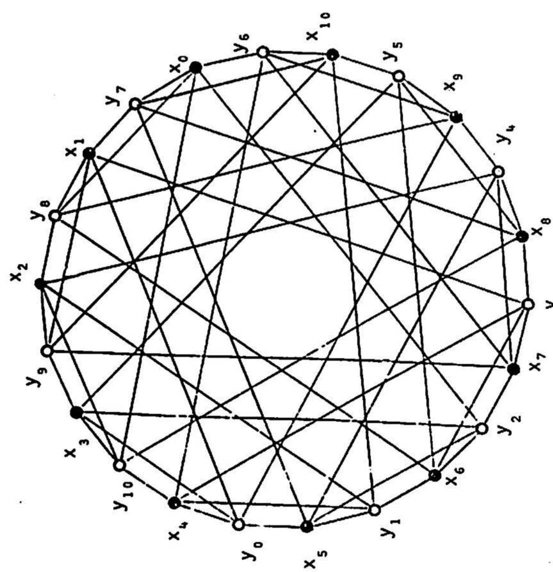


FIGURE 2

$$y_0,\quad y_1,\quad y_2,\quad y_3$$
$$y_4,\quad y_5,\quad y_6$$
$$y_7,\quad y_8,\quad y_9,\quad y_{10}$$

of that 10-dimensional skew polytope. Moreover, the 6 vertices of the facet $y_\mu$ represent the 6 quadrics $x_\nu = 0$ into which the quadric $y_\mu = 0$ can be reciprocated [6, p. 649].

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Received August 2, 1983

C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No.5, October 1983 octobre

# SUR LES FONCTIONS SIMULTANEMENT SURADDITIVES ET SURMULTIPLICATIVES

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*Presented by J. Aczél, F.R.S.C.*

Résumé

On établit que sous des conditions assez peu contraignantes les fonctions simultanément suradditives et surmultiplicatives sont en fait additives et multiplicatives.

---

L'objet de cette Note est de présenter une démonstration simple du théorème suivant et d'en déduire quelques résultats récemment proposés.

Théorème 1

Soit A un anneau et R un anneau ordonné. On suppose que R possède la propriété suivante :

(1) Si $z \in R$ et $z \neq 0$, alors $z^2 > 0$.

Soit $P : A \to R$ une application satisfaisant pour tous x,y de A,

(2) $P(x+y) \geqslant Px + Py$ (suradditivité)

et

(3) $P(xy) \geqslant Px.Py$ (surmultiplicité).

Alors P est à la fois additive et multiplicative, c'est-à-dire que l'on a les signes d'égalité dans (2) et (3).

DEMONSTRATION

On commence par noter que $P(0) \leqslant 0$ en faisant $x = y = 0$ dans (2). D'ailleurs, on vérifie que $P(O) = 0$ puisque (3) fournit $P(0) \geqslant P(O).P(0)$. Donc $0 \geqslant P(0) \geqslant P^2(0)$, d'où $P(0) = 0$.

On procède ensuite à une cascade d'inégalités. On a $-P^2(x) \geqslant -P(x^2)$ d'après (3). Puis $-P(x^2) \geqslant -P(-x^2)$ car avec (2)

$$0 = P(x^2 - x^2) \geqslant P(x^2) + P(-x^2).$$

Enfin $P(-x^2) \geqslant P(x)\, P(-x)$ d'après (3). On résume donc

$$0 \geqslant P(x)\,[P(-x) + P(x)].$$

Cette inégalité étant valable pour tout x de A, on a aussi

$$0 \geqslant P(-x)\,[P(x) + P(-x)].$$

Soit en additionnant

$$0 \geqslant [P(x) + P(-x)]^2.$$

Grâce à l'hypothèse (1) faite sur R, il vient $P(-x) = -P(x)$.

On écrit maintenant

$$P(x+y) \geqslant P(x) + P(y) = -P(-x) - P(-y) \geqslant -P(-x-y) = P(x+y).$$

D'où les égalités partout, donc (4). De même

$$P(xy) \geqslant P(x)\,P(y) = -[P(x)\,P(-y)] \geqslant -P(-xy) = P(xy).$$

D'où les égalités partout. Ce qui termine la démonstration du théorème.

Remarques

1. On notera que nous n'avons pas supposé la commutativité des anneaux A ou R, ni d'ailleurs l'existence d'une unité pour ces anneaux. On peut même diminuer les hypothèses faites sur A, puisque l'on a seulement besoin des propriétés suivantes : l'existence d'une addition $x + y$ , d'une multiplication $x.y$ , d'un élément 0 dans A et d'un opposé $-x$ de sorte que les relations suivantes aient lieu :

$$x + (-x) = 0\ ,\ -(x+y) = (-x) + (-y)\ ,\ -(-x) = x\ ,$$
$$x(-y) = -(xy)\ ,\ 0.0 = 0.$$

2. La question de la nécessité de la condition (1) relative à R pour la validité du théorème 1 reste ouverte.

Corollaire 2 (M. Radulescu, 1980).

Soit $\mathbb{R}$ l'axe réel. L'identité est la seule fonction $f : \mathbb{R} \to \mathbb{R}$ , non identiquement nulle, suradditive et surmultiplicative.

L'anneau $\mathbb{R}$ possède en effet la propriété (1) ce qui permet l'application du Théorème 1. Un résultat classique (cf Aczél, 1966) sur les endomorphismes de $\mathbb{R}$ permet de conclure. Ce corollaire 2 contraste avec le fait qu'il existe beaucoup de fonctions $f : \mathbb{R} \to \mathbb{R}$ simultanément sousadditives et sousmultiplicatives (par exemple : $f(\lambda) = c|\lambda|^{\alpha}$, où $0 < \alpha < 1 < c$), beaucoup de fonctions sousadditives et surmultiplicatives (par exemple: $f(\lambda) = c|\lambda|^{\alpha}$ , où $0 < \alpha < 1$, $0 < c < 1$) et beaucoup de fonctions suraddives et sousmultiplicatives (par exemple : $f(\lambda) = (1+c)\lambda - c|\lambda|$, où $c > 0$).

Dans un travail à paraître (P. Volkmann, 1983) le théorème est établi pour un anneau A muni d'une unité (laquelle intervient de façon essentielle dans la preuve) et pour un anneau produit $R = \mathbb{R}^E$. C'est ce résultat qui a suggéré la validité du théorème 1.

Corollaire 3

Soient X et Y deux espaces compacts non vides. On désigne par C(X) (respectivement C(Y)) l'algèbre de toutes les fonctions réelles et continues définies sur X (respectivement sur Y), algèbre munie de l'ordre naturel. Supposons qu'il existe une application $P : C(X) \to C(X)$ possédant les cinq propriétés suivantes

a) Pour tout y de Y, il existe g dans P(C(X)) telle que $g(y) \neq 0$,

b) P(C(X)) sépare les points de Y, c.-à-d. que pour $y_1 \neq y_2$, il existe $g \in P(C(X))$ et $g(y_1) \neq g(y_2)$,

c) Pour tout ensemble ouvert, nonvide U dans X, il existe une fonction f de C(X), dont le support est contenu dans U, et telle que Pf ne soit pas identiquement nulle.

d) P est suradditive,

e) P est surmultiplicative.

Alors X et Y sont des espaces homéormorphes.

Démonstration

Puisque C(Y) possède la propriété (I), les conditions d) et e) et le théorème 1 impliquent que P soit additive et multiplicative. Fixons $y \in Y$. Soit $g = Pf$ une fonction fournie par a). On a $g = P(f.1) = g\, P(1)$ donc $P(1)(y) = 1$. Dès lors la fonction $F : \mathbb{R} \to \mathbb{R}$ définie par $F(\lambda) = P(\lambda.1)(y)$ est additive, multiplicative et non identiquement nulle. D'après le corollaire 2, $F(\lambda) = \lambda$. Il en résulte que l'application $f \to P\, f(y)$ est linéaire et multiplicative. Puisque toute $f \geqslant 0$ de C(X) est un carré, cette application est non négative, donc définit une mesure de Radon positive non nulle sur X. Celle-ci d'après la multiplicativité a son support réduit à un point $\phi(y)$ de X. Nous avons défini $\phi : Y \to X$. Soient $y_1 \neq y_2$. D'après b), il existe f dans C(X) et $P\, f(y_1) \neq P\, f(y_2)$.

Soit $f(\phi(y_1)) \neq f(\phi(y_2))$. Donc $\phi(y_1) \neq \phi(y_2)$. L'application $\phi$ est donc injective. Elle est continue comme il est facile de le voir par contradiction en utilisant la compacité de X et le fait que $P\, f = f \circ \phi \in C(Y)$. Le corollaire 3 est acquis, par compacité, si l'on prouve l'injectivité de $\phi$. Supposons par contradiction que le compact $\phi(y)$ ne coïncide pas avec X. D'après c), il existe f de C(X) dont le support est dans le complémentaire de $\phi(Y)$ et $y \in Y$ de sorte que $P\, f(y) \neq 0$. Mais la condition du support fournit $P\, f(y) = f(\phi(y)) = 0$, ce qui est contradictoire.

Corollaire 4

Avec les notations du corollaire 3, il existe une bijection suradditive et surmultiplicative entre C(X) et C(Y) si et seulement si X et Y sont homéomorphes.

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Received August 3, 1983

C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No.5, October 1983 octobre

# TWISTED HONEYCOMBS $\{3, 5, 3\}_t$

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*Presented by H.S.M. Coxeter, F.R.S.C.*

Abstract

When $t \leq 9$, twisted honeycombs $\{3, 5, 3\}_t$ are finite. When $t = 6$ the honeycomb is reflexible.

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For every regular polyhedron or tessellation $\{p, q\}$ (with p-gonal faces, q round each vertex) we define Petrie polygon as a skew polygon such that every two consecutive edges, but no three, belong to a face ([2], pp. 128). For a regular honeycomb $\{p, q, r\}$ (with cells $\{p, q\}$, r round each vertex), a Petrie polygon is then defined to be a skew polygon such that every three consecutive edges, but no four, belong to a cell ([2], pp. 145). Each regular honeycomb has two enantiomorphic (left and right) kinds of Petrie polygons ([3], pp. 26). It is sometimes possible to derive a contracted honeycomb from $\{p, q, r\}$ by identifying pairs of points separated by $t$ steps along every right-handed Petrie polygon to obtain a twisted honeycomb $\{p, q, r\}_t$. Let $t'$ denote the number of sides of left-handed Petrie polygons.

If $t = t'$, $\{p, q, r\}_t$ is said to be reflexible and its group of symmetries $[p, q, r]_t$ is given by

$$R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1R_2)^p = (R_2R_3)^q = (R_3R_4)^r = (R_1R_2R_3R_4)^t = 1 \, , \qquad (1)$$
$$R_3 \rightleftarrows R_1 \rightleftarrows R_4 \rightleftarrows R_2$$

([5], pp. 89). The rotation subgroup of this group is generated by

$$L = R_2R_4 \ , \quad M = R_4R_1 \quad \text{and} \quad N = R_3R_1 \ , \tag{2}$$

and has the presentation

$$L^2 = M^2 = N^2 = (LM)^p = (LMN)^q = (MN)^r = (LN)^t = 1 \ . \tag{3}$$

If $t \neq t'$ ($t'$ is the order of $LMNM$), $\{p, q, r\}_t$ is not symmetric by a reflection and its symmetry group $((t, p, r; q))$ is given by (3) ([2], pp. 142). In this case, the honeycomb is said to be chiral. For detailed account see [3] and [6].

$\{3, 5, 3\}$ is a 3-dimensional regular map or a hyperbolic honeycomb. Its cells are regular icosahedra $\{3, 5\}$. When all points of $\{3, 5, 3\}$ that are separated by five steps along Petrie polygons of a cell are identified we obtain a finite reflexible honeycomb having

11 vertices, 55 edges, 55 faces, 11 cells

denoted by ${}_5\{3, 5, 3\}_5$ [6]. It cells are hemi-icosahedra $\{3, 5\}_5$ and its symmetry group ${}_5[3, 5, 3]_5$ is

$$\begin{gathered} R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1R_2)^3 = (R_2R_3)^5 = (R_3R_4)^3 = (R_1R_2R_3)^5 = 1 \ , \\ R_3 \rightleftarrows R_1 \rightleftarrows R_4 \rightleftarrows R_2 \ . \end{gathered} \tag{4}$$

Coxeter in [6] showed that the group is isomorphic to PSL(2, 11) of order 660, and the generators $R_1$, $R_2$, $R_3$, $R_4$ can be represented by

$$\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} -3 & 3 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} -5 & -3 \\ 5 & 5 \end{bmatrix} \pmod{11} \tag{5}$$

respectively. The relation $(R_1R_2R_3)^5 = 1$ implies $R_1 = (R_2R_3R_1)^4R_2R_3$ or, in terms of $L$, $M$ and $N$ given by (2), $R_1 = (LMNMLN)^2LMN$. Furthermore $R_2 = (LMNMLN)^2LMNML$ , $R_3 = (LMNMLN)^2LM$ and $R_4 = (LMNMLN)^2LMNM$ . Hence ${}_5[3, 5, 3]_5 \cong PSL(2, 11)$ is generated by L, M, N which can be represented by

$$\begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 4 & -3 \\ 2 & -4 \end{bmatrix} \qquad (6)$$

respectively.

Chiral honeycombs $\{3, 5, 3\}_4$ and $\{3, 5, 3\}_5$ are enantiomorphic (reflected images of each other) ([3], pp. 34). Their cells are collapsed icosahedra, obtained from icosahedra by identifying opposite vertices while still distinguishing opposite edges ([8], pp. 783).

In the group ((6, 3, 3; 5)) , given by

$$L^2 = M^2 = N^2 = (LM)^3 = (LMN)^5 = (MN)^3 = (LN)^6 = 1 , \qquad (7)$$

the icosahedral subgroup generated by LM and N has 11 cosets. Hence the whole group is of order $11 \cdot 60 = 660$ and since matrices (6) satisfy relations (7) it follows that

$$((6, 3, 3; 5)) \cong PSL(2, 11) .$$

Furthermore, since $(LMNM)^6 = 1$ , the honeycomb $\{3, 5, 3\}_6$ is reflexible. It has

11 vertices, 110 edges, 110 faces, 11 cells.

Its cells are collapsed icosahedra. The complete group $[3, 5, 3]_6$ of the honeycomb is (1) with $p = r = t/2 = 3$ and $q = 5$ , and its order is $2 \cdot 660 = 1320$ . This group is isomorphic to the direct product

$$[3, 5, 3]_6 \cong PSL(2, 11) \times C_2 ,$$

which can be verified directly from the representation

$$R_1 = \tau\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}, \quad R_2 = \tau\begin{bmatrix} -3 & 3 \\ 4 & 3 \end{bmatrix}, \quad R_3 = \tau\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}, \quad R_4 = \tau\begin{bmatrix} -5 & -3 \\ 5 & 5 \end{bmatrix},$$

where $\tau^2 = 1$ .

Matrix

$$\begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix}$$

of determinant -1 transforms $R_1, R_2, R_3, R_4$ into $R_4, R_3, R_2, R_1$ and hence reciprocates the honeycomb. This implies that the total symmetry group of a pair of reciprocal $\{3, 5, 3\}_6$'s is isomorphic to $PGL(2, 11) \times C_2$ . We adopt the symbol $[[3, 5, 3]]_6$ for this group ([4], p. 323) and thus have

$$[[3, 5, 3]]_6 \cong PGL(2, 11) \times C_2 .$$

This group is the complete symmetry group of the 3-valent bipartite graph with 220 vertices and 330 edges consisting of the midpoints of all edges and midpoints of all faces of $\{3, 5, 3\}_6$ (compare with [4], pp. 323). Since the order of the group is $1320 \cdot 2 = 330 \cdot 2^3$ it follows that the graph is 3-regular and hence different from the graph with 220 vertices in the Foster's list [7]. Furthermore this graph is two-fold covering of the graph with 110 vertices obtained from $_5\{3, 5, 3\}_5$ [6].

At the present there are only two more known finite twisted honeycombs $\{3, 5, 3\}_t$ : for $t = 7$ and $t = 9$ . The details are given in the table below. These were both found by computer enumeration of cosets of the icosahedral subgroup generated by N and LM in ((t, 3, 3; 5)) [1]. It might be interesting to mention that, when $t = 9$ the computer defined total of 12654 cosets before finally collapsing to 57 .

Setting $t = 8$ in (3) implies $(LN)^4 = 1$ and hence $[3, 5, 3]_8$ is isomorphic to $[3, 5, 3]_4$ .

Table

| Honeycomb | Order of the group | t' | Vertices | Edges | Faces | Cells |
|---|---|---|---|---|---|---|
| $\{3, 5, 3\}_4$ | 360 | 5 | 6 | 60 | 60 | 6 |
| $\{3, 5, 3\}_6$ | 1320 | 6 | 11 | 110 | 110 | 11 |
| $\{3, 5, 3\}_7$ | 12180 | 29 | 203 | 2030 | 2030 | 203 |
| $\{3, 5, 3\}_9$ | 3420 | 10 | 57 | 570 | 570 | 57 |

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Received August 12, 1983

C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No. 5, October 1983 octobre

# THE CURVATURE TENSOR OF LORENTZ MANIFOLDS WITH SPIN STRUCTURE PART II

W.H. Greub and H.R. Petry

*Presented by P. Scherk, F.R.S.C.*

Abstract

In this paper the results of part I[1)] are applied to the case where the spin bundle $\xi$ admits a parallel cross-section. In particular, a simple expression for the metric tensor of a Ricci-flat Lorentz manifold is deduced, which corresponds to a known class of gravitational fields including the gravitational waves.

## 1. Spin structures with a parallel cross-section (covariantly constant sections)

Let M be a Lorentz manifold with a spin structure $\xi$; i.e. M is a 4-dimensional manifold with a metric g of signature (-,+++), oriented and time-oriented, and $\xi$ is complex vector bundle of rank 2 over M subject to the following conditions[1)]:

a) $\xi$ admits a complex-valued two-form $\varepsilon$ on each fibre.

b) There is a strong bundle map $\Gamma: \tau_M \to \xi_S$ from the tangent space $\tau_M$ to the bundle $\xi_S$ of self-adjoint antilinear transformations of the fibres of $\xi$.

c) For any two vector fields X,Y on M we have the relation

$$g(X,Y) = \frac{1}{2}\operatorname{tr}((\Gamma(X)\cdot\Gamma(Y)) \quad . \tag{1}$$

Recall that $\lambda$ is self-adjoint in the fibre $F_x$ $\xi$ if and only if

$$\varepsilon_x(\lambda a,b) = \bar{\varepsilon}_x(a,\lambda b)$$

holds for all $a, b \in F_x$.
The Levi-Civitȧ connection $\tilde{\nabla}$ on $\tau_M$ induces a unique connection $\nabla$ on $\xi$ with the properties

$$\nabla_X \varepsilon = 0 \, , \tag{2}$$

$$\nabla_X \Gamma(Y) - \nabla_Y \Gamma(X) - \Gamma(Y)\nabla_X + \Gamma(X)\nabla_Y - \Gamma[X,Y] = 0 \quad . \tag{3}$$

Suppose now that $\xi$ admits a covariantly constant section $\sigma_1$ without zeros. (If M is connected it will already be non-zero if it is non-zero at a single point). Locally we can always find a second section $\sigma_2$ such that

$$\varepsilon(\sigma_1, \sigma_2) = 1 \, . \tag{4}$$

Hence

$$\varepsilon(\sigma_1, \nabla_X \sigma_2) = X(\varepsilon(\sigma_1, \sigma_2)) - \varepsilon(\nabla_X \sigma_1, \sigma_2) = 0$$

whence

$$\nabla_X \sigma_2 = A(X) \cdot \sigma_1 \tag{5}$$

for some complex-valued 1-form A on M. Note that $\sigma_2$ is unique up to a scalar multiple of $\sigma_1$, i.e. if $\lambda$ is an arbitrary complex function on M then

$$\sigma_2' = \sigma_2 + \lambda \sigma_1$$

satisfies equation (4) and equation (5) is replaced by

$$\nabla_X \sigma_2' = A' \sigma_1 \tag{6}$$

with

$$A' = A + d\lambda \, . \tag{7}$$

Next write

$$\Gamma(X)\sigma_\alpha = \Gamma_\alpha^\beta(X)\sigma_\beta \qquad (\alpha = 1,2) \, . \tag{8}$$

Since $\Gamma(X)$ is self-afjoint we find the relations

(9) $\Gamma_1^2 = \bar{\Gamma}_1^2 \quad , \quad \Gamma_2^1 = \bar{\Gamma}_2^1 \quad , \quad \Gamma_2^2 = -\bar{\Gamma}_1^1 \quad ,$

which state that the 1-forms $\Gamma_2^1$ and $\Gamma_1^2$ must be real-valued. From equs. (3) and (5), we then derive the formulae:

(10a) $d\Gamma_1^2 = 0,$ (10b) $d\Gamma_1^1 = -A\wedge\Gamma_1^2,$ (10c) $d\Gamma_2^1 = \bar{A}\wedge\Gamma_1^1 + A\wedge\bar{\Gamma}_1^1$ .

## 2. The canonical coordinates

To proceed further we need the following

Lemma 1: Let $\Lambda,\sigma$ and $\psi$ be 1-forms such that

(11) $$d\phi = \psi\wedge\Lambda \ .$$

Assume that $\Lambda$ is closed and non-zero everywhere. Then there are local functions $\alpha,\beta,\gamma$ such that

$$\phi = d\alpha + \beta\Lambda \quad , \qquad \psi = d\beta + \gamma\Lambda \ .$$

Proof: We can choose local coordinates $x^1 \dots x^n$ on M with $dx^n = \Lambda$. Write $\phi = \phi_i dx^i$, $\psi = \psi_i dx^i$. Equations (11) reads then as follows:

(11a) $\dfrac{\partial\phi_i}{\partial x^n} - \dfrac{\partial\phi_n}{\partial x^i} = 0 \ ,$ (11b) $\dfrac{\partial\phi_n}{\partial x^i} - \dfrac{\partial\phi_i}{\partial x^n} = \psi_i, \quad i,n < n \ .$

By equation (11a) $\phi_i = \dfrac{\partial\alpha}{\partial x^i}$ for some locally defined function $\alpha$.
Set $\beta = \phi_n \cdot \dfrac{\partial\alpha}{\partial x^n}$ .
Thus the first formula of the lemma holds. Moreover, (11b) yields

$$\psi_i = \frac{\partial^2}{\partial x^i\partial x^n}\alpha + \frac{\partial\beta}{\partial x^i} - \frac{\partial^2}{\partial x^n\partial x^i}\alpha = \frac{\partial\beta}{\partial x^i}$$

By setting $\gamma = \psi_n - \dfrac{\partial\beta}{\partial x_n}$ we obtain the second formula of the lemma, q.e.d.

Lemma 2: The local cross-section $\sigma_2$ can be chosen such that A takes the form

(12) $$A = -\gamma\Gamma_1^2$$

for some locally defined function $\gamma$.

Proof: Equs. (10a) and (10b) imply by lemma 1 that $A = -d\beta-\gamma\Gamma_1^2$. Set $\sigma_2' = \sigma_2+\beta\sigma_1$. Then $A' = -\gamma\Gamma_1^2$ by formula (7), q.e.d.

We assume from now on that $\sigma_2$ has been chosen this way, and insert (12) into the equations (10a-c). The result is

(13a) $d\Gamma_1^2 = 0$ , (13b) $d\Gamma_1^1 = 0$ , (13c) $d\Gamma_2^1 = (\bar{\gamma}\Gamma_1^1+\gamma\bar{\Gamma}_1^1)\wedge\Gamma_1^2$ .

Theorem I: M admits local coordinates u,v,x,y such that

(14a) $\Gamma_1^2 = du$ , (14b) $\Gamma_1^1 = d(x+iy)$ , (14c) $\Gamma_2^1 = dv + fdu$ .

Here f is a real-valued function which depends only on u,x,y and satisfies

$$\bar{\gamma} = \frac{\partial f}{\partial z}$$

where $z = x+iy$.

Proof: Equations (13a-b) yield locally a real function u and a complex function $z = x+iy$ such that (14a) and (14b) hold. By proposition 1 the equations (13a) and (13c) yield real functions v,f,k such that $\Gamma_2^1 = dv + fdu$, and

(15) $$\bar{\gamma}dz + \gamma d\bar{z} = df + kdu .$$

Since $\Gamma$ is a strong bundle isomorphism, it follows that the 1-forms du, dv, dx, dy are linearly independent at each point. Thus u,v,x,y are indeed coordinate functions. Equation (15) shows that $k = -\frac{\partial f}{\partial u}$ , $\frac{\partial f}{\partial v} = 0$ and that $\frac{\partial f}{\partial z} = \bar{\gamma}$ , q.e.d.

The next theorems express the Lorentz metric on M and its Riemannian curvature in terms of these coordinates.

Theorem II: In terms of the coordinates in theorem I g has the form:

(16) $$g = \frac{1}{2}(du\otimes dv+dv\otimes du) + fdu\otimes du + dx\otimes dx + dy\otimes dy$$

Proof: Let X and Y be arbitrary vector fields on M. Formula (1) yields

$$g(X,Y) = \frac{1}{2}\,\Gamma_\alpha^\beta(X)\bar{\Gamma}_\beta^\alpha(Y)$$

Using theorem I we find that

$$g(X,Y) = \frac{1}{2}\,(du(X)(dv(Y)+f dv(Y)) + (dv(X)+f du(X))du(Y)$$

$$+\frac{1}{2}\,(dz(X)d\bar{z}(Y) + d\bar{z}(X)dz(Y))\ ,$$

and thus (16) is established.

Theorem III: In terms of the coordinates in theorem I the Riemannian curvature R is expressed as follows:

$$(17)\quad R = -[dx{\wedge}du)\otimes(dx{\wedge}du) + (dy{\wedge}du)\otimes(dy{\wedge}du)]\cdot\frac{\partial^2 f}{\partial z\partial\bar{z}}$$

$$-[dx{\wedge}du)\otimes(dx{\wedge}du) - (dy{\wedge}du)\otimes(dy{\wedge}du)]\mathrm{Re}\,\frac{\partial^2 f}{\partial\bar{z}^2}$$

$$-[dx{\wedge}du)\otimes(dy{\wedge}du) + (dy{\wedge}du)\otimes(dx{\wedge}du)]\mathrm{Re}\,\frac{\partial^2 f}{\partial\bar{z}^2}$$

Moreover, the Ricci tensor $\hat{R}$, is given by

$$(18)\qquad \hat{R} = -\frac{\partial^2 f}{\partial z\partial\bar{z}}\cdot du\otimes du\ .$$

Proof: From ref. 1) we know that for arbitrary vector fields X,Y,V,W the following equation holds

$$R(X,Y;V,W) = \mathrm{Re}(\mathrm{tr}R_\xi(X,Y)\Gamma(V)\Gamma(W))\ ,$$

where $R_\xi$ denotes the curvature form of $\nabla$ in the spin-bundle $\xi$. Now

$$R_\xi(X,Y)\sigma_1 = 0$$

$$R_\xi(X,Y)\sigma_2 = dA(X,Y)\sigma_1\ .$$

Hence

$$\mathrm{tr}R_\xi(X,Y)\Gamma(V)\Gamma(W) = dA(X,Y)(\bar{\Gamma}_1^1(W)\Gamma_1^2(V)-\Gamma_1^2(W)\bar{\Gamma}_1^1(V)) .$$

Using the coordinate expressions of $A,\Gamma_1^1$ and $\Gamma_1^2$ given by theorem I, gives us immediately formula (17): To obtain the Ricci tensor we observe that the metric (16) allows only diagonal contractions of the curvature with respect to dx and dy.

Corollary: Assume that g is Ricci-flat. Then f is the real part of a function h of z,u which is analytic in z. In this case the curvature takes the form

$$R = -\mathrm{Re}\,\frac{\partial^2 h}{\partial z^2}\,(dz\wedge du)\otimes(dz\wedge du) \tag{19}$$

Proof: Since $\hat{R} = 0$, it follows that $\frac{\partial^2}{\partial z \partial \bar{z}} f = 0$, i.e. f is a harmonic function of x and y and thus the real part of a function h with the desired properties. Equation (19) follows then immediately from (17) q.e.d.

Conclusion:

The form of the Ricci-flat metric which we have deduced above, corresponds to a known class of gravitational fields including the gravitational waves[2]. Our result characterizes these fields intrinsically and shows that these are the only ones which admit parallel spinor fields.

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Received August 15, 1983

C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No.5, October 1983 octobre

# SPECTRAL MEASURES AND AUTOCORRELATION VIA TRANSMUTATION

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*Presented by M. Shinbrot, F.R.S.C.*

Abstract. A model one dimensional geophysical problem is studied with transmission readout leading directly to the spectral measure as the Fourier transform of an autocorrelation function. The solution of the inverse problem is then obtained in the form of impedance as a function of travel time and stability estimates are indicated. A new type of extended Gelfand-Levitan (G-L) equation is also obtained. The details will appear in [4].

1. Basic constructions. Consider the inverse problem for $\rho(x)v_{tt} = (\mu(x) v_x)_x$ ($v = 0$ for $t < 0$) where some initial impulse disturbance is supplied at (0,0) and from the readout or response at some point $x = \tilde{x}$ (transmission data) one wants to find the acoustic impedance $A = (\rho\mu)^{\frac{1}{2}}$ as a function of travel time $y = \int_0^x (\rho/\mu)^{\frac{1}{2}} d\xi$. One can think of SH waves in a vertically stratified earth, where $\rho$ = density and $\mu$ is a shear modulus, and with readout $v(0,t) = G(t)$ (reflection data) this has been treated in various ways (cf. [1;2;3;7]). We are motivated here by certain problems of wave propagation in a spherically symmetric medium with impulse disturbance at the origin and readout at $r = \tilde{r}$. With the change of variable to y our equation is (•) $(Av_y)_y/A = Q(D_y)v = v_{tt}$ and we assume $\rho,\mu \in C^1$ with $\rho$ and $\mu$ constant for $x \geq \tilde{x}$ (thus $A' = 0$ and $A = A_\infty$ eventually while $0 < a \leq A(y) \leq \tilde{A} < \infty$ - by rescaling if necessary we assume $1 = A(0)$). Take now (•) with impulse data (♦) $v_t(y,0) = \delta(y)$ whose solution in

* Work partially supported by ONR Contract #N00014-83-K0051

the model case $A = 1$ is $v(y,t) = Y(t-y)$ ($y,t \geq 0$ where $Y$ is the Heavyside function). We refer now to [1] and let $\varphi_\lambda^Q$ denote "spherical functions" satisfying (*) $Q\varphi = -\lambda^2\varphi$ with $\varphi_\lambda^Q(0) = 1$ and $D_y\varphi_\lambda^Q(0) = 0$. One can express Riemann functions for example in terms of transmutation kernels and we write $S(y,t,\eta) = \langle\varphi_\lambda^Q(y)\varphi_\lambda^Q(\eta), \text{Cos}\lambda t\rangle_\omega$ with $R(y,t,\eta) = \langle\varphi_\lambda^Q(y)\varphi_\lambda^Q(\eta), \text{Sin}\lambda t/\lambda\rangle_\omega$ where $\omega$ denotes the Q-spectral pairing. For $A$ of the type treated here one has (cf. [1;2;7]) $d\omega = d\omega_Q = \omega_Q(\lambda)d = \omega d\lambda$ with $\omega = 1/2\pi|c_Q(\lambda)|^2$ where, defining the Jost solutions of (*) by $\Phi_{\pm\lambda}^Q(y) \sim A_\infty^{-\frac{1}{2}}\exp(\pm i\lambda y)$ as $y \to \infty$, one has $\varphi_\lambda^Q = c_Q(\lambda)\Phi_\lambda^Q + c_Q(-\lambda)\Phi_{-\lambda}^Q$ ($A_\infty$ is known). Now for the solution of (•) with Cauchy data data $v(y,0) = f(y)$ and $v_t(y,0) = F(y)$ one has $v(y,t) = \langle S(y,t,\eta), A(\eta)f(\eta)\rangle + \langle R(y,t,\eta), A(\eta)F(\eta)\rangle$ so for $f(\eta) = 0$ and $F(\eta) = \delta(\eta)/A_0 = \delta(\eta)$ (cf. (♦)) we obtain $v(y,t) = R(y,t) = \langle\varphi_\lambda^Q(y), \text{Sin}\lambda t/\lambda\rangle_\omega$. For $y = 0$ the readout is $G(t) = \langle 1, \text{Sin}\lambda t/\lambda\rangle_\omega$ from which the spectral density $\omega$ is determined by $\omega(\lambda) = (2\lambda/\pi)\int_0^\infty G(t)\text{Sin}\lambda t dt$. From this one obtains $A$ by use of the (G-L) machine (cf. [1;2;7]) and the nature of the map $G \to A$ is fairly well understood. Now pick some sufficiently large $\tilde{y}$ (this is discussed below) and write $H(t) = v(\tilde{y},t) = \langle\varphi_\lambda^Q(\tilde{y}), \text{Sin}\lambda t/\lambda\rangle_\omega$ so that $\varphi_\lambda^Q(\tilde{y})\omega/\lambda = (2/\pi)\int_0^\infty H(t)\text{Sin}\lambda t dt$ and hence

$$(1) \qquad \varphi_\lambda^Q(\tilde{y})\int_0^\infty G(t)\text{Sin}\lambda t dt = \int_0^\infty H(t)\text{Sin}\lambda t dt$$

The function $\varphi_\lambda^Q(\tilde{y})$ is an even entire function of exponential type $\tilde{y}$ and the expression of $G$ in terms of $H$ in (1) can be regarded in the context of deconvolution (cf. [6]). Indeed by the Paley-Wiener theorem we write ($\Phi(t)$ even) $\varphi_\lambda^Q(\tilde{y}) = \hat{\Phi}(\lambda) = 2\int_0^{\tilde{y}}\Phi(t)\text{Cos}\lambda t dt$ ($\hat{\Phi} = F\Phi$ = Fourier transform). Taking $\tilde{G}$ and $\tilde{H}$ to be odd extensions of $G$ and $H$ it follows that $\hat{\Phi}(\lambda)\tilde{G}^\wedge = \tilde{H}^\wedge$.

<u>Theorem 1.1</u>. <u>The readouts $G$ at $y = 0$ and $H$ at $y = \tilde{y}$ satisfy</u> $\Phi * \tilde{G} = \tilde{H}$.

2. <u>Splitting techniques</u>. Let us use the transmutation machine of [1] to split up everything in order to make "deconvolution" optimal. Recall that $\omega =$

$1/2\pi|c_Q(\lambda)|^2$ is even and $\varphi_\lambda^Q(y)$ is even in $\lambda$. Also for calculation it will be convenient to remove a $1/\lambda$ factor and write $H'(t) = \langle\varphi_\lambda^Q(y), \text{Cos}\lambda t\rangle_\omega = \int_0^\infty \varphi_\lambda^Q(y)\omega(\lambda)\text{Cos}\lambda t d\lambda = \frac{1}{2}\int_{-\infty}^\infty \varphi_\lambda^Q(y)\omega(\lambda)e^{i\lambda t}d\lambda$. Further $\omega(\lambda)\varphi_\lambda^Q(y) = (1/2\pi)\{\Phi_\lambda^Q(y)/c_Q(-\lambda) + \Phi_{-\lambda}^Q/c_Q(\lambda)\} = (1/2\pi)\{\Psi_\lambda^Q(y) + \Psi_{-\lambda}^Q(y)\}$ where $\Psi_\lambda^Q(y) = \Phi_\lambda^Q(y)/c_Q(-\lambda)$ is analytic in the upper half plane. Hence set $H_1(t) = (1/4\pi)\int_{-\infty}^\infty \Psi_{-\lambda}^Q(y)\exp(i\lambda t)d\lambda = (1/4\pi)\int_{-\infty}^\infty \Psi_\lambda^Q(y)\exp(-i\lambda t)d\lambda$ and it follows that $H'(t) = H_1(t) + H_1(-t)$. We remark that in using the Fourier theory in various forms one automatically introduces various odd and even extensions of the quantities H, G, H', etc. We note that by contour integral arguments as in [1;2;7] $H_1(t) = 0$ for $t < \tilde{y}$ and thus $H_1(t)$ provides the readout H' for $t > \tilde{y}$ ($H_1(-t)$ contributes nothing to H' for $t > 0$). Now we can write $\frac{1}{2}\Psi_\lambda^Q(\tilde{y}) = \int_{\tilde{y}}^\infty H'(t)\exp(i\lambda t)dt$ from the above. Further let us take our readout point $\tilde{x}$ large enough so that $\rho$ and $\mu$ are constant for $x \geq \tilde{x}$ and consequently $A(y) = A_\infty$ for $y \geq \tilde{y}$ ($\tilde{y} = y(\tilde{x})$). But $\Phi_\lambda^Q(\tilde{y})$ is the Jost solution $\Phi_\lambda^Q(y) \sim A_\infty^{-\frac{1}{2}}\exp(i\lambda y)$ as $y \to \infty$ and for $\tilde{y}$ as indicated we must have then $\Phi_\lambda^Q(\tilde{y}) = A_\infty^{-\frac{1}{2}}\exp(i\lambda\tilde{y})$. Therefore we have

<u>Theorem 2.1. Under the hypotheses indicated for $\tilde{y}$ suitably large</u>

$$(2)\qquad 1/c_Q(-\lambda) = 2A_\infty^{\frac{1}{2}}e^{-i\lambda\tilde{y}}\int_{\tilde{y}}^\infty H_1(t)e^{i\lambda t}dt = 2A_\infty^{\frac{1}{2}}e^{-i\lambda\tilde{y}}\hat{H}'(\lambda);\ \omega = (2/\pi)A_\infty|\hat{H}'|^2$$

<u>from which one can recover A by the methods of [1;2;3;7]</u>.

<u>Remark 2.2</u>. This formula seems "striking" because it directly exhibits the spectral measure in terms of the Fourier transform of an autocorrelation type function $H(t) = \int_{\tilde{y}}^\infty H'(t+\tau)H'(\tau)d\tau$. One knows of course that there is an intimate and profound connection between vibrating string problems and problems of extrapolation and interpolation for stationary time series (cf. [5]) and the formula (2) seems to fit into that context very neatly. In fact it seems to provide a new link directly connected with the geophysical problem and perhaps this will lead in directions connecting the traditional time series analysis in geophysics with exact techniques for the inverse problem.

In the same spirit one can split $G'(t) = G_1(t) + G_1(-t)$ where $G_1(t) = (1/4\pi)\int_{-\infty}^{\infty}\Psi_{-\lambda}^{Q}(0)\exp(i\lambda t)d\lambda = (1/4\pi)\int_{-\infty}^{\infty}\Psi_{\lambda}^{Q}(0)\exp(-i\lambda t)d\lambda$ ($G_1(t) = 0$ for $t < 0$) and hence after some calculation one obtains

<u>Theorem 2.3. Let</u> $K(t-\tau) = (A_\infty^{1/2}/2\pi)\int_{-\infty}^{\infty} e^{-i\lambda(t-\tau)}\Phi_\lambda^Q(0)e^{-i\lambda y}d\lambda$. <u>Then</u> $K(t-\tau) = 0$ <u>for $t < 0$ and $\tau > t+\tilde{y}$ with</u> $G_1(t) = \int_{\tilde{y}}^{\tilde{y}+t} H_1(\tau)K(t-\tau)d\tau$. <u>This gives a domain of dependence relation between</u> $H_1$ <u>and</u> $G_1$ <u>but we do not know</u> $\Phi_\lambda^Q(0)$.

3. <u>Recovery of A and stability</u>. The recovery of A via $\omega$ indicated in Theorem 2.1 by methods of [1;2;3;7] goes as follows. Recall $G(t) = \int_0^\infty (\mathrm{Sin}\lambda t/\lambda)\,\omega(\lambda)d\lambda$ and set $d\sigma = d\omega - (2/\pi)d\lambda$ with $T(y,x) = \int_0^\infty (\mathrm{Sin}\lambda x/\lambda)\mathrm{Cos}\lambda y d\lambda$. Then $G(t) = \int_0^\infty (\mathrm{Sin}\lambda t/\lambda)\{d\sigma + (2/\pi)d\lambda\} = 1 + G_r(t)$ and depending on whether $y > x$ or $y < x$ one obtains $T(y,x) = \frac{1}{2}\{G_r(y+x) - G_r(y-x)\}$ or $T(y,x) = \frac{1}{2}\{G_r(y+x) + G_r(x-y)\}$. The G-L equation can then be written as $(x < y)$ (†) $K(y,x) + \frac{1}{2}\{G_r(t+x) - G_r(y-x)\} = \frac{1}{2}\int_0^y K(y,s)\{G_r'(x+s) - G_r'(|x-s|)\}ds$. We recall also that $A^{-1/2}(y) = 1 - K(y,y)$ so A is determined from the unique solution of (†). For stability one first takes approximate data $G_r^*$ close enough to $G_r$ on $[0,2y]$ and then one has $\|\Delta K(y,\cdot)\|_{\infty,y} \le c\{\|\varepsilon\|_{\infty,2y} + \|\varepsilon'\|_{L^1(2y)}\}$ where $\varepsilon = G_r^* - G_r = \Delta G$ (cf. [3]). In this context the stability question has also been investigated numerically in [8;9] with good results. Now with $G'$ considered even because of the cosine representation (and adjusting a factor of 2) we can write from Theorem 2.1 (‡) $G'(t) = \int_0^\infty \omega(\lambda)\mathrm{Cos}\lambda t d\lambda = (A_\infty/\pi)\int_{-\infty}^{\infty}|\hat{H}'|^2\exp(-i\lambda t)d\lambda = A_\infty \mathcal{H}(t) = A_\infty\int_{\tilde{y}}^{\infty} H'(t+\tau)H'(\tau)d\tau$. We conclude then that

<u>Theorem 3.1. For $t > 0$ one has</u> $G'(t) = A_\infty \mathcal{H}(t)$.

We factor out the delta functions in (‡) now by working with $G = 1 + G_r$, $A^{1/2}H_1 = \delta(t-\tilde{y}) + h_1$, etc. for $t > 0$ and one obtains (note $\tilde{y}-t < \tilde{y}$ for $t > 0$ and $h_1(\tilde{y}-t) = 0$), $G_r' = h_1(\tilde{y}+t) + \int_{\tilde{y}}^{\infty} h_1(t+\xi)h_1(\xi)d\xi$. Set $h(t) = \int_{\tilde{y}}^{t} h_1(\tau)d\tau$ and $G_r(t) = \int_0^t G_r'(\tau)d\tau$; after some calculation we obtain then

<u>Theorem 3.2. Assume</u> $\|h_1\|_\infty$, $\|h\|_\infty$, $\|h_1^*\|_\infty$, <u>and</u> $\|h^*\|_\infty \le M$ <u>and that</u> $\|h_1\|_{L^1}$ <u>is suitably small. Then</u> $|\Delta A|$ <u>on</u> $[0,\tilde{y}]$ <u>can be estimated as indicated above via estimates</u> $\|\varepsilon'\|_{L^1(0,2T)} \le c\|\Delta h_1\|_{L^1(\tilde{y},\infty)}$ and $\|\varepsilon\|_{\infty,2T} \le \|\Delta h\|_{\infty,\tilde{y}+2T} + M\{\|\Delta h_1\|_{L^1(\tilde{y},\infty)} + \|\Delta h\|_{L^1(\tilde{y},\infty)}\}$.

4. <u>Complements</u>. Let us go back to (1), multiply by $(2/\pi)\mathrm{Sin}\lambda\tau$, and integrate to obtain $\varphi_\lambda^Q(\tilde{y}) = \int_0^{\tilde{y}} K_\eta(\tilde{y},\eta)\mathrm{Cos}\lambda\eta d\eta + A_\infty^{-\frac{1}{2}}\mathrm{Cos}\lambda\tilde{y}$. Here we recall that $1 - K(y,y) = A^{-\frac{1}{2}}(y)$, $A = A_\infty$ at $\tilde{y}$, $K(y,\eta) = 0$ for $\eta > y$, and $K(y,0) = 0$ (cf. [1], p. 282). Then from (1) it follows after some calculation that (■) $H(\tau) = \frac{1}{2}A_\infty^{-\frac{1}{2}}\{G(\tilde{y}+\tau) + G(\tilde{y}-\tau)\} + \frac{1}{2}\int_0^\infty G(t)\{K_2(\tilde{y},t-\tau) + K_2(\tilde{y},\tau-t) - K_2(\tilde{y},t+\tau)\}dt$. Take now $\tau > \tilde{y}$ so $G(\tilde{y}-\tau) = 0$ and $K_2(\tilde{y},t+\tau) = 0$ ($K_2 \sim K_\eta(\xi,\eta)$) to arrive at

<u>Theorem 4.1. For $\tau > \tilde{y}$,</u> $H(\tau) = \frac{1}{2}\{G(\tilde{y}+\tau) + G(\tau-\tilde{y})\} + \frac{1}{2}\int_0^{\tilde{y}} K(\tilde{y},s)\{G'(\tau-s) - G'(\tau+s)\}ds$.

Let us note now that for $\tau < \tilde{y}$ (where $H(\tau) = 0$) (■) reduces to the G-L equation (†). Indeed $G(\tau-\tilde{y}) = 0$ in (■) while $-G(\tilde{y}-\tau)$ and $K_2(\tilde{y},t+\tau)$ remain and after some calculation one obtains (†). Let us think now of $G$ as odd and $G'$ as even (via their sine and cosine representations) and write Theorem 4.1 and (†) together (treating $K(\tilde{y},\xi)$ as odd in $\xi$ with $K(\tilde{y},\xi) = 0$ for $|\xi| > \tilde{y}$ and $K(\tilde{y},0) = 0$). There results

<u>Theorem 4.2. For $\tau > 0$ one has an extended G-L equation</u> $H(\tau) - K(\tilde{y},\tau) = \mathfrak{G}(\tilde{y},\tau) + \frac{1}{2}K(\tilde{y},\cdot) * G'$ <u>where</u> $\mathfrak{G}(\tilde{y},\tau) = \frac{1}{2}\{G(\tau+\tilde{y}) + G(\tau-\tilde{y})\}$.

<u>Remark 4.3</u>. The dependence indicated in Theorem 4.1 between G and H again involves only finite intervals but in a different manner than in Theorem 2.3. One hopes to use Theorem 4.1 and Theorem 4.2 in conjunction to develop a numerical scheme based on fixed point ideas to relate A on $[0,\tilde{y}]$ with H on $[\tilde{y},3\tilde{y}]$.

<u>Remark 4.4</u>. The derivation of the G-L equation (†) in [2;7] was largely ad hoc in nature and we can give a canonical derivation based on general

transmutation procedures as in [1]. Thus the canonical G-L equation has the form $\langle \beta(y,t), A(t,x)\rangle = \tilde{\beta}(y,x)$ where $\beta$ and $\tilde{\beta}$ are the kernels of certain transmutations $D^2 \to Q$ and $A(t,x) = \int_0^\infty \omega(\lambda)\mathrm{Cos}\lambda x\mathrm{Cos}\lambda t d\lambda = \langle \mathrm{Cos}\lambda x, \mathrm{Cos}\lambda t\rangle_\omega$. In fact $\tilde{\beta}(y,x) = \langle \mathrm{Cos}\lambda x, \varphi_\lambda^Q(y)\rangle_\omega = 0$ for $x < y$ and $\beta(y,t) = (2/\pi)\int_0^\infty \varphi_\lambda^Q(y)\mathrm{Cos}\lambda t d\lambda$. Now for $x < y$ we integrate formally to obtain $\langle \beta(y,t), A(t,x)\rangle = 0$ for $A(t,x)$ expressible in terms of G. An analysis of kernels as in [1], pp. 332-333 then allows us to write $\beta(y,t) = A^{-\frac{1}{2}}(y)\delta(y-t) + K_t(y,t)$ and some routine calculations (using $K(y,y) = 1 - A^{-\frac{1}{2}}(y)$) leads to (†).

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Received August 30, 1983

C.R. Math. Rep. Acad. Sci. Canada - Vol. V, No.5, October 1983 octobre

# FACTORIZATION OF ISOMETRIES IN $O^+$ INTO HALF-TURNS

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## 1. Introduction

It is well known that every isometry of a finite dimensional regular vector space V is a product of at most n reflections, where n is the dimension of V. This theorem is often called the Cartan-Dieudonné theorem (see [1], p.129). Similarly, every isometry in the proper orthogonal group $O^+$ is a product of at most n half-turns (see [1], p.134). A number of questions arises: What can be said if V is not regular, or if dim V is not finite, and is it possible to find for each isometry $\pi$ a natural number s such that $\pi$ is the product of s reflections, but not a product of fewer than s reflections? Similar questions can be posed for elements in $O^+$ considered as products of half-turns.

Some of the answers are known: For the orthogonal group of a finite dimensional regular vector space see P.Scherk [7], for the nonregular case see M.Götzky [5]. E.Ellers [3] also includes infinite dimensional vector spaces. For the group $O^+$ of a finite dimensional regular vector space see H.Ishibashi [6].

In this paper we deal with the proper orthogonal group $O^+$ generated by half-turns. The vector space need not be finite dimensional and it need not be regular. We shall state our main result in section 2. The proof will be published elsewhere.

* Research supported in part by NSERC Canada grant no.A7251

In [2], F.Bachmann gives a characterization of the group of motions of a Euclidean or non-Euclidean plane. This group is isomorphic to the group $O_3^+$. In order to extend his characterization to geometries of higher dimensions, suitable axioms have to be found. To pursue this goal, Bachmann suggests ([2], p.341) to solve the length problem first. In our main theorem we come very close to a solution of the length problem. We show that the length of each isometry has to be one of two possible values. We also provide an example which indicates that the geometry given by the path and the fix of an isometry alone is not sufficient to give a criterion to distinguish between the two possible values.

## 2. Products of Half-Turns

Let V be a (possibly infinite dimensional) vector space over the commutative field K such that char $K \neq 2$.

For $\pi \in GL(V)$ we define $F(\pi) = \ker(\pi - 1)$ and $B(\pi) = \mathrm{im}(\pi - 1)$ and call $F(\pi)$ the fix and $B(\pi)$ the path of $\pi$. We shall always assume that $\dim B(\pi) < \infty$. Then we can define $\det \pi = \det_{B(\pi)} |\pi$ (see [4], p.300).

Let f be a symmetric bilinear form on V. Then (V,f) is called an orthogonal vector space. For any subset $M \subset V$ we define $M^{\perp} = \{y \in V;\ f(M,y) = \{0\}\}$. A vector $v \in V$ is called isotropic if $v \in v^{\perp}$. For any subspace W of V the space $W \cap W^{\perp}$ is called the radical of W; we write $W \cap W^{\perp} = \mathrm{rad}W$. If $\mathrm{rad}W = \{0\}$, we call W regular. The radical $V^{\perp}$ of V will be denoted by R.

The group $O(V) = \{\pi \in GL(V);\ f(x\pi, y\pi) = f(x,y),\ F(\pi) \supset R\}$ is called the (weak) orthogonal group of (V,f). An element in $O(V)$ is called an isometry. If $\sigma \in O(V)$, $\dim B(\sigma) = 1$, and $B(\sigma) \not\subset R$, then $\sigma$ is called a reflection. Every reflection is involutory and $B(\sigma)$ is regular.

The subgroup $O^+(V) = \{\pi \in O(V);\ \det \pi = 1\}$ is called the proper orthogonal group. A product of two reflections is called a rotation. Clearly every rotation is in $O^+(V)$. If $\eta = \sigma\varrho$, where $\sigma$ and $\varrho$ are reflections, and $B(\sigma) \subset B(\varrho)^{\perp}$, then the rotation $\eta$ is called a half-turn. Clearly a half-turn $\eta$ is an involution, $\dim B(\eta) = 2$, and $B(\eta)$ is regular.

THEOREM. Assume $\dim V/R \geqq 3$, $K \neq GF(3)$, $\pi \in O^+(V)$, and $\dim F(\pi)^{\perp}/R \geqq \dim B(\pi)$.
Then there are half-turns $\eta_1, \ldots, \eta_s$ such that $\pi = \eta_1 \cdots \eta_s$.
Put $\dim B(\pi) + \dim(B(\pi) \cap R) = d$.
For the minimal s we get
$2s = d$ or $2s = d+2$ if $\dim F(\pi)^{\perp}/\mathrm{rad}F(\pi)^{\perp} \geqq 2$ and
$2s = d+2$ or $2s = d+4$ if $\dim F(\pi)^{\perp}/\mathrm{rad}F(\pi)^{\perp} < 2$.

The assumption $\dim V/R \geqq 3$ is necessary for $\pi \in O^+$ to be a product of half-turns. For $R = \{0\}$ this is obvious since for a 2-dimensional vector space there is only one half-turn, namely $-1$, and there are clearly many elements in $O^+(V)$.

The assumption $\dim F(\pi)^{\perp}/R \geqq \dim B(\pi)$ is necessary for $\pi \in O(V)$ to be a product of reflections (see [3]). Since $\dim F(\pi)^{\perp}/R \leqq \dim B(\pi)$ is true for every $\pi \in O(V)$, we obtain $\dim F(\pi)^{\perp}/R = \dim B(\pi)$. This last equality is always true if $\dim V < \infty$. Assume $\pi \in O(V)$ and $\pi$ is a product of reflections; then $\pi \in O^+(V)$ if and only if $\dim B(\pi)/R = \dim B(\pi) - \dim(B(\pi) \cap R)$ is even.

It is interesting to observe that the spaces $B(\pi)$ and $F(\pi)$ alone are not sufficient to determine the minimal number s of half-turns in our theorem. In order to see that, let V be a regular vector space such that $\dim V = 2$, and let $\sigma, \varrho$ be reflections with $B(\sigma) \neq B(\varrho)$. Put $\tau = \sigma\varrho$. Then dim $B(\tau) = 2$ and $s = 1$ if and only if $B(\sigma) \subset B(\varrho)^{\perp}$; otherwise $s = 2$.

During the course of the proof we have to establish the following factorization for $\pi \in O(V) \setminus O^{+}(V)$:

LEMMA. Assume $K \neq GF(3)$, $B(\pi) \cap R = \{0\}$, dim $B(\pi) = 2t+1$, and dim $B(\pi)/\mathrm{rad}B(\pi) \geq 2$. Then there are half-turns $\eta_1, \dots, \eta_t$ and a reflection $\omega$ such that $\pi = \eta_1 \cdots \eta_t \omega$.

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Received September 13, 1983

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