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A DUALITY THEOREM FOR UNBOUNDED CLOSED OPERATORS

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The duality theorem for a bounded linear operator T on a complex Banach space X asserts that T is decomposable, in the sense of Foiaş [4] (more generally S-decomposable, in the sense of Vasilescu [7]) iff the conjugate T^* is decomposable (resp. S-decomposable) [9]. In this study, we extend the spectral duality theory to an unbounded closed operator $T : D_T(\subset X) \to X$.

We may assume that every finite open cover $\{G_i\}_{i=0}^n$ of the spectrum $\sigma(T)$, in symbols $\{G_i\}_{i=0}^n \in \text{cov}\,\sigma(T)$, has, at most, one unbounded set G_0. A set $G \subset \mathbb{C}$ is said to be a neighborhood of ∞, in symbols $G \in V_\infty$, if for $r > 0$ sufficiently large, $\{\lambda \in \mathbb{C} : |\lambda| > r\} \subset G$. If T has the single valued extension property, we shall make an extensive use of the spectral manifold $X(T,S) = \{x \in X : \sigma(x,T) \subset S\}$, where $\sigma(x,T)$ denotes the local spectrum of T at $x \in X$. One should note that, contrary to the bounded case, $\sigma(x,T) = \emptyset$ does not necessarily imply that $x = 0$.

We write Inv T for the lattice of all subspaces of X, that are invariant under T. For $Y \in \text{Inv}\,T$, $T|Y$ is the restriction of T to Y and T/Y denotes the coinduced operator on the quotient space X/Y. $A_T(\Omega)$ is the class of functions $f : \Omega(\subset \mathbb{C}) \to \mathbb{C}$, which are analytic in a neighborhood Ω of $\sigma_\infty(T) = \sigma(T) \cup \{\infty\}$ and regular at ∞.

1. DEFINITION. T is said to have the spectral decomposition property (SDP) if, for any $\{G_i\}_{i=0}^n \in \text{cov}\,\sigma(T)$ with $G_0 \in V_\infty$, there exists a system $\{Y_i\}_{i=0}^n \subset \text{Inv}\,T$, satisfying the following conditions:

(I) $Y_i \subset D_T$ if G_i $(1 \le i \le n)$ is relatively compact;

(II) $X = \sum_{i=0}^{n} Y_i$ and $\sigma(T|Y_i) \subset G_i$, $0 \leq i \leq n$.

2. DEFINITION. T is said to be decomposable if, for any $\{G_i\}_{i=0}^n \in \text{cov}\, \sigma(T)$ with $G_0 \in V_\infty$, there exists a system $\{Y_i\}_{i=0}^n$ of spectral maximal spaces [4] of T satisfying conditions (I) and (II) of Definition 1.

In the bounded case, there is no distinction between the decomposable operators and the operators with the SDP ([1], [5], [6]). For unbounded closed operators, the two concepts no longer coincide.

As a preliminary result, a well-known property of the spectral manifold admits an extension to the case of an unbounded closed operator T:

3. THEOREM. If T has the SDP then, for any closed $F \subset \mathbb{C}$, $X(T,F)$ is a spectral maximal space of T and $\sigma[T|X(T,F)] \subset F \cap \sigma(T)$.

The following direct sum decomposition of $X(T,F)$ will play a central role in the subsequent theory.

4. THEOREM. If T has the SDP then, for any compact $F \subset \mathbb{C}$, there exists a subspace $\Xi(T,F)$ with the following properties:

(i) $$X(T,F) = \Xi(T,F) \oplus X(T,\emptyset);$$

(ii) $T|\Xi(T,F)$ is bounded and $\sigma[T|\Xi(T,F)] = \sigma[T|X(T,F)]$.

The proof [2] of the Theorem reveals that $\Xi(T,F) = PX(T,F)$, where P is the spectral projection $P = \frac{1}{2\pi i} \int_\Gamma R[\lambda;T|X(T,F)d\lambda$ which is independent of the admissible contour Γ surrounding F. Theorem 4 is instrumental in characterizing an unbounded decomposable operator in terms of the following

5. THEOREM. Given T, the following assertions are equivalent:

(i) T is decomposable;

(ii) T has the SDP and $X(T,\emptyset) = \{0\}$, or equivalently,

T has the SDP and $\{0\}$ is a spectral maximal space of T;

(iii) T has the SDP and $T|X(T,F)$ is bounded for some compact F;

(iv) T has the SDP and every T-bounded spectral maximal space [2] is a spectral maximal space of T.

Some further consequences of Theorem 4, now follow.

6. COROLLARY. Given T, let F_0 be closed and F_1 be compact. Then

$$\Xi(T,F_0 \cap F_1) = X(T,F_0) \cap \Xi(T,F_1).$$

Moreover, if F_0 and F_1 are disjoint then

$$X(T,F_0 \cup F_1) = X(T,F_0) \oplus \Xi(T,F_1).$$

7. COROLLARY. If T has the SDP then, for every open $G \in V_\infty$, the coinduced $T/X(T,\overline{G})$ is bounded and $\sigma[T/X(T,\overline{G})] \subset G^c$, (^{c} stands for the complement). If G is open and relatively compact then $T/\Xi(T,\overline{G})$ is closed and

$$\sigma[T/\Xi(T,\overline{G})] \subset G^c.$$

Although the next corollary is not a direct consequence of Theorem 4, it fits into this sequel of properties.

8. COROLLARY. Suppose that T, T* and T** are densely defined. If T* has the SDP then

(i) for every closed F, $X^*(T^*,F)$ is closed in the weak*-topology of X*;

(ii) for every compact F, $\Xi^*(T^*,F)$ is closed in the weak*-topology of X*.

The spectral duality theorem will be achieved by two different approaches: (I) via functional calculus, (II) through the successive conjugates T*, T** and T*** of the given closed T.

In the first approach we assume that $\rho(T) \neq \emptyset$. The following preliminary results [3] are instrumental in the proof of the spectral duality theorem.

9. THEOREM. If T has the SDP then

(I) for every $f \in A_T(\Omega)$, f(T) is decomposable;

(II) for every closed F, $X[f(T);F] = \begin{cases} X[T,f^{-1}(F) \cap \sigma(T)], & \text{if } f(\infty) \in F; \\ \Xi[T,f^{-1}(F) \cap \sigma(T)], & \text{if } f(\infty) \in F^c. \end{cases}$

10. THEOREM. Given T, let $f \in A_T(\Omega)$ be injective on Ω. If f(T) is decomposable then T has the SDP.

11. COROLLARY. If T has the SDP and $f \in A_T(\Omega)$ is injective on Ω, then

$$X(T,\emptyset) = X[f(T),f(\infty)].$$

12. COROLLARY. Given T, let $A = R(\lambda;T)$ for some $\lambda \in \rho(T)$. T has the SDP iff A is decomposable. Furthermore, if T has the SDP then the following assertions are equivalent:

(i) $\sigma(x,T) = \emptyset$; (ii) $x \in X(A,\{0\})$; (iii) $\lim_{n\to\infty} \|A^n x\|^{1/n} = 0$

13. THE SPECTRAL DUALITY THEOREM (I).

Let T be densely defined and let $A = R(\lambda_0,T)$ for some $\lambda_0 \in \rho(T)$. The following properties hold:

(i) T has the SDP iff T* has the SDP;

(ii) T (resp. T*) has the SDP and $\overline{X(A,\mathbb{C}-\{0\})} = X$, (resp. $\overline{X^*(A^*,\mathbb{C}-\{0\})}^w$ is total in X*) iff T* (resp. T) is decomposable. ($\overline{\ \ }^w$ is the weak*-closure).

In the second approach, the hypothesis $\rho(T) \neq \emptyset$ is no longer needed but an appropriate domain-density condition, such as

(*) T and T* are densely defined;

(**) T, T* and T** are densely defined,

will be assumed. We shall avail of the following direct sum decomposition [9]:

$$X^{***} = KX^* \oplus (JX)^a$$

where J and K denote the natural embeddings of X into X** and of X* into X***, respectively. We use the notations Y^a and ${}^aZ$ for the annihilator of $Y \subset X$ in X* and the preannihilator of $Z \subset X^*$ in X, respectively.

Among the preliminary results of this approach to the spectral duality theorem, we mention the following [8]:

(I). Assumptions: (*) and $x \in D_T$. Then $Jx \in D_{T^{**}}$ and $T^{**}Jx = JTx$.

(II). Assumptions: (**), $x^{***} \in KX^*$ and $<T^{**}Jx,x^{***}>$ is a bounded linear functional of $Jx \in JD_T$. Then $x^{***} \in KD_{T^*}$ and $T^{***}x^{***} = KT^*K^{-1}x^{***}$.

Furthermore, (**) implies: $KD_{T^*} = KX^* \cap D_{T^{***}}$, $JD_T = JX \cap D_{T^{**}}$ and, for $Jx \in JD_T$, we have $JTx = T^{**}Jx$.

Moreover, if $Y_1 \subset X^*$ is closed for the weak*-topology of X^* then, in terms of the subspaces $Y_0 = {}^aY_1$, $Y_2 = Y_1{}^a$ and $Y_3 = Y_1{}^{aa} \subset X^{***}$, several equivalences between certain quotient spaces, under topological isomorphism, can be established which ultimately lead us to the following

14. THEOREM. Given T, assume that (**) holds and T* has the SDP. Let $G \subset \mathbb{C}$ be open and $F = G^c$.

(A) If $G \in V_\infty$ and $Y = {}^aX^*(T^*,\overline{G})$ then

(i) $Y \subset D_T$, $Y \in \text{Inv } T$, $\sigma(T|Y) \subset F$;

(ii) T/Y is closable. If $\overline{T/Y}$ is the minimal closed extension of T/Y, then $\sigma(\overline{T/Y}) \subset \overline{G}$.

(B) If G is relatively compact and $Y = {}^a\Xi(T^*,\overline{G})$ then

(i) $Y \in \text{Inv } T$, $\sigma(T|Y) \subset F$;

(ii) T/Y is bounded with domain $D_{T/Y} = X/Y$ and $\sigma(T/Y) \subset \overline{G}$.

15. THE SPECTRAL DUALITY THEOREM (II): T has the SDP iff T* has the SDP.

To prove the theorem, a two-set open cover $\{G_0,G_1\}$ of $\sigma(T)$ is considered.

The "only if" part, for a two-summand decomposition of X follows easily from Theorem 4. For the "if" part, an annihilator of an appropriate subspace of X* is defined and used for a spectral decomposition of X/Y. The isometric isomorphism $(\overline{T/Y})^* \cong T^*|Y^a$, in terms of Theorem 14 is used to relate the spectral decomposition of X to that of X/Y. The proof reaches its conclusion by the equivalence of the two-summand spectral decomposition and the SDP, as obtained in [2].

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THE COEFFICIENT FIELD OF A SEMIGROUP ALGEBRA

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A commutative ring R and a semigroup S determine the semigroup ring RS of S over R. The questions arise as to whether RS and S determine R and whether RS and R determine S. More specifically, consider the following questions.

(Q1) If $RS_1 \simeq RS_2$, does it follow that $S_1 \simeq S_2$?

(Q2) If $R_1S \simeq R_2S$, is $R_1 \simeq R_2$?

The first question has been extensively investigated for groups S_1 and S_2 (see [7, Chapter 14] and [8, Chapter III]). Our results concern (Q2). The isomorphism question for polynomial rings (see [4], [1]) is a special case of (Q2), for each ring of the form $R[\{X_\lambda\}]/A$, where A is an ideal generated by a family of difference binomials $X_{\lambda_1}^{e_1}\ldots X_{\lambda_n}^{e_1} - X_{\lambda_1}^{f_1}\ldots X_{\lambda_n}^{f_n}$, is a monoid ring over R. It is known that isomorphism of $R_1[X]$ and $R_2[X]$ need not imply isomorphism of R_1 and R_2, and each of the papers [9] and [5] contains examples of non-isomorphic two-dimensional Noetherian domains R_1 and R_2 such that the torus extensions $R_1[X,X^{-1}] \simeq R_1Z$ and $R_2[X,X^{-1}] \simeq R_2Z$ are isomorphic. For fields R_1 and R_2, however, we obtain the following result.

THEOREM A. Assume that F and K are fields, S and T are semigroups, S contains a periodic element, and $FS \simeq KT$. Then $F \simeq K$.

Theorem A is striking for two reasons. First, there are few results in the literature that have been proved for such general semigroups S and T; for example, each monoid contains a periodic element, so the conclusion of Theorem A holds for arbitrary monoids S and T. Second, the case where the coefficient ring is a field has historically been of strong interest in studying isomorphism questions of this type — for example, in work on (Q1).

We outline a proof of Theorem A; details appear elsewhere [3]. The case of Theorem A where S and T are groups is due to Adjaero and Spiegel [2]. The proof proceeds by reducing the question for general S and T to the case where S and T are periodic commutative semigroups.

To reduce to the case where S and T are commutative, we use the fact that there exists a smallest congruence $\sim_S$ on S such that the factor semigroup $S/\sim_S$ is commutative. Moreover, the kernel I_S of the canonical homomorphism of FS onto $F[S/\sim_S]$ is the unique minimal ideal of FS with commutative residue class ring. Hence the isomorphism $FS \simeq KT$ induces an isomorphism $F[S/\sim_S] \simeq K[T/\sim_T]$, where $S/\sim_S$ and $T/\sim_T$ are commutative.

For commutative S and T, the isomorphism $FS \simeq KT$ induces an isomorphism $FS/N(FS) \simeq KT/N(KT)$, where $N(FS)$ and $N(KT)$ denote the nilradicals of FS and KT, respectively. Results of [6] show, however, that for a congruence $\sim$ depending upon $\mathrm{char}(F)$, we have $FS/N(FS) \simeq F[S/\sim]$. Hence, in considering the isomorphism

$FS \simeq KT$, we may assume without loss of generality that FS is commutative and reduced. This implies that the semigroup S is free of asymptotic torsion — a condition that is crucial in the next reduction of the problem. That reduction is to the case of periodic semigroups S and T , and a key result in the reduction process is the following proposition.

PROPOSITION B. Assume that S is a commutative semigroup such that FS is reduced. If $f \in FS\backslash(0)$ is such that the ideals (f) and $(f-f^2)$ are idempotent, then the set S^* of periodic elements of S is nonempty and $f \in F[S^*]$.

We sketch a proof of Proposition B. If S is cancellative, a proof can be based on the fact that for a torsion-free abelian group G , the group ring FG has only trivial units. In the general case we consider the Archimedean decomposition $S = \cup S_a$ of S induced by the congruence η defined by $s\eta t$ if and only if $rad(s + S) = rad(t + S)$. To prove the result, we use induction on m , the number of equivalence classes under η represented by elements of $Supp(f)$. Since S is free of asymptotic torsion, each S_a is a cancellative subsemigroup of S . If $m = 1$, we have $f \in FS_a$ for some a , and the desired conclusion follows from the cancellative case and the fact that f and $f - f^2$ generate idempotent ideals in FS_a . At the inductive step, we assume that $Supp(f)$ contains an aperiodic element b . The inductive hypothesis yields a contradiction if there exists

$s \in \text{Supp}(f)$ and a prime ideal P of S containing s but not b. Hence, we assume that $\text{Supp}(f)$ is contained in the subsemigroup U of S defined by $U = \{t \in S \mid b + S \subseteq \text{rad}(t + S)\}$. If $u \in U$ and $t_1, t_2 \in S_b$ are such that $t_1 + u = t_2 + u$, then $t_1 = t_2$. Using this fact, we can find a prime ideal Q of S disjoint from U such that Q contains the ideal $I = \{x \in S \mid t_1 + x = t_2 + x$ for some $t_1, t_2 \in S_b$ with $t_1 \neq t_2\}$. We consider f as an element of FT, where $T = S \setminus Q$. Upon passage to $F[T/\sim]$, where $\sim$ is the cancellative congruence on T, we obtain a contradiction to the assumption that b is aperiodic.

Theorem C follows from Proposition B; it provides the desired reduction to the case of periodic commutative semigroups.

THEOREM C. Assume that S is a commutative semigroup with $S^* \neq \phi$, and that FS is reduced. Let $W = \{f \in FS \mid f$ and $f - f^2$ generate idempotent ideals of $FS\}$. Then $W = FS^*$ and W is the unique maximal von Neumann regular subring of FS.

Finally, if S and T are periodic commutative semigroups and if $\phi: FS \longrightarrow KT$ is an isomorphism of FS onto KT, then ϕ induces a bijection ϕ^* of the family $\{\Delta_\alpha\}$ of residue fields of FS onto the residue fields of KT in such a way that Δ_α and $\phi^*(\Delta_\alpha)$ are isomorphic for each α. Periodicity of S implies, however, that F is naturally imbedded in each Δ_α in such a way that Δ_α is generated over F by roots of unity. Since F is a residue field of FS and K is a residue field of KT, it follows that there exist imbeddings $\sigma: F \longrightarrow K$ and

$\psi: K \longrightarrow F$ in such a way that K is generated over $\sigma(F)$ by roots of unity. This implies that $K = \sigma(F)$, however, and yields the desired isomorphism $F \simeq K$.

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RINGS ALL OF WHOSE TORSION QUASI-INJECTIVE MODULES ARE INJECTIVE

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ABSTRACT: In this paper we study rings whose torsion quasi-injective modules are injective, in the context of Goldie torsion theory. It is shown that such rings are precisely those for which each direct sum of torsion quasi-injective modules is quasi-injective. In the commutative case, these rings are characterized by the property that all modules over them split. Rings whose torsion quasi-injectives are Σ-quasi-injective will also be considered.

Throughout this note it is assumed that rings are associative, have the identity element, and modules are left unital. R will denote a ring with identity and R-Mod the category of left R-modules. For fundamental definitions and results related to torsion theories, we refer to [10] and [11]. A module M is quasi-injective if every homomorphism from a submodule of M into M can be extended to an endomorphism of M. A ring R is called left QI if each quasi-injective left R-module is injective. These rings were originally introduced in [2] and later studied by many authors (see, for example [3,4,5,7,8,9]). In [1], the present authors have studied rings all of whose (Goldie) torsion quasi-injective modules are injective. A summary of the main results proved in [1] is given below.

A ring R will be called a left TQI-ring if each torsion quasi-injective left R-module is injective. If (T,F) is a hereditary torsion theory and $F(T)$ denotes the associated filter of left ideals of R, i.e. $F(T) = \{I \mid I$ is a left ideal of R, and $R/I \in T\}$, then a left ideal of R which is a member of $F(T)$ will be called an F-ideal. The following results have been proved.

THEOREM 1. Let (G,F) be the Goldie torsion theory for R-Mod. Then the following are equivalent:

(1) R is left TQI.

(2) Each direct sum of torsion quasi-injective modules is quasi-injective.

THEOREM 2. Let R be a ring with an essential socle, and (G,F) be the Goldie torsion theory for R-Mod. Then the following are equivalent:

(1) R is left TQI.

(2) R has ACC on F-ideals, each F-ideal is the intersection of maximal left ideals, and R/socle(R) is a left QI-ring.

THEOREM 3. Let R be a ring whose non-zero torsion cyclic modules have non-zero socle, and (G,F) be the Goldie torsion theory for R-Mod. Then the following are equivalent:

(1) R is left TQI.

(2) R has ACC on F-ideals and each F-ideal is the intersection of maximal left ideals of R.

(3) The torsion class coincides with the class of semi-simple modules.

EXAMPLE 1. Let F be a field and let $R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$. Then R is a left TQI-ring but not a left QI-ring.

THEOREM 4. Let R be a semilocal ring. Then the following are equivalent:

(1) R is left TQI.

(2) R has ACC on F-ideals and each F-ideal is the intersection of maximal left ideals of R.

The next theorem gives a characterization of commutative TQI-rings. Let us first recall that a ring R is said to have SP if every left R-module splits (i.e. if the torsion submodule of each left R-module M is a direct summand of M).

THEOREM 5. Let R be a commutative ring and (G,F) be the Goldie torsion theory for R-Mod. Then the following are equivalent:

(1) R is TQI.

(2) R has SP.

Below is given an example of a commutative TQI-ring which is not QI.

EXAMPLE 2. ([6, p.161]). Let K be a field and A an infinite indexing set. Let $Q = \prod_{\alpha\in A} K^{(\alpha)}$, where $K^{(\alpha)} = K$ and $R = \sum_{\alpha\in A} \oplus K^{(\alpha)} + 1\cdot K \subseteq Q$, $1 \in Q$. Then R is a TQI-ring but not QI.

Finally we state a characterization of rings whose torsion quasi-injective modules are Σ-quasi-injective. Recall that a quasi-injective module M is called Σ-quasi-injective in case each direct sum of arbitrarily many copies of M is quasi-injective.

THEOREM 6. Let R be a ring and (G,F) be the Goldie torsion theory for R-Mod. Then the following are equivalent:

(1) Each torsion quasi-injective left R-module is Σ-quasi-injective.

(2) R has ACC on F-ideals.

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RADAR RECEPTION AND NILPOTENT HARMONIC ANALYSIS V

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It is an important aspect of the harmonic analysis on the real Heisenberg nilpotent group $\tilde{A}(\mathbb{R})$ that the irreducible unitary linear representation U of $\tilde{A}(\mathbb{R})$ which subduces the unitary central character

$$\left(\begin{bmatrix}0\\0\end{bmatrix},s\right)\to e(s) = e^{2\pi i s}$$

and which is uniquely determined up to unitary isomorphy by this property (Stone-von Neumann theorem) can be realized in various rather different looking ways. Taking into account Theorem 1 in the first part [4] of this series of papers, the aforementioned characteristic of the theory of $\tilde{A}(\mathbb{R})$ implies a variety of different forms of expressing group-theoretically the radar cross-ambiguity function $H(f,g;.,.)$ with respect to envelope functions $f,g \in \mathcal{S}(\mathbb{R})$; cf. [5]. It is the purpose of the present part to restate some of these versions and to mention a few of their direct consequences. Thus the present paper completes the preceding parts [4-7]. Further details and proofs may be found in the forthcoming paper [8].

1. The Polarized Cross-Section

According to the terminology introduced by Howe [2], the polarized cross-section to the center

$$\tilde{Z} = \{(\begin{bmatrix}0\\0\end{bmatrix},s)|s \in \mathbb{R}\} = [\tilde{A}(\mathbb{R}),\tilde{A}(\mathbb{R})]$$

in $\tilde{A}(\mathbb{R})$ is the set

$$\{(\begin{bmatrix}x\\y\end{bmatrix},0)|x \in \mathbb{R}, y \in \mathbb{R}\}.$$

Consider the complex vector space $\mathcal{S}(\mathbb{R})$ of smooth vectors for

U as an everywhere dense vector subspace of the complex Hilbert space $L^2(\mathbb{R})$. In particular, $\mathcal{S}(\mathbb{R})$ inherits the scalar product $<.|.>$ from $L^2(\mathbb{R})$. For any two functions $f \in \mathcal{S}(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})$ let $f \otimes \bar{g}$ denote their dyadic tensor product with respect ot the prehilbert space structure of $\mathcal{S}(\mathbb{R})$ (cf. Schatten [3]). Then we have the following representation-theoretic characterization of the radar cross-ambiguity function.

Theorem 1. For any two envelope functions $f \in \mathcal{S}(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$ the radar cross-ambiguity function $H(f,g;.,.)$ coincides with the restriction of the trace evaluation

$$tr_U(f \otimes \bar{g})$$

to the polarized cross-section to $\tilde{Z}$ in $\tilde{A}(\mathbb{R})$.

The strong Stone-von Neumann Theorem shows that the linear Schrödinger representation U of $\tilde{A}(\mathbb{R})$ is square integrable modulo $\tilde{Z}$, i.e., U belongs to the discrete series of $\tilde{A}(\mathbb{R})$. Thus the Schur orthogonality relations (cf. Borel [1]) imply, for instance, the identity

$$\int_{\mathbb{R}} \int_{\times \mathbb{R}} H(f,g;x,y)\bar{H}(f',g';x,y)dxdy = <f|f'>.<g'|g>$$

where f,f',g,g' are elements of the Schwartz-Bruhat space $\mathcal{S}(\mathbb{R})$ ("Moyal's identity"; cf. [5], [8]).

2. The Isotropic Cross-Section

Another useful cross-section to the center $\tilde{Z}$ in $\tilde{A}(\mathbb{R})$ is the so-called isotropic cross-section

$$\{(\begin{bmatrix} x \\ y \end{bmatrix}, \tfrac{1}{2}xy) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}.$$

Since this cross-section is also describable as the set $\exp(\log \mathbb{R} \oplus \log \mathbb{R})$ it is adapted to the exponential coordinates of $\tilde{A}(\mathbb{R})$. If we denote by $\begin{bmatrix} p \\ q \end{bmatrix}$ the coordinates of the vector subspace $\log \mathbb{R} \oplus \log \mathbb{R}$ of the Lie algebra $\mathfrak{n}$ of $\tilde{A}(\mathbb{R})$ with respect to its standard basis then we get by means of the isotropic cross-section the following

Theorem 2. Let f,g be elements of the vector space $\mathcal{S}(\mathbb{R})$. Then the identity

$$H(f,g;p,q) = \mathrm{tr}_{\exp}\, U(\bar{g} \otimes f)\begin{bmatrix}-p\\-q\end{bmatrix}$$

holds for all $(p,q) \in \mathbb{R} \times \mathbb{R}$.

Recall that the radar autoambiguity function $H(f;.,.)$ is radially symmetric if and only if there are numbers $\zeta \in \mathbb{T}$ and $m \in \mathbb{N}$ such that

$$f = \zeta W_m$$

holds [5], where $(W_m)_{m\geq 0}$ denotes the family of Hermite-Weber functions (harmonic oscillator wave functions). Let $(L_m^{(\alpha)})_{m\geq 0}$ denote the Laguerre-Weber functions of order $\alpha>-1$. Then the preceding theorem implies the following

Corollary. For all natural numbers $m\geq 0$ and $n\geq 0$ we have

$$H(W_m,W_n;p,q) = \sqrt{\frac{n!}{m!}}(\sqrt{\pi}(p+iq))^{m-n}\, L_n^{(m-n)}(\pi(p^2+q^2)) \quad (m\geq n),$$

$$H(W_m,W_n;p,q) = \sqrt{\frac{m!}{n!}}(\sqrt{\pi}(-p+iq))^{n-m}\, L_m^{(n-m)}(\pi(p^2+q^2)) \quad (n\geq m)$$

where $(p,q) \in \mathbb{R} \times \mathbb{R}$.

3. The Lattice Model

Denote by D the image of the cubic integer lattice $\mathbb{Z}^3$ under the embedding map

$$\mathbb{Z}^3 \ni (\mu,\nu,\sigma) \longrightarrow \left(\begin{bmatrix}\mu\\\nu\end{bmatrix},\sigma\right) \in \tilde{A}(\mathbb{R}).$$

Then D is a discrete and cocompact subgroup of $\tilde{A}(\mathbb{R})$ and the homogeneous space $D\backslash\tilde{A}(\mathbb{R})$ of right cosets modulo D forms a two-step compact nilmanifold, the Heisenberg nilmanifold. Decompose the complex Hilbert space $L^2(D\backslash\tilde{A}(\mathbb{R}))$ relative to the unique probability measure on the compact nilmanifold $D\backslash\tilde{A}(\mathbb{R})$ into the Hilbert sum

$$L^2(D\backslash\tilde{A}(\mathbb{R})) = \bigoplus_{n \in \mathbb{Z}} M_n$$

of Hilbert subspaces $(M_n)_{n \in \mathbb{Z}}$ which are completions of the vector subspaces

$$\{f \in \mathscr{C}^\infty(D\backslash\tilde{A}(\mathbb{R})) \mid f(v,s+s') = e(ns)f(v,s'), v \in \mathbb{R} \oplus \mathbb{R}, s,s' \in \mathbb{R}\} \quad (n \in \mathbb{Z})$$

of smooth left-$\mathbb{Z}^3$-periodic complex-valued functions on $\tilde{A}(\mathbb{R})$. The complex Hilbert spaces $(M_n)_{n \in \mathbb{Z}}$ are primary summands with respect to the right quasi-regular representation δ of $\tilde{A}(\mathbb{R})$ on $L^2(D\backslash\tilde{A}(\mathbb{R}))$ and the subduced representation δ_n forms a unitary linear representation of $\tilde{A}(\mathbb{R})$ in M_n which contains exactly $|n|$ copies of the single irreducible unitary linear representation of $\tilde{A}(\mathbb{R})$ with central character

$$(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, s) \longrightarrow e(ns) = e^{2\pi i n s} \quad (n \in \mathbb{Z})$$

when $n \neq 0$. In view of the Stone-von Neumann Theorem, the irreducible unitary linear representation δ_1 of $\tilde{A}(\mathbb{R})$ acting in M_1 is unitarily isomorphic to the linear Schrödinger representation U and the Weil-Brezin isomorphism given by the formula

$$w: \quad \mathcal{S}(\mathbb{R}) \ni f \longrightarrow ((\begin{bmatrix} x \\ y \end{bmatrix}, s) \longrightarrow e(s) \sum_{\mu \in \mathbb{Z}} f(\mu+x)e(\mu y)) \in \mathscr{C}^\infty(D\backslash\tilde{A}(\mathbb{R}))$$

extends to a unitary isomorphism of U onto δ_1. From this result we infer the following

Theorem 3. The identity

$$H(f,g;x,y) = \operatorname{tr}_{\delta_1}(w(f) \otimes \overline{w(g)})(\begin{bmatrix} x \\ y \end{bmatrix}, 0)$$

holds for the envelope functions f,g in the vector space $\mathcal{S}(\mathbb{R})$ and all pairs $(x,y) \in \mathbb{R} \times \mathbb{R}$.

Corollary. Keeping to the same notation, the absolutely convergent Fourier expansion

$$w(f).\overline{w(g)}(\begin{bmatrix} x \\ y \end{bmatrix}, 0) = \sum_{(\mu,\nu) \in \mathbb{Z} \times \mathbb{Z}} H(f,g;\mu,\nu)e(-\nu x+\mu y)$$

holds for all points $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R} \oplus \mathbb{R}$ with respect to the topology of uniform $\mathscr{C}^{\infty}$-convergence.

An application of the L^2-Plancherel theorem yields

Theorem 4. For all pairs $(n,m) \in \mathbb{N} \times \mathbb{N}$ the identities

$$\sum_{(\mu,\nu) \in \mathbb{Z} \times \mathbb{Z}} H(W_n;\mu,\nu).H(W_m;\mu,\nu) = \sum_{(\mu,\nu) \in \mathbb{Z} \times \mathbb{Z}} |H(W_n,W_m;\mu,\nu)|^2$$

hold ("Poisson-Plancherel identities" for radially symmetric radar ambiguity functions).

An application of the Corollary of Theorem 2 supra finally yields the following

Corollary. In the case $m \geq n$ the Poisson-Plancherel identities for Laguerre-Weber functions of different orders

$$\sum_{(\mu,\nu) \in \mathbb{Z}\times\mathbb{Z}} L_n(\pi(\mu^2+\nu^2))L_m(\pi(\mu^2+\nu^2)) = \frac{n!}{m!}\pi^{m-n} \sum_{(\mu,\nu) \in \mathbb{Z}\times\mathbb{Z}} (\mu^2+\nu^2)^{m-n}(L_n^{(m-n)}(\pi(\mu^2+\nu^2)))^2$$

hold ($L_n = L_n^{(0)}$, as usual).

The most simplest case occurs when m=1, n=0. In this case the aforementioned identity reduces to the equation

$$\frac{1}{4\pi} \sum_{\mu \in \mathbb{Z}} e^{-\pi\mu^2} = \sum_{\mu \in \mathbb{Z}} \mu^2 e^{-\pi\mu^2}$$

which is well known from the theory of classical Jacobi theta functions. These functions are deeply connected with the nil-theta functions living in a natural way on the Heisenberg compact nilmanifold $D \backslash \tilde{A}(\mathbb{R})$.

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VARIATIONAL INEQUALITIES FOR A CLASS OF CONTACT PROBLEMS WITH FRICTION IN ELASTOSTATICS

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ABSTRACT.
The existence and uniqueness of the solution of a class of variational inequalities related with Signorini problem arising in elastostatics with non-local friction is considered.

1. INTRODUCTION.

It is well known that variational concepts play a very important and basic role in applied mathematics. Variational formulations can be used not only to unify diverse fields, but also to suggest new theories. Variational methods are generally used for approximations. Recently variational theory has been enriched by the development of the theory of variational inequalities, which has become a rich source of inspiration both in pure and applied mathematics. Variational inequalities have stimulated new and deep results dealing with nonlinear partial differential equations. Also variational inequalites have been used in a large variety of problems in mechanics, physics, optimization and control, nonlinear programming, engineering sciences, etc. Today, variational inequalities are considered as an indispensible tool in various fields of mathematics and engineering sciences.

The general problem of equilibrium of elastic bodies in contact with rigid foundation on which frictional forces are developed is one of the most difficult problem in solid mechanics. The complete study of the boundary value problem arising in the formulation of Signorini problem with friction is an interesting problem both in mechanics and mathematical point of view. In 1981, Oden and Pires[8] have shown that the Signorini problem with non-local friction in elastostatics can be characterized by a class of variational inequalities. The issue of existence of solutions to such problems in cases in which Coulomb's friction law is assumed to hold is still open, see[1]. In this paper, using the technique of the fixed point theory, initiated by Lions and Stampacchia[3] and developed by Noor[5 , 6], we prove the existence of solution of such problems. Several special cases that can be derived from the general problem are also discussed.

It should be remarked that the formulation of such problems as variational inequalities were originally investigated by Duvaut and Lions[2], but they were unable to prove the existence of solutions of such problems except in special cases.

2. PRELIMINARIES:

Let H be a Hilbert space with its dual H', whose norm and inner product are denoted by $\|.\|$ and $(.,.)$ respectively. The pairing between H' and H is denoted by $<.,.>$. Let M be a non-empty closed convex subset of H and a(u.v) be a coercive and continuous bilinear form on H, that is there exist constants $\alpha > 0$, $\beta > 0$ such that

$$a(v,v) \geq \alpha\| v\|^2, \qquad \text{for all } v \in H \tag{1}$$

and

$$a(u,v) \leq \beta\|u\|\|v\|, \qquad \text{for all } u,v \in H \tag{2}$$

In particular, it follows that $\alpha \leq \beta$.

Finally, let the form b: HxH ————> R satisfy the following properties.

(i). b(u,v) is linear in the first argument.

(ii). b(u,v) is bounded, that is there exists a constant $\gamma > 0$ such that

$$b(u,v) \leq \gamma\|u\|\|v\| \qquad \text{for all } u,v \in H \tag{3}$$

(iii). b(u,v) is sublinear in the secod argument, that is for all $u,v \in H$,

$$b(u,u-v) \leq b(u,u)-b(u,v) \tag{4}$$

PROBLEM 1:

For given $f \in H'$, find $u \in M$ such that

$$a(u,v-u)+b(u,v)-b(u,u) \geq <f,v-u> \qquad \text{for all } v \in M. \tag{5}$$

REMARKS:

I. The variational inequality (5) characterizes the Signorini problem with non-local friction. If Ω is a open bounded domain in R^n with regular boundary $\partial\Omega$, representing the interior of an elastic body subjected to external forces and if part of the boundary may come into contact with a rigid foundation, then the variational inequality (5) is simply a statement of the virtual work for an elastic body restrained by frictional forces, assuming that a non-local law of friction holds. The strain energy of the body corresponding to an admissible displacement v is 1/2a(v,v). Thus a(u,v-u) is the work produced by the stresses through strains caused by the virtual displacement v-u. The external and frictional forces are represented by the linear continuous functional f and by the form b(u,v) respectively. For physical and mathematical formulation of the inequality, see Oden and Pires [8].

II. If we restrict the dependence of the form b(u,v) to its second variable only, that is, if b(u,v) = j(v), then problem 1 becomes:

For given $f\in H'$, find $u\in M$ such that

$a(u,v-u)+j(v)-j(u)\geq \langle f,v-u\rangle$ for all $v\in M$, for all v M,

a problem originally considered and investigated by Duvaut and Lions[2]. The existence of its solution has been proved recently by Necas, Jarusek, and Haslinger[4].

III. If the frictional forces are zero, then problem 1 reduces to the classical Signorini problem of elastostatics, that is the analysis of deformations of a linear elastic body in contact with a rigid frictionless foundation. In this case problem 1 becomes:

For given $f\in H'$, find $u\in M$ such that

$a(u,v-u)\geq \langle f,v-u\rangle$ for all $v\in M$.,

a problem studied by Lions and Stampacchia[3]. Furthermore, if M = H, then we have the classsical problem of linear elasticity.

Since a(u,v) is a continuous bilinear form on H, then by the Riesz-Frechet representation theorm, we have

$$a(u,v) = \langle Tu,v\rangle \qquad \text{for all } v\in H. \qquad (6)$$

It has been shown that $\|T\|\leq\beta$, see [5]. Finally, we define Λ , a canonical isomorphism from H' onto H by

$$\langle f,v\rangle = (\Lambda f,v), \qquad \text{for all } v\in H,\ f\in H'. \qquad (7)$$

Then $\|\Lambda\|_{H'}=1 = \|\Lambda^{-1}\|_{H}$.

We make the following hypothesis.

CONDITION N.

We assume that $\gamma<\alpha$, where α is the coercivity constant of a(u,v) and γ is the boundedness constant of the form b(u,v).

3. MAIN RESULTS.

We now state and prove the main result of this paper.

THEOREM 1.

Let $a(u,v)$ be a coercive, continuous bilinear form and $b(u,v)$, the form satisfying the properties (i)-(iii). If condition N holds, then there exists a unique $u \in M$ such that, for given $f \in H'$,

$$a(u,v-u)+b(u,v)-b(u,u) \geq \langle f,v-u \rangle, \qquad \text{for all } v \in M. \tag{8}$$

We need the following results for its proof. The first is a generalization of a result of Lions and Stampacchia [3].

Lemma 1:

Let ζ be a number such that $0<\zeta<\frac{2(\alpha-\gamma)}{\beta^2-\gamma^2}$ and $\zeta<\frac{1}{\gamma}$. Then there exists a θ with $0<\theta<1$ such that

$$\|\phi(u_1)-\phi(u_2)\| \leq \theta \| u_1-u_2 \| , \qquad \text{for all } u_1,u_2 \in H,$$

where for $u \in H$, $\phi(u) \in H'$ is defined by

$$\langle \phi(u),v \rangle = (u,v)-\zeta a(u,v)+\zeta b(u,v)+\zeta \langle f,v \rangle \qquad \text{for all } v \in H, \tag{9}$$

and β is the boundedness constant of the bilinear form $a(u,v)$.

Proof:

For all $u_1,u_2 \in H$, consider

$$\begin{aligned}\langle \phi(u_1)-\phi(u_2),v \rangle &= (u_1-u_2,v)-\zeta a(u_1-u_2,v)+\zeta b(u_1-u_2,v)\\ &= (u_1-u_2,v)-\zeta \langle Tu_1-Tu_2,v \rangle +\zeta b(u_1-u_2,v), \quad \text{by (6)}\\ &= (u_1-u_2-\zeta\Lambda(Tu_1-Tu_2),v)+\zeta b(u_1-u_2,v), \text{ by (7)},\end{aligned}$$

from which it follows that

$$|\langle \phi(u_1)- (u_2),v \rangle| \leq \|u_1-u_2-\zeta\Lambda(Tu_1-Tu_2)\|\|v\| +\zeta\gamma\|u_1-u_2\|\|v\|$$

From (6) and (7), we have

$$\begin{aligned}\|u_1-u_2-\zeta\Lambda(Tu_1-Tu_2)\|^2 &\leq \|u_1-u_2\|^2 +\zeta^2\| T\|^2\|u_1-u_2\|^2-2\zeta a(u_1-u_2,u_1-u_2)\\ &\leq (1+\zeta^2\beta^2-2\alpha\zeta)\| u_1-u_2\|^2\end{aligned}$$

Thus

$$|\langle\phi(u_1)-\phi(u_2),v\rangle| \le (\sqrt{1+\zeta^2\beta^2-2\zeta\alpha} + \gamma\zeta)\|u_1-u_2\| \|v\|$$
$$= \theta \|u_1-u_2\| \|v\|,$$

where

$$\theta = \sqrt{1+\zeta^2\beta^2-2\zeta\alpha} + \gamma\zeta<1 \quad \text{for} \quad 0<\zeta< \frac{2(\alpha-\gamma)}{\beta^2-\gamma^2} \quad \text{and} \quad \zeta< \frac{1}{\gamma} \quad \text{by condition N.}$$

Hence

$$\|\phi(u_1)-\phi(u_2)\| = \sup_{v\in H} \frac{|\langle\phi(u_1)-\phi(u_2),v\rangle|}{\|v\|} \le \theta\|u_1-u_2\|$$

REMARK:

Note that in the absence of the frictional force, that is $b(u,v)= 0$, lemma 1 is exactly the same as proved by lions and Stampacchia.

Lemma 2. [6]

Let M be a convex subset of H. Then, given $z\in H$, we have

$$x = P_M z$$

if and only if

$$x\in M: \quad (x-z,y-x)\ge 0 \quad \text{for all } y\in M,$$

where P_M is a projection of H into M.

Lemma 3. [6].

P_M is nonexpansive, that is $\| P_M z_1 - P_M z_2\| \le \|z_1-z_2\|$ for all $z_1,z_2\in H$.

Using the technique of Lions and Stampacchia[3] and Noor[6,7], we now prove theorem 1.

PROOF OF THEOREM 1.

Uniqueness; see Oden and Pires[8].

Existence:

For a fixed ζ as in lemma 1, and $u\in H$, define $\phi(u)\in H'$ by (9). By lemma 2, there exists a unique $w\in M$ such that

$$(w,v-w)\ge \langle\phi(u),v-w\rangle, \qquad \text{for all } v\in M,$$

and w is given by

$$w = P_M\Lambda\phi(u) = Tu,$$

which defines a map from H into M.

Now for all $u_1, u_2 \in H$,

$$\| Tu_1 - Tu_2 \| = \| P_M \Lambda\phi(u_1) - P_M \Lambda\phi(u_2) \| \le \| \Lambda\phi(u_1) - \Lambda\phi(u_2) \| \quad \text{by lemma 3.}$$

$$\le \theta \| u_1 - u_2 \| , \quad \text{by lemma 1.}$$

Since $\theta<1$, Tu is a contraction and has a fixed point $Tu=u$, which belongs to M, a closed convex set of H and satisfies

$$(u,v-u) \ge \langle\phi(u),v-u\rangle = (u,v-u)-\zeta a(u,v-u)+\zeta b(u,v-u)+\zeta\langle f,v-u\rangle.$$

Thus for $\zeta > 0$ and using the properties (iii) of b(u,v), we have

$$a(u,v-u)+b(u,v)-b(u,u) \ge \langle f,v-u\rangle \quad \text{for all } v\in M,$$

showing that u is a unique solution of problem 1.

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SIGNATURES AND ALEXANDER POLYNOMIALS OF TWO-BRIDGE KNOTS

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Let K be a knot in S^3 and let $\Delta_K(t)$ be the Alexander polynomial of K. Let $\sigma(K)$ denote the signature of K.

It is well-known [1] that if all the roots of $\Delta_K(t)$ are real then $\sigma(K) = 0$. In fact, R. Riley observed

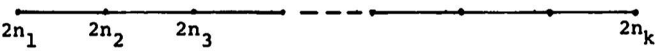
Proposition 1 *Suppose that K is a special alternating knot. Then the number of roots ξ of $\Delta_K(t)$ with $|\xi| = 1$ is exactly the absolute value of $\sigma(K)$.*

A proof follows easily from [1,2].

Proposition 1 is not true in general. For example, the granny knot and the square knot have the same Alexander polynomial $(1-t+t^2)^2$, but the signature of the former is 4 and the other is 0.

A quite while ago, R. Riley asked if Proposition 1 holds true for a 2-bridge knot K. In this short note, we will show that it does not.

Let K be a 2-bridge knot. As an algebraic knot, K is associated with a graph:



where n_i are non-zero integers. The type (n,q) of K is determined by the value of the continued fraction

$$\frac{n-q}{n} = \frac{1}{2n_1} - \frac{1}{2n_2} + \frac{1}{2n_3} \cdots \frac{1}{+(-1)^{k-1} 2n_k} .$$

To compute $\Delta_K(t)$, consider the principal minor M of the knot matrix of K [2, p.393]. M is of the form:

$$M = \begin{bmatrix} n_1 & -1 & & & 0 \\ & n_2 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & 0 & & & n_k \end{bmatrix}$$

Then the Alexander polynomial is given by

$$\Delta_K(t) = \det(M - tM^t) ,$$

where M^t denotes the transpose of M . Further, $\sigma(K)$ is given by [2, p.397. Theorem 5.3]

$$\sigma(K) = \sum_{i=1}^{k} \frac{n_i}{|n_i|} .$$

Now we are ready to construct a knot required.

Let K be the 2-bridge knot with $n_1 = n_2 = 1$, $n_3 = -3$ and $n_4 = -5$. Then $\sigma(K) = 0$. On the other hand, $\Delta_K(t) = 15 - 49t + 69t^2 - 49t^3 + 15t^4$ and $\Delta_K(t)$ has four complex roots on the unit circle in the complex plane $\mathbb{C}$. K is a 2-bridge knot of type (197, 69).

Although Proposition 1 is generally not true, I suspect that it may be true for a fibred 2-bridge knot. Note that if a 2-bridge knot is fibred, then all $n_i = 1$ or -1.

In fact, if K is a fibred 2-bridge knot of genus 2, then the principal minor of K is S-equivalent to one of the following:

$$\begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ 0 & & 1 & -1 \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ 0 & & 1 & -1 \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ 0 & & -1 & -1 \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ 0 & & -1 & -1 \\ & & & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & & \\ & -1 & -1 & \\ 0 & & 1 & -1 \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & & \\ & -1 & -1 & \\ 0 & & -1 & -1 \\ & & & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & & \\ & -1 & -1 & \\ 0 & & 1 & -1 \\ & & & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & & \\ & -1 & -1 & \\ 0 & & -1 & -1 \\ & & & -1 \end{bmatrix} .$$

For each case, it is easy to show that $\sigma(K)$ is exactly the number of roots of $\Delta_K(t)$ with absolute value 1.

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THE SYMPLECTIC MOTION GROUPS OVER GF(2)

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Abstract. The lower level character values and all the degrees are found for the ordinary representations of the symplectic motion groups over GF(2), that are semi-direct products of an abelian translation group by the symplectic group $Sp_{2n}(2)$, that is isomorphic with the orthogonal group $O_{2n+1}(2)$.

1. Introduction. The symplectic motion group M_n over GF(2) is the semi-direct product of an elementary abelian translation group T_n of order $N^2 = 2^{2n}$ by the symplectic group $G_n = Sp_{2n}(2)$. It is representable by (2n+1)-dimensional matrices M over GF(2) having the partitioned factorizations

$$M = \begin{bmatrix} S & C \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & C+C_S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & C_S \\ 0 & 1 \end{bmatrix} \quad (1.1)$$

where S is symplectic, C is an arbitrary nx1 column vector, and C_S is a column determined by S so that the last factor in (1.1)is a matrix of the orthogonal group $O_{2n+1}(2)$ with quadratic invariant

$$Q_n(Z) = \sum_{i=1}^{n} z_{2i-1}z_{2i} + z_{2n+1}^2 \text{, for row vectors } Z \quad (1.2)$$

The motion groups M_n^{σ} described in [5] with $\sigma = \pm 1$ are subgroups of M_n of index $(N+\sigma)N/2$, in which the last factor of (1.1) is restricted to matrices of $O_{2n}^{\sigma}(2) = G_n^{\sigma}$. As in [5] we denote by J_n the direct sum of n transposition matrices that interchange z_{2i-1} and z_{2i}. Then matrices $\hat{M}$ of G_{n+1} satisfy the relations

$$\hat{M}\, J_{n+1}\, \hat{M}^T = J_{n+1}\ , \quad \text{or} \quad \hat{M}^T = J_{n+1}\, \hat{M}^{-1}\, J_{n+1} \tag{1.3}$$

The stabilizer $\hat{M}_n$ in G_{n+1} of the column vector $(0^{2n+1},1)^T$ has index $4N^2-1$ and consists of symplectic matrices $\hat{M}$ having last column $(0^{2n+1},1)^T$. Hence by (1.3) the $(2n+1)^{th}$ column of $\hat{M}$ is $(0^{2n},1,0)^T$, and each matrix $\hat{M}$ in $\hat{M}_n$ has the partitioned form

$$\hat{M} = \begin{bmatrix} S & C & 0 \\ 0 & 1 & 0 \\ C^T J_n S & x & 1 \end{bmatrix} = \begin{bmatrix} I & C & 0 \\ 0 & 1 & 0 \\ C^T J_n & x & 1 \end{bmatrix} \begin{bmatrix} S & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \hat{T}\,\hat{S} \tag{1.4}$$

We call this stabilizer group $\hat{M}_n$ the extended symplectic motion group. It is the semi-direct product of the abelian translation group $\hat{T}_n$ by the symplectic group $\hat{S}_n$. Its center is generated by the involution τ for which $C = 0$, $x = 1$, and $S = I$. The factor group $\hat{M}_n/\langle\tau\rangle$ is isomorphic with the motion group M_n.

In this report we determine for all the absolutely irreducible complex (AIC) representations of $\hat{M}_n$ (and hence of M_n) the character values on all classes that contain either an element g_i of G_n or the element τg_i. For the ten level 1 and 38 level 2 characters of $\hat{M}_n$ we express the values on all classes in terms of G_n characters, such as the sum α and difference β of the characters induced in G_n by the 1-characters of G_n^- and G_n^+. The four types of AIC characters are: 1) those of the factor group $\hat{M}_n/\hat{T}_n \cong G_n$; 2) the other characters a_i of M_n related to those of $\hat{M}_{n-1}$; 3) characters b_i related to those of G_n^-; and 4) characters c_i related to those of G_n^+.

2. <u>Class sizes</u>. Let g_i be a representative of class C_i of G_n with a matrix S_i such that S_i-I has rank ρ_i. Then the coset $\hat{T}_n g_i$ in $\hat{M}_n$ contains $2N^2$ elements in two or more classes $C_{i\lambda}$ whose

sizes $|C_{i\lambda}|$ are integral multiples of the size of the first listed class C_{i1} in the coset, namely the "principal class" that contains the conjugates of g_i in $\hat{M}_n$. The class C_{i2} contains τg_i, and

$$|C_{i1}| = |C_{i2}| = |C_i| \, 2^{\rho_i} = |C_i| \, N^2/\alpha(g_i) \qquad (2.1)$$

The groups G_n, M_n, and $\hat{M}_n$ contain respectively N^{2n}, N^{2n+2}, and $2N^{2n+2}$ unipotent elements. Those in $\hat{M}_2$ lie in classes whose sizes, grouped by cosets $\hat{T}_2 g_i$, with $|C_i|$ as outer factor, are

$$1(1+1+15+15)+15(2+2+12+8+8)+15(4+4+24)+45(4+4+4+4+16)+ \\ 90(8+8+16) + 90(8+8+16) = 2^{13}. \qquad (2.2)$$

3. Degrees and character values on principal classes. Type 1 characters of $\hat{M}_n$ are characters of its factor group G_n whose degrees are given in [4]. They have equal values on each class in the coset $\hat{T}_n g_i$, and are labeled for $\hat{M}_n$ by a G_n character symbol with a subscript p referring to the principal class containing g_i. Type 2 characters a_i include all other AIC characters of M_n, and have equal values on classes containing elements y and τy. Characters b_i of type 3 and c_i of type 4 are faithful characters that have opposite signs for elements y and τy. The restrictions to G_n, (i.e., to principal classes) of characters of types 2, 3, and 4 are respectively the characters of G_n induced by the AIC characters of its subgroups $\hat{M}_{n-1}$, G_n^-, and G_n^+ with indices N^2-1, $(N-1)N/2$, $(N+1)N/2$ respectively. Thus the sum of squared degrees of the four types is

$$(1 + (N^2-1) + (N-1)N/2 + (N+1)N/2) \, |G_n| = |\hat{M}_n| \qquad (3.1)$$

We have now accounted for all AIC characters of $\hat{M}_n$ and can calculate their degrees and their values on classes C_{i1} and C_{i2} in each coset $\hat{T}_n g_i$. The degrees for $\hat{M}_1$ and $\hat{M}_2$ are as follows:

	$\hat{M}_1$ degrees	$\hat{M}_2$ degrees	(3.2)
Type 1:	$\chi_1^i(G_1) = 1,1,2;$	$\chi_1^i(G_2) = 1,1,5,5,5,5,9,9,10,10,16$	
Type 2:	$3\chi_1^i(\hat{M}_0) = 3,3\ ;$	$15\chi_1^i(\hat{M}_1) = 15,15,15,15,30,30,45,45,45,45$	
Type 3:	$\chi_1^i(G_1^-) = 1,1,2:$	$6\chi_1^i(G_2^-) = 6,6,24,24,30,30,36$	
Type 4:	$3\chi_1^i(G_1^+) = 3,3;$	$10\chi_1^i(G_2^+) = 10,10,10,10,20,40,40,40,40$	

As explained in [3,4] for the orthogonal groups, we denote a generic character by a degree symbol that is the ratio of a word, representing a monic degree polynomial P(N), divided by an integer independent of $N = 2^n$. The r^{th} letter in the word is h, k, g, or i according as $N - 2^{r-1}$, $N + 2^{r-1}$, both, or neither is a factor of P(N), except that a final letter N appears whenever $N \mid P(N)$.

4. <u>Level 1 character formulas</u>. Expressed in terms of the basic characters α, β, γ and their associates $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ defined in [1], the five level 1 characters of type 1 (G_n characters) are (4.1)

Formula: $(\bar{\alpha}-\bar{\gamma})/3, (\bar{\alpha}-3\bar{\beta}+2\bar{\gamma})/6, (\bar{\alpha}+3\bar{\beta}+2\bar{\gamma})/6, \tfrac{1}{2}(\alpha-\beta)-1, \tfrac{1}{2}(\alpha+\beta)-1$

Degree symbol: $(g/3)_p$, $(hh/6)_p$, $(kk/6)_p$, $(kh/2)_p$, $(hk/2)_p$ (4.2)

where the subscripts p refer to values on principal classes. The sign function $\bar{1}_p$ is such that $\bar{\alpha}_p = \bar{1}_p \alpha_p$, etc., and a second sign function ε is defined to be 1 on classes C_{i1}, -1 on C_{i2}, and zero on all other classes in a coset $\hat{T}_n g_i$. The permutation characters of M_n and $\hat{M}_n$ on the cosets of G_n split as $1+a_o$ and $1+a_o+b_o+c_o$ respectively, where a_o is the level 1 character of type 2, b_o and $\bar{b}_o = \bar{1}_p b_o$ are of type 3, and c_o and $\bar{c}_o = \bar{1}_p c_o$ are of type 4.

<u>Theorem 4.1. The level 1 characters</u> a_o, b_o, <u>and</u> c_o <u>of</u> M_n <u>of types 2,3, and 4 are given on all classes of</u> M_n <u>by the formulas</u>

$$a_o = \varepsilon^2\alpha_p - 1 = g. \qquad \text{Degree } N^2 - 1 \qquad (4.3a)$$

$$b_o = \tfrac{1}{2}(\varepsilon\alpha_p + \beta_p) - \beta \quad = hN/2. \quad \text{Degree } (N-1)N/2 \qquad (4.3b)$$

$$c_o = \tfrac{1}{2}(\varepsilon\alpha_p - \beta_p) + \beta \quad = kN/2. \quad \text{Degree } (N+1)N/2 \qquad (4.3c)$$

Proof: Since $1+a_o+b_o+c_o$ has the value $2\alpha_p$ on principal classes C_{i1}, but vanishes on all others, while $1+a_o$ has the value α_p on classes C_{i1} and C_{i2} and vanishes on the other classes, it follows that the values of $1+a_o$ and b_o+c_o are $\varepsilon^2\alpha_p$ and $\varepsilon\alpha_p$ respectively. Restricting hk/2 and kh/2 from G_{n+1} to $\hat{M}_n$, we obtain

$$(hk/2)_{n+1}\downarrow \; = (hk/2)_p + 1 + a_o + c_o. \quad \text{Degree } (2N-1)(N+1) \qquad (4.4a)$$

$$(kh/2)_{n+1}\downarrow \; = (kh/2)_p + 1 + a_o + b_o. \quad \text{Degree } (2N+1)(N-1) \qquad (4.4b)$$

Since the difference of induced characters $\beta = hk/2 - kh/2$ for G_n is twice as large for G_{n+1}, we conclude that

$$2\beta = \beta_p + c_o - b_o \qquad (4.5)$$

Therefore a_o, b_o, and c_o are determined on all classes by (4.3).

5. Level 2 characters. The group M_n has 38 generic AIC characters of level 2 (distinct when $n > 3$) of which 16 of type 1 are characters of $O_{2n+1}(2) = G_n$ described in [2]. The ten of type 2 are denoted a_i and correspond to the ten level 1 characters of $\hat{M}_{n-1}$. Three associated pairs b_i and $\bar{b}_i = \bar{1}_p b_i$ of type 3, and three associated pairs c_i and $\bar{c}_i = \bar{1}_p c_i$ correspond to level 1 characters of $O^-_{2n}(2)$ and $O^+_{2n}(2)$ respectively. Corresponding degree symbols are:

$$a_1 = gg/12,\ a_3 = ghN/8,\ a_5 = gh\bar{N}/8,\ a_7 = ghh/24,\ a_9 = gkh/8$$

$$a_2 = gg/4\ ,\ a_4 = gkN/8,\ a_6 = gk\bar{N}/8,\ a_8 = gkk/24,\ a_{10} = ghk/8$$

$$\left.\begin{aligned} b_1 &= hgN/6,\ b_2 = gkN/12,\ b_3 = gihN/12 \\ c_1 &= kgN/6,\ c_2 = ghN/12,\ c_3 = gikN/12 \end{aligned}\right\} \qquad (5.1)$$

We calculate first the characters b_i of type 3 and c_i of type 4 from the equations

$$b_1+b_o = b_o(\alpha_p-\gamma_p)/3, \quad b_2+b_3-b_1 = b_o\gamma_p, \quad b_2-b_3-b_o = b_o\beta_p \quad (5.2)$$

$$c_1+c_o = c_o(\alpha_p-\gamma_p)/3, \quad c_2+c_3-c_1 = c_o\gamma_p, \quad c_2-c_3-c_o = c_o\beta_p$$

Then three level 2 characters are calculated and checked, setting

$$a_2 = b_oc_o-a_o, \quad a_3 = b_o^{[1^2]}, \quad a_4 = c_o^{[1^2]}, \quad a_2+a_3+a_4 = a_o^{[1^2]} \quad (5.3)$$

Next we restrict to M_n the G_{n+1}-characters U'=ghih/120, V'=gkik/120:

$$a_7 = U'_{n+1} - U'_p-(hh/6)_p-\bar{b}_3, \quad a_8 = V'_{n+1} - V'_p - (kk/6)_p - \bar{c}_3 \quad (5.4)$$

Finally, the remaining five level 2 type 2 characters are found by decomposing products of a_o with the five level 1 type 1 characters.

$$a_5 = a_o(hh/6)_p - a_7, \quad a_6 = a_o(kk/6)_p - a_8 \quad (5.5)$$

$$a_9 = a_o(kh/2)_p - a_o-a_2-a_3, \quad a_{10} = a_o(hk/2)_p - a_o-a_2-a_4 \quad (5.6)$$

$$a_1 = a_o(g/3)_p - a_5 - a_6 . \quad (5.7)$$

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A PARAMETRIC REPRESENTATION OF THE SPHERE WITH p HANDLES

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Introduction. It is well known that a compact connected orientable 2-manifold is diffeomorphic to the sphere with p handles ($p = 0,1,\dots$). There are many parametric representations of the sphere ($p = 0$) and the torus ($p = 1$) to $\mathbb{R}^3$. However, it is less trivial to find such a representation, if $p \geq 2$. In this paper we shall construct a parametric representation for the sphere with p handles for all p with the help of the Cassini oval.

1. The Cassini oval is the locus of points in the Euclidean plane such that the product of distances from two points $(e^{\frac{1}{2}}, 0)$ and $(-e^{\frac{1}{2}}, 0)$ is constant. In complex notation,

$$|z - e^{\frac{1}{2}}| \cdot |z + e^{\frac{1}{2}}| = a$$

Thus, if we denote the Cassini oval by Γ, the equation is

$$z^2 \cdot \bar{z}^2 - e(z^2 + \bar{z}^2) = a^2$$

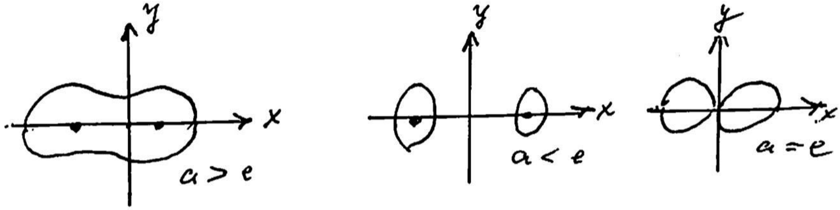
The shape of Γ depends on e and a (see fig. below).

1) If $e < a$, Γ is a simple closed curve.

2) If $e > a$ consists of two simple closed curves Γ^+ and Γ^-.

3) If $e = a$, and $e > 0$, Γ is the "figure 8".

4) If $e = a = 0$, then Γ reduces to the origin.



These curves look exactly like the curves in which a sphere with handles intersects a plane. Therefore it is natural to use the Cassini ovals to obtain a parametric representation for these surfaces. We shall start the construction by defining a parametric representation of the oval.

2. The parametric representation of Γ. Consider the map $z \longrightarrow z^2$ and set $z^2 = \omega$. Then the image of Γ is given by

$$\tilde{\Gamma}: |\omega|^2 - e(\omega + \bar{\omega}) = a^2 - e^2$$

Setting $\omega = u + iv$ we obtain

$$u^2 + v^2 - 2eu = a^2 - e^2$$

Thus $\tilde{\Gamma}$ is a circle with radius a centered at e. The origin is inside $\tilde{\Gamma}$, if $a > e$, outside $\tilde{\Gamma}$ if $a < e$ and on $\tilde{\Gamma}$ if $a = e$.

Parametrize the circle $\tilde{\Gamma}$ in the form

$$\omega(t) = e + a \exp(it), \quad 0 \leqq t \leqq 2\pi.$$

Now suppose that $a > e$. Then 0 is not on $\tilde{\Gamma}$ and so the angle function θ for $\tilde{\Gamma}$ is defined by

$$\theta(t) = \int_0^t \frac{u\dot{v} - \dot{u}v}{u^2 + v^2} dt$$

and so we can write

$$\omega(t) = |\omega(t)| \exp(i\theta(t))$$

This shows that the function

$$f(t) = |\omega(t)|^{\frac{1}{2}} \exp\left(\frac{i}{2}\,\theta(t)\right) \qquad -2\pi \leq t \leq 2\pi$$

is a parametric representation for Γ.

If $a < e$, the two ovals can be parametrized in the form

$$f^{+}(t) = (e + a \exp i\theta(t))^{\frac{1}{2}}, \quad \kappa > 0$$

and

$$f^{-}(t) = (e + a \exp i\theta(t))^{\frac{1}{2}}, \quad \kappa < 0$$

3. The polynomials P_n. Consider the polynomials in a real variable τ given by $P_0 = 1$ and

$$P_n(\tau) = (-1)^n p_1(\tau) \ldots p_n(\tau)$$

where

$$p_k(\tau) = 1 - \frac{\tau^2}{k^2}, \quad k = 1, \ldots, n.$$

Then $P_n(0) = (-1)^n$ and $P_n(k) = 0$, $k = \pm 1, \ldots \pm n$. Now set

$$e(\tau) = p_n(\tau)$$

and

$$a(\tau) = p_n(\tau)(1 + C_n P_{n-1}(\tau)),$$

where C_n is a positive constant such that

$$C_n \cdot |P_n(\tau)| < 1, \quad |\tau| \leq n.$$

It follows that $e(\tau) > 0$ and $a(\tau) > 0$ for $|\tau| < n$.

Moreover,

$$a(\tau) - e(\tau) = -C_n P_n(\tau)$$

and so

$$a(\tau) > e(\tau), \quad n-1 < \tau < n.$$

4. The 2-manifold M. Let Q be the rectangle $-2\pi \le t \le 2\pi$, $-n \le \tau \le n$ and consider the function

$$\omega_\tau(t) = e(\tau) + a(\tau)\exp(it), \qquad (\tau, t) \in Q.$$

It is easy to check that ω vanishes exactly at the points $(t, \pm n)$ and $(\pm\pi, k)$, $k = \pm 1, \dots, \pm(n-1)$.
Let Ω be the domain obtained from Q by removing these points and define a map from Ω to $\mathbb{R}^3$ by

$$\varphi(\tau, t) = f_\tau(t) + \tau e_3,$$

where

$$f_\tau(t) = \omega_\tau(t)^{\frac{1}{2}}.$$

A straightforward computation shows that

$$\frac{\partial\varphi}{\partial\tau} \times \frac{\partial\varphi}{\partial t} \neq 0$$

and thus φ defines a surface $\dot{M}$ in $\mathbb{R}^3$. This surface satisfies the equation $\Phi(z, \tau) = 0$ where Φ is given by

$$\Phi(z, \tau) = z^2 \cdot \bar{z}^2 - \{e(\tau)(z^2 - \bar{z}^2) - a(\tau)^2.$$

Let M denote the (compact) surface in $\mathbb{R}^3$ defined by $\Phi = 0$.
Then $\dot{M} \subset M$ and $M - \dot{M}$ consists of the "exceptional points"

$$a_k = (0, k), \quad k = \pm 1, \dots \pm n.$$

Since

$$\frac{\partial\Phi}{\partial z}(0, \tau) = 0 \quad \text{and} \quad \frac{\partial\Phi}{\partial\bar{z}}(0, \tau) = 0,$$

it follows that the tangent plane of M at these points is parallel to the (e_1, e_2)-plane.

5. The Euler characteristic of M. To show that M is the sphere with (n-1) handles we shall construct a tangent vector

field Z on M and show that its index sum is $2(1-n)$.

Consider the smooth vector field Z in $\mathbb{R}^3$ given by

$$Z(z,\tau) = i\left(-z\cdot r^2 + \bar{z}\cdot e(\tau)\right), \quad r^2 = |z|^2.$$

Thus $Z(z,\tau)$ is parallel to the (e_1, e_2)-plane for all (z,τ). Moreover, Z is tangent to M and the zeros of Z on M are exactly the points

$$a_k, \quad k = \pm 1, \dots \pm n.$$

Let $j_Z(a_k)$ denote the index of Z at a_k. We shall show that

$$j_Z(a_k) = -1 \qquad k = \pm 1, \dots \pm(n-1)$$

and

$$j_Z(a_k) = +1, \quad k = \pm n$$

Fix a plane $\tau = k$ and let Y_k be the restriction of Z to $\tau = k$. Since the tangent plane of M at a_k is parallel to the (e_1, e_2)-plane, we have

$$j_Z(a_k) = j_{Y_k}(0).$$

The vector field Y_k is given by

$$Y_k(z) = -z\cdot r^2 + \bar{z}\cdot p_n(k)$$

Now suppose that $k \neq \pm n$. Then $p_n(k) \neq 0$ and so Rouché's theorem implies that

$$j_{Y_k}(0) = -1, \quad k = \pm 1, \dots \pm(n-1).$$

On the other hand, since $p_n(n) = 0$,

$$j_{Y_n}(0) = +1.$$

This shows that

$$j_Z(a_k) = -1, \quad k = \pm 1, \dots \pm(n-1)$$

and

$$j_Z(a_k) = +1, \quad k = \pm n$$

Hence the index sum of Z (and so the Euler characteristic of M) is given by

$$j_Z(M) = 2(n-1)(-1) + 2 = 2(1-n)$$

and so M is indeed diffeomorphic to the sphere with $(n-1)$ handles.

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