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STRUCTURE PROPERTIES OF CERTAIN Q-MATRICES

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Presented by G.de B. Robinson, F.R.S.C.

This paper contains a study of those matrices $M \in R^{nxn}$ that satisfy $M \not\geq 0$. The paper deals with the question of when such a matrix is a Q-matrix. Several Sufficient conditions are given.

For a matrix $M \in R^{nxn}$ and a column vector $q \in R^{nx1}$ the linear complementarily problem, denoted by LCP(q,M), is to find column vectors $w \in R^{nx1}$ and $z \in R^{nx1}$ so that

$$Iw - Mz = q,\ w \geq 0,\ z \geq 0 \text{ and } w^T z = 0.$$

The matrix M is said to be a Q-matrix if the linear complementarity problem LCP(q,M) has a solution for every $q \in R^{nx1}$. If the solution is unique for each $q \in R^{nx1}$, then M is called a P-matrix. The books by Berman and Plemmons [1] and by Murty [3] both contain a good review of these subjects.

In [4] Murty proved that a matrix is a P-matrix if and only if all its principal subdeterminants are positive. But as Murty points out in [3] , no simple necessary and sufficient conditions are known for a matrix to be a Q-matrix. Any regular matrix is a Q-matrix [1]. Another known sufficient condition is that the matrix be strictly copositive [3]. When $M \geqq 0$, Murty [5] has shown that M is a Q-matrix if and only if $m_{ii} > 0$ for each $i = 1,\dots,n$. Hence, we shall restrict our attention to the case where $M \not\geqq 0$. Throughout the paper, $M^{(k)}$ denotes the kth column of matrix M.

The following observations are easily verified. First, M is a Q-matrix if and only if PMP^T is a Q-matrix, whenever P is a permutation matrix. When M is nonsinguar, then M is a Q-matrix if and only if M^{-1} is a Q-matrix.

Each row of a Q-matrix must contain a positive entry.

Concentrating on the case where $n = 2$, we can complete the characterization of those matrices $M \not\geq 0$ which are Q-matrices.

Theorem 1 If $M \in R^{2x2}/R_{+}^{2x2}$ is a Q-matrix, then M is nonsingular. When M is nonsingular then M is a Q-matrix if and only if $I^{(k)} = Mz$, $z > 0$ has a solution, whenever $M^{(k)} \not\leq 0$.

Now consider the more general case where $M \in R^{nxn}$ such that $M \not\geq 0$. Recall that a square matrix is monotone whenever it has a nonnegative inverse.

Theorem 2 If $M \in R^{nxn}$ is such that $m_{ij} \geq 0$ whenever $i \neq j$ and each $(n-1)x(n-1)$ principal submatrix of M is a monotone Q-matrix, then M is a Q-matrix.

Theorem 2 can be used to generate examples of Q-matrices that are not regular matrices.

Now let M be a nonsingular Q-matrix and define $\bar{M}$ by

$$\bar{M} = \begin{bmatrix} M & y \\ x^T & \beta \end{bmatrix}.$$

Theorem 3. Let $M \in R^{nxn}$, $x \in R_{-}^{nx1}$, $y \in R^{nx1}$ and $\beta > 0$. If

(i) $M - (1/\beta)yx^T$ is a Q-matrix and

(ii) $I^{(n+1)} \in pos(D^{(1)},\ldots,D^{(n)}, -[y^T\ \beta]^T)$ for all possible complementary cones such that $[D^{(1)}\vdots\cdots\vdots D^{(n)}] \neq [I^{(1)}\vdots\cdots\vdots I^{(n)}]$ and $[D^{(1)}\vdots\cdots\vdots D^{(n)}] \neq [-M^{(1)}\vdots\cdots\vdots -M^{(n)}]$, then $\bar{M}$ is a Q-matrix.

The matrices in Theorem 2 and Theorem 3 do not have a zero diagonal. In general,

Theorem 4. If $M \in R^{nxn}$ has a zero diagonal and $m_{ij} \geq 0$ whenever $j > i$, then M is not a Q-matrix.

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L'INDÉCIDABILITÉ D'UNE CLASSE DE CORPS DES FONCTIONS MÉROMORPHES

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Résumé.

On montre entre autres choses que pout tout entier t le corps des fonctions méromorphes dans t variables à coefficients réels est indécidable.

Pour un sous-corps K du corps $\mathbb{C}$ des nombres complexes désignons par H_K le sous-anneau de $K[[X]]$ formé des séries formelles dont le rayon de convergence est infini. Soit M_K le corps des fractions de H_K. Nous appelons M_K le corps des fonctions méromorphes à coefficients dans K.

Rappelons qu'un corps K est dit pythagoricien si $K^2 + K^2 = K^2$, c.à d. si toute somme de deux carrés de K est un carré de K. Par exemple, tout corps algébriquement clos est pythagoricien et tout corps réellement clos est pythagoricien.

Nous allons démontrer que le corps des constantes K de M_K est élémentairement définissable dans M_K lorsque K est un sous-corps pythagoricien de $\mathbb{C}$. De plus, si K est un sous-corps pythagoricien du corps $\mathbb{R}$ des nombres réels, alors M_K est un corps indècidable.

Dans le même ordre d'idées nous établirons des résultats concernant la dimension de Pfister de H_K et M_K. Rappelons que la dimension de Pfister $d(A)$ d'un anneau A est le plus petit entier d tel que toute somme de carrés d'éléments de R est une somme de d carrés d'éléments de R, ou ∞ s'il n'existe pas de tel entier.

Proposition 1. Si K est un sous-corps pythagoricien de $\mathbb{C}$, alors K est *élémentairement définissable (sans paramètres) dans* M_K *par une formule que ne dépend pas de* K.

Démonstration. Puisque le genre de la courbe $X^2 = 1 + Y^8$ est plus grand que 1, le théorème d'uniformisation de Picard [voir 1, p. 268] implique qu'il n'y a pas de fonctions méromorphes non-constantes ξ et η telles que $\xi^2 = 1 + \eta^8$. Par conséquent on obtient la définition élémentaire

$$K = \{\xi \in M_K \mid \exists \eta \in M_K \ (\xi^2 = 1 + \eta^8)\}.$$

Avant de considérer la dimension de Pfister nous aurons besoin d'un résultat technique.

Lemme 1. *Pour toute fonction entière* f *avec* $f(0) = 1$ *il existe une fonction entière* φ *avec* $\varphi(0) = 0$ *telle que* $f(X)e^{\varphi(x)} \in H_{\mathbb{Q}(\sqrt{-1})}$. *De plus, si* $f \in H_{\mathbb{R}}$ *et* $f(0) = 1$ *il existe une fonction entière* $\varphi \in H_{\mathbb{R}}$ *telle que* $f(X)e^{\varphi(x)} \in H_{\mathbb{Q}}$ *et* $\varphi(0) = 0$.

Démonstration. Si l'on écrit la fonction cherchée $\varphi(X) = \sum_{i=1}^{\infty} a_i X^i$, alors

$$e^{\varphi(x)} = 1 + \sum_{i=1}^{\infty} b_i X^i$$

où $b_i - a_i$ est un polynôme à coefficients rationnels de $a_1, \ldots, a_{i-1}$. Si $f(X) = 1 + \sum_{i=1}^{\infty} c_i X^i$, alors le $n^{\text{ième}}$ coefficient de fe^{φ} est

$$c_n + c_{n-1}b_1 + \ldots + c_1 b_{n-1} + b_n .$$

Puisque $\mathbb{Q}(\sqrt{-1})$ (resp. $\mathbb{Q}$) est dense dans $\mathbb{C}$ (resp. $\mathbb{R}$) on construit des nombres a_i par recurrence sur i tels que φ est une fonction entière avec les propriétés désirées.

Proposition 2. *Soit* K *un sous-corps de* $\mathbb{R}$ *qui n'admet qu'un seul ordre (en* tant que corps ordonné). Si f est une fonction entière de H_K telle que $f(X) \geq 0$ pout tout $x \in \mathbb{R}$, alors f est une somme de carrés de fonctions de H_K. Si la dimension de Pfister $d(K)$ est finie, alors f est une somme de $2d(K)$ carrés de fonctions de H_K.

Démonstration. Soient (α_i) les zéros réels $(\neq 0)$ de f et (a_i) les multiplicités correspondantes. Soit a (≥ 0) la multiplicité de 0 en tant que zéro des f. Alors la condition $f(x) \geq 0$ pour tout $x \in \mathbb{R}$ implique que les nombres a et (a_i) sont pairs. Les zéros non-réels de f sont de la forme (β_j), $(\overline{\beta}_j)$, où $\overline{\beta}_j$ est le complexe conjugué de β_j. Pour chaque j β_j et $\overline{\beta}_j$ ont la même multiplicité, disons b_j.

En vertu du théorème de Weierstrass il existe une fonction entière ψ avec $\psi(0) = 0$ et des entiers k_i, l_j tels que

$$f(X) = c\, e^{\psi(x)} X^a\, \Pi_1^{\,2}\, \Pi_2\, \Pi_3$$

où c est un nombre positif de K et

$$\Pi_1 = \prod_i \left[\left(1 - \frac{x}{\alpha_i}\right) e^{\frac{x}{\alpha_i} + \frac{1}{2}\left(\frac{x}{\alpha_i}\right)^2 + \dots + \frac{1}{k_i}\left(\frac{x}{\alpha_i}\right)^{k_i}}\right]^{\frac{a_i}{2}}$$

$$\Pi_2 = \prod_j \left[\left(1 - \frac{x}{\beta_j}\right) e^{\frac{x}{\beta_j} + \frac{1}{2}\left(\frac{x}{\beta_j}\right)^2 + \dots + \frac{1}{l_j}\left(\frac{x}{\beta_j}\right)^{l_j}}\right]^{b_j}$$

$$\Pi_3 = \prod_j \left[\left(1 - \frac{x}{\overline{\beta}_j}\right) e^{\frac{x}{\overline{\beta}_j} + \frac{1}{2}\left(\frac{x}{\overline{\beta}_j}\right)^2 + \dots + \frac{1}{l_j}\left(\frac{x}{\overline{\beta}_j}\right)^{l_j}}\right]^{b_j}$$

Ici $\Pi_1 \in H_{\mathbb{R}}$ et $\Pi_2 = \overline{\Pi}_3 \in H_{\mathbb{C}}$. (On convient que $\overline{g} = h$, $g,h \in H_{\mathbb{C}}$, si les coefficients de h sont les complexes conjugués de ceux de g.)

Par le lemme 1 il existe des fonctions $\varphi_1 \in H_{\mathbb{R}}$ et $\varphi_2 \in H_{\mathbb{C}}$ telles que

$$\Pi_1 e^{\varphi_1} \in H_{\mathbb{Q}} \quad \text{et} \quad \Pi_2 e^{\varphi_2} \in H_{\mathbb{Q}(\sqrt{-1})} .$$

Donc

$$f = c\,(e^{\psi-2\varphi_1-\varphi_2-\overline{\varphi}_2})x^a \cdot (\Pi_1 e^{\varphi_1})^2 (\Pi_2 e^{\varphi_2})(\overline{\Pi_2 e^{\varphi_2}})$$

où $c \cdot e^{\psi-2\varphi_1-\varphi_2-\overline{\varphi}_2} \in H_K$. En vertu du fait que

$$(\Pi_2 e^{\varphi_2})(\overline{\Pi_2 e^{\varphi_2}}) = (\tfrac{1}{2}(\Pi_2 e^{\varphi_2} + \overline{\Pi_2 e^{\varphi_2}}))^2 + (\tfrac{1}{2\sqrt{-1}}(\Pi_2 e^{\varphi_2} - \overline{\Pi_2 e^{\varphi_2}}))^2$$

et

$$e^{\psi-2\varphi_1-\varphi_2-\overline{\varphi}_2} = (e^{\frac{1}{2}(\psi-2\varphi_1-\varphi_2-\overline{\varphi}_2)})^2 \in H_K^2$$

il s'ensuit que si c est la somme de d carrés de nombres de K, alors f est la somme de $2d$ carrés de fonctions de H_K.

En utilisant un résultat dans [2, p. 184] on obtient aisément:

Corollaire 1. *Pour tout sous-corps K de $\underline{\underline{R}}$ on a $d(M_K) \leq 2^t$, où 2^t est la plus petite puissance de 2, qui est $\geq \max(2, d(K))$.*

Corollaire 2. *Pour le corps des nombres rationnels $\underline{\underline{Q}}$ on a $d(H_{\underline{\underline{Q}}}) = d(M_{\underline{\underline{Q}}}) = 4$. Pour tout sous-corps pythagoricien K de $\underline{\underline{R}}$ on a $d(H_K) = d(M_K) = 2$.*

En utilisant des arguments connus de l'analyse classique on obtient

Lemme 2. *Soient f une fonction de $H_{\underline{\underline{R}}}$ et α un nombre réel. Alors $f(\alpha) = 0$ si et seulement s'il existe un nombre réel c tel que*

$$\frac{(f(x))^2}{1+(f(x))^2} \leq c(x-\alpha)^2$$

pour tout $x \in \underline{\underline{R}}$.

Dorénavant, soit K un sous-corps pythagoricien de $\underline{\underline{R}}$. Dans le langage des corps considérons la formule $\Phi(\alpha, f, T)$ pour M_K, définie par

$$\exists c \in K,\ c \neq 0 \quad \exists p,q \in M_K \ (C(T-\alpha)^2 - \frac{f^2}{1+f^2} = p^2 + q^2\ )$$

où l'on note que la quantification par rapport à K est possible grâce à la proposition 1.

En vertu des résultats précédents nous obtenons

Lemme 3. *Soient* f *une fonction de* H_K *et* $\alpha \in \underline{\underline{Q}}$. *Alors*

$$\Phi(\alpha,f,X) \Leftrightarrow f(\alpha) = 0.$$

D'autre part pour tout $\alpha \in K$

$$\Phi(\alpha,f,X) \Rightarrow f(\alpha) = 0.$$

Théorème 1. *Pour tout sous-corps pythagoricien* K *de* $\underline{\underline{R}}$ *l'ensemble* $\underline{\underline{N}}$ *des nombres naturels est élémentairement définissable (sans paramètres) dans* M_K. *En particulier,* M_K *est un corps indécidable.*

Démonstration. Puisque K est élémentairement définissable dans M_K le théorème est une conséquence de l'assertion suivante:

Si α est un nombre de K, alors $\alpha \in \underline{\underline{N}} \Leftrightarrow \forall g,h \in M_K$

$$[\Phi(0,g,h) \wedge (\forall k \in K(\Phi(k,g,h) \rightarrow \Phi(k+1,g,h))] \rightarrow \Phi(\alpha,g,h).$$

Ici l'implication "$\Rightarrow$" est claire. Pour prouver $\Leftarrow$ il suffit de choisir $g = X$ et de prendre pour h une fonction entière de $H_{\underline{\underline{Q}}} \subseteq H_K$, dont les seuls zéros sont les nombres $0,1,2,\ldots$.

On construit une telle fonction en appliquant le lemme 1 à la fonction $(\Gamma(-x)x)^{-1}$,

Remarque. La démonstration du théorème 1 montre que M_K est indécidable pour *tout* sous-corps (pas forcément pythagoricien) K de $\underline{\underline{R}}$ tel que K est définissable dans M_K.

Le résultat suivant est une conséquence immédiate de la proposition 1.

Proposition 2. Si K *et* L *sont des sous-corps pythagoriciens de* $\mathbb{C}$, *alors* $M_K \equiv M_L$ *implique* $K \equiv L$. (Ici "$\equiv$" désigne équivalence élémentaire.)

La réciproque de la proposition 2 est fausse. Le corps L des nombres algébriques réels est réellement clos et par suite $L \equiv \mathbb{R}$. Cependant, puisque $\mathbb{N}$ d'après le théorème 1 est définissable dans M_L et $M_{\mathbb{R}}$ par la même formule, un argument bien connu de la théorie des modèles montre que $M_L \not\equiv M_{\mathbb{R}}$.

Pour terminer, nous considérons les corps des fonctions méromorphes en plusieurs variables. Pour un sous-corps K de $\mathbb{C}$ désignons par H_K^t, ($t \in \mathbb{N}$), le sous-anneau de $K[[X_1,\dots,X_t]]$ formé des séries formelles qui sont convergentes dans tout l'espace $\mathbb{C}^t$. Alors H_K^t est un anneau intègre, dont le corps des fractions M_K^t est le corps des fonctions méromorphes en t variables à coefficients dans K.

Si K est pythagoricien une modification facile de la proposition 1 montre que K est définissable dans M_K^t par une formule qui ne dépend ni de K ni de t. De plus, la formule avec laquelle on définit $\mathbb{N}$ dans M_K marche aussi dans le cas général. On en déduit

Théoreme 2. Pour tout $t \in \mathbb{N}$ *et tout sous-corps pythagoricien* K *de* $\mathbb{R}$ *l'ensemble* $\mathbb{N}$ *des nombres naturels est élémentairement définissable (sans paramètres) dans* M_K^t *par une formule qui ne depend ni de* K *ni de* t. *En particulier,* M_K^t *est indécidable.*

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ON THE COVARIANCE OF THE MOORE-PENROSE INVERSE OF A MATRIX

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1. In view of the covariance of the inverse A^{-1} of a regular n n matrix A, namely $(TAT^{-1})^{-1} = TA^{-1}T^{-1}$ for all regular (invertible) matrices T the following question arises: Let A be a singular $n \times n$ matrix of rank r over the field $\mathbb{C}$ of the complex numbers and denote by A^+ the Moore-Penrose (M.P.) inverse of A. For which regular $n \times n$ matrices T over $\mathbb{C}$ does hold the covariance condition

(1) $$(TAT^{-1})^+ = T A^+ T^{-1} \quad ?$$

Denote by $\mathscr{C}(A)$ the class of these matrices T. It is readily seen that for all singular matrices A this class includes the group $\mathscr{W}$ consisting of the non-zero multiples of all $n \times n$ unitary matrices. In general the class $\mathscr{C}(A)$ is not a group. The polar representation of a regular matrix in the form $T = UH$ where U is unitary and H is positive definite hermitean (p.d.h.) leads to the following equivalent of the covariance condition:

Theorem 1. The matrix $T = UH$ is contained in $\mathscr{C}(A)$ if and only if the p.d.h. matrix $H^2 = T^*T$ commutes with the two hermitean matrices AA^+ and A^+A, that is: H^2AA^+ and H^2A^+A are hermitean.

The basic question is hereby reduced to the problem: Determine all p.d.h. matrices $H \in \mathscr{C}(A)$.

2. First let us assume that A is a matrix of rank one. It can be represented in the form $A = ab^*$ where a and b are two non-zero column vectors and b^* the row vector conjugate transpose to b. These are unique up to a non-zero complex factor γ: If $ab^* = cd^*$ then $c = \gamma a$ and $d^* = b^*/\gamma$. The M.P. inverse of A is found to be $A^+ = ba^*/(a^*a \cdot b^*b)$ and if $\underline{a}$ and $\underline{b}$ denote the unit vectors of a and b (i.e. $a = |a|\underline{a}$ etc.) then $AA^+ = \underline{a}\,\underline{a}^*$ and $A^+A = \underline{b}\,\underline{b}^*$ are two non-negative hermitean matrices of rank one. Since the matrix ca^* is hermitean if and only if $c = \varrho a$, ϱ real, Theorem 1 implies that $H^2a = \lambda^2 a$, $H^2b = \mu^2 b$.

The class $\mathscr{C}(A)$ is described by the following theorem:

Theorem 2. Every p.d.h. matrix $H \in \mathscr{C}(A)$, $A = ab^* \neq 0$, has a and b as eigen vectors with positive eigen values λ, μ. If

(i) a, b are orthogonal, $b^*a = 0$, then $A^2 = 0$ and λ, μ are arbitrary positive numbers and

(2) $\quad H = \lambda \underline{a}\, \underline{a}^* + \mu\, \underline{b}\, \underline{b}^* + H_{n-2}\ , \quad H_{n-2}a = 0\ , \quad H_{n-2}b = 0\ ,$

where H_{n-2} is a non-negative hermitean matrix of rank $n-2$;

(ii) a, b are not orthogonal: $b^*a \neq 0$. Then all $\alpha a + \beta b$ (α, β complex) are eigen vectors of H with the same eigen value λ. A complex number γ can be found such that $c = a + \gamma b$ is orthogonal to a, i.e. $c^*a = 0$ and

(2') $\quad H = \lambda(\underline{a}\, \underline{a}^* + \underline{c}\, \underline{c}^*) + H_{n-2} \in \mathcal{T}(A)\ , \quad H_{n-2}a = 0\ . \quad H_{n-2}c = 0\ .$

(iii) a, b are linearly dependent; then

(2") $\quad H = \lambda\, \underline{a}\, \underline{a}^* + H_{n-1}\ , \quad H_{n-1}a = 0\ .$

3. To simplify the discussion let us now consider the case of an $n \times n$ - matrix A of rank 2 ($n > 2$). It can be represented in the form $A = (a\ b)\begin{pmatrix} c^* \\ d^* \end{pmatrix}$ where a, b and c, d are two pairs of linearly independent column vectors in $\mathbb{C}^n$. If $(a\ b)(c\ d)^* = (\hat{a}\ \hat{b})(\hat{c}\ \hat{d})^*$ then there is a 2×2 regular matrix S such that $(\hat{a}\ \hat{b}) = (a\ b)S^{-1}$ and $\begin{pmatrix} \hat{c}^* \\ \hat{d}^* \end{pmatrix} = S\begin{pmatrix} c^* \\ d^* \end{pmatrix}$.

Lemma 1. The matrix S can always be chosen so that $\hat{b}^*\hat{a} = 0$, $\hat{d}^*\hat{c} = 0$.

In this case $(\hat{a}\ \hat{b})(\hat{c}\ \hat{d})^*$ may be called the orthogonal representation of A. The planes $\langle a,b \rangle$, $\langle c,d \rangle$ will be called the characteristic planes of the matrix A.

We introduce the 2×2 matrix $\Gamma(A) = \Gamma = \begin{pmatrix} c^*a & c^*b \\ d^*a & d^*b \end{pmatrix}$. The elements of $\Gamma(A)$ are similarity invariants of the matrix A: With an arbitrary regular matrix P one has $\Gamma(PAP^{-1}) = \Gamma(A)$.

Lemma 2. The matrix $A = (a\ b)(c\ d)^*$ of rank 2 is nilpotent of index 2, i.e. $A^2 = 0$ if and only if $\Gamma(A) = 0$, that is if the characteristic planes of A are totally orthogonal.

We note that A is hermitean if and only if $(c\ d) = (a\ b)S$ and S is regular hermitean.

The class $\mathcal{T}(A)$ of all matrices T satisfying the covariance condition (1) depends on the two matrices $A\,A^+ = a\,a^* + b\,b^*$ and $A^+A = c\,c^* + d\,d^*$.

Lemma 3. If X is a matrix of rank 2 with the characteristic planes $\langle a,b \rangle$, $\langle c,d \rangle$ then $\mathcal{T}(X) = \mathcal{T}(A)$. In particular if $\underline{A} = (\underline{a}\ \underline{b})(\underline{c}\ \underline{d})^*$, then $\mathcal{T}(\underline{A}) = \mathcal{T}(A)$.

These lemmata can all be generalized to $n \times n$ matrices A of rank $r > 2$.

4. A 2×2 matrix Λ is said to be an eigen matrix of the $n\times n$ matrix H if there is a pair of linearly independent vectors a, b such that

$$H(a\ b) = (a\ b)\Lambda \quad .$$

The plane $\langle a,b\rangle$ is then called the eigen plane of H with the eigen matrix Λ (cf.[1]). The matrix Λ is defined up to singularity: Let S be a regular 2×2 matrix; Λ and $S^{-1}\Lambda S$ are eigen matrices of H for the same eigen plane.

If $A = (a\ b)(c\ d)^*$ is a matrix of rank 2, then its characteristic plane $\langle a,b\rangle$ is an eigen plane of A with the eigen matrix $\Gamma(A)$.

Two linearly independent eigen vectors of the matrix H generate an eigen plane of H; the corresponding eigen matrix is diagonal. If the eigen matrix of a plane can be diagonalized a pair of eigen vectors can be determined which generates this plane.

Lemma 4. If an $n\times n$ matrix H has Λ as an eigen matrix the conjugate transpose H^* has the conjugate $\bar{\Lambda}$ of Λ as eigen matrix. Thus $\bar{\Lambda}' = \Lambda^* = S^{-1}\Lambda S$ is an eigen matrix of H^* (cf.[1]). If H is hermitean it follo: that with Λ also $\bar{\Lambda}$ and Λ^* are eigen matrices of H.

Lemma 5. If $\langle a,b\rangle$ is an eigen plane of the matrix H and P is a reg' $n\times n$ matrix, then $\hat{a} = Pa$, $\hat{b} = Pb$ generate an eigen plane of the matrix PHP^{-} If $b^*a = 0$ the eigen matrix Λ of the eigen plane $\langle a,b\rangle$ is p.d.h.

5. After these preliminaries the p.d.h. matrices $H \in \mathcal{C}(A)$ can be explicitly determined In the simplest case the result is analogous to (2) in Theorem 2 (i):

Theorem 3. Let $A = (a\ b)(c\ d)^*$ be a matrix of rank 2 and $A^2 = 0$ (cf. Lemma 2). The every p.d.h. matrix $H \in \mathcal{C}(A)$ appears in the form

$$H = (a\ b)\Lambda(a\ b)^* + (c\ d)M(c\ d)^* + H_{n-4} \tag{3}$$

where Λ, M are two p.d.h. eigen matrices of H and H_{n-4} is a non-negative hermitean matrix such that

$$H_{n-4}(a\ b\ c\ d) = 0\ .$$

This Theorem can readily be extended to the case of a matrix A of rank $r > 2$ $(n \geqslant 2r)$ which is nilpotent of index 2.

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SCALES

Errata

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Presented by J.H.H. Chalk, F.R.S.C.

The table on p. 280 (C.R. Math. Rep. Acad. Sci. Canada, vol. IV, No. 5, Oct. 1982) contains some errors. These have been corrected and in the revised table below the corrected entries are distinguished.

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First sixty degrees of the scales of Pythagoras, Zarlino and Zarlino A

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	$\frac{2^8}{3^5}$	$\frac{3^2}{2^3}$	$\frac{2^5}{3^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$	$\frac{3^6}{2^9}$	$\frac{3}{2}$	$\frac{2^7}{3^4}$	$\frac{3^3}{2^4}$	$\frac{2^4}{3^2}$	$\frac{3^5}{2^7}$	2	$\frac{2^9}{3^5}$	$\frac{3^2}{2^2}$	$\frac{2^6}{3^3}$	$\frac{3^4}{2^5}$	$\frac{2^3}{3}$	$\frac{3^6}{2^8}$	3	$\frac{2^8}{3^4}$
1	$\frac{2^4}{3.5}$	$\frac{3^2}{2^3}$	$\frac{2.3}{5}$	$\frac{5}{2^2}$	$\frac{2^2}{3}$	$\boxed{\frac{5^2}{2.3^2}}$	$\frac{3}{2}$	$\frac{2^3}{5}$	$\frac{5}{3}$	$\boxed{\frac{3^2}{5}}$	$\frac{3\times5}{2^3}$	2	$\frac{2^5}{3.5}$	$\frac{3^2}{2^2}$	$\frac{2^2.3}{5}$	$\frac{5}{2}$	$\frac{2^3}{3}$	$\frac{5\times3^2}{2^4}$	3	$\frac{2^4}{5}$
1	$\frac{3.5}{2.7}$	$\boxed{\frac{2^3}{7}}$	$\frac{2.3}{5}$	$\frac{5}{2^2}$	$\frac{2^2}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{2^3}{5}$	$\frac{5}{3}$	$\boxed{\frac{7}{2^2}}$	$\frac{3\times5}{2^3}$	2	$\frac{3.5}{7}$	$\frac{3^2}{2^2}$	$\boxed{\frac{7}{3}}$	$\frac{5}{2}$	$\frac{2^3}{3}$	$\frac{2.7}{5}$	3	$\frac{2^4}{5}$

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
	$\frac{3^3}{2^3}$	$\frac{2^5}{3^2}$	$\frac{3^5}{2^6}$	2^2	$\frac{2^{10}}{3^5}$	$\frac{3^2}{2}$	$\frac{2^7}{3^3}$	$\frac{3^4}{2^4}$	$\frac{2^4}{3}$	$\frac{3^6}{2^7}$	2.3	$\frac{2^9}{3^4}$	$\frac{3^3}{2^2}$	$\frac{2^6}{3^2}$	$\frac{3^5}{2^5}$	2^3	$\frac{2^{11}}{3^5}$	3^2	$\frac{2^8}{3^3}$	$\boxed{\frac{3^4}{2^3}}$
	$\frac{2.5}{3}$	$\frac{2.3^2}{5}$	$\frac{3.5}{2^2}$	2^2	$\boxed{\frac{5^2}{2.3}}$	$\frac{3^2}{2}$	$\frac{2^3.3}{5}$	5	$\frac{2^4}{3}$	$\frac{5.3^2}{2^3}$	2.3	$\boxed{\frac{5^2}{2^2}}$	$\boxed{\frac{2^2.5}{3}}$	$\boxed{\frac{2^2.3^2}{5}}$	$\frac{3.5}{2}$	2^3	$\frac{5^2}{3}$	3^2	$\frac{2^4.3}{5}$	2.5
	$\frac{2.5}{3}$	$\boxed{\frac{7}{2}}$	$\frac{3.5}{2^2}$	2^2	$\boxed{\frac{3.7}{5}}$	$\frac{3^2}{2}$	$\boxed{\frac{2.7}{3}}$	5	$\frac{2^4}{3}$	$\frac{2^2.7}{5}$	2.3	$\boxed{\frac{5^2}{2^2}}$	$\boxed{\frac{2^2.5}{3}}$	$\boxed{7}$	$\frac{3.5}{2}$	2^3	$\frac{5^2}{3}$	3^2	$\boxed{\frac{2^2.7}{3}}$	2.5

	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
	$\frac{2^5}{3}$	$\frac{3^6}{2^6}$	$2^2.3$	$\frac{2^{10}}{3^4}$	$\frac{3^3}{2}$	$\frac{2^7}{3^2}$	$\frac{3^5}{2^4}$	2^4	$\frac{2^{12}}{3^5}$	2.3^2	$\frac{2^9}{3^3}$	$\frac{3^4}{2^2}$	$\frac{2^6}{3}$	$\frac{3^6}{2^5}$	$2^3.3$	$\frac{2^{11}}{3^4}$	3^3	$\frac{2^8}{3^2}$	$\frac{3^5}{2^3}$	2^5
	$\frac{2^5}{3}$	$\frac{5.3^2}{2^2}$	$2^2.3$	$\boxed{\frac{5^2}{2}}$	$\frac{3^3}{2}$	$\frac{2^3.3^2}{5}$	3.5	2^4	$\boxed{\frac{5^2.2}{3}}$	2.3^2	$\frac{2^5.3}{5}$	$2^2.5$	$\frac{2^6}{3}$	$\frac{5.3^2}{2}$	$2^3.3$	5^2	3^3	$\frac{2^4.3^2}{5}$	2.3.5	2^5
	$\boxed{\frac{3.7}{2}}$	$\frac{5.3^2}{2^2}$	$2^2.3$	$\boxed{\frac{5^2}{2}}$	$\frac{3^3}{2}$	$\boxed{2.7}$	3.5	2^4	$\boxed{\frac{7^2}{3}}$	2.3^2	$\boxed{\frac{2^3.7}{3}}$	$2^2.5$	$\boxed{3.7}$	$\frac{5.3^2}{2}$	$2^3.3$	5^2	3^3	$\boxed{2^2.7}$	2.3.5	2^5

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FLAT MODULES AND NON-COMMUTATIVE LOCALIZATIONS

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Presented by P. Ribenboim, F.R.S.C.

If A is a commutative ring and M an A-module, then it is well-known that M is A-flat exactly if M_β is A_β-flat for every prime ideal β of A, which in turn is equivalent to M_m is A_m-flat for every maximal ideal m of A. [BAC]

Though there are many very useful notions of non-commutative localizations available in the literature (see for example [S_1] or [S_2]), unfortunately a result similar to the one mentioned above is not found in the case when the ring is not commutative.

In this note, we shall introduce a special type of localization, similar to the one introduced by Ribenboim [R] and show that this localization possesses the desired local-global principle for a class of flat modules.

Let A be an associative ring with identity and M an A-module on both sides. Then $Z(M) = \{x \mid x \in M: ax = xa \text{ for } a \in A\}$ is called the <u>center</u> of M. It is a module over the commutative subring Z(A) of A. M is called an A-<u>bimodule</u> if $M = AZ(M) = Z(M)A$, that is M is generated by Z(M) as an A-module. This notion is due to M. Artin [A]. These modules behave very much like modules over a commutative ring: for example if M and N are A-bimodules then $M \otimes N$ and $N \otimes M$ are both

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A-bimodules and they are canonically isomorphic. (All tensor products shall be over A). For a detailed discussion the reader is referred to [A] or [R].

Let $\phi : A \to B$ be a homomorphism of rings, ϕ is called an extension if B is a A-bimodule under ϕ.

An analysis of the proof of the local-global principle in the commutative case reveals that the following more general result is true.

1. Proposition: Let A be a ring, M an A-bimodule and $\{\phi_i : A \to A_i \mid i \in I\}$ be a family of extensions of A such that $\oplus A_i$ is right faithfully flat, then M is right flat if and only $M \otimes A_i$ is a right flat A_i-module for every $i \in I$.

Proof: Necessity follows easily from Corollary 2 of Proposition 8, page 34 of [BAC]. For sufficiency, first we remark that $M \otimes A_i$, is a A-flat right module (Corollary 3, page 35.ibid). Now let $0 \to N \to P$ be a monomorphism of left A-modules. Then $(M \otimes A_i) \otimes N \longrightarrow (M \otimes A_i) \otimes P$ is a monomorphism for each i. However the A-bimodules $M \otimes A_i$ and $A_i \otimes M$ are isomorphic (see remarks above). Hence $(A_i \otimes M) \otimes N \longrightarrow (A_i \otimes M) \otimes P$ is a family of monomorphisms. By forming the direct sum over I and using the associativity of the tensor product one concludes that $M \otimes N \to M \otimes P$ is a monomorphism, showing that M is a right flat A-module.

Now the right module $\oplus A_i$ is faithfully flat if and only if $\oplus A_i$ is flat and for every maximal left ideal m of A , we have $(\oplus A_i)m \cong \oplus (A_i m) \neq \oplus A_i$ (page 44, Prop. 1. [BAC]). Hence in order to establish the desired non-commutative local-global principle for flat modules, we must show that there exists a family of extensions $\{A_i\}$ of A , which are right flat modules and which "distinguish" the family of maximal left ideals of A.

However the following more general result is true.

2. Theorem: Given a ring A and I a left ideal of A, there exists a ring A_I and an extension $\alpha_I : A \to A_I$ with the following properties:

(i) α_I is an epimorphism of rings.

(ii) A_I is a right flat A-module.

(iii) $A_I I \neq A_I$.

and (iv) if $\beta : A \to B$ is another extension with the above three properties then there exists a unique extension (up to isomorphism) $\gamma : B \to A_I$ such that $\gamma \circ \beta = \alpha_I$.

Proof: Let $\mathcal{U} = \{(A_i, \alpha_i)\}$ be the family of pairs, where $\alpha_i : A \to A_i$ is an extension with the three properties mentioned above. $\mathcal{U}$ is non empty since (A, id) belongs to $\mathcal{U}$. By [P-S: thérème 2.7] corresponding to each such pair, there is a Gabriel topology G_i of left ideals on A such that $A_i \cong A_{G_i}$ as rings, where A_{G_i} is the ring of left quotients with respect to G_i.

Define $(A_i, \phi_i) \equiv (A_j, \phi_j)$ if there is a ring isomorphism $\phi_{ij} : A_i \to A_j$ such that $\phi_{ij} \circ \phi_j = \phi_j$. Then $\mathcal{U}/\equiv\ = [\mathcal{U}]$ is a set, since each element $[A_i, \phi_i]$ of $[\mathcal{U}]$ corresponds to a unique left Gabriel topology over A and this correspondence is injective.

Define $[A_i, \phi_i] < [A_j, \phi_j]$ if there exists an extension $\sigma_{ij} : A_i \to A_j$ such that $\sigma_{ij} \circ \phi_i = \phi_j$. This is a well defined partial order on $[\mathcal{U}]$ (page 237, 238 of [R]). Moreover, since ϕ_i's are epimorphisms σ_{ij}, if it exists, is unique and is an epimorphism.

$([\mathcal{U}], <)$ is a directed set. Given $[A_i, \phi_i]$ and $[A_j, \phi_j]$. Consider $A_i \otimes A_j$ and the map $\delta : A \to A_i \otimes A_j$ given by $\delta = j_i \circ \phi_i = j_j \circ \phi_j : A \xrightarrow{\phi_i} A_i \xrightarrow{j_i} A_i \otimes A_j$. Then δ is an extension, $A_i \otimes A_j$ is a right flat A-module and δ is an epimorphism of rings (page 238 [R]).

We claim $[A_i \otimes A_j, \delta]$ belongs to $[\mathcal{U}]$. It remains to verify the third condition only.

By the flatness of A_i and A_j it suffices to prove the following

Lemma: Let A be a ring I a left ideal of A, and B and B' two extensions of A such that they are right flat; we suppose $B \otimes A/I \neq (0)$ and $B' \otimes B \otimes A/I = (0)$ then $B' \otimes A/I = (0)$.

Proof: The left B-module $B \otimes A/I$ is cyclic generated by $1 \otimes \bar{1}$ (1 is the identity of B, $\bar{1}$ is the image of the identity of A). Hence $1 \otimes \bar{1} \neq 0$. However for every $b' \in B'$, we have $b' \otimes (1 \otimes \bar{1}) = 0$. Then (Lemma 10, page 41 [BAC]) there exist a finite family of elements $a_j \in A$ and $b'_j \in B'$ such that $\Sigma b'_j a_j = b'$ and $a_j(1 \otimes \bar{1}) = 0$ for every j. Therefore $a_j \in I$ and consequently $b' \in B'I$ which means $B' \otimes A/I = (0)$.

Now let us consider the inductive family $\{\Phi_{[A_i,\phi_i]}, \gamma_{[A_i,\phi_i][A_j,\phi_j]}\}$ of rings indexed by $[\mathcal{U}]$ where $\Phi_{[A_i,\phi_i]}$ is the ring A_i of $[A_i,\phi_i]$ and if $[A_i,\phi_i] < [A_j,\phi_j]$ then $\gamma_{[A_i,\phi_i],[A_j,\phi_j]}$ is the (unique) extension $\gamma_{ij} : A_i \longrightarrow A_j$. By the uniqueness of γ's we have for $[A_i,\phi_i] < [A_j,\phi_j] < [A_k,\phi_k]$,

$$\gamma_{[A_i,\phi_i][A_k,\phi_k]} = \gamma_{[A_i,\phi_i][A_j,\phi_j]} \circ \gamma_{[A_j,\phi_j][A_k,\phi_k]}$$

Let $A_I = \varinjlim \Phi_{[A_i,\phi_i]}$ and $\alpha_I : A \longrightarrow A_I$ be $\varinjlim \phi_i : A \longrightarrow A_i = \Phi_{[A_i,\phi_i]}$. Then α_I is an extension.

Clearly (A_I,α_I) has properties (i) and (ii) of the theorem. To show that A_I has property (iii), it is enought to show $A_I \otimes A/I \cong (\varinjlim A_i) \otimes A/I \cong \varinjlim (A_i \otimes A/I)$ is not zero. Since each

$A_I \otimes A/I$ is a non zero cyclic left A_I module with generator $\bar{1}$ and the morphisms send generator into generator, one concludes that $A_I \otimes A/I$ is a non zero cyclic left A_I-module with generator $\bar{1}$ (which is the image of the generators). Hence $A_I \otimes A/I$ is not equal to zero.

Now let $\beta : A \longrightarrow B$ is another extension with properties (i) to (iii) mentioned in the theorem. Then $[B,\beta] \in \mathcal{U}$. Hence we have a canonical extension $\gamma : B \longrightarrow A_I$ such that $\gamma \circ \beta = \alpha_I$ and since β is an epimorphism γ is unique up to isomorphism.

Let us call (A_I,α_I) the <u>left localization of</u> A <u>at the left ideal</u> I <u>of</u> A.

Now we have the following

3. <u>Theorem</u>. <u>Let A be a ring M an A-bimodule then the following are equivalent.</u>

(a) <u>M is right A-flat.</u>

(b) $M \otimes A_m$ <u>is right</u> A_m<u>-flat for every left maximal ideal m of A.</u>

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ON THE CONNECTION OF DIFFERENTIAL OPERATORS VIA SCATTERING INPUT

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Abstract. For certain Fourier type operators $\hat{P}$ a transform calculus is developed on the full line and used to derive a generalized Marčenko equation connecting $\hat{P}$ and more general $\hat{Q}$.

1. Basic ingredients. We consider Fourier type operators following [8; 10] to be of the form $\hat{P}u = u'' - q(x)u$ where q is real, even, positive, and continuous with $q\exp 2Hx \in L^1(0,\infty)$. The hypotheses are stronger than necessary for our purposes but provide a good model. Let $\hat{Q}u = (\Delta_Q u')'/\Delta_Q + \hat{Q}(x)u$ for suitable Δ_Q and $\hat{Q}(x)$ as in [1]. $\hat{Q}$ is considered only on $[0,\infty)$ and may either be modeled on a radial Laplace-Beltrami operator or be of a type often encountered in applications as in [1;5]. For the equation (*) $\hat{Q}u = -\lambda^2 u$ one defines spherical functions as solutions of (*) satisfying $\varphi_\lambda^Q(0) = 1$ and $D_x\varphi_\lambda^Q(0) = 0$ and Jost solutions $\Phi_{\pm\lambda}^Q(x)$ satisfying $\Phi_{\pm\lambda}^Q(x) \sim \Delta_Q^{-\frac{1}{2}}(x)e^{\pm i\lambda x}$ as $x \to \infty$. One has $\varphi_\lambda^Q = c_Q(\lambda)\Phi_\lambda^Q + c_Q(-\lambda)\Phi_{-\lambda}^Q$. Under "normal" hypotheses (cf. [1]) Φ_λ^Q, $\lambda c_Q(-\lambda)$, and $1/c_Q(-\lambda)$ are analytic for $\mathrm{Im}\lambda > 0$ and φ_λ^Q is entire in λ. For (‡) $\hat{P}u = -\lambda^2 u$ (cf. [8;10;11]) in addition to spherical functions and Jost solutions φ_λ^P and $\Phi_{\pm\lambda}^P$ as above we define $\chi_\lambda^P(x)$ as the solution of (‡) satisfying $\chi_\lambda^P(0) = 0$ and $D_x\chi_\lambda^P(0) = -1$. For real λ as $x \to \infty$ one has $\varphi_\lambda^P(x) \to M_1(\lambda)e^{-i\lambda x} + M_1(-\lambda)e^{i\lambda x}$ and $\chi_\lambda^P(x) \sim M(\lambda)e^{-i\lambda x} + M(-\lambda)e^{i\lambda x}$ where M and M_1 are analytic for $\mathrm{Im}\lambda > -H$ (except for a simple pole at $\lambda = 0$) and neither vanishes for $\mathrm{Im}\lambda > -\delta$. The function $\rho(\lambda) = 1/4i\lambda MM_1$ is analytic for $\mathrm{Im}\lambda > -\delta$ and one defines $A(\lambda) = -\{MM_1^- + M^-M_1\}/2MM_1$ for real λ ($M^- = M(-\lambda) = M(\lambda)$ for λ real). We can write $\Phi_\lambda^P(x) = 2i\lambda\{M\varphi_\lambda^P(x) - M_1\chi_\lambda^P(x)\}$ and one defines $\Sigma_\lambda^P(x) =$

$2i\lambda\{M\varphi_\lambda^P(x) + M_1\chi_\lambda^P(x)\}$ which are defined for all x with $\Phi_\lambda^P(-x) = \Sigma_\lambda^P(x)$ (also note that $\Phi_{-\lambda}^P(x) = \rho\Sigma_\lambda^P(x) + A\Phi_\lambda^P(x)$). The functions Φ_λ^P and Σ_λ^P play a natural role in the full line scattering theory for $\hat{P}$ (cf. [6;7;9]) and in fact as $x \to \infty$, $\rho\Sigma_\lambda^P(x) \sim e^{-i\lambda x} - A(\lambda)e^{i\lambda x}$, so that -A is a classical reflection coefficient. One can show also directly that $M_1(\lambda) = c_P(-\lambda)$ and $F(\lambda) = 2i\lambda M(\lambda)$ where F is a classical Jost function arising from $\chi_\lambda^P = \{F\Phi_- - F^-\Phi_+\}/2i\lambda$. The spectral measure for the eigenfunction expansion relative to φ_λ^P (resp. χ_λ^P) is $d\nu = d\lambda/2\pi|c_P|^2$ (resp. $d\sigma = d\lambda/2\pi|M|^2$). We remark also that $2\mathrm{Re}\,\Psi_\lambda^P = 2\pi\hat{\nu}_P(\lambda)\varphi_\lambda^P$ where $\Psi_\lambda^P = \Phi_\lambda^P/c_P(-\lambda)$ ($d\nu = \hat{\nu}_P d\lambda$) and $\Psi_\lambda^P = 2i\lambda\{(M/M_1)\varphi_\lambda^P - \chi_\lambda^P\}$ with $\mathrm{Im}\,\Psi_\lambda^P = -2\lambda\chi_\lambda^P - (2\lambda AM/M_1^-)\varphi_\lambda^P$ (the decomposition of Ψ_λ^P into real and imaginary parts was stated incorrectly in some illustrative material in [1], pp. 325-329 and [5], pp. 54-57 - the appropriate modifications are obvious).

2. Results. We recall that B: $\hat{P} \to \hat{Q}$ is a transmutation if $B\hat{P} = \hat{Q}B$ acting on suitable objects. There are various transmutations discussed in [1] characterized by their action on eigenfunctions. We assume $\hat{Q}$ has absolutely continuous spectrum and B: $\varphi_\lambda^P \to \varphi_\lambda^Q$ and $\tilde{B} = (B^{-1})^{\#}$: $\varphi_\lambda^P \to |c_P/c_Q|^2\varphi_\lambda^Q$ can be represented in terms of spectral kernels $\beta(y,x) = \langle\varphi_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_\nu$ and $\tilde{\beta}(y,x) = \langle\varphi_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_\omega$ ($d\omega = d\lambda/2\pi|c_Q|^2$). Using a variation on methods developed in [3] we prove an inversion of Kontorovič-Lebedev type, namely, for suitable f and $\Omega_\lambda^Q = \Delta_Q\varphi_\lambda^Q$, if $\hat{f}(\lambda) = \int_0^\infty f(x)\Omega_\lambda^Q(x)dx$ then

$$(1) \qquad f(x) = (1/2\pi)\int_{-\infty}^{\infty}\hat{f}(\lambda)\Psi_\lambda^Q(x)d\lambda$$

Using this formula and associated technique we also give a new proof of the fact that $\tilde{B}\Psi_\lambda^P = \Psi_\lambda^Q$ for $\hat{P}$ and $\hat{Q}$ of $\hat{Q}$ type above (cf. [1]). Next we define a transmutation $\check{B}$: $\hat{P} \to \hat{Q}$: $\Phi_\lambda^P \to \Phi_\lambda^Q$ with spectral kernel $\check{\beta}(y,x) = \langle\Psi_\lambda^P(x),\Phi_\lambda^Q(y)\rangle_\lambda/2\pi$ (note $\check{B}$ and e.g. B transmute on different objects in general).

Now the idea of the Marčenko (M) equation is to relate B and $\tilde{B}$ via $\check{B}$.

When $\hat{P} = D^2$ we gave a development of this in [1;4] based on [7] by systematic use of the full Fourier transform on $(-\infty,\infty)$. In the present situation for Fourier type $\hat{P}$ we can establish analogous machinery (and our result reduces to the situation of [1;4] when $\hat{P} = D^2$). First following [8;10] one can prove that for suitable f, $\Phi(f) = \int_{-\infty}^{\infty} f(x)\Phi_\lambda^P(x)dx$ can be inverted in the form $f(x) = (1/2\pi)\int_{-\infty}^{\infty}\Phi(f)\rho(\lambda)\Sigma_\lambda^P(x)d\lambda$ (note our confirmation of this sort of inversion in [1], Chap. 2, Remark 10.12 and at the end of [2] was badly phrased and we have now a correct version). One can define then a kind of generalized translation

$$(2)\qquad \mathfrak{T}_x^y f(x) = (1/2\pi)\int_{-\infty}^{\infty} (f)\Phi_\lambda^P(y)\Sigma_\lambda^P(x)\rho(\lambda)d\lambda$$

and a generalized convolution $(g * f)(x) = (f * g)(x) = \langle g(y), \mathfrak{T}_x^y f(x)\rangle = (1/2\pi)\int_{-\infty}^{\infty}\Phi(f)\Phi(g)\Sigma_\lambda^P(x)\rho(\lambda)d\lambda$ so that $\Phi(f * g) = \Phi(f)\Phi(g)$ etc. It is then possible to express the result $\tilde{B}\Psi_\lambda^P = \Psi_\lambda^Q$ in the form

Theorem 1. $\tilde{\beta}(y,x) = \{\check{\beta}(y,\cdot) * H\}(x)$ where $\Phi(H) = M_1/c_Q(-\lambda)$ and we can write then $\tilde{B} = \check{B}\mathcal{H}$ where $\ker\mathcal{H} = (1/2\pi)\int_{-\infty}^{\infty}\Phi(H)\Sigma_\lambda^P(y)\Phi_\lambda^P(t)\rho(\lambda)d\lambda$.

Taking now the generalized Gelfand-Levitan equation of [1;4] in the form $B = \tilde{B}\tilde{W}$ where $\ker\tilde{W} = \tilde{W}(x,y) = \langle\varphi_\lambda^P(x)\varphi_\lambda^P(y), \hat{\upsilon}^2/\hat{\omega}\rangle_\lambda$ ($\hat{\upsilon}_P = \hat{\upsilon}$, $d\omega = \hat{\omega}d\lambda$) one selects an operator $\tilde{\mathcal{H}}$ with $\ker\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(x,s) = (1/2\pi)\int_{-\infty}^{\infty}(M_1/c_Q^-)\Phi_\lambda^P s)\Sigma_\lambda^P(x)\rho d\lambda$ so that the operator M equation $B\tilde{\mathcal{H}} = \check{B}(\mathcal{H}\widetilde{W\mathcal{H}})$ has a "nice" form. In particular with this choice of $\tilde{\mathcal{H}}$, $L(y,x) = \ker(B\tilde{\mathcal{H}}) = 0$ for $x > y$ and one obtains after considerable calculation a generalized M equation for kernels

Theorem 2. Let $s_Q = c_Q/c_Q^-$ be the "scattering" input from the $\hat{Q}$ operator and set $S(t,x) = (1/2\pi)\int_{-\infty}^{\infty}\Phi_\lambda^P(t)\Phi_\lambda^P(x)(1+\rho/\rho^-)(s_Q\rho/2)d\lambda$ with $T(t,x) = (1/2\pi)\int_{-\infty}^{\infty}\rho\Phi_\lambda^P(t)\Sigma_\lambda^P(x)\{1 - (1+\rho/\rho^-)(\rho/2)\}d\lambda - (1/4\pi)\int_{-\infty}^{\infty}\rho\Phi_\lambda^P(t)\Phi_\lambda^P(x)(1+\rho/\rho^-)Ad\lambda$. Then for $x > y$

$$\check{\beta}(y,x) = \int_y^\infty \check{\beta}(y,t)\{T(t,x) - S(t,x)\}dt \tag{3}$$

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THE MASLOV CLASS OF A LAGRANGIAN IMMERSION IN AN ALMOST KÄHLER MANIFOLD

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Presented by P.C. Greiner, F.R.S.C.

Abstract. The Maslov class of a Lagrangian immersion in an almost Kähler manifold is shown to be represented by the symplectic dual of the mean curvature field of the immersion.

Let (M^{2n},J,g,ω) be an almost Kähler manifold, i.e. J is an almost complex structure on M^{2n} with g a compatible Hermitian metric ($g(JX,JY) = g(X,Y)$), and the two-form

$$\omega(X,Y) = g(X,JY)$$

a symplectic structure on M^{2n}.

An immersion $f : N^n \longrightarrow M^{2n}$ of an n-dimensional manifold N^n in M^{2n} is Lagrangian iff

$$f^*\omega = 0.$$

The second fundamental form of f is a section of $T^\perp N \otimes (T^*N \times T^*N)$ defined by

$$\xi_f(X,Y) = \hat{\nabla}_X Y - \nabla_X Y$$

where $\hat{\nabla}$, resp. ∇, is the Levi-Cevita connection of g, resp. f^*g (whereby we identified N^n with its image $f(N^n)$ in M^{2n}). The normal vector field

$$H_f = \mathrm{Tr}\ \xi_f$$

is the mean curvature field of f. f is minimal iff H_f is identically

zero.
As each tangent space to M^{2n} is a symplectic vector space, a bundle $\mathcal{L}(M^{2n})$ with fibre $\mathcal{L}_x(M^{2n})$ = the Lagrangian Grassmannion of $(T \times M^{2n}, \omega_x)$ over $x \in M^{2n}$, may be constructed. The Lagrangian Grassmann bundle $\mathcal{L}(f)$ of the immersion f is the bundle

$$\mathcal{L}(f) = f^{*}\mathcal{L}(M^{2n}).$$

Proposition 1. $\mathcal{L}(f)$ is a trivial fibre bundle over N^n, with fibre $U(n)/O(n)$.

Proof. A result of Arnol'd [AR] shows that the fibres of $\mathcal{L}(f)$ are homogeneous spaces, isomorphic with $U(n)/O(n)$. The result follows from the fact that $\mathcal{L}(f)$ admits a canonical global section

$$\sigma_f : N^n \longrightarrow \mathcal{L}(f) : y \longmapsto T_y N^n.$$

As a consequence of this proposition, a construction due to J.M. Morvan for $M^{2n} = E^{2n}$ [MO] can be generalized.
Using the two-to-one map

$$\phi : U(n)/O(n) \longrightarrow S^1 : \lambda \longmapsto \det^2\lambda$$

and the fibre projection

$$\rho : \mathcal{L}(f) \longrightarrow U(n)/O(n)$$

a cocycle m_f' in $\mathcal{H}_1(N^n, \mathbb{Z})$ is defined by assigning to each loop $\gamma : S^1 \longrightarrow N^n$ the degree of the composition

$$S^1 \xrightarrow{\gamma} N^n \xrightarrow{\sigma_f} \mathcal{L}(f) \xrightarrow{\rho} U(u)/O(n) \xrightarrow{\phi} S^1$$

This cocycle is represented by the one-form

$$\beta_f = (\tilde{\sigma}_f \circ \phi)^*\left(\frac{1}{2\pi i}\frac{dz}{z}\right) = \tilde{\sigma}_f^*\alpha$$

(whereby $\tilde{\sigma}_f = \rho \circ \sigma_f$, and $\alpha = \phi^*(\frac{1}{2\pi i}\frac{dz}{z})$).

The cohomology class $m_f = [\beta_f]$ in $\mathcal{H}^1(N^n, \mathbb{Z})$ is the Maslov class of the Lagrangian immersion f [GST]. The integer $m_f(\gamma)$,

$$m_f(\gamma) = \int_\gamma \beta_f$$

is the Maslov index of a curve γ in N^n.

Proposition 2. Let $f : N^n \longrightarrow M^{2n}$ be a Lagrangian immersion of a manifold N^n in an almost Kähler manifold (M^{2n}, J, g, ω), and let H_f be the mean curvature field of f, then

$$m_f = [\frac{1}{\pi} H_f \lrcorner\, \omega]$$

Proof. The proof is done is several steps. Observe firstly the isomorphism

$$\tilde{\sigma}_f^* T(U(n)/O(n)) \cong T^*N^n \hat{\otimes} T^\perp N^n = (T^*N^n \otimes T N^n) \cap f^*\mathrm{Symp}(M^{2n},\omega)$$

Hereby is $\mathrm{Symp}(M^{2u},\omega)$ a subbundle of $T^*M^{2n} \otimes TM^{2n}$, whose fibre over $x \in M^{2u}$ consists of the vector space of the symplectic Lie algebra of $(T_x M^{2u}, \omega_x)$. In other words, $L_y \in T_y^*N^n \hat{\otimes} T_y^\perp N^n$ iff

$$\omega_y(L_y X, Y) + \omega_y(X, L_y Y) = 0 \qquad (X, Y \in T_y N^n)$$

The isomorphism is constructed with respect to an orthonormal base $(e^1, \ldots, e^n)$ for $T_y N^n$, with dual base $(\theta^1, \ldots, \theta^n)$ for $T_y^* N^n$ $(y \in N^n)$, by

$$\psi_y : \begin{cases} \theta^j \otimes e_j \longrightarrow \theta^j \otimes Je_j \\ i(\theta^j \otimes e_k - \theta^k \otimes e_j) \longrightarrow \theta^j \otimes Je_k - \theta^k \otimes Je_j \end{cases}$$

From [KN] and the fact that ω is closed, the tangent map $T\tilde{\sigma}_f$ in the diagram

$$\begin{array}{ccc} TN^n & \xrightarrow{\;T\tilde{\sigma}_f\;} & T^*N^n \hat{\otimes} T^*N^n \\ \downarrow & & \downarrow \\ N^n & \xrightarrow{\;\tilde{\sigma}_f\;} & \sigma_f(N^n) \end{array}$$

is identified with the second fundamental form :

$$T\sigma_f(X) = \xi_{fX} : TN^n \longrightarrow T^{\perp}N^n : Y \longmapsto \xi_f(X,Y).$$

Observe now that J is a section of $T^{\perp}N^n \hat{\otimes} T^{\perp}N^n$. Consequently, for all $y \in N^n$, the dual $J_y^{\#}$ is a one-form on $T_y^*N^n \hat{\otimes} T_y^{\perp}N^n \cong \Lambda U(u)/\Lambda O(u)$.
As in [MO], a direct calculation shows that

$$[\tfrac{1}{\pi}\, \tilde{\sigma}_f{}^* J^{\#}] = [\beta_f].$$

Choosing at a point $y \in N^n$ the bases as above, with (Je_k) a base for $T_y^{\perp}N^n$, and $(-\theta^k J)$ the corresponding base for $T^{\perp *}_y N^n$, we write

$$J_y = \sum_{k=1}^{n} \theta^k \otimes Je_k \quad ; \quad J_y^{\#} = -\sum_{k=1}^{n} e_k \otimes \theta^k J$$

and

$$\xi_{fX} = \sum_{k,\ell=1}^{n} \xi^k{}_{X\ell}\, \theta^{\ell} \otimes J_{ek}.$$

The proposition is then obtained from

$$\tilde{\sigma}_{fy}^{::} J^{\#}(X) = \xi_{fX} \lrcorner J^{\#} = \sum_{k=1}^{n} \xi_{Xk}^{k} = \sum_{k=1}^{n} g_y(\xi_f(X,e_k),Je_k)$$

$$= \sum_{k=1}^{n} g_y(\xi_f(e_k,e_k),JX) = \omega_y(H_f,X).$$

Corollary 3. The Maslov class of a minimal Lagrangian immersion in an almost Kähler manifold is zero.

As examples, one easily can construct non-minimal Lagrangian surfaces in Thurstons example [TH] of a non-Kählerian almost Kähler manifold.

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ON THE CATENARIAN PROPERTY OF THE POLYNOMIAL RINGS OVER A PRÜFER DOMAIN

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Presented by P. Ribenboim, F.R.S.C.

Summary. We sketch herein the proof of the following theorem: If R is a locally finite dimensional Prüfer domain then $R[T_1,T_2,\ldots,T_r]$ is catenarian, for every $r \geq 1$.

A *catenarian ring* R is a ring such that, for every pair $p \subset q$ of prime ideals of R, the length of every saturated chain of prime ideals of R between p and q is finite and constant. The starting point of our work is the following question posed by P. Ribenboim : Is $V[T_1,T_2,\ldots,T_r]$ catenarian , when V is a finite dimensional valuation ring? If r=1, it is well-known that the answer is affirmative (cf. [1], [2], [3]). The next step to several indeterminates is fairly delicate and the aim of this report is to give an account of the techniques we use to show the theorem stated in the Summary. Details of the proofs may appear elsewhere.

LEMMA 1. Let A and B be two integral domains, P a prime ideal of A. Suppose that : (i) there exists a multiplicative set S of B such that $S^{-1}B \cong A/P$; (ii) the canonical projection $u: A \twoheadrightarrow A/P$ induces a surjective map from the group of the units of A onto that of A/P. Let $v: B \hookrightarrow C \overset{\text{not}}{=} S^{-1}B$ be the canonical embedding. In this situation, we consider the following pull-back of canonical homomorphisms:

$$\begin{array}{ccc} R & \xrightarrow{u'} & B \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow{u} & A/P \cong C \end{array}$$

and we suppose, moreover, that the following ("glueing") condition holds:

(γ) Let $\mathcal{Q}_2$ be a prime ideal of B and $\mathcal{Q}_1$ a prime ideal of A such that $v'^{-1}(\mathcal{Q}_1) \subset u'^{-1}(\mathcal{Q}_2)$, then there exists a prime ideal $\mathcal{Q}$ of C such that $v'^{-1}(\mathcal{Q}_1) \subset (u \circ v')^{-1}(\mathcal{Q}) \subset u'^{-1}(\mathcal{Q}_2)$.

In this situation, A and B are catenarian rings if, and only if, R is a catenarian ring.

We notice that, using Lemma 1, we can show easily, for instance, that $\mathbb{Z} + (T_1,T_2,\dots,T_r)\mathbb{Q}[T_1,T_2,\dots,T_r]$ is a catenarian (non Noetherian) integrally closed domain of dimension r+1.

DEFINITION 2. Let R be an integral domain. Let $\mathcal{Q}$ be a prime ideal of the polynomial ring $R[T_1,T_2,\dots,T_r]$. For every permutation σ of the set $\{1,2,\dots,r\}$, we consider :

$$q^{(\sigma(k))} = \mathcal{Q} \cap R[T_{\sigma(1)},T_{\sigma(2)},\dots,T_{\sigma(k)}],\ 1 \leq k \leq r,$$

and we define :

$$\delta_{R,\sigma}\,\mathcal{Q} = \#\{0 \leq k \leq r-1 : q^{(\sigma(k+1))} \supsetneq q^{(\sigma(k))}[T_{\sigma(k+1)}]\}$$

where $q^{(\sigma(0))} = \mathcal{Q} \cap R$. We indicate simply $\delta_R\,\mathcal{Q}$, or $\delta\,\mathcal{Q}$, when, having fixed the coefficient ring, σ is the identical permutation.

LEMMA 3. Let V be a finite dimensional valuation domain and $\mathcal{Q}$ a prime ideal of the polynomial ring $V[T_1,T_2,\dots,T_r]$.

(a) For every pair of permutations σ, τ of the set $\{1,2,\dots,r\}$,

$$\delta_{V,\sigma}\,\mathcal{Q} = \delta_{V,\tau}\,\mathcal{Q}$$

(b) Let $0 \leq s \leq t \leq r$, then:

$$(\delta_{V[T_1,T_2,\dots,T_s]}\mathcal{Q}) + ht(q^{(s)}) = (\delta_{V[T_1,T_2,\dots,T_t]}\mathcal{Q}) + ht(q^{(t)}).$$

(c) Let $h = \delta_V\,\mathcal{Q}$, then there exists a permutation σ of the set $\{1,2,\dots,r\}$ such that :

$$\delta_{V,\sigma}\,q^{(\sigma(i))} = \begin{cases} 0 & \text{if } 0 \leq i \leq r-h \\ \delta_{V,\sigma}\,q^{(\sigma(i-1))}+1, & \text{if } r-h+1 \leq i \leq r. \end{cases}$$

Using the previous Lemma, we are able to prove the following key theorem.

THEOREM 4. Let (V, m) be a finite dimensional valuation domain. Take $\mathcal{Q}_1 \subset \mathcal{Q}_2$ two prime ideals of the polynomial ring $V[T_1, T_2, \ldots, T_r]$. Suppose that $\mathcal{Q}_2 \cap V = m$ and $\mathcal{Q}_1 \cap V = (0)$. Then, there exist a prime ideal $\mathcal{Q}_3$ of $V[T_1, T_2, \ldots, T_r]$ and a permutation σ of the set $\{1, 2, \ldots, r\}$ such that:

(a) $\mathcal{Q}_1 \subset \mathcal{Q}_3 \subset \mathcal{Q}_2$, with $\mathcal{Q}_3 \cap V = m$;

(b) ${}_{V,\sigma}^{\delta} Q_3^{(\sigma(i))} = {}_{V,\sigma}^{\delta} Q_1^{(\sigma(i))}$, for every $1 \leq i \leq r$.

COROLLARY 5. Let R be a Prüfer domain such that every maximal ideal of R has finite height. Then the polynomial ring $R[T_1, T_2, \ldots, T_r]$ is catenarian, for every $r \geq 1$.

Proof. (Sketch). By reduction to the local case, we can suppose that $R = V$ is a valuation ring with $\dim(V) = d < \infty$. From Theorem 4, we deduce easily that the statement holds when $d = 1$. Thus we suppose $d \geq 2$ and we induce. The main point is to show that, if p is a non-zero non-maximal prime ideal of the valuation domain V, then the following pull-back of canonical ring homomorphisms:

$$\begin{array}{ccc} V[T_1, T_2, \ldots, T_r] & \longrightarrow\!\!\!\!\!\rightarrow & (V/p)[T_1, T_2, \ldots, T_r] \\ \downarrow & & \downarrow \\ V_p[T_1, T_2, \ldots, T_r] & \longrightarrow\!\!\!\!\!\rightarrow & (V_p/pV_p)[T_1, T_2, \ldots, T_r] \end{array}$$

satisfies the glueing condition (γ). This is proven inductively on r. The conclusion then follows immediately from Lemma 1.

REMARK (March, 1983). In preparing a typed draft of the manuscript we learned that a statement analogous to Corollary 5 has been proven indipendently by S. Malik and J.L. Mott, in the preprint "Strong S-domains". We thank J. Mott for having supplied us this paper.

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