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EXPONENTIAL OBJECTS IN CATEGORIES OF (PRE)TOPOLOGICAL SPACES AND THEIR NATURAL FUNCTION SPACES

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Presented by P.G. Rooney, F.R.S.C.

ABSTRACT. A characterization of the exponential objects in epireflective subcategories of the (pre)topological spaces is given; it is shown that the corresponding natural function spaces are always endowed with the limitierung of continuous convergence.

1. Introduction. In view of the fact that there is no non-trivial cartesian closed epireflective subcategory of PrT (pretopological spaces) [10: 3.3] it is natural to ask for the exponential objects in such categories and for a description of the corresponding function spaces. In connection with these problems, we present results from [11: Sections 5,6] and obtain considerable improvements, in particular by application of [12: Section 2]. Here the essence of a construction due to Arens [1: proof of Thm. 3] plays an important rôle. The fact cited above [10: 3.3] turns out as a corollary.

For definitions and terminology not given here, we refer to [11] and [12].

2. Exponential objects in epireflective subcategories of PrT and the limitierung of continuous convergence. Every non-trivial epireflective subcategory $\underline{A}$ of PrT contains $ZDimT_0$ (zero-dimensional T_0-spaces) [10: 3.1] and is, consequently, finally dense in LIM (limit spaces) [11: 4.3]. Hence, if there is a natural $\underline{A}$-structure on $\underline{A}(X,Y)$, it coincides with the limitierung of continuous convergence $c(X,Y)$ (2.3).

Let $\underline{A}$ be an initially structured category. (In particular, every epireflective subcategory of LIM is initially structured.) For $X,Y \in \underline{A}$, denote by $p(X,Y)_{\underline{A}}$ the smallest splitting $\underline{A}$-structure on $\underline{A}(X,Y)$. (Following Dugundji [4: 10.1], we use "splitting" and "conjoining" instead of "proper" and "admissible", as we have already done in [12].) $\underline{A}_p(X,Y) = (\underline{A}(X,Y),p(X,Y)_{\underline{A}})$ is called power. An object $X \in \underline{A}$ is said to be exponential in $\underline{A}$ iff the functor $- \times X : \underline{A} \longrightarrow \underline{A}$ has a right adjoint. $\underline{A}$ is cartesian closed iff every $\underline{A}$-object is exponential in $\underline{A}$.- In case of LIM, the powers are given by the limitierung of continuous convergence: $p(X,Y)_{LIM} = c(X,Y)$. Instead of $(LIM(X,Y),c(X,Y))$, we write $LIM_c(X,Y)$.

A subcategory of LIM is called non-trivial iff it contains a non-indiscrete space. Every non-trivial epireflective subcategory of TOP is coproductive in TOP [5: 1.1.3]. Similarly one can prove:

2.1. PROPOSITION. Every epireflective subcategory of LIM containing all discrete spaces is closed under formation of coproducts in LIM.

2.2. COROLLARY. Every non-trivial epireflective subcategory $\underline{A}$ of PrT is stable under coproducts in LIM.

PROOF. $\underline{A}$ contains $ZDimT_0$, hence all discrete spaces. □

Applying 2.2 to [11: 5.1,5.4], we obtain:

2.3. THEOREM. Let $\underline{A}$ be a non-trivial epireflective subcategory of PrT, $\underline{D}$ an initially dense subclass of $\underline{A}$, and $X \in \underline{A}$. Then the following are equivalent:

(1) X is exponential in $\underline{A}$. (2) $- \times X$ preserves quotient maps in $\underline{A}$.

(3) For each $Y \in \underline{D}$, there is a (unique) splitting conjoining $\underline{A}$-structure on $\underline{A}(X,Y)$ (namely $c(X,Y)$).

(4) For each $Y \in \underline{D}$, $p(X,Y)_{\underline{A}}$ is conjoining.

(5) For each $Y \in \underline{D}$, $\underline{A}_p(X,Y) = LIM_c(X,Y)$. (6) For each $Y \in \underline{D}$, $LIM_c(X,Y) \in \underline{A}$.

If, in addition, $\underline{A}$ contains the Sierpinski space $\mathcal{L}$ (i.e. T_0TOP) or only T_1-limit spaces, we have the further equivalence

(7) For each $Y \in \underline{D}$, there exists a largest conjoining $\underline{A}$-structure on $\underline{A}(X,Y)$ (namely $c(X,Y)$).

If X is exponential in $\underline{A}$, we have $\underline{A}_p(X,Y) = LIM_c(X,Y)$ for all $Y \in \underline{A}$, i.e. all corresponding natural function spaces are endowed with the limitierung of continuous convergence.

By [10: 3.3] and 2.3(1),(6) it follows that no non-trivial epireflective subcategory of PrT (TOP) is stable under powers in LIM.– 2.3 holds still for epireflective subcategories of LIM containing a finite non-indiscrete space. In that case, application of condition (6) shows easily the cartesian closedness of many well-known epireflective subcategories of LIM. (For examples see [11: 5.2].)

The subcategories of TOP which are excluded by the additional assumptions of 2.3(7) are just the bireflective ones distinct from TOP. However, it is possible to extend (7) to all epireflective subcategories of TOP.

2.4. THEOREM [11: 6.1]. Let $\underline{A}$ be an epireflective subcategory of TOP, $\underline{D}$ an initially dense subclass of $\underline{A}$, and $X \in \underline{A}$. Denote by Q the epireflector from TOP in T_0TOP. The following are equivalent:

(1) X is exponential in $\underline{A}$.

(2) For each $Y \in \underline{A}$, there is a largest conjoining $\underline{A}$-structure on $\underline{A}(X,Y)$ (namely $c(X,Y)$).

(3) For each $Y \in Q(\underline{D})$, there is a largest conjoining $\underline{A}$-structure on $\underline{A}(X,Y)$ (namely $c(X,Y)$).

Note that for any possible $\underline{A}$ in 2.4, there is an initially dense class consisting of T_0-spaces, e.g. $\underline{A} \cap T_0TOP$. Hence condition (3) in 2.4 is only a slight restriction of (7) in 2.3, as will also become clear from applications.

The following result, which is easily deducible from 2.4, shows that one knows the exponential objects in all epireflective subcategories of TOP, if one has this information for the bireflective ones.

2.5. COROLLARY [11: 6.3]. Let $\underline{A}$ be a bireflective subcategory of TOP, $\underline{A}' = \underline{A} \cap T_0TOP$. Then $\{ X \mid X \text{ is exponential in } \underline{A}' \} =$
$= \{ X \mid X \text{ is exponential in } \underline{A} \} \cap \underline{A}' = \{ X \mid X \text{ is exponential in } \underline{A} \} \cap T_0TOP$.

3. Exponential objects in some epireflective subcategories of TOP and their natural function spaces. We characterize the exponential objects in TOP and some epireflective subcategories of TOP by local compactness conditions and observe that in special cases, the corresponding natural function space structures coincide with well-known topologies, e.g. the compact-open topology.

In order to describe the exponential objects in TOP (3.3), we give a technical lemma (3.1) based on an idea of Arens and classes of topological spaces fulfilling the assumptions of this lemma (3.2).— As in [12] - but in contrast to [11]! - "(completely) regular" does not include T_0. The corresponding categories are denoted by Reg and CReg; and $T_3TOP = \text{Reg} \cap T_0TOP$, $\text{Tych} = \text{CReg} \cap T_0TOP$.

3.1. LEMMA. Let X be regular, Y a T_0-space with at least two elements. If for any closed subset $A \subset X$, $z \in X - A$ and points $r,s \in Y$ such that r has a neighborhood which does not contain s, there is a function $h \in TOP(X,Y)$ with $h(A) \subset \{r\}$ and $h(z) = s$, then $LIM_c(X,Y) \in TOP$ implies that X is locally compact.

For the basic ideas of the proof, cf. [1: proof of Thm. 3], [12: proof of 2.15(2) $\Rightarrow$ (9)].

3.2. PROPOSITION. The assumptions of 3.1 are fulfilled in the following cases:
(1) $X \in \text{Reg}$, $Y = \mathfrak{L}$. (2) $X \in \text{CReg}$, $Y = [0,1]$.
(3) $X \in \text{ZDim}$, $Y = \mathbb{D}_2$ (= discrete space with two elements).

PROOF. Define h by: (1) $h(X - \overline{\{z\}}) = \{0\}$, $h(\overline{\{z\}}) = \{1\}$;
(3) $h(U) = \{s\}$, $h(X - U) = \{r\}$ for a clopen neighborhood U of z with $U \subset X - A$. □

For any complete lattice L, denote by s(L) the Scott topology on L [6: p.292] (cf. [3: p.554]). For $X,Y \in TOP$, let co(X,Y) denote the compact-open topology, bo(X,Y) Lambrinos's bounded-open topology [9: p.50] on TOP(X,Y); $TOP_{co}(X,Y)$ and $TOP_{bo}(X,Y)$ are used in the obvious way. $\mathcal{O}(X)$ is the lattice of open sets of X. We call X (weakly) locally compact iff every point of X has a neighborhood base consisting of compact sets (a compact neighborhood), core-compact iff for all $x \in U \in \mathcal{O}(X)$, there is a $V \in \mathcal{O}(X)$ with $x \in V \subset U$ such that every open cover of U has a finite subsystem covering V.

3.3. THEOREM. For $X \in$ TOP, the following are equivalent:

(1) X is exponential in TOP.

(2) - (7) as in 2.3 for $\underline{A}$ = TOP, $\underline{D} = \{\mathcal{L}\}$.

(8) $LIM_c(X,\mathcal{L})$ is injective in TOP (T_0TOP).

(9) $LIM_c(X,\mathcal{L}) = (TOP(X,\mathcal{L}), s(TOP(X,\mathcal{L})))$.

(10) $s(TOP(X,\mathcal{L}))$ is conjoining.

(11) $\{ (U,x) \mid U$ open in X and $x \in U \}$ is open in $(\mathcal{O}(X), s(\mathcal{O}(X))) \times X$.

(12) $\mathcal{O}(X)$ $(TOP(X,\mathcal{L}))$ is a continuous lattice.

(13) X is core-compact.

(14) The sober reflection of X is locally compact.

If X is regular or sober, these are equivalent to

(15) X is locally compact.

If X is regular or Hausdorff, these are equivalent to

(16) X is weakly locally compact.

(17) $LIM_c(X,\mathcal{L}) = TOP_{co}(X,\mathcal{L})$. (18) $co(X,\mathcal{L})$ is conjoining.

If X is regular, these are equivalent to

(19) $LIM_c(X,\mathcal{L}) = TOP_{bo}(X,\mathcal{L})$.

If X is completely regular, these are equivalent to

(20) $LIM_c(X,[0,1]) = TOP_{co}(X,[0,1])$ $(LIM_c(X,\mathbb{R}) = TOP_{co}(X,\mathbb{R}))$.

(21) $co(X,[0,1])$ is conjoining ($co(X,\mathbb{R})$ is conjoining).

(22) $LIM_c(X,[0,1]) \in$ TOP $(LIM_c(X,\mathbb{R}) \in$ TOP$)$.

(23) $LIM_c(X,[0,1]) \in$ Tych $(LIM_c(X,\mathbb{R}) \in$ Tych$)$.

If X is zero-dimensional, these are equivalent to

(24) $LIM_c(X,\mathbb{D}_2) = TOP_{co}(X,\mathbb{D}_2)$. (25) $co(X,\mathbb{D}_2)$ is conjoining.

(26) $LIM_c(X,\mathbb{D}_2) \in$ TOP.

Whenever X is locally compact, we have $TOP_p(X,Y) = TOP_{co}(X,Y) = LIM_c(X,Y)$ for all $Y \in$ TOP.

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7): 2.3.

(1) $\Leftrightarrow$ (11) $\Leftrightarrow$ (12) $\Leftrightarrow$ (13) $\Leftrightarrow$ (15) $\Leftrightarrow$ (16): [11: 6.5].

(10) $\Leftrightarrow$ (11): immediate.

(10) $\Rightarrow$ (9): It follows from [3: Prop. 6] that $s(TOP(X,\mathcal{L}))$ is always splitting. Apply (3).

(9) $\Rightarrow$ (8): (9) $\Rightarrow$ (7) $\Rightarrow$ (13). Apply [8: 2.3], [13: Prop. 2].

(8) $\Rightarrow$ (7): obvious.

(14) ⇔ (13): $\mathcal{O}$(X) and $\mathcal{O}$(Q(X)) are isomorphic lattices. Hence by (12) ⇔ (13), X is core-compact iff Q(X) is. Apply [6: 4.5(1),(4)].
(15) ⇒ (18) ⇒ (17) ⇒ (7), (15) ⇒ (21) ⇒ (20) ⇒ (22), (15) ⇒ (25) ⇒ (24) ⇒ (26): [12: 2.7].
(19) ⇔ (17): [9: 2.1(iv)].
(20) ⇒ (23) ⇒ (22): [1: Thm. 1].
(22) ⇒ (15), (26) ⇒ (15): 3.1, 3.2.
The rest follows from 2.3 and [12: 2.7]. □

Of course in 3.3, the classes {$\mathcal{Z}$},{[0,1]},{$\mathbb{D}_2$} may be replaced by other classes which are initially dense in TOP,CReg,ZDim, respectively.– The additional assumptions for (15),(16),(17),(18) in 3.3 may not be omitted: Isbell [7: 2.11] gave a core-compact T_0-space which is not (weakly) locally compact with the properties that $LIM_c(X,\mathcal{Z}) \neq TOP_{co}(X,\mathcal{Z})$ and co(X,$\mathcal{Z}$) is not conjoining. (A similar example may be found in [6: Section 7].)

Application of 2.5 shows that 3.3 provides also a characterization theorem for the exponential objects in T_0TOP.

In connection with 2.3, the following results are more or less corollaries of 3.3.

3.4. COROLLARY [10: 3.3]. The only cartesian closed epireflective subcategories of PrT (TOP) are the indiscrete and the indiscrete T_0-spaces.

PROOF. Let $\underline{A}$ be a non-trivial epireflective subcategory of PrT. Then $\mathbb{Q} \in ZDimT_0 \subset \underline{A}$ ($\mathbb{Q}$ = rationals). Since $\mathbb{Q}$ is not locally compact, $LIM_c(\mathbb{Q},\mathbb{D}_2) \notin TOP$ by 3.3(15),(26). Hence $LIM_c(\mathbb{Q},\mathbb{D}_2) \notin PrT$, because $LIM_c(\mathbb{Q},\mathbb{D}_2)$ is a limit group [2: Satz 13] and every pretopological group is topological. By 2.3, $\mathbb{Q}$ is not exponential in $\underline{A}$. □

By application of [1: Thm. 1], we deduce from 2.3 and 3.3:

3.5. COROLLARY. Every locally compact regular space (T_3-space) is exponential in Reg (T_3TOP).

3.6. COROLLARY. Every exponential object in ZDim (ZDimT_0) is locally compact.

3.7. COROLLARY. For X ∈ CReg (Tych), the following are equivalent:

(1) X is exponential in CReg (Tych).
(2) – (7) of 2.3 with $\underline{A}$ = CReg (Tych), $\underline{D}$ = {[0,1]}.
(8) co(X,[0,1]) is conjoining. (9) $LIM_c(X,[0,1]) = TOP_{co}(X,[0,1])$.
(10) $LIM_c(X,[0,1]) \in TOP$. (11) X is locally compact.

In particular, the exponential objects in CReg and Tych are just the locally compact spaces in these categories.

Corollary 3.7 is an improvement of [11: 6.8]. It remains true if CReg and {[0,1]} are replaced by T_2TOP; this improves [11: 6.9]. In particular:

3.8. THEOREM [11: 6.9]. The exponential objects in T_2TOP are just the locally compact Hausdorff spaces.

The results obtained in this section for TOP, T_0TOP, T_2TOP, CReg, Tych (and the indiscrete (T_0-)spaces) motivate the following problem:

3.9. PROBLEM. Let $\underline{A}$ be an epireflective subcategory of TOP. Are the exponential objects in $\underline{A}$ just the core-compact $\underline{A}$-spaces?

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ON THE MONOTONE UNION AND MONOTONE INTERSECTION PROPERTIES OF TOPOLOGICAL MANIFOLDS

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Presented by P.G. Rooney, F.R.S.C.

Introduction M. Brown [2] has established that if N is an open n-cell, then N has the monotone union property. K.W.Kwun [3] proved that if M is a closed PL manifold whose dimension is not four, then M-p has the monotone union property where p is any point of M.

O.Bierman [1] proved that if a manifold has the monotone intersection property, then it also has the monotone union property, and he raised the following question. Are the monotone union and monotone intersection properties equivalent for compact topological manifolds with boundary?

§1. Definitions

(1) A compact topological manifold M with boundary $\partial M \neq \emptyset$ has the monotone union property provided that whenever $\{M_i\}$ is a sequence of manifolds such that for each i.

(a) M_i is homeomorphic to M and

(b) M_i is contained in the interior of M_{i+1},

then $\bigcup_{i=1}^{\infty} M_i$ is homeomorphic to the interior of M.

(2) A compact topological manifold N with boundary $\partial N \neq \emptyset$ has the monotone intersection property provided that whenever $\{N_i\}$ is a sequence of manifolds such that

(a) $\operatorname{int} N_1 \supset N_2 \supset \operatorname{int} N_2 \supset N_3 \supset \ldots$

(b) for each i, N_i is homeomorphic to N, then $N_1 - \bigcap_{i=1}^{\infty} N_i$ is homeomorphic to $\partial N_1 \times [0,1)$

(3) A compact, connected topological manifold M without boundary is rigid whenever

$M' \subset M\times(0,1) \subset M\times[0,1]$
separates $M\times 0$ from $M\times 1$ and M' is homeomorphic to M, $M\times[0,1]-(M\times 0 \cup M\times 1 \cup M')$ is the disjoint union of two open sets U,V homeomorphic to $M\times(0,1)$. (It follows that $\bar{U}-M\times 0 \approx M\times(0,1] \approx \bar{U}-M'$. Similarly for $\bar{V}$).

Theorem 1.

(a) All closed surfaces (2-manifolds) are rigid.

(b) All closed n-manifolds, $n\geq 6$, with finite fundamental group are rigid.

Theorem 2.

Let M be a compact topological manifold having the monotone union-property such that ∂M is connected and rigid. Then M has the monotone intersection property".

Therefore in particular,

Theorem 3.

The monotone union and monotone intersection properties are equivalent for compact 3-dimensional manifolds M with ∂M connected.

Theorem 4.

Let M be a compact, topological 3-dimensional manifold embedded in R^3, having the monotone union property. Then, M is homeomorphic to D^3, a 3-cell.

Proof

Let M_o be an embedding of M in R^3. Let B_1 be a big 3-cell containing M_o in its interior. Let B_o be a little 3-cell in $\mathrm{int}M_o$. Let h be a homeomorphism of R^3 such that $h(B_o)=B_1$. Define $M_i=h^i M_o$ and $B_i=h^i B_o$.

Then,

$$B_o \subset \mathrm{int}M_o \subset M_o \subset \mathrm{int}B_1 \subset B_1 \subset \mathrm{int}M_1 \subset M_1 \subset \ldots$$

and $M_i=M_{i+1}=M$. But $\cup M_i=\cup B_i$ and is homeomorphic to R^3.

Q.E.D.

Theorem 5.

Let M^n be a compact, topological n-dimensional manifold having the monotone union property with $\dim M^n = n \geq 6$ such that ∂M is connected and has a finite fundamental group. Then M has the monotone intersection property.

§ 2. Let M be an irreducible, simple 3-manifold with ∂M an incompressible torus ("simple" means that every incompressible torus in M is parallel (i.e. Isotopic) to ∂M).

For example, let M be the complement of a neighborhood of a knot in S^3, having no nontrivial companions, which is not a torus knot or a cable knot.

Claim that M has the monotone union property. (Let N be compact, irreducible. If ∂N contains a compressible torus T, then N is a solid torus).

(Proof. There is a disk D in N with $\partial D \neq 0$ on T. if DxI is a regular neighborhood of D in N, the 2-sphere $D\times 0 \cup D\times 1 \cup \overline{T - D\times I}$ bounds a 3-ball in N).

Suppose $M_i = M = M_{i+1}$ and $M_i \subset \operatorname{int} M_{i+1}$

Let $N = \overline{M_{i+1} - M_i}$. Now, ∂M_i is incompressible in M_{i+1}.

(Otherwise ∂M_i would be compressible in N and thus would be a solid torus, which can not be since $\partial N = \partial M_i \cup \partial M_{i+1}$.).

Therefore ∂M_i is parallel in M_{i+1} to ∂M_{i+1}, i.e.

$$M_{i+1} = M_i \cup (\partial M_i \times I) \text{ and } \bigcup_1 M_i = M_1 \cup (\partial M_1 \times R^1) = \operatorname{int} M.$$

The above example shows the existence of a compact topological 3-dimensional manifold having the monotone union property such that ∂M has a component different from S^2, in the piecewise linear category.

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DUALITY OF C^*-ALGEBRA FIBRE BUNDLES

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Abstract. We prove a duality between the category of C^*-algebra fibre bundles having the same fibre A and the category of topological fibre bundles having the same fibre the structure space of A.

In [1,2] there are discussions of C^*-algebra fibre bundles. A C^*-algebra fibre bundle $\Sigma = (E,X,A,p,\mathcal{U},\mathcal{G}_u,G)$ is a Steenrod fibre bundle [3] over a locally compact Hausdorff space X with a C^*-algebra A as fibre and group G a group of $*$-automorphisms of A. The continuous sections of the bundle that vanish at infinity constitute a C^*-algebra D under the natural algebraic operations. The main results of [1,2] identify the structure space of D with a topological fibre bundle over the same space X with fibre the structure space of A. Therefore a fibre bundle $\tilde{\Sigma} = (\hat{D},X,\hat{A},\tilde{p},\mathcal{U},\tilde{\mathcal{G}}_u,\hat{G})$ is obtained.

Let $\mathcal{C}$ be the category of C^*-algebra fibre bundles having the same fibre C^*-algebra A and the same group G. A morphism

$$h : \Sigma=(E,X,A,p,\mathcal{U}',\mathcal{G}_u,G) \longrightarrow \Sigma'=(E',X',A,p',\mathcal{U}',\mathcal{G}'_u,G)$$

in $\mathcal{C}$ is given by a continuous map $h : E \longrightarrow E'$ satisfying the following properties:

(1) h carries each fibre $p^{-1}(x)$ of E homeomorphically onto a fibre $p'^{-1}(x')$ of E' thus inducing a continuous map $\bar{h}: X \longrightarrow X'$ such that $p'h=\bar{h}p$, and $\bar{h}(u) \in \mathcal{U}'$ for each $u \in \mathcal{U}$.

(2) If $x \in u \cap \bar{h}^{-1}(u')$ and $h_x : p^{-1}(x) \longrightarrow p'^{-1}(x')$ is the map induced by h $(x'=\bar{h}(x))$ then the map $\bar{g}_{uu'}(x) \equiv \varphi'^{-1}_{u'x'} h_x \varphi_{ux} \in G$, where $\varphi_{ux}(a) = \varphi_u(x,a)$, $a \in A$.

(3) The map $\bar{g}_{uu'} : u \cap \bar{h}^{-1}(u') \longrightarrow G$ is continuous.

It is readily proved that the identity map $E' \longrightarrow E'$ induces the identity morphism $1_{\Sigma'} : \Sigma' \longrightarrow \Sigma'$ in the sense that if $h_1 : \Sigma \longrightarrow \Sigma'$ and $h_2 : \Sigma' \longrightarrow \Sigma''$ are morphisms in $\mathcal{C}$ then $1_{\Sigma'} h_1 = h_1$ and $h_2 1_{\Sigma'} = h_2$. If $x \in u \cap \overline{h_2h_1}^{-1}(u'')$ and if $(h_2h_1)_x$ is the map induced by h_2h_1 then the map $\bar{g}_{uu''}(x) \equiv \varphi''^{-1}_{u''x''}(h_2h_1)_x\varphi_{ux}$ $= \varphi''^{-1}_{u''x''} (h_2)_x(h_1)_x \varphi_{ux} = \varphi''^{-1}_{u''x''}(h_2)_x\varphi'_{u'x'}\varphi'^{-1}_{u'x'}(h_1)_x \varphi_{ux}$ $= \bar{g}_{u'u''}(x)\, \bar{g}_{uu'}(x) \in G$ and $\bar{g}_{uu''}: u \cap \overline{h_2h_1}^{-1}(u'') \longrightarrow G$ is continuous. Thus $h_2h_1 : \Sigma \longrightarrow \Sigma''$ is a morphism in $\mathcal{C}$ satisfying $(h_3h_2)h_1 = h_3(h_2h_1)$ whenever $h_3 : \Sigma'' \longrightarrow \Sigma'''$ is a morphism in $\mathcal{C}$.

Let A be a fixed C^*-algebra and G be a group of *-automorphisms of A. Let $\mathcal{T}$ be the category of topological fibre bundles having the same fibre $\hat{A}$ the structure space of A and the same group $G_1 = \{g^* :$ there is a $g \in G$ such that $g^*(\pi)(a)=\pi(g(a))$, $a \in A$, $\pi \in \hat{A}\}$. The topology on G_1 is derived from the anti-epimorphism $g \in G \longmapsto g^* \in G_1$. A morphism in $\mathcal{T}$ is defined similarly as a morphism in $\mathcal{C}$.

Let Σ be an object in $\mathcal{C}$. Let $T(\Sigma)$ be the topological fibre bundle $\tilde{\Sigma}$ constructed in [1,2]. Then $T(\Sigma)$ is an object in $\mathcal{T}$. For each morphism $h: \Sigma \longrightarrow \Sigma'$ in $\mathcal{C}$, a morphism $T(h): T(\Sigma) \longrightarrow T(\Sigma')$ in $\mathcal{T}$ is defined as follows:

$T(h): \hat{D} \longrightarrow \hat{D}'$ is defined by $T(h)(\sigma)(\gamma') = \pi t_{u'}(\gamma'(x'))$,

where σ corresponds to (x,u,π) and satisfies $\sigma(\gamma) = \pi(t_u(\gamma(x)))$ [1;TH.3.3] and where $x' = \bar{h}(x)$, $u' = \bar{h}(u)$.

If $x \in u$, $(T(h))_x : p^{-1}(x) \longrightarrow p'(x')$ is defined by associating to the σ corresponding to (x,u,π) the σ' corresponding to $(\bar{h}(x),\bar{h}(u),\pi)$. Then $(T(h))_x$ is a homeomorphism and satisfies $p'T(h) = \overline{T(h)}p$.

If $x \in u \cap \overline{T(h)}^{-1}(u')$ then the map $\tilde{\mathcal{G}}^{-1}_{u'x'}.T(h)_x\tilde{\mathcal{G}}_{ux}(\pi) =$ $\tilde{\mathcal{G}}^{-1}_{u'x'}.T(h)_x(x,u,\pi) = \tilde{\mathcal{G}}^{-1}_{u'x'}.(x',u',\pi) = \pi$. Thus

$$\tilde{\mathcal{G}}^{-1}_{u'x'}.(T(h))_x\tilde{\mathcal{G}}_{ux} \in G_1,$$

therefore $T(h): T(\Sigma) \longrightarrow T(\Sigma')$ is a morphism in $\mathcal{T}$.

<u>Theorem 1</u>. $T : \mathcal{C} \longrightarrow \mathcal{T}$ is a covariant functor.

<u>Proof</u>: It is obvious that $T(1_{\Sigma'}) = 1_{T(\Sigma')}$.

If $h_1: \Sigma \longrightarrow \Sigma'$ and $h_2: \Sigma' \longrightarrow \Sigma''$ are morphisms in $\mathcal{C}$ then $T(h_2h_1)(\sigma)\gamma'' = \pi(t_{h_2h_1(u)}(\gamma''(h_2h_1(x))))$ corresponds to $(h_2h_1(x),h_2h_1(u),\pi) \in \hat{D}''$. On the other hand, $(T(h_2)T(h_1))(\sigma)$ $= T(h_2)(T(h_1)\sigma) = T(h_2)(h_1(x),h_1(u),\pi) = (h_2h_1(x),h_2h_1(u),\pi)$.

Thus $T(h_2h_1) = T(h_2)T(h_1)$, and T is a covariant functor.

If $\mathcal{B} = (B, X, A, p, \mathcal{U}, \Phi_u, G_1)$ is an object in $\mathcal{T}$ if $u, v \in \mathcal{U}$, and if $x \in u \cap v$ then $(x,\pi) = \Phi_u^{-1}\Phi_v(x, h_{vu}(x)(\pi))$, $\pi \in \hat{A}$. Then $h_{vu}(x) \in G_1$ and hence there exists $g_{uv}(x) \in G$ such that

$$h_{vu}(x)(\pi)(a) = \pi(g_{uv}(x)(a)), \quad a \in A, \quad \pi \in \hat{A}.$$

Since $h_{vu}(x)h_{uw}(x) = h_{vw}(x)$ it follows that the cocycle property $g_{wu}(x)g_{uv}(x) = g_{wv}(x)$ holds. Hence Steenrod's construction [3] yields a fibre bundle $\Sigma_{\mathcal{B}}$ over the base space X with fibre A and group G.

For $\mathcal{B} \in \mathcal{T}$ let $S(\mathcal{B})$ be the C^*-algebra fibre bundle $\Sigma_{\mathcal{B}}$ obtained by the above argument. If $\alpha : \mathcal{B} \longrightarrow \mathcal{B}'$ is a morphism in $\mathcal{T}$, a morphism $S(\alpha): S(\mathcal{B}) = \Sigma_{\mathcal{B}} \longrightarrow S(\mathcal{B}') = \Sigma_{\mathcal{B}'}$ in $\mathcal{C}$ is defined as follows:

$$\alpha : p^{-1}(x) \longrightarrow p'^{-1}(x') \text{ induces a map } \bar{\alpha} : X \longrightarrow X'.$$

If $x \in u \cap \bar{\alpha}^{-1}(u')$, $G_{uu'}(x) \equiv \Phi_{u'x'}^{-1}\, \alpha_x \Phi_{ux} \in G_1$. Hence there is a $g_{u'u}(x) \in G$ such that $G_{uu'}(x) = (g_{u'u}(x))^*$. Then the map $S(\alpha): B \longrightarrow B'$ defined by $S(\alpha)[x,a] \equiv [\bar{\alpha}(x), g_{u'u}(x)(a)] = [x', g_{u'u}(x)(a)]$ induces a map $S(\alpha)_x : p^{-1}(x) \longrightarrow p'^{-1}(x')$ by $S(\alpha)_x(x,a) \equiv (x', g_{u'u}(x)(a))$ which ia a homeomorphism and satisfies $p'S(\alpha) = \overline{S(\alpha)}p$ on B, where B is the disjoint union of $\{u \times A\}$ under an equivalence relation and $[x,a]$ is the equivalence class of (x,a).

If $x \in u \cap \overline{S(\alpha)}^{-1}(u')$ then $\bar{g}_{uu'}(x)(a) \equiv ((\varphi'^{-1}_{u'x}\, S(\alpha)_x\, \varphi_{ux})(a)$
$= (\varphi'^{-1}_{u'x'}\, S(\alpha)_x)\varphi_u(x,a) = \varphi'^{-1}_{u'x'}\, S(\alpha)_x\, [x,a] = \varphi'^{-1}_{u'x'}[x', g_{u'u}(x)(a)]$
$= g_{u'u}(x)(a)$. Thus $\bar{g}_{uu'}(x) = g_{u'u}(x) \in G$ and

$$\bar{g}_{uu'} : u \cap \overline{S(\alpha)}^{-1}(u') \longrightarrow G \text{ is continuous.}$$

$S(\alpha)$ is therefore a morphism in $\mathcal{C}$.

<u>Theorem 2</u>. $S : \mathcal{T} \longrightarrow \mathcal{C}$ is a covariant functor.

<u>Proof</u>: If $\alpha : \mathcal{B} \longrightarrow \mathcal{B}'$ is a morphism in $\mathcal{T}$ then $S(\alpha)$: $S(\mathcal{B}) \longrightarrow S(\mathcal{B}')$ is a morphism in $\mathcal{C}$ satisfying :

(1) $S(1_{\mathcal{B}'}) = 1_{S(\mathcal{B}')}$.

(2) If $\alpha_1 : \mathcal{B} \longrightarrow \mathcal{B}'$ and $\alpha_2 : \mathcal{B}' \longrightarrow \mathcal{B}''$ are morphisms in $\mathcal{T}$, then $S(\alpha_2\alpha_1) = S(\alpha_2)S(\alpha_1)$. Indeed $S(\alpha_2\alpha_1)[x,a] = [x'', g_{u''u}(x)(a)]$ $= [x'', g_{u''u'}(x)(g_{u'u}(x)(a))] = S(\alpha_1)[x', g_{u'u}(x)(a)] = S(\alpha_2)S(\alpha_1)$ $[x,a]$. Hence S is a covariant functor.

<u>Theorem 3</u>. (S,T) is a duality between the categories $\mathcal{T}$ and $\mathcal{C}$.

<u>Proof</u>: (1) For every $\mathcal{B} \in \mathcal{T}$, let $\eta_{\mathcal{B}}(\mathcal{B}) = \tilde{\Sigma}_{\mathcal{B}}$, where $\tilde{\Sigma}_{\mathcal{B}}$ is a topological fibre bundle constructed from the C^*-algebra fibre bundle $\Sigma_{\mathcal{B}}$ as in [1,2] and $\Sigma_{\mathcal{B}}$ is the C^*-algebra fibre bundle derived from $\mathcal{B}$ as above. If $\alpha : \mathcal{B} \longrightarrow \mathcal{B}'$ is a morphism in $\mathcal{T}$ then

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\eta_{\mathcal{B}}} & TS(\mathcal{B}) \\ \alpha \downarrow & & \downarrow TS(\alpha) \\ \mathcal{B}' & \xrightarrow{\eta_{\mathcal{B}'}} & TS(\mathcal{B}') \end{array}$$

is commutative, and therefore $\eta : I_{\mathcal{T}} \approx TS$, where $I_{\mathcal{T}}$ denotes the identity functor of $\mathcal{T}$.

(2) If $h: \Sigma \longrightarrow \Sigma'$ is a morphism in $\mathcal{C}$ then the map $\zeta_{\Sigma}[x,a] \equiv (x,a)$ from $ST(\Sigma)$ into Σ makes the following diagram

$$\begin{array}{ccc} ST(\Sigma) & \xrightarrow{\zeta_{\Sigma}} & \Sigma \\ ST(h) \downarrow & & \downarrow h \\ ST(\Sigma') & \xrightarrow{\zeta_{\Sigma'}} & \Sigma' \end{array}$$

commutative, and therefore $\zeta : ST \approx I_{\mathcal{C}}$, where $I_{\mathcal{C}}$ is the identity functor of $\mathcal{C}$.

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SUR UNE CLASSE D'ORDRES MAXIMAUX

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Presenté par P. Ribenboim, F.R.S.C.

On dégage une classe d'ordres maximaux noethériens à gauche dans un anneau artinien simple qui contient strictement ceux à identité polynomiale et qui possèdent certaines propriétés de ces anneaux.

PRELIMINAIRES. - Soit R un anneau premier noethérien à gauche admettant un anneau de fractions artinien simple Q (donc de Goldie à droite). Si R est un ordre maximal de Q nous savons que le centre Z de R est un domaine complètement intégralement clos dans son corps des fractions K (par exemple [8] ch. 8, lemme 3.1). Nous dirons que R est localement fini sur son centre Z si le sous-anneau engendré par Z et par un nombre fini d'éléments de R est un Z-module de type fini. Il est clair que R est alors entier sur son centre Z. On sait alors que R étant noethérien à gauche, entier sur son centre, Q existe bien et que l'on a Q = RK = KR ([3] prop. 1.16) et que Z est un domaine de Krull ([2] ; [8] ch. 8, lemme 3.1). Nous dirons que R est centralement localement fini s'il est localement fini sur son centre Z et si toute famille finie d'éléments de R appartient à un sous-anneau de R qui est un Z-ordre maximal (au sens de Fossum ; [4], [8] chap. I fin du paragraphe 7).

EXEMPLES.

1) Un ordre maximal premier noethérien PI-anneau R est centralement localement fini sur son centre Z : d'après ([2] ou [8] chap. 8, th. 3.2), R est un Z-ordre maximal et Z est un domaine de Krull ([8] chap. 8, lemme 3.1) puisque R est entier sur Z. D'après un résultat de Sirsov (par exemple [8] chap. 8, th. 2.3) R est localement fini sur Z. Il est clair que R étant lui-même un Z-ordre maximal, R est centralement localement fini.

2) Si K' est un corps de Köthe de dimension infinie sur son centre k ([8] chap. I, définition avant le lemme 8.5 ; et [7]), et si $X_1,\dots,X_n$ sont des inconnues commutant entre elles et avec tout élément de K', $K'[X_1,\dots,X_n]$ est un ordre maximal sans diviseurs de zéro noethérien de son corps des fractions Q. Si

on considère un nombre fini d'éléments de $K'[X_1,\dots,X_n]$ c'est-à-dire de polynômes f_i en $X_1,\dots,X_n$ à coefficients dans K', leurs coefficients appartiennent à un corps K" de type fini sur k qui est aussi le centre de K" et ces polynômes f_i appartiennent à $K''[X_1,\dots,X_n]$ de type fini sur son centre $k[X_1,\dots,X_n]$ qui est aussi celui de $K'[X_1,\dots,X_n]$. Donc $K'[X_1,\dots,X_n]$ est centralement localement fini et il est facile de voir qu'il n'est pas un PI-anneau. Ceci montre que la classe des ordres maximaux premiers noethériens à gauche centralement localement finis contient strictement la classe des ordres maximaux noethériens et PI-anneaux.

<u>THEOREME</u> 1. - Un ordre maximal noethérien à gauche et premier R localement fini sur son centre Z vérifie le going-down théorème par rapport à Z.

<u>DEMONSTRATION</u>. - On applique une proposition de [2] et sa démonstration ([8] chap. 8, prop. 1.1). Il suffit de vérifier que pour tout idéal premier P de R, $R/_P$ est un anneau de Goldie. Mais R/p est un anneau premier noethérien à gauche, entier sur le sous-anneau de son centre $Z/_{Z\cap P}$ qui de ce fait admet un anneau de fractions artinien simple ([3] prop. 1.3) et ainsi $R/_P$ déjà noethérien à gauche, est de Goldie à droite.

THEOREME 2. - Soit R un anneau noethérien à gauche et premier R centralement localement fini sur son centre Z, ordre maximal de son anneau des fractions Q. Il y a bijection de l'ensemble $\mathcal{P}$ des idéaux premiers de hauteur 1 de R sur l'ensemble $\overline{\mathcal{P}}$ des idéaux premiers de hauteur 1 de Z : pour tout $p \in \overline{\mathcal{P}}$ il existe un unique $P \in \mathcal{P}$ avec $p = P \cap Z$.

<u>DEMONSTRATION</u>. - R est entier sur Z et si $p \in \overline{\mathcal{P}}$ il existe un idéal premier P de R avec $P \cap Z = p$. Le théorème d'incomparabilité ([1]; [8] chap. 3, prop. 4.3) montre que l'on a $P \in \mathcal{P}$. Réciproquement, si $P \in \mathcal{P}$, $P \cap Z = p$ appartient à $\overline{\mathcal{P}}$ d'après le théorème 1. Il reste à démontrer que si $P \in \mathcal{P}$, $P' \in \mathcal{P}$ sont tels que $P \cap Z = p = P' \cap Z$ on a $P = P'$. Considérons les générateurs $f_1,\dots,f_n$ de l'idéal à gauche P et les générateurs $f'_1,\dots,f'_m$ de l'idéal à gauche P' et considérons le Z-ordre maximal (au sens de Fossum) R' contenant $f_1,\dots,f_n$, $f'_1,\dots,f'_m$ et contenu dans R. Il est clair que $P_1 = P \cap R'$ et $P'_1 = P' \cap R'$ sont des idéaux bilatères de R'. L'anneau R'_p est un Z_p-ordre maximal (au sens de Fossum) et c'est même un Z_p-ordre maximal classique ([8] chap. 1, fin du paragraphe 7) puisque Z_p est noethérien ; $(P_1)_p$ et $(P'_1)_p$ sont des idéaux bilatères de R'. Mais on sait que R'_p est un ordre d'Asano régulier admettant un seul c-idéal premier ([8] pour les définitions) et dont les idéaux bilatères sont en chaîne. On a donc par exemple $(P_1) \subseteq (P'_1)_p$ et $sf_i \in P'_1$ pour un $s \in S$, $S = R - p$. On a donc $sf_i \in P'$ avec $s \notin P'$ et $f_i \in P'$, $\forall\, i = 1,\dots,n$ et $P \subseteq P'$. Le théorème d'incomparabilité donne alors $P = P'$. □

RAPPEL. - Etant donné un idéal premier P d'un anneau noethérien des deux côtés, premier R dont on note Q l'anneau des fractions. On pose

$$\mathcal{F}_P = \{I \text{ idéal à gauche de } R \mid I \text{ contient un idéal bilatère } N \text{ de } R,\ N \not\subseteq P\}$$

et

$$\mathcal{F}^P = \{I \text{ idéal à gauche de } R \mid I^{\cdot}.a = \{x \in R \mid xa \in I\} \text{ coupe } \mathcal{C}(P)\}.$$

Les familles $\mathcal{F}^P$ et $\mathcal{F}_P$ sont topologisantes et idempotentes et les localisés correspondants de R sont des anneaux notés $R_{\mathcal{F}_P}$ (ou R_P) et $R_{\mathcal{F}^P}$: ce sont des sous-anneaux de Q, [8].

THEOREME 3. - Soit R un anneau premier noethérien des deux côtés, ordre maximal de son anneau des fractions Q et centralement localement fini sur son centre Z. Alors si P est un idéal premier quelconque de R, on a :

$$R_P = R_{\mathcal{F}^P} = R_{Z \cap P}$$

et R_P est égal à l'anneau classique des fractions de R selon $\mathcal{C}(P)$ si et seulement si P est le seul idéal premier sur $Z \cap P$. C'est en particulier le cas si hauteur P = 1.

DEMONSTRATION. - On sait que R est régulier (en anglais "bounded") et même totalement borné ([3] prop. 1.4). D'après ([8] chap. 4, prop. 1.7), on a $R_P = R_{\mathcal{F}^P}$. Tout élément $s \in Z - P \cap Z$ s'inverse dans R_P car $Rss^{-1} = R$ et $Rs \in \mathcal{F}_P$. De plus, soit $x \in R_P$, il existe N idéal bilatère de R avec $N \not\subseteq P$ tel que $Nx \subseteq R$, si on pose $N^{\times} = R.\cdot(R^{\cdot}.N)$, on a $N^{\times}x \subseteq R$, et ou bien $N^{\times} = R$ alors $x \in R$, ou bien $N^{\times} \neq R$ et $P_i \supseteq N^{\times} \supseteq \prod_{i=1}^{r} P_i^{n_i}$ avec $P_i \supseteq N^{\times}$, $i = 1,\dots,r$, où les P_i sont des idéaux premiers de hauteur 1 de R (voir [8] pour ces propriétés classiques) non contenus dans P. Soit q un idéal premier de Z tel que $0 \subsetneq q \subseteq p = P \cap R$ et hauteur q = 1. D'après le théorème 1, il existe un idéal premier Q de R de hauteur 1 avec $Q \subseteq P$ et $Q \cap Z = q$. Il n'y en a qu'un (théorème 2) : ainsi tous les P_i, $i=1,\dots,r$, n'étant pas dans P sont tels que $P_i \cap Z \not\subset p$ et par suite pour $i = 1,\dots,r$, $P_i \cap Z$ contient un élément $s_i \in \{Z - p\}$ et il existe $t \in \{Z - p\}$ avec $tx \in R$ et ceci prouve que $R_P = R_{P \cap Z}$.

D'après [8] chap. 8, lemme 1.5 et 1.6, les idéaux premiers P_i de R avec $P_i \cap R = p$ sont en nombre fini $n > 1$: Ils sont en bijection avec les idéaux maximaux M_i de R_p et l'on a $M_i \cap R = P_i$ et $M_i = (P_i)_p$. S'il y a un seul P sur p, il y a un seul idéal maximal donc maximum P_p dans R_p. On a alors, d'après [3] prop. 2.3, la condition de Ore des deux côtés par rapport à $\mathcal{C}(P)$ et $R_{\mathcal{C}(P)} = R_p$. Réciproquement, supposons qu'on ait la condition de Ore des deux côtés de R par rapport à $\mathcal{C}(P)$, alors $\mathcal{F}^P$ est l'ensemble des idéaux à gauche coupant $\mathcal{C}(P)$ de R ; de $cx = 0$, $c \in \mathcal{C}(P)$, $x \in R$, on déduit $Rcx = 0$ donc $x \in \mathcal{F}^P(R)$, puisque $Rc \in \mathcal{F}^P$ et par suite $x = 0$, ([8] chap. 4, prop. 1.1) donc $cy = 0$ avec $y \in Q$ implique $y = 0$ donc c est non diviseur de zéro à droite dans l'anneau artinien simple Q donc aussi à gauche. D'après ([6] ; [8] chap. 4, th. 1.5) $\mathcal{F}^P$ vérifie la condition (T) de Goldmann [5] et $P_{\mathcal{F}^P}$ est l'idéal maximum de R_p. De plus on a $P_{\mathcal{F}^P} = PR_p$ et $PR_p \cap R = P$. Si Q est un idéal bilatère premier de R tel que $Q \cap Z = p = P \cap Z$ on obtient $Q_p \subseteq P_p$ car $Q_p \neq R_p$ et il vient $Q \subseteq Q_p \cap R \subseteq P_p \cap R = P$ et donc $Q = P$ d'après le théorème d'incomparabilité et finalement P est le seul idéal bilatère premier de R sur P.

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ON RIGHT-LEFT C^∞-SUFFICIENCY OF JETS

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ABSTRACT

We give a necessary and sufficient condition for the right-left sufficiency of jets; this improves some results of J.N. Mather and G. Wasserman.

1. INTRODUCTION

Let $\mathcal{E}_n$ denote the ring of germs at 0 of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\underline{m}_n$ its maximal ideal. For $f = (f_1,\ldots,f_p) \in \bigoplus_p \underline{m}_n$, $j^r(f)$ is the r-jet at 0 of f, and $J^r(n,p)$ is the space of the r-jets of the elements of $\bigoplus_p \underline{m}_n$. Let $L(n)$ denote the group of germs of smooth diffeomorphisms h from a neighborhood of $0 \in \mathbb{R}^n$ onto a neighborhood of $0 \in \mathbb{R}^n$ such that $h(0)=0$. For $f \in \underline{m}_n$, let I_f be the ideal generated in $\mathcal{E}_n$ by the partial derivatives of f and let $f^*: \mathcal{E}_1 \longrightarrow \mathcal{E}_n$ be the homomorphism of $\mathbb{R}$-algebras defined by $f^*(g) = g \circ f$.

Definition 1.1. An r-jet $z \in J^r(n,p)$ is right-left sufficient (respectively right sufficient) if for any $f \in \bigoplus_p \underline{m}_n$ such that

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$j^r(f)=z$ there exists $(h,k)\in L(n)\times L(p)$ (resp. $h\in L(n)$) such that $z=k\circ f\circ h^{-1}$ (resp. $z=f\circ h^{-1}$).

In [1] (see also [2]) we proved:

Theorem 1.2. $z\in J^r(n,1)$ is right sufficient if and only if for any $w\in J^{r+1}(n,1)$ such that $j^r(w)=0$

$$\underline{m}_n^{r+1}\subset \underline{m}_n.I_{z+w}.$$

We will now show:

Theorem 1.3. Let $z\in J^r(n,1)$. The following statements are equivalent:

(a) the r-jet z is right-left sufficient.

(b) $\underline{m}_n^{r+1}\subset \underline{m}_n.I_f+f^*(\underline{m}_1)$ for each $f\in\underline{m}_n$ such that $j^r(f)=z$.

(c) $\underline{m}_n^{r+1}\subset \underline{m}_n.I_{z+w}+(z+w)^*(\underline{m}_1)+\underline{m}_n^{r+3}$ for each $w\in J^{r+2}(n,1)$ such that $j^r(w)=0$.

Note: $f^*(\underline{m}_1)$ is the image of $\underline{m}_1$ under f^*, and is not the ideal generated in $\mathcal{E}_n$ by the image.

The implication a)=>b) is proven in [4, p.43]. From theorem 1.3 we deduce the following result of J.N. Mather [3] stated as follows in [4, chapter 2]:

Theorem 1.4. Let $f\in\underline{m}_n$. If

$$m_n^{r-1}\subset \underline{m}_n.I_f+f^*(\underline{m}_1)+\underline{m}_n^{r+1},$$

then the r-jet $z = j^r(f)$ is right-left sufficient.

Remarks 1.5.

(a) In theorem 1.4, the condition is not necessary. For example, using Theorem 1.2, one checks that $z(x,y) = x^3 + xy^3 \in J^4(2,1)$ is right sufficient and, a fortiori, right-left sufficient, but $\underline{m}_2^3 \not\subset \underline{m}_2 . I_z + z^*(\underline{m}_1) + \underline{m}_2^5$ since $y^4 \notin \underline{m}_2 . I_z + z^*(\underline{m}_1) + \underline{m}_2^5$ as can be easily shown.

(b) Let $f \in \underline{m}_n$. The condition $\underline{m}_n^{r+1} \subset \underline{m}_n I_f + f^*(\underline{m}_1)$ does not imply that $z = j^r(f)$ is right-left sufficient as the following example shows. Let $f(x) = x^{k+1}$ $(k \geq 1)$. We have $\underline{m}_1^{k+1} \subset \underline{m}_1 \cdot I_f + f^*(\underline{m}_1)$ but $z = j^k(x^{k+1})$ is not right-left sufficient since $g \equiv 0$ is not right-left equivalent to x^{k+1}, although $j^k(g) = j^k(x^{k+1})$.

2. THE PROOF

Let $L^r(n) \times L^r(1)$ be the analytic Lie group of the r-jets of the elements of $L(n) \times L(1)$. For each $r \in \mathbb{N}$, the group action of $L(n) \times L(1)$ on $\underline{m}_n$ induces a well defined analytic action of $L^r(n) \times L^r(1)$ on $J^r(n,1)$. For $z \in J^r(n,1)$ let $O_z^r = \{\gamma . z \mid \gamma \in L^r(n) \times L^r(1)\}$ be the orbit of z under the action of $L^r(n) \times L^1(n)$. One can easily show:

<u>Lemma 2.1.</u> [4, p.41] The tangent space $T_zO_z^r$ of O_z^r at z is equal to $\pi_r(\underline{m}_n.I_z + z^*(\underline{m}_1))$ where $\pi_r:\underline{m}_n \longrightarrow J^r(n,1)$ is defined by $\pi_r(f) = j^r(f)$.

2.2. Proof of theorem 1.3.

(a) => (b) is proven in [4, p.43]

(b) ⇒ (a) . Let $f \in \underline{m}_n$ such that $z = j^r(f)$. We must show that there exists $(h,k) \in L(n) \times L(1)$ such that $k o f o h^{-1} = z$. By hypothesis we have

$$\underline{m}_n^{r+1} \subset \underline{m}_n.I_f + f^*(\underline{m}_1) \quad .$$

We deduce from [3, p.141] that there exists $N \in \mathbb{N}$ such that $j^N(f)$ is right-left sufficient.

We now show that $j^N(f) \in O_z^N$. This will finish the proof since $j^N(f)$ is right-left sufficient. Let $j^N(f) = z + w$ where $j^r(w) = 0$. For any $t \in \mathbb{R}$, we have by hypothesis

$$\underline{m}_n^{r+1} \subset \underline{m}_n.I_{z+tw} + (z+tw)^*(\underline{m}_1) \quad .$$

Thus

$$\pi_N(\underline{m}_n.I_{z+tw} + (z+tw)^*(\underline{m}_1)) \supset \underline{m}_n^{r+1}/\underline{m}_n^N \quad .$$

From lemma 2.1 and the implicit function theorem, it follows that

$$O_{z+tw}^N \supset \{z + w' \mid w' \in J^N(n,1), j^r(w') = 0 \text{ , and } w' \text{ is near } tw\} \quad .$$

In particular, this is true for each $t \in [0,1]$. Therefore the compactness of $[0,1]$ implies that $j^N(f) = z+w \in O_z^N$.

(b) <=> (c) .

(b) => (c) is obvious. To prove the converse we will use the following version of the Malgrange-Mather Preparation Theorem:

<u>Preparation theorem</u>. [4, p.12]. Let $f\in \underline{m}_n$. Let C be a finitely-generated $\mathcal{E}_n$-module. Then C is also an $\mathcal{E}_1$-module via f^*. Let A be a cyclic $\mathcal{E}_1$-submodule of C and let B be an $\mathcal{E}_n$-submodule of C . Let D be an $\mathcal{E}_n$-submodule of C such that $\dim_{\mathbb{R}} C/D<\infty$. If

$$A+B+(f^*(\underline{m}_1)+\underline{m}_n^2).D\supset D$$

then

$$A+B\supset D .$$

(c) => (b)

Let $f\in\underline{m}_n$ such that $j^r(f)=z$. By hypothesis we have

$$(*) \qquad \underline{m}_n^{r+1}\subset \underline{m}_n\, I_{j^{r+2}(f)}+(j^{r+2}(f))^*(\underline{m}_1)+\underline{m}_n^{r+3}.$$

Since

$$\underline{m}_n.I_{j^{r+2}(f)}\subset \underline{m}_n.I_f+\underline{m}_n^{r+3}$$

and

$$(j^{r+2}(f))^*(\underline{m}_1)\subset f^*(\underline{m}_1)+\underline{m}_n^{r+3} ,$$

we deduce from (*) and the preparation theorem that

$$\underline{m}_n^{r+1}\subset \underline{m}_n.I_f+f^*(\underline{m}_1) .$$

□

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GENERALIZED INVERSES FOR LINEAR MANIFOLDS AND APPLICATIONS TO BOUNDARY VALUE PROBLEMS IN BANACH SPACES

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Presented by G.de B. Robinson, F.R.S.C.

ABSTRACT

We describe some results of a comprehensive theory of generalized inverses and operator parts for linear manifolds in Banach spaces. An application is given to ordinary differential subspaces.

§ 1. INTRODUCTION

Generalized inverses for densely defined (single-valued) linear operators have been studied extensively (see [9],[10]). In this paper, we will compute all possible operator parts of an inverse linear relation as well as define and study the generalized inverse of a (nondensely defined) linear manifold in a Banach space. The concept of a special class of generalized inverse (an analogue of a Moore-Penrose generalized inverse) of a linear manifold in Hilbert space was first introduced in [3]. The approach of this paper places the notion of the "operator part" in a natural setting, freeing it from the unnecessary restrictions on domains and ranges. The theory also extends some earlier work of the second author on generalized inverses of operators to those of linear manifolds. An application to ordinary differential subspaces is given. Unexplained notation or definitions can be found in [2].

§ 2. GENERALIZED INVERSES AND OPERATOR PARTS OF LINEAR MANIFOLDS IN BANACH SPACE

Let X_1, X_2 be Banach spaces. Denote by X_1^* the Banach space of conjugate-linear continuous functionals on X_1. A linear map P from X_1 into itself is called a projector (or algebraic projector) if $P^2 = P$. A linear map $P^\dagger$ from X_1^* into itself is called a w*- projector if $(P^\dagger)^2 = P^\dagger$, and $P^\dagger$ is w*- continuous. We say that a w*- closed linear manifold $Y^\dagger$ in X_1^* is w*- complemented in X_1^* if X_1^* is the algebraic direct sum of $Y^\dagger$ and a w*- closed linear manifold in X_1^*. Let M be an arbitrary, but fixed <u>closed</u> linear manifold in $X_1 \oplus X_2$. Following Coddington et al.[2] (see also [1]), we say that a vector space R in $X_2 \oplus X_1$ is an operator part of M^{-1} if R is the graph of a closed linear operator such that M^{-1} is the algebraic direct sum of R and $(\{0\} \oplus \text{Null } M)$.

The following theorem characterizes all possible operator parts of M^{-1} provided that Null M is complemented in X_1.

<u>THEOREM 1</u>. Assume that Null M is topologically complemented in X_1 and let P_0 be a given continuous projector from X_1 onto Null M. Let

$$R_0 := \Big\{ \{ g, y - P_0(y) \} \mid \{ y, g \} \in M \Big\}.$$

Then

(1) R_0 is an operator part of M^{-1}.

(2) R is an operator part of M^{-1} if and only if

$$R := \Big\{ \Big\{ g, R_0(g) - A\Big(\{ g, R_0(g) \} \Big) \Big\} \mid g \in \text{Range } M \Big\}$$

for some continuous linear operator A from $X_2 \oplus X_1$ into Null M such that $A(\{0,y\}) = 0$ for all $y \in$ Null M.

Aside from some technical aspects, the proof makes use of Sobyczk's lemma about a characterization of all possible projectors defined on a given vector space whose range is a prescribed subspace; see [11], also [10], Proposition 1.7.

DEFINITION. Assume that Null M (Null M^*) is topologically complemented (w^*- complemented) in X_1 (X_2^*), respectively. Then a vector space $M^\#$ in $X_2 \oplus X_1$ is called a generalized inverse of M if it has the form:

(1) $M^\# := \left(\text{graph } (I-P)\right) M^{-1} \left(\text{graph } (I-{}^*P^\dagger)\right)$, where P is a continuous projector form X_1 onto Null M, and $P^\dagger$ is a w^*- projector from X_2^* onto Null M^*.

THEOREM 2. Let $M^\#$ be as (1). Then Dom $M^\#$ = Range M $\dot{+}$ Range ${}^*P^\dagger$, direct sum

$$M^\# = \left(\text{graph } (I-P)\right) M^{-1} \dot{+} \left(\text{Range } {}^*P^\dagger \oplus \{0\}\right), \text{ direct sum.}$$

Since $\left(\text{graph } (I-P)\right) M^{-1}$ is an operator part of M^{-1}, $M^\#$ can be viewed as an operator extension of the operator part (by enlarging its domain).

In the following, we will find all possible generalized inverses of M via a pair of known projectors. The derivation uses the Sobyczk's lemma [11].

THEOREM 3. Let P_0 be an arbitrary, but fixed continuous projector from X_1 onto Null M and let $P_0^\dagger$ be an arbitrary, but fixed w^*- continuous projector from X_2^* onto Null M^* (thus we are assuming such existences). Then $M^\#$ is a generalized inverse of M if and only if

$$M^{\#} := \left(\text{graph } (I-P_0 - A)\right) M^{-1} \left(\text{ graph } (I-{}^*P_0{}^{\dagger} - {}^*A^{\dagger})\right)$$

for same continuous linear operator A from X_1 into Null M such that $A(y) = 0$ for all y in Null M and a w*- continuous linear operator $A^{\dagger}$ from X_2^* into Null M^* such that $A^{\dagger}(y) = 0$ for all $y \varepsilon$ Null M^*.

An important problem in boundary value problems for differential operators is to find Green's functions or generalized Green's functions. We will answer this problem in an abstract setting. The proof consists of using Theorem 1 together with generalized inverses of finite matrices, a coordinatized extension theory of linear manifold [4], [5]and finally a representation theory for the operator A appearing in the Theorem 1.

THEOREM 4. Let T_0, T_1 be known closed linear manifolds such that

$$T_0 \subset T_1 \subset X_1 \oplus X_2,$$

$\dim T_1/T_0 < \infty$, dim Null $T_1 < \infty$, dim Null $T_0^* < \infty$. Let M be an arbitrary, but fixed closed linear manifold such that $T_0 \subset M \subset T_1$.

Then

(1) Some operator part of T_1^{-1} is compact (an integral operator) if and only if any operator part of M^{-1} is compact (an integral operator).

(2) Some operator part of $(T_0^*)^{-1}$ is compact (an integral operator) if and only if any operator part of $(M^*)^{-1}$ is compact (an integral operator).

A special case of the following theorem is well known (for example, see [3]).

THEOREM 5. Let T be the graph of a maximal closed operator generated by a regular, ordinary linear differential expression such that

$$T \subset L_p\,[a,b] \oplus L_q\,[a,b] \ (1 \leq p,q < \infty).$$

Let W be any given finite dimensional vector space contained in $L_p\,[a,b] \oplus L_q\,[a,b]$. Let M be any closed linear manifold such that $M \subset T \dot{+} W$, $\dim\,(T \dot{+} W)/M < \infty$. Then any operator part of M^{-1} and any generalized inverse of M are compact integral operators.

Detailed proofs and related results are given in [6]. The theory of least-squares solutions of multi-valued linear operators has been developed by the authors in [7], and an application of the theory of operator parts to the formulation and convergence of the method of steepest descent for closed linear operators is given in [8].

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ON THE WIGNER QUASI-PROBABILITY DISTRIBUTION FUNCTION I

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Presented by P. Greiner, F.R.S.C.

The mixed Wigner quasi-probability distribution function $P(f,g;.,.)$ of quantum mechanical states $f \in \mathcal{S}(R^n)$ and $g \in \mathcal{S}(R^n)$ is the Euclidean Fourier transform on the phase space $R^n \oplus R^n$ of the radar crossambiguity function $H(f,g;.,.)$ associated with the pair (f,g) of pulse envelopes. In other words, $P(f,g;.,.)$ is a dualized version of the function

$$H(f,g;x,y) = \int_{R^n} f(t+{}^1/_2 x)\bar{g}(t-{}^1/_2 x)e^{2\pi i \langle t|y\rangle}dt.$$

In a series of previous papers [6], [7], [8], [9], the author has used harmonic analysis of the real Heisenberg nilpotent group $\tilde{A}(R^n)$ to investigate the "geometric" properties of $H(f,g;.,.)$. In particular, the symplectic invariance of the radar ambiguity surface has been pointed out. It is the aim of the present paper to transfer these geometric results and some of its consequences by means of the Euclidean Fourier transform to the Wigner phase space distribution function $P(f,g;.,.)$. - The second part of the paper will be concerned with roughly speaking those symplectic transformations that give rise to the same invariants of both the radar crossambiguity function H and its dual counterpart P.

1. The Wigner Quasi-Probabilitiy Distribution Function

Let $\mathcal{S}(R^n)$ denote the Schwartz-Bruhat space of rapidly decreasing complex-valued $\mathcal{C}^\infty$-functions on the configuration space R^n. For any pair (f,g) of functions belonging to the space $\mathcal{S}(R^n) \times \mathcal{S}(R^n)$ the mixed Wigner quasi-probability distribution function of statistical quantum mechanics is defined according to the formula

$$P(f,g;p,q) = \int_{\mathbf{R}^n} f(q+{}^1/_2 t)\bar{g}(q-{}^1/_2 t)e^{-2\pi i\langle p|t\rangle}dt.$$

If f = g then $P(f;.,.) := P(f,f;.,.)$ is called the Wigner quasi-probability distribution function (cf. de Bruijn [1]). Define the radar autoambiguity function $H(f;.,.)$ similarly. Notice that $P(f,g;.,.)$ occurs as the density function in the Weyl quantization $Op_1(a)$ of observables a that belong to the space $\mathscr{S}(\mathbf{R}^n \oplus \mathbf{R}^n)$.

Let $\tilde{A}(\mathbf{R}^n)$ denote the (2n+1)-dimensional real Heisenberg nilpotent group with (one-dimensional) center Z and U the Schrödinger model of the (up to unitary isomorphisms unique) faithful irreducible unitary linear representation of $\tilde{A}(\mathbf{R}^n)$ which induces the character $z \rightsquigarrow e^{2\pi i z}id$ on Z. Thus Planck's constant is normalized to the value 1. Recall that $\mathscr{S}(\mathbf{R}^n)$ is the space of $\mathscr{C}^\infty$-vectors of the realization U in the complex Hilbert space $L^2(\mathbf{R}^n)$. If c_U denotes the coefficient functions of U and $\mathscr{F}_{\mathbf{R}^{2n}}$ the Euclidean Fourier transform of the phase space $\mathbf{R}^n \oplus \mathbf{R}^n$, then we obtain

Theorem 1. Let the functions $f \in \mathscr{S}(\mathbf{R}^n)$ and $g \in \mathscr{S}(\mathbf{R}^n)$ be given. Then the identity

$$\boxed{P(f,g;.,.) = \mathscr{F}_{\mathbf{R}^{2n}} c_U(f,g;.,.,0)}$$

holds on the phase space $\mathbf{R}^n \oplus \mathbf{R}^n$.

The preceding result takes its value from the following facts:

(i) Since $\tilde{A}(\mathbf{R}^n)$ admits a lot of symmetry, there are a lot of different realizations of U which can be checked conveniently by an application of the uniqueness theorem of Stone-von Neumann.

(ii) The irreducible unitary linear representation U of $\tilde{A}(R^n)$ has some very pleasant properties. For instance, the geometric property that the orbit associated to the isomorphy class of U under the Kirillov correspondence is a linear variety in the dual vector space $\mathfrak{n}^*$ of the Lie algebra $\mathfrak{n}$ of $\tilde{A}(R^n)$ implies that U is square integrable mod Z, i.e., that U belongs to the discrete series of $\tilde{A}(R^n)$ (Segal's theorem). Therefore, the closed vector subspace of $L^2(\tilde{A}(R^n))$ which is spanned topologically by the coefficient functions c_U has a structure that is well understood.

(iii) For the coefficient functions c_U a long list of computation rules (for instance, the Schur orthogonality relations and various convolution identities) is known.

Let us mention the following geometric consequence of Theorem 1 which depends upon the Shale-Weil theorem which on his part is based on the uniqueness theorem of Stone-von Neumann:

Theorem 2. Let any function $f \in \mathscr{S}(R^n)$ of L^2-norm $\|f\| = 1$ be given. Suppose that for each pair $(p,q) \in R^n \oplus R^n$ there exists a pair $(p',q') \in R^n \oplus R^n$ such that the Wigner quasi-probabiliy distribution function satisfies the identity

$$P(f;p,q) = P(f';p',q')$$

for a function $f' \in \mathscr{S}(R^n)$ with $\|f'\| = 1$. Then there exist a unique symplectic automorphism $\sigma \in Sp(n,R)$ of the phase space $R^n \oplus R^n$, a unitary linear mapping $T_{\check{\sigma}}$ of the complex Hilbert space $L^2(R^n)$ associated uniquely with the contragredient automorphism $\check{\sigma} \in Sp(n,R)$ and a number $\zeta \in C$ of modulus $|\zeta| = 1$ such that

$$\boxed{\sigma(p',q') = (p,q), \quad \zeta T_{\check{\sigma}}(f) = f'}$$

holds.

The preceding theorem states that the real symplectic group Sp(n,R) which is a subgroup of the group of automorphisms of $\tilde{A}(R^n)$ forms the invariants of the Wigner-Woodward ambiguity surface (cf. [3]). In terms of statistical quantum mechanics, the expectation value

$$\langle Op_1(a)f|f\rangle$$

of any observable $a \in \mathcal{S}(R^n \oplus R^n)$ is constant on the orbit of states

$$\{ \zeta T_{\breve{\sigma}}(f) | \zeta \in T, \sigma \in Sp(n,R) \}$$

generated by $f \in \mathcal{S}(R^n)$, $\|f\| = 1$.

2. The Projective Representation T of Sp(n,R)

As in the preceding section, let $\mathfrak{n}$ denote the Lie algebra of $\tilde{A}(R^n)$, i.e., the (2n+1)-dimensional real Heisenberg Lie algebra with (one-dimensional) center log Z = R. Then the symplectic group Sp(n,R) forms also a subgroup of the group of automorphisms of the Lie algebra $\mathfrak{n}$ which acts trivially on the center R of $\mathfrak{n}$.

Let M denote a n-dimensional vector subspace of $\mathfrak{n}$. Then $\mathfrak{l} = M \oplus R$ and $\breve{\sigma}(\mathfrak{l})$ for any $\sigma \in Sp(n,R)$ form polarizations in $\mathfrak{n}$, i.e., both are totally isotropic vector subspaces with respect to the Lie bracket $[.,.]$ of $\mathfrak{n}$ of maximal dimensions n+1. By Kirillov theory we may assume that the irreducible unitary linear representation U of $\tilde{A}(R^n)$ will be induced from $\exp \mathfrak{l}$ to $\tilde{A}(R^n)$ by the unitary character $z \rightsquigarrow e^{2\pi i z}$ of Z. Then the elements F of the representation space $L^2(R^n)$ of U may be considered as functions on $\tilde{A}(R^n)$ which satisfy the covariance property. Define the partial Fourier cotransform

$$\bar{\mathcal{F}}_{\mathfrak{l},\breve{\sigma}(\mathfrak{l})}F(v,z) = \int_{\exp\mathfrak{l}/\exp\breve{\sigma}(\mathfrak{l})\cap\mathfrak{l}} F((v,z)(v',z'))e^{2\pi i z'}d\mu(v',z')$$

where $\mu \neq 0$ is a positive measure on $\exp \mathfrak{l}/\exp \breve{\sigma}(\mathfrak{l}) \cap \mathfrak{l}$ which is invariant under the action of $\exp \mathfrak{l}$. Let the Lebesgue measure μ be standardized such that $\bar{\mathcal{F}}_{\mathfrak{l},\breve{\sigma}(\mathfrak{l})}$ becomes a unitary operator on $L^2(R^n)$. If γ denotes the natural left action of Sp(n,R) on the functions F on $\tilde{A}(R^n)$ we have the following result:

<u>Theorem 3.</u> The unitary linear operator $T_{\check{\sigma}}$ occurring in Theorem 2 supra satisfies

$$\boxed{T_{\check{\sigma}} = \bar{\mathcal{F}}_{1,\check{\sigma}(1)} \circ \gamma(\check{\sigma})}$$

for all $\sigma \in Sp(n,R)$.

The assignment $\sigma \rightsquigarrow T_{\check{\sigma}}$ defines a projective linear representation of Sp(n,R). Its 2-cocycle on Sp(n,R) can be computed explicitly in terms of the Maslov index.

<u>3. Non-Negative Wigner Quasi-Probability Distribution Functions</u>

Consider the case n = 1 and let $f \in \mathcal{S}(R)$ be fixed. If we suppose $P(f;.,.) \geqq 0$, then the theorem of Mathias and Bochner (cf. Stewart [10]) implies that the radar autoambiguity function H(f;.,.) is of positive type on the time-frequency plane R^2. In view of the identity (cf. [6])

$$H(f;.,.) = c_U(f,f;.,.,0),$$

the coefficient function $c_U(f,f;.,.,.)$ which a priori is of positive type on $\tilde{A}(R)$ is of positive type on R^3 also. Taking into account a recent paper by Fischer [2] we get the following result (cf. Hudson [4], Janssen [5]):

<u>Theorem 4.</u> For $f \in \mathcal{S}(R)$ the condition $P(f;.,.) \geqq 0$ is satisfied if and only if f is a Gabor function, i.e., if and only if the identity

$$\boxed{f(t) = ce^{-at^2+bt}}$$

holds for all $t \in R$ with complex numbers a, b, c such that Re a > 0.

Thus the Wigner quasi-probability distribution function $P(f;.,.)$ admits only for the special case of Gabor functions $f \in \mathcal{S}(\mathbf{R})$ an interpretation as a proper probability density. Moreover, if $P(f;.,.) \geq 0$ is $SO(2,\mathbf{R})$-invariant, we get $\operatorname{Im} a = 0$ and $b = 0$, i.e., the functions f and $P(f;.,.)$ are Gaussian densities on $\mathbf{R}$ and $\mathbf{R}^2$, respectively. This is in accordance with the fact that the Gaussian distribution function can be characterized as the sole probability distribution function which is invariant under rotations in the Euclidean plane $\mathbf{R}^2$. Further details belonging to this circle of ideas will be given in the forthcoming second part of this paper.

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AN APPROXIMATION THEOREM FOR COSINE OPERATOR FUNCTIONS

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Presented by J. Aczél, F.R.S.C.

Abstract: Operator valued solutions C of d'Alembert's functional equation are approximated by functions constructed by means of powers of the resolvent of the infinitesimal generator of C .

Let X denote a Banach space and let $B(X)$ be the algebra of bounded linear operators on X .

An operator cosine function on X is a strongly continuous function $C : \mathbb{R} \to B(X)$ fulfilling d'Alembert's functional equation

$$C(t+s) + C(t-s) = 2\,C(t)C(s)\ , \quad s,t \in \mathbb{R}\ ,$$
$$C(0) = I\ .$$

Then the infinitesimal generator A of C is a closed linear operator given by

$$A\,x := C''(0)x$$

with its natural domain $D(A)$. It is well known that $\overline{D(A)} = X$.

Now assume that C is an operator cosine function with $\|C(t)\| \leq 1$ for all $t \in \mathbb{R}$. Then all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ are in the resolvent set $\rho(A)$ of A and we have

$$(1) \qquad z\,R(z^2,A)x = \int_0^\infty e^{-zs}\,C(s)x\,ds\ , \qquad x \in X\ ,$$

and

$$(2) \qquad \left\|\frac{d^n}{dz^n}\,z\,R(z^2,A)\right\| \leq \frac{n!}{\operatorname{Re} z}\ , \qquad n \in \mathbb{N} \cup \{0\}\ ,$$

$R(z,A)$ being the resolvent operator of A .

Since the derivatives of $z\,R(z^2,A)$ play a crucial role in the theory of operator cosine functions ([2]) the following representations could be of interest not only in this paper.

Lemma: For $z \in \rho(A)$ put $g(z) := z\,R(z^2,A)$. Then for $m \in \mathbb{N} \cup \{0\}$

$$\frac{d^{2m}}{dz^{2m}} g(z) = (2m)!\left[\sum_{k=0}^{m} \binom{2m+1}{2k} z^{2m+1-2k} A^k\right] R(z^2,A)^{2m+1}$$

and

$$\frac{d^{2m+1}}{dz^{2m+1}} g(z) = -(2m+1)!\left[\sum_{k=0}^{m+1} \binom{2m+2}{2k} z^{2m+2-2k} A^k\right] R(z^2,A)^{2m+2} \quad .$$

Note that these are bounded operators by the fact that $A\,R(z^2,A)$ is bounded. The proof of the lemma is by induction and can be omitted. Heuristically these formulae can be found by differentiating

$$\frac{z}{z^2-a} = \frac{1}{2}\left(\frac{1}{z-\sqrt{a}} + \frac{1}{z+\sqrt{a}}\right),$$

though we do not assume the existence of $\sqrt{A}$.

Now define for $t \in \mathbb{R}$, $t \neq 0$, and $m \in \mathbb{N} \cup \{0\}$, $C_m(t) \in B(X)$ by

$$(3) \qquad C_m(t) := \left[\sum_{j=0}^{m} \binom{2m}{2j}\left(\frac{t}{2m}\right)^{2j} A^j\right]\left[\left(\frac{2m}{t}\right)^2 R\left(\left(\frac{2m}{t}\right)^2, A\right)\right]^{2m} \quad .$$

Theorem: $\lim_m C_m(t)x = C(t)x$ for all $x \in X$,

$$\|C_m(t)x - C(t)x\| \leq \frac{t^2}{\sqrt{m}} \|A x\| \quad \text{for all } x \in D(A),\ m \geq 2 \quad .$$

The use of $C_m(t)$ as an approximation is motivated by the following.

Remark: If B is the generator of a strongly continuous operator group T on X then $A := B^2$ is the generator of a cosine function C given by

$$(4) \qquad C(t) := \frac{1}{2}[T(t) + T(-t)] \quad .$$

On the other hand ([1])

$$(5) \qquad T(t)x = \lim\left[\frac{n}{t} R\left(\frac{n}{t},B\right)\right]^n x \quad , \qquad x \in X \quad .$$

Inserting (5) in (4), replacing B^2 by A and n by $2m$ we get

$$C(t)x = \lim_m C_m(t)x$$

with $C_m(t)$ defined as above. Since not all cosine functions can be represented by groups as in (4), this observation does not yield a proof of our theorem.

<u>Proof</u>: Since C_m and C are even functions we may restrict attention to $t>0$.
According to the lemma, $C_m(t)$ can be represented by

$$C_m(t)x = -\left(\frac{2m}{t}\right)^{2m} \frac{1}{(2m-1)!} \left(\frac{d^{2m-1}}{dz^{2m-1}} z\, R(z^2,A)\right)\Big|_{z=\frac{2m}{t}} x \quad ,$$

which gives by (2)

$$\| C_m(t) \| \leq 1$$

and by (1) after some partial integrations

$$C_m(t)x = \left(\frac{2m}{t}\right)^{2m} \frac{1}{(2m-1)!} \int_0^\infty e^{-\frac{2m}{t}s} s^{2m-1} C(s)x\, ds \quad .$$

On the other hand

$$C(t)x = \frac{2m}{t} \int_0^\infty e^{-\frac{2m}{t}s} C(t)x\, ds$$

$$= \left(\frac{2m}{t}\right)^{2m} \frac{1}{(2m-1)!} \int_0^\infty e^{-\frac{2m}{t}s} s^{2m-1} C(t)x\, ds \quad .$$

So

(6) $$C_m(t)x - C(t)x = \left(\frac{2m}{t}\right)^{2m} \frac{1}{(2m-1)!} \int_0^\infty e^{-\frac{2m}{t}s} s^{2m-1} [C(s)x - C(t)x]\, ds \quad .$$

To find a bound for $C(s)x - C(t)x$ assume that <u>$x \in D(A)$</u> .
Then, by [3], Prop. 2.2.,

$$C(s)x - C(t)x = \int_s^t S(u)A\,x\, du$$

where S denotes the sine function associated with C defined by

$$S(u)x = \int_0^u C(t)x\, dt, \quad x \in X, \quad u \in \mathbb{R} \quad .$$

So
$$\|C(s)x - C(t)x\| \leq |t-s| \sup_u \left\|\int_o^u C(t)\, A x\, dt\right\| \quad (u \text{ between } s \text{ and } t)$$
$$\leq |t-s| \cdot \max(s,t) \cdot \|A x\|$$
$$= \|A x\| \cdot \begin{cases} (t-s)t & \text{if } t \geq s \\ (s-t)s & \text{if } s \geq t \end{cases} .$$

Inserting these bounds in (6) and putting $c_m := \left(\frac{2m}{t}\right)^{2m} \frac{1}{(2m-1)!}$ we get

$$\|C_m(t)x - C(t)x\| \leq c_m \cdot \|A x\| \cdot \left(\int_o^t e^{-\frac{2m}{t}s} s^{2m-1} (t^2-st)ds + \int_t^\infty e^{-\frac{2m}{t}s} s^{2m-1} (s^2-st)ds\right)$$
$$= t^2 \|A x\| e^{-2m}\left[\frac{(2m)^{2m}}{(2m)!}\left(\frac{2m}{2m+1} + 1\right) + \frac{1}{2m} \sum_{j=o}^{2m+1} \frac{(2m)^j}{j!}\right]$$
$$\leq t^2 \|A x\| e^{-2m}\left[2 \cdot \frac{(2m)^{2m}}{(2m)!} + \frac{1}{2m} e^{2m}\right]$$
$$\leq t^2 \left(\frac{1}{\sqrt{\pi m}} + \frac{1}{2m}\right) \|A x\| , \quad m \in \mathbb{N}$$
$$\leq \frac{t^2}{\sqrt{m}} \|A x\| , \quad m \geq 2 .$$

Therefore, our assertions are true for $x \in D(A)$. The density of $D(A)$ and the uniform boundedness of $\|C_m(t)\|$ then gives the convergence of $C_m(t)x$ to $C(t)x$ for all $x \in X$, $t \neq 0$.

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ON THE FREE SPECTRA OF MAXIMAL CLONES

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Presented by G.A. Grätzer, F.R.S.C.

Let P_k be the algebra of finitary functions over the base set $E_k = \{0,\ldots,k-1\}$ and let D be a subclone of P_k. The free spectrum of D is defined as the sequence S_n (D), $n \geq 0$, where S_n (D) is the cardinality of the free algebra on n free generators in Var ($\langle E_k, D\rangle$). It is easy to see that $S_n(D)$ equals the number of all n-ary functions contained in D. The free spectrum is an important invariant of the clone D.

In [2] , J. Berman investigates the free spectra of clones in P_3 and determines $S_n(D)$ (or several elements of the sequence $S_n(D)$) for the most important subclones of P_3. In the present paper the free spectra of the maximal subclones will be investigated.

1. Maximal clones of monoton functions

In this case D = Pol (ρ) where ρ is a bounded partial ordering of E_k of determining it. The exact value of $S_n(D)$ is not known, and the problem seems to be very hard. The best result in this direction was given by V.B. Alekseev [1] :

$$S_n(D) = \exp\left(\frac{c}{\sqrt{2\pi}} \cdot \frac{k^n}{\sqrt{n}} (1+\varepsilon(n)) \right),$$

where $\varepsilon(n) \to 0$ if $n \to \infty$ and c is an effective constant depending on ρ. This result gives an asymptotical equation for $\log S_n(D)$.

2. Selfdual clones

In this case $D = \mathrm{Pol}\ (\rho)$ where ρ is the graph of a permutation π and π has $\frac{k}{p}$ cycles of the same prime length p.

For an arbitrary permutation-group on E_k we can define D_G as the set of all functions which are selfdual with respect to all $\pi \in G$.

Theorem 1.

Suppose that G acts semiregularly on E_k and has ℓ orbits. Then

$$S_n(D_G) = k^{\ell \cdot k^{n-1}} .$$

The proof is based on the extension properties of partial selfdual functions ([4]).

For a maximal selfdual clone we have $k = p.\ell$, $G = \langle \pi \rangle$ and G acts semiregularly and has $\frac{k}{p}$ orbits. Thus we obtain

$$S_n(D) = k^{\frac{k^n}{p}} .$$

For k=3 this gives $3^{3^{n-1}}$ in agreement with a result of J. Berman [3] .

3. Linear clones

Using representation theorem of I.G. Rosenberg [5] we obtain the following.

Theorem 2.

If D is a linear maximal clone in P_k and $k=p^m$, then

$$S_n(D) = k^{mn+1}.$$

4. Clones defined by an equivalence relation

D= Pol (ρ) where ρ is a nontrivial equivalence relation on E_k. Suppose that ρ has ℓ equivalence classes on E_k and the i-th class has d_i elements $1 \le i \le \ell$. Then by enumerating the induced partition on E_k^n we obtain the following.

Theorem 3.

$$S_n(D) = \prod_{\substack{j_1,\dots,j_\ell \in N \\ j_1+\dots+j_\ell = n}} \left(\sum_{i=1}^{\ell} d_i^{d_1^{j_1} \dots d_\ell^{j_\ell}} \right)^{\frac{n!}{j_1! \dots j_\ell!}}.$$

From this formula it's easy to deduce an answer to a question of J. Berman [3] : if k=3 and $\rho = \{ (0,0); (1,1); (2,2); (0,1); (1,0) \}$ then $\ell = 2$, $d_1=1$, $d_2=2$. Now for D= Pol (ρ)

$$S_n(D) = \prod_{i=0}^{n} (1 + 2^{2^i})^{\binom{n}{i}}.$$

5. Clones defined by a central relation

The problem in this case does not seem to be easy. The smallest non-unary central relation is the following binary relation on E_3

$\rho = \{(0,0); (1,1); (2,2); (0,1); (1,0); (0,2); (2,0)\}$

In this case S_o (Pol ρ)=3, S_1(Pol ρ)=17, S_2(Pol ρ)=1361 but S_n for n>2 not known. For this relation ρ we can only prove the following:

$$2^{3^n} \leq S_n(D) \leq 1361^{3^{n-2}} \approx 2^{1.16 \cdot 3^n}$$

We conjecture that $\lim_{n \to \infty} \frac{\log_2 S_n(D)}{3^n} = 1.$

6. Clones defined by a regular relation

Here we deal only with the Slupecki clone. The Slupecki clone consists of all essentially unary functions and of all non surjective functions. Thus

$$S_n(D) = k!n + \sum_{i=1}^{k-1} (-1)^{i+1} \binom{k}{i} (k-i)^{k^n}.$$

For k=3, this reduces to Berman's formula in [3] :

$$S_n(D) = 6n + 3.2^{3^n} - 3.$$

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ON THE FUNCTIONAL EQUATION $\phi(x) = \phi(px) + \phi(qx+p)$

Norbert Steinmetz

Presented by F.V. Atkinson, F.R.S.C.

Abstract. Let $p,q > 0$ be fixed real constants with $p+q = 1$. It is shown that any absolutely continuous solution of the functional equation $\phi(x) = \phi(px) + \phi(qx+p)$, $0 \le x \le 1$, is linear.
On the other hand, there exist continuous nonlinear solutions.

1. The functional equation

$$\psi(x) = \frac{1}{2}\psi\left(\frac{x}{2}\right) + \frac{1}{2}\psi\left(\frac{x+1}{2}\right) \tag{1}$$

plays some role in deducing the duplication formula of the Gamma function (Artin [1], p.33) and the resolution of the cotangent into partial fractions (Mohr [3], Walter [4]) by real variable methods.

Equation (1) is a special case of

$$\psi(x) = p\psi(px) + q\psi(qx+p) , \quad 0 \le x \le 1 , \tag{2}$$

(with fixed $p,q > 0$, $p+q = 1$) which may be derived from Cauchy's equation

$$f(x+y) = f(x) + f(y) , \quad x \text{ and } y \text{ real} , \tag{3}$$

as follows: Setting $x = pt-1$, $y = qt+1$, equation (3) changes into the one variable functional equation

$$f(t) = f(pt-1) + f(qt+1) . \tag{4}$$

Defining $\phi(x) = f\left(\frac{x-p}{pq}\right)$ for x real, this gives

$$\phi(x) = \phi(px) + \phi(qx+p) , \quad p,q > 0 , \quad p+q = 1 , \tag{5}$$

and thus equation (2) by differentiation.

2. One might expect that all continuous solutions of (5) are linear. This is not the case as is shown by the following example. Let ϕ_o be an arbitrary continuous solution of

$$(6) \qquad \phi_o(px) + \phi_o(qx+p) = 0\,, \quad 0 \le x \le 1\,,$$

such that $\phi_o(0) = \phi_o(p) = 0$ (ϕ_o may be prescribed in $0 < x < p$), and let ϕ_n be defined inductively by

$$(7) \qquad \phi_n(x) = \begin{cases} \frac{1}{2}\phi_{n-1}\left(\frac{x}{p}\right)\,, & 0 \le x \le p\,, \\ \frac{1}{2}\phi_{n-1}\left(\frac{x-p}{q}\right)\,, & p < x \le 1\,. \end{cases}$$

Then

$$(8) \qquad \phi(x) = \sum_{n=0}^{\infty} \phi_n(x)\,, \quad 0 \le x \le 1\,,$$

is a continuous solution of (5), which is nonlinear, if, e.g., $\phi_o(p^2) \neq 0$. In the case $p = q = \frac{1}{2}$, a much simpler example is

$$(9) \qquad \phi(x) = \sum_{n=1}^{\infty} 2^{-n} \sin(2^n \pi x)$$

(Artin [1], p.35).

3. We shall prove:

Theorem. Any absolutely continuous solution of (5), $0 \le x \le 1$, is linear: $\phi(x) = c(x-p)$.

Remark. In the case $p = q = \frac{1}{2}$, Mohr [3] proved the Theorem for solutions with Riemann-integrable first derivative.

The proof of the Theorem is based on a lemma dealing with equation (2).

Lemma. Let ψ_o be Lebesgue-integrable over $[0,1]$ and let (ψ_n) be defined inductively by

(10) $$\psi_n(x) = p\psi_{n-1}(px) + q\psi_{n-1}(qx+p) .$$

Then,

(11) $$\int_0^1 |\psi_n(x)-c|dx \to 0 \quad \text{as } n\to\infty ,$$

where

(12) $$c = \int_0^1 \psi_o(x)dx ,$$

and even

(13) $$\psi_n(x) \to c \quad \text{as } n\to\infty ,$$

uniformly in $0\le x\le 1$, if ψ_o is continuous.

Remark. In the case $p = q = \frac{1}{2}$, a somewhat stronger result of Jessen [2] could be used, which states that $\psi_n(x) = 2^{-n} \sum_{k=0}^{2^n-1} \psi_o\left(\frac{x+k}{2^n}\right)$ tends to $\int_0^1 \psi_o(t)dt$ almost everywhere in $0\le x\le 1$ as $n\to\infty$.

4. To prove the lemma we will first assume that ψ_o is continuous in $0\le x\le 1$. If

(14) $$\omega_n(h) = \max \{|\psi_n(x)-\psi_n(y)|: 0\le x,y\le 1, |x-y|\le h\}$$

denotes the smallest modulus of continuity of ψ_n, then from (10) easily follows

(15) $$\omega_n(h) \le p\omega_{n-1}(ph) + q\omega_{n-1}(qh) \le \omega_{n-1}(rh) ,$$

where $r = \max(p,q) < 1$. Thus, by mathematical induction,

(16) $$\omega_n(h) \le \omega_o(r^n h) \le \omega_o(r^n) ,$$

which tends to zero as $n \to \infty$. Especially,

(17) $$\psi_n(x)-\psi_n(0) \to 0 , \quad \text{uniformly as } n \to \infty ,$$

and so, since, by (10), $c = \int_0^1 \psi_n(x)dx$ is independent of n,

(18) $$\psi_n(x) \to c , \quad \text{uniformly as } n \to \infty .$$

To prove the main part, given $\varepsilon > 0$, we choose a continuous function θ_o in $0 \le x \le 1$, such that

(19) $$\int_0^1 |\psi_o(x)-\theta_o(x)|dx < \varepsilon$$

and

(20) $$\int_0^1 \theta_o(x)dx = c ,$$

and define the sequence (θ_n) in the same way as (ψ_n). Since $\int_0^1 |\psi_n(x)-\theta_n(x)|dx$ is nonincreasing as n increases, (19) gives together with $\theta_n(x) \to c$, uniformly in $0 \le x \le 1$,

(21) $$\int_0^1 |\psi_n(x)-c|dx < \varepsilon$$

for sufficiently large n, and the lemma is proved.

5. To prove our Theorem, we assume that ϕ is an absolutely continuous solution of (5) in $0 \le x \le 1$. Then $\psi = \phi'$ exists and is a Lebesgue-integrable solution of (2) almost everywhere. The preceding lemma shows that ψ is necessarily constant almost everywhere (set $\psi_o = \psi$). Thus, ϕ is linear and, since $\phi(p) = 0$ (set $x = 0$ in (5)), $\phi(x) = c(x-p)$. This proves the theorem.

Finally, I would like to thank Professor P. Volkmann for stimulating my interest in the topics of this note.

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UNE PROPRIETE DES COMMUTATEURS DANS LES GROUPES LOCAUX

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Presenté par P. Ribenboim, F.R.S.C.

Résumé

Nous montrons qu'une certaine propriété des éléments d'un groupe local se transmet aux racines. Nous en déduisons ensuite quelques conséquences.

Si π est un ensemble de premiers, on dit qu'un groupe G est π-local si pour tout premier p, $p \notin \pi$, l'application $x \to x^p$ est une bijection de G sur lui-même.

On désigne par Z^cG les termes de la suite centrale ascendante de G; en particulier, Z^0G est le sous-groupe trivial de G et Z^1G est le centre de G.

Le commutateur $[x,y]$ représente, dans la suite du texte, $xyx^{-1}y^{-1}$.

Soit G un groupe et H un sous-groupe de G. Le π'-isolateur de H dans G est défini (voir Ribenboim [4]) de la façon suivante:

$I_0(G,H)_{\pi'} = H$.

$I_1(G,H)_{\pi'}$ est le sous-groupe de G engendré par l'ensemble des x dans G tels que $x^n \epsilon H$ pour un certain n premier à π.

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$$I_k(G,H)_{\pi'} = I_1(G,I_{k-1}(G,H)_{\pi'})_{\pi'} .$$
$$I_\infty(G,H)_{\pi'} = \bigcup_{k=0}^{\infty} I_k(G,H)_{\pi'} .$$

Puisque par la suite le groupe G et l'ensemble π de premiers seront fixes, nous écrirons, afin de simplifier la notation I_kH au lieu de $I_k(G,H)_{\pi'}$, $(k=0,1,2,\ldots,\infty)$.

<u>Proposition</u>: Soit G un groupe π-local et $H \leq G$. Si $x\in I_kH$, $y^n\in I_{k-1}H$ et $[x,y^n]\in Z^cI_{k-1}H$ pour un certain n premier à π, alors $[x,y]\in Z^cI_kH$.

<u>Démonstration</u>: Il suffit de prouver le résultat pour $k = 1$, les autres cas résultant de la définition de I_kH $(k=2,3,\ldots,\infty)$ et du fait que pour tout sous-groupe H d'un groupe π-local, on a $H\cap Z^cI_kH = Z^cH$ pour $c = 0,1,2,\ldots$ et $k = 0,1,2,\ldots,\infty$ (voir à ce sujet [3] et [5]) .

La preuve pour $k = 1$ se fait par induction sur c. Si $c = 0$, le résultat est trivial puisque $y^n = xy^nx^{-1} = (xyx^{-1})^n$ entraîne $y = xyx^{-1}$ dans G qui est π-local.

Supposons la proposition vraie pour $i = 0,1,\ldots,c-1$. Nous montrons d'abord que si $u^n\in Z^cH$, alors $u\in Z^cI_1H$ pour n premier à π. En effet, si $u^n\in Z^cH$, alors $[u^n,z]\in Z^{c-1}H$ pour tout $z\in H$. D'après l'hypothèse d'induction, $[u,z]\in Z^{c-1}I_1H$ pour tout $z\in H$. En particulier, si $w\in I_1H$ est tel que $w^m\in H$ pour un certain m premier à π, on a $[u,w^m]\in Z^{c-1}I_1H$ et d'après l'hypothèse d'induction $[u,w]\in Z^{c-1}I_2H$. Comme $[u,w]\in I_1H$, il s'ensuit que $[u,w]\epsilon Z^{c-1}I_1H$. Tous les éléments de I_1H étant des produits de tels w, il est clair que $[u,y]\epsilon Z^{c-1}I_1H$ pour tout $y\epsilon I_1H$ et ainsi $u\epsilon Z^cI_1H$.

Supposons maintenant que $x\epsilon I_1H$, $y^n\epsilon H$ et $[x,y^n]\epsilon Z^cH$. On a:

$$(xyx^{-1})^n = xy^nx^{-1} = t_1y^n \text{ , pour un } t_1\epsilon Z^cH$$
$$= u_1^ny^n \text{ , puisque G est } \pi\text{-local}$$
$$= t_2(u_1y)^n \text{ , où } t_2\epsilon Z^{c-1}I_1H \text{ puisque } u_1\epsilon Z^cI_1H$$
d'après l'hypothèse d'induction et ce qui précède
$$= u_2^n(u_1y)^n \text{ , puisque G est } \pi\text{-local}$$
$$= t_3(u_2u_1y)^n \text{ , où } t_3\epsilon Z^{c-2}I_2H \text{ puisque } u_2\epsilon Z^{c-1}I_2H$$
$$\vdots$$
$$= (u_cu_{c-1}\ldots u_2u_1y)^n \text{ , où chaque } u_i\epsilon Z^{c-i+1}I_iH \text{ .}$$

Remarquons qu'il faut utiliser alternativement le fait que G est π-local et l'hypothèse d'induction qui entraîne que si la proposition est vraie pour $i = 0,1,\ldots,c-1$, alors $u^n\epsilon Z^cH$ implique $u\epsilon Z^cI_1H$ pour tous les sous-groupes de G. Il s'ensuit maintenant que $xyx^{-1} = u_cu_{c-1}\ldots u_2u_1y$ puisque G est π-local et il reste à montrer que le produit $u = u_cu_{c-1}\ldots u_2u_1\epsilon Z^cI_1H$. Remarquons d'abord que $u_c = xyx^{-1}y^{-1}u_1^{-1}u_2^{-1}\ldots u_{c-1}^{-1}$ appartient à $I_{c-1}H\cap Z^1I_cH = Z^1I_{c-1}H$; de même, $u_cu_{c-1} = xyx^{-1}y^{-1}u_1^{-1}\ldots u_{c-2}^{-1}$ appartient à $I_{c-2}H\cap Z^2I_{c-1}H = Z^2I_{c-2}H$ et ainsi de suite. Il est donc possible de conclure que $u = xyx^{-1}y^{-1}\epsilon Z^cI_1H$ et ainsi $[x,y]\epsilon Z^cI_1H$.

<u>Remarques</u>: Bien que le résultat précédent soit purement technique, il peut servir de point de départ pour une nouvelle démonstration de quelques résultats intéressants. Par exemple, il en découle par une induction facile sur la classe de nilpotence que si G est π-local et si N est un sous-groupe nilpotent de G de classe $\leq c$, alors $I_\infty N$ est aussi nilpotent de classe $\leq c$; de même si N est localement nilpotent (alors $I_\infty N$ est localement nilpotent (voir à ce sujet [1], cor. 15.1, p. 233).

Nous remarquons de plus que cela entraîne que la localisation d'un groupe nilpotent N, en tant que groupe (voir [4]), coincide avec sa localisation en tant que groupe nilpotent (voir [2]). En effet, si $\ell_\pi : N \to \hat{N}_\pi$ est la localisation de N en tant que groupe, $\ell_\pi(N)$ est nilpotent et d'après [4], $\hat{N}_\pi = I_\infty \ell_\pi(N)$; il découle donc de la remarque précédente que $\hat{N}_\pi$ est nilpotent, ce qui suffit pour montrer que les deux théories de la localisation coincident pour les groupes nilpotent. Cette question avait été soulevée par Warfield ([6],p. 73).

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GROUPOID VARIETIES SUCH THAT EVERY 2-GENERATED GROUPOID IN THE VARIETY HAS FIXED FINITE ORDER

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Abstract.

A connection between finite projective planes and a class of qroupoid varieties is obtained. This gives some information on the possibility of embbeding certain partial finite planes in complete finite planes.

1. Introduction.

C. C. Lindner [3] has shown that a quasigroup variety defined by a collection of 2-variable identities together with $x^2 = x$ and having the property that every 2-generated quasigroup in the variety has order n has the finite embeddability property.

In this paper we prove two theorems.

Theorem 1. Let $\mathcal{V}$ be a non-trivial variety of idempotent groupoids such that every 2-generated groupoid in the variety has finite order n. Then n is the order of a finite projective plane.

Theorem 2. Let $n = p^r$ where p is a prime. Then there is a variety of groupoids defined by a finite number of 2-variable identities and $x^2 = x$, such that every 2-generated groupoid in the variety has order n.

2. Proof of the Theorems.

Proof of Theorem 1. Let G be a non-trivial groupoid in the variety $\mathcal{V}$. Let H be the subgroupoid of G generated by two of its elements. Then $|H| = n$. Consider $H \times H$. This belongs to $\mathcal{V}$ and $|H \times H| = n^2$. Form a B.I.B.D. whose points are the elements of $H \times H$ and such that for every pair of points, the block containing those points are the elements of the subgroupoid generated by those points. This design has parameters $v = n^2, k = n, \lambda = 1, b = n^2 + n, r = n + 1$. By Theorem 12.3.3 page 176 in [2], it is shown that this design is an affine plane of order n. Hence a projective plane of order n exists.

Corollary. By the Bruck-Ryser Theorem [1], the variety does not exist if $n \equiv 1, 2 \mod 4$ and n is not expressible as the sum of squares of two integers.

Proof of Theorem 2. Let $n = p^r$ and let λ be a primitive generating element of $GF(p^r)$. Define a quasigroup $(G, *)$ by $a * b = \lambda a + (1 - \lambda)b$ for $a, b \in GF(p^r)$. A direct calculation shows that the automorphism group of $(G, *)$ is doubly-transitive on its elements. (Note the mappings $x \to u x + v$; $u, v \in GF(n)$, $u \neq 0$, are automorphisms.) Now define $W_0(0, 1) = 0$, $W_1(0, 1) = 1$, $W_2(0, 1) = 1 * 0$, and $W_i(0, 1) = W_{i-1}(0, 1) * 0$ for $i \geq 2$. Then $W_i(0, 1) = \lambda^{i-1}$ for $i = 2, 3, \cdots p^r - 1$. Since λ is primitive, the elements of G are $W_0(0, 1), W_1(0, 1), \ldots, W_{p^r-1}(0,1)$ and hence 0 and 1 generate G. From the multiplication table of G, one obtains $W_i(0, 1) * W_j(0, 1) = W_k(0, 1)$ where $k = k(i, j)$. Now since the automorphism group of $(G, *)$ is doubly transitive, it follows that $W_i(x, y) * W_j(x, y) = W_k(x, y)$ is an identity. It is clear by induction, that any word $W(x, y) = W_i(x, y)$ for some value of i. Now take the collection of identities $W_i(x, y) * W_j(x, y) = W_k(x, y)$

$i = 0, 1, \cdots, p^r - 1$ and $j = 0, 1, \cdots, p^r - 1$ and $k = k(i, j)$. These define a variety of groupoids such that every 2-generated groupoid in the variety is isomorphic to the constructed groupoid.

Remarks.

1. We may get other constructions by replacing $GF(p^r)$ by any near-field of order p^r and defining $a * b = a + \lambda(b - a)$ where λ generates the near-field.

2. If we put on the condition that a 2-generated groupoid of the variety has a doubly transitive group of automorphisms the 2-generated groupoids must be of the type mentioned in remark 1.(See S. K. Stein [4].)

3. The definition given here uses $n^2 - n + 1$ identities (i.e. $W_i(x, y) * W_j(x, y) = W_k(x, y)$ $i = 0, 1, \cdots, n-1$; $j = 0, 1, \cdots, n-1$, $i \neq j$, and $x^2 = x$). It is conjectured that in general 2 identities suffice. This conjecture has been proved for n a prime and for $n = 4, 8, 9$ in unpublished (as yet) work by N. S. Mendelsohn and D. M. Johnson.

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THE STRUCTURE OF LOCALLY FINITE GROUPS WITH MIN-P

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Presented by P. Ribenboim, F.R.S.C.

The recent classification of the finite simple groups will have a major impact on mathematics. Within group theory this impact is already quite spectacular. In chapter 4 of [2] and in [1] some consequences of the classification theorem to the structure of simple locally finite groups were outlined. We state here one such consequence:

Theorem 1: If for the prime p the p-subgroups of the simple locally finite group G do not generate the variety of all groups then G is a linear group.

If, in the situation of Theorem 1, the group G is infinite it must be countable and possess a local system Σ consisting of finite Chevalley groups of fixed type (possibly twisted) and rank (cf. [2], 4.6). Very recently S. Thomas [4] has obtained that such a group G is indeed a Chevalley group of that type (possibly twisted) and rank over some locally finite field. (This result has also been announced by Borovik, Omsk and by G. Shute, East Lansing, Michigan).

In [5] Wehrfritz has given a theory of locally finite groups satisfying the minimum condition for p-subgroups (min-p) for the prime p (cf. [2], Chapter 3). From this it follows that the p-subgroups of a locally finite group with min-p generate a variety of soluble groups. If such a group does not involve any infinite simple group then [2], 3.29 shows that the factor group $G/O_{p'}G$ has an abelian normal p-subgroup of finite index. ($O_{p'}G$ is the largest normal subgroup of G not containing any element of order p.) Using Theorem 1 we can now extend this structural result to all locally finite groups with min-p.

Denote by S_pG the maximal normal locally p-soluble subgroup of the locally finite group G.

<u>Theorem 2: If for the locally finite group G with min-p one has $G \neq S_pG$, then the factor group $H = G/S_pG$ has minimal normal subgroups and their join, the socle $\mathit{soc}\, H$ of H, is a direct product of finitely many simple groups. The group $H/\mathit{soc}\, H$ has an abelian normal subgroup of finite rank and finite index.</u>

<u>Proof</u>: That H has mininal normal subgroup generated by their p-elements follows directly from the properties of the p-size of a locally finite group with min-p, which directly generalizes the p-part of the order of a finite group. Let $\mathit{soc}\, H$ be the join of these minimal normal subgroups of H. Since all these also satisfy min-p, they are themselves direct products of finitely many simple groups. As H has min-p, $\mathit{soc}\, H$ is a direct product of finitely many minimal normal subgroups and thus a direct product of finitely many simple groups S_i, i=1,...,k, satisfying min-p. The centralizer $C_H \mathit{soc}\, H$ intersects $\mathit{soc}\, H$ trivially, thus will be locally p-soluble; hence $C_H \mathit{soc}\, H = \langle 1 \rangle$. Thus H is a subgroup of the automorphism group of $\mathit{soc}\, H$. This is essentially the direct product of the groups Aut S_i extended by a subgroup of the symmetric group of degree k permuting those of the S_i which are isomorphic.

For finite S_i the group Aut S_i/S_i of outer automorphisms is soluble and has a cyclic normal subgroup of index bounded in terms of the Chevally structure of S_i . If S_i is infinite it is linear by Theorem 1 by virtue of min-p . Now, either arguing locally or using S. Thomas' result [4], one obtains from [3] that Aut S_i/S_i is soluble, a finite extension of a pro-cyclic group. As the torsian subgroup of a pro-cyclic group is locally cyclic, one now has, that H has a normal subgroup H_0 of index dividing k! such that each of the simple groups S_i is normal in H . One has $H_0 \subseteq \prod_{i=1}^{k} \text{Aut } S_i$, and thus $H_0/soc\ H$ has an abelian subgroup of finite index (depending on the Chevalley structures of the S_i) of rank $\leq k$.

Remark: D. Winter [6] has pointed out that a simple locally finite group that is linear must be countable (cf. [2], 1.L.2). Thus one obtains in the preceding proof that H is countable. Combining this with the structural result for S_pG one obtains:

If the locally finite group G satisfies the minimum condition for p-subgroups then the factor group $G/O_{p'}G$ is countable.

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ON THE PRODUCT OF TWO FERMAT CURVES OVER FINITE FIELDS

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Presented by P. Ribenboim, F.R.S.C.

1. Let $k = \mathbb{F}_q$ be the finite field with $q = p^f$ elements where p is a prime > 2. Let C_m denote the Fermat curve over k defined by the projective equation

$$C_m : X_0^m + X_1^m + X_2^m = 0, \quad m \geq 3, \quad (m,p) = 1.$$

Put $X_m = C_m \times C_m$. Then X_m defines a projective smooth algebraic surface over k of geometric genus $p_g = g^2$ where g is the genus of C_m $(g = \frac{(m-1)(m-2)}{2})$. The zeta-function of X_m has the form

$$Z(X_m,T) = \prod_{i=0}^{4} P_i(X_m,T)^{(-1)^{i+1}}$$

where $P_i(X_m,T)$ is a polynomial with integer coefficients of degree B_i (the ith Betti number of X_m) whose reciprocal roots have the complex absolute value $q^{i/2}$ (Deligne [1]).

The zeta-function $Z(X_m,T)$ is one of the most important arithmetical invariants of X_m, and almost all arithmetical properties of X_m should be derived from it. Indeed, our purpose here is to determine explicitly

(A) the Néron-Severi group $NS_k(X_m)$ of X_m,

(A') the isogeny structure of the Jacobian variety $J(C_m)$ of C_m, and

(B) the intersection matrix of $NS_k(X_m)$

by making use of p-adic analysis of the polynomial $P_2(X_m,T)$.

For the detailed and expanded version of this note, see Yui [5].

2. The zeta-function $Z(X_m,T)$ of X_m can completely be determined from the zeta-function $Z(C_m,T)$ of C_m. First we recall the results of Weil [3,4] expressing $Z(C_m,T)$ in terms of Jacobi sums:

$$Z(C_m,T) = \frac{f(T)}{(1-T)(1-qT)} \quad \text{with} \quad f(T) = \prod_{\underline{a} \in \mathfrak{A}_m} (1 - j(\underline{a})T)$$

where $\underline{a}$ runs over the set

$$\mathfrak{A}_m = \left\{ \underline{a} = (a_0,a_1,a_2) \;\middle|\; \begin{array}{l} a_i \in (\mathbb{Z}/m\mathbb{Z}),\ a_i \not\equiv 0 \pmod m \\ a_0 + a_1 + a_2 \equiv 0 \pmod m \end{array} \right\}$$

and $j(\underline{a})$ denotes the Jacobi sum

$$j(\underline{a}) = \sum_{\substack{1+u_1+u_2=0 \\ u_i \in k^\times}} \chi(u_1)^{a_1} \chi(u_2)^{a_2}$$

χ being a certain fixed multiplicative character of order m of the multiplicative group $k^\times$ of k. Each Jacobi sum $j(\underline{a})$ is an algebraic integer in the mth cyclotomic field $K_m = \mathbb{Q}(e^{2\pi i/m})$ with the complex absolute value $q^{\frac{1}{2}}$. (Here we choose the character χ as follows: Selecting a generator u for $k^\times$ once and for all, let $\chi(u) = \omega(u)^{q-1/m}$ where $\omega(u)$ is the Teichmüller character of $k^\times$. With this choice for χ We assume that $q = p^f$ is the least power of p satisfying the congruence $q = p^f \equiv 1 \pmod m$. View this integer f as the order of p in K_m.)

LEMMA. *The zeta-function* $Z(X_m,T)$ *is of the form*

$$Z(X_m,T) = \frac{[f(T)]^2[f(qT)]^2}{(1-T)\,P_2(X_m,T)\,(1-q^2T)}$$

<u>with</u>

$$P_2(X_m,T) = (1 - qT)^2 \prod_{(\underline{a},\underline{b}) \in \mathcal{B}_m} (1 - j(\underline{a})j(\underline{b})T)$$

<u>where</u> $\mathcal{B}_m = \mathfrak{A}_m \times \mathfrak{A}_m$.

3. Let $G = \mathrm{Gal}(K_m/\mathbf{Q}) \tilde{=} (\mathbb{Z}/m\mathbb{Z})^\times$ be the Galois group of K_m over $\mathbf{Q}$, and let H be the cyclic subgroup of G generated by $p \bmod m$, i.e, $H = \{ p^i \bmod m \mid 0 \leq i < f \}$. Define the arithmetical function

$$A_H : \mathfrak{A}_m \longrightarrow \mathbb{Z} \quad \text{by} \quad A_H(\underline{a}) = \sum_{t \in H} \left[\sum_{i=1}^{2} \left\langle \frac{ta_i}{m} \right\rangle \right]$$

where $[\lambda]$ denotes the greatest integer function and $\langle\lambda\rangle = \lambda - [\lambda]$ for any $\lambda \in \mathbf{Q}$. $A_H(\underline{a})$ is integral with the range $[0,f]$ and is invariant under permutation of components of $\underline{a}$.

<u>KEY LEMMA.</u> <u>For any integer</u> $d \in [0,2f]$, <u>let</u>

$$\mathcal{B}_m^d = \{ (\underline{a},\underline{b}) \in \mathcal{B}_m \mid A_H(\underline{a}) + A_H(\underline{b}) = d \}.$$

<u>Then</u>

$$\mathcal{B}_m = \bigcup_{d \in [0,f)} (\mathcal{B}_m^d \cup \mathcal{B}_m^{2f-d}) \cup \mathcal{B}_m^f \quad \underline{\text{(disjoint)}}$$

<u>with</u> <u>Card</u>$(\mathcal{B}_m^d)$ = <u>Card</u>$(\mathcal{B}_m^{2f-d})$ <u>for any</u> $d \in [0,f]$.

<u>Consequently</u>,

$$B_2 = 2 + 4g^2 = 2(\sum_{d \in [0,f)} \mathrm{Card}(\mathcal{B}_m^d)) + \mathrm{Card}(\mathcal{B}_m^f).$$

4. Now consider the Néron-Severi group $NS_k(X_m)$ of X_m, that is, the group of divisor classes defined over k on X_m, modulo algebraic equivalence. $NS_k(X_m)$ is a finitely generated abelian group, and furthermore, is free by a theorem

of Lefschetz. Thus, $NS_k(X_m) \cong \mathbb{Z}^{\rho(X_m)}$ and the rank $\rho(X_m)$ is the Picard number of X_m. As the Tate conjecture holds true for X_m (Tate [2]), $\rho(X_m)$ can be explicitly determined.

THEOREM. *With* G *and* $\mathcal{B}_m^f$ *as above, let*

$$[\mathcal{B}_m^f]^G = \{(\underline{a},\underline{b}) \in \mathcal{B}_m^f \mid A_H(t\underline{a}) + A_H(t\underline{b}) = f \text{ for all } t \in G\}.$$

Then

$$\rho(X_m) = 2 + \mathrm{Card}([\mathcal{B}_m^f]^G).$$

In particular, $\rho(X_m) \equiv 0 \pmod 2$.

EXAMPLES.

m	p mod m	f	$\rho(X_m)$	m	p mod m	f	$\rho(X_m)$
7	1	1	128	16	3,11	4	10118
	2,4	3	236		5,13	4	13988
	3,5,6	6	902		7	2	7238
16	1	1	3044		9	2	5924
					15	2	44102

The upper bound for $\rho(X_m)$ is $B_2 = 2 + 4g^2$. When and how often is $\rho(X_m)$ equal to B_2? Recalling that f is the order of p in K_m, we have the following results.

PROPOSITION. (a) $\rho(X_m) = B_2$, *if and only if,* $2|f$ *and* $p^{f/2} + 1 \equiv 0 \pmod m$.

(b) *Fixing* m, *define*

$$\delta_s(m) = \frac{\{\, p : \text{prime} \mid (p,m) = 1 \text{ and } \rho(X_m) = B_2\}}{\{\, p : \text{prime} \mid (p,m) = 1 \,\}} .$$

Then

$$\delta_s(m) = \begin{cases} \dfrac{1}{2^d} & \text{if } 4|m \\[2ex] \dfrac{2^{rc} - 1}{2^d(2^r-1)} & \text{if } 4\nmid m \end{cases}$$

where 2^d *is the highest power of* 2 *dividing the Euler function* $\phi(m)$, *and when* $4\nmid m$ r *denotes the number of odd* primes p_i dividing m and 2^c is the highest power of 2 dividing all $p_i - 1$.

5. Let $J(C_m)$ be the Jacobian variety of C_m defined over k. Then the endomorphism ring $\mathrm{End}_k(J(C_m))$ of $J(C_m)$ is a finitely generated, torsion-free $\mathbb{Z}$-module of rank $r(f,f)$ (Tate [2]). Furthermore, there is a relation between $r(f,f)$ and $\rho(X_m)$, due to Tate [2]:

$$(*) \qquad \rho(X_m) = 2 + r(f,f).$$

Combining Key Lemma with the equality $(*)$, we obtain the inequality corresponding to the Igusa inequality for $\rho(X_m)$ in characteristic 0 $(\rho(X_m) \leq B_2 - 2p_g = 2 + 2g^2)$.

PROPOSITION. Suppose that $\rho(X_m) \neq B_2$ $(= 2 + 4g^2)$. Then

$$2g + 2 \leq \rho(X_m) \leq 2 + 4g^2 - 2 \sum_{d \in [0,f)} \mathrm{Card}(\mathcal{B}_m^d).$$

With explicit knowledge of $\rho(X_m)$ (and hence of $r(f,f)$) at our disposal, we can determine, in some cases, the structure of $J(C_m)$, up to isogeny.

EXAMPLE. Let $m = 7$ and let $p \equiv 1 \pmod 7$, then $J(C_7)$ is isogenous to $E^6 \times A^3$ where E is an ordinary elliptic curve and A is a simple abelian variety of dimension 3, both defined over $k = \mathbb{F}_p$.

6. Finally we shall compute the intersection matrix of $NS_k(X_m)$.

THEOREM. Let $D = C_m \times o$, $D' = o \times C_m$ (o is the identity of $J(C_m)$, and for each generator $\gamma_i \in \mathrm{End}_k(J(C_m))$ $(1 \leq i \leq r(f,f) = \rho(X_m) - 2)$, let

$$\Gamma_{\gamma_i} = \{ (x, \gamma_i(x)) \mid x \in J(C_m) \}.$$

Put

$$D_i = \Gamma_{\gamma_i} - D - \deg(\gamma_i)D' \quad \text{for} \quad 1 \le i \le r(f,f).$$

Then D, D' *and* D_i $(1 \le i \le r(f,f))$ *form a basis for* $NS_k(X_m)$ *and the intersection matrix of* $NS_k(X_m)$ *is of the form*

$$r(f,f)\left\{\overbrace{\left(\begin{array}{c|cc} (D_i . D_j) & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & 1 \\ & 1 & 0 \end{array}\right)}^{r(f,f)}\right.$$

with

$$(D_i . D_j) = \deg(\gamma_i - \gamma_j) - \deg(\gamma_i) - \deg(\gamma_j)$$

for $1 \le i \le j \le r(f,f)$.

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