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ON GROUPS WITH PROFINITE ARITHMETIC

S.K. Grosser and W.N. Herfort

Presented by P. Ribenboim, F.R.S.C.

The well known example by Adian-Novikov [1] invalidates both the Burnside conjecture and a conjecture by O.Ju. Schmidt: "There is an infinite abelian subgroup in every infinite group". Thus the Schmidt conjecture defines a restrictive condition. One is far from knowing all groups that satisfy it; it is, however, satisfied (e.g.) for locally finite groups (Theorem of Hall-Kulatilaka [16]). For locally compact [IN]-groups (see [4]) the question reduces to one for compact groups. Even for compact groups the general answer is unknown; in this case, however, the stronger condition that non-trivial elements possess infinite centralizers effects a reduction to a problem for pro-p-groups, as follow: If G is a compact group in which all non-trivial elements have finite centralizers then G is either finite or is a pro p-groups for an odd prime p. (The question of whether or not an infinite pro-p-group with the above centralizer-condition actually exists is still open).

In general, the method of topologizing problems of the discrete theory provides additional perspective and allows for the development of a more comprehensive theory in which techniques of Lie-theory and of the discrete theory can profitably be combined. Thus the topological versions of finiteness conditions (see especially [16, ch. 4]) have given rise to the rather extensive theory of compactness conditions (see [13]). In this sense, one may formalize the conditions refered to above as follows:

[AF] = class of discrete groups G whose abelian subgroups are finite.

[CF] = class of discrete groups G with finite centralizers $C_G(x)$ of the elements $x \neq e$.

$[AF]^-$ = class of locally compact groups G whose closed abelian subgroups are compact.

$[CF]^-$ = class of locally compact groups G with compact centralizers $C_G(x)$ of the elements $x \neq e$.

One has $[CF] \subset [AF]$ and $[CF]^- \subset [AF]^-$. Let [K] be the class of compact groups. The two questions of whether or not $[CF] \cap [K] = [AF] \cap [K]$ = class of finite groups, are still open [Mc M]; they are indeed closely related to the Burnside-problem for compact groups. (See, however, the result quoted above).

For the formulation of our results we need the classes [IN] and [SIN] of locally compact groups G possessing, respectively, a compact G-invariant neighborhood of e and a fundamental system of such neighborhoods; [M] denotes the class of locally compact groups G all of whose irreducible Hilbert space respresentations are finite-dimensional ("Moore-groups" [11],[18]).

Theorem 1 Assume $G \in [IN] \cap [AF]^-$. Then there is a sequence $K \rightarrow G \twoheadrightarrow D$, in which K is compact and open, and D is a discrete torsion group not possessing any locally cyclic subgroup. If $D \notin [AF]$ then D contains the weak direct product, over an infinite index set, of cyclic groups C_p, for a fixed prime p.

For a proof one first employs the structure theorem for [IN]-groups. The crucial step in the proof then is to show that locally cyclic groups in D would give rise to non-compact closed abelian subgroups of G; this demonstration is accomplished by induction on certain families of subgroups.

<u>Theorem 2</u> For a Lie-group G the following holds:
$G \in [AF]^- \Leftrightarrow G_0$ is compact and $G/G_0 \in [AF]$.

The proof of ($\Rightarrow$) employs the Lie-theory of compact groups together with the Hall-Kulatilaka-Theorem [8] in order to construct a subgroup H of G and a sequence $T \rightarrowtail H \twoheadrightarrow V$, where T is a maximal torus of G_0, V is an infinite-dimensional GF(p)-vector space (p a prime) and $H' \leq T \leq Z(H)$. Choosing H to be non-compact if $G \notin [AF]^-$ one arrives at a contradiction by means of arguments from linear algebra applied to the bilinear form $(\dot{x},\dot{y}) \to [x,y]$ on the GF(p)-vector space $V \times V$.

<u>Theorem 3</u> If $G \in [CF]^- \cap [SIN]$ then G is either totally disconnected or compact.

Assuming $G_0 \neq E$ one shows that G_0 is compact and, with the help of a maximal torus in G_0, constructs a non-compact closed subgroup H with H_0 abelian. Then, since $G \in [SIN]$, one can find a non-compact Lie-group M, epimorphic image of H, and an element $m \in (M \setminus E)$ with $C_M(m)$ non-compact. Application of a lifting theorem for fixed points finally leads to an element $x \in (G \setminus E)$ with $C_G(x)$ non-compact.

Theorem 4 If $G \in [CF]^- \cap [M]$ then G is either compact or is a finite extension of a p-group in $[M]$ (p a prime).

By Theorem 3, if G is non-compact, it is periodic (i.e., each element lies in a compact subgroup). Next one shows that $K \rightarrowtail G \twoheadrightarrow A$, may be assumed, where K is profinite and A is an abelian discrete torsion group. In addition one may assume that K has a G-invariant system of e-neighborhoods. If G fails to satisfy the structure theorem a contradiction can be derived by means of a series of technical lemmas involving profinite group theory (e.g., a profinite version of Glauberman's theorem on the lifting of fixed points, and an application of Theorem 1).

Two examples are given to delimit the scope of the results obtained in the paper.

Example 1 (a nilpotent group of class 2 in $[AF]^- \setminus [CF]^-$) shows that the distinction between $[AF]^-$ and $[CF]^-$ is meaningful. Let p be an odd prime, $W := C_p^{N}$, with its natural compact topology, $V := (C_p^{N})^*$ endowed with the discrete topology. Let $G := W \times \Lambda^2(W)$ with the operation

$$(v,\sigma)(v',\sigma') := (v+v', \sigma+\sigma'+2^{-1} v \wedge v').$$

Topologize G by the sequence $\Lambda^2(W) \rightarrowtail G \twoheadrightarrow V$, in which $\Lambda^2(W)$ is clearly compact.

Example 2 (a non-compact p-Moore-group in $[CF]^-$) elucidates the second part of Theorem 4. Let $F := \langle x_1, x_2, \dots \rangle$ be the free group in countably many variables (x_i), and let p be a fixed prime. From F one constructs the $\hat{F}_p(\aleph_0)$, the restricted free pro-p-group in countably many variables (see [15]). Let $\hat{F}_p^*$ be the Frattini-subgroup and let $G := F\,F_p^*$. This gives rise to the algebraic sequence $\hat{F}_p^* \rightarrowtail G \twoheadrightarrow (C_p^{N})^*$, by means of which G is topologized as a locally compact, non-compact group with $\hat{F}_p^*$ as a compact open subgroup. The proof that G possesses the properties required rests on the fact that in any free pro-p-group the centralizers of non-trivial elements are procyclic.

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GAPS IN THE SPECTRUM OF AN ALMOST PERIODIC SCHRÖDINGER OPERATOR

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In [1], [3], and [4], the spectrum of the one-dimensional Schrödinger operator is shown to be totally disconnected for a dense G_δ of limit periodic potentials. These authors consider the continuous case, i.e., the operator

$$\frac{-d^2}{dx^2} + V(x) \quad \text{on} \quad L_2(\mathbb{R}) \ ,$$

but it is pointed out in [5,§6] that a similar result holds in the discrete case, i.e., for the operator

$$h_0 + V \quad \text{on} \quad \ell_2(\mathbb{Z})$$

where $(h_0)_{ij} = \delta_{i,j+1} + \delta_{i+1,j}$ and $V_{ij} = V(j)\delta_{ij}$.

The present note takes a first step towards extending this result to the class of all almost periodic potentials. Recall that an almost periodic function on $\mathbb{R}$ (resp. on $\mathbb{Z}$) is limit periodic precisely when its frequencies generate a rank one subgroup of $\mathbb{R}$ (resp. a torsion subgroup of $\mathbb{R}/\mathbb{Z}$). In general the frequencies form an arbitrary countable subset of $\mathbb{R}$ (resp. $\mathbb{R}/\mathbb{Z}$) .

Lemma 1. Every continuous section of normal elements of a C^*-algebra bundle has continuous spectrum if either all fibres are nonunital or if all fibres are unital and the section of units is continuous.

Proof. That the spectrum of the normal continuous section $\alpha \mapsto a(\alpha)$ is continuous means that for each open set $U \subseteq \mathbb{C}$, the set of α with $\mathrm{Sp}\, a(\alpha) \subseteq U$ is open (upper semicontinuity), and the set of α with $U \cap \mathrm{Sp}\, a(\alpha) \neq \emptyset$ is open (lower semicontinuity). That these sets of α are open follows by continuity of the norm from the following characterizations in the unital and nonunital cases respectively:

$$\mathrm{Sp}\, x \subseteq U \iff \|f(x)-1\| < 1 \quad \text{for some} \quad f \in C_{00}(U)$$
$$\text{(resp. } 0 \in U \text{ and this holds with } f(0) = 1\text{)} \ ,$$

$$U \cap \operatorname{Sp} x \neq \emptyset \iff f(x) \neq 0 \text{ for some } f \in C_{00}(U)$$
$$\text{(resp. either this holds, or } 0 \in U) ,$$

where x denotes $a(\alpha)$, and $C_{00}(U)$ denotes the set of continuous functions $\mathbb{C} \to \mathbb{C}$ with compact support contained in U .

Lemma 2. Let G be a subgroup of $\mathbb{R}$ (resp. $\mathbb{R}/\mathbb{Z}$) and let $(\alpha_1, \alpha_2, \ldots)$ be a sequence of homomorphisms of G into $\mathbb{R}$ (resp. $\mathbb{R}/\mathbb{Z}$) converging pointwise to the identity. Denote by A the C^*-algebra on $L_2(\mathbb{R})$ (resp. $L_2(\mathbb{R}/\mathbb{Z})$) generated by the operators $fu(g)$ where f is in $C_0(\mathbb{R})$ (resp. $C(\mathbb{R}/\mathbb{Z})$) and g is in G (here $u(g)$ denotes translation by g). Denote by $A_1, A_2, \ldots$ the analogous C^*-algebras with $\alpha_1(G), \alpha_2(G), \ldots$ in place of G . Then there is a unique C^*-algebra bundle over the compact space $\{1,2,\ldots,\infty\}$ with fibres $A_1, A_2, \ldots, A$ such that for each f in $C_0(\mathbb{R})$ (resp. $C(\mathbb{R}/\mathbb{Z})$) and each g in G the section $(fu(\alpha_1(g)), fu(\alpha_2(g)), \ldots, fu(g))$ is continuous.

Proof. Denote by B the C^*-algebra of sections of $(A_1, A_2, \ldots, A)$ generated by the sections $(f_1u(\alpha_1(g)), f_2u(\alpha_2(g)), \ldots, fu(g))$ where $g \in G$ and $f_n \to f$ in $C_0(\mathbb{R})$ (resp. $C(\mathbb{R}/\mathbb{Z})$) . We must show that each section in B has continuous norm. If $b = (a_1, a_2, \ldots, a)$ is such a section, which we may assume to be positive (since $\|b\|^2 = \|b^*b\|$) , then,.by the functional calculus, to show upper semicontinuity it is enough to show that a_n is small for large n if $a = 0$, and to show lower semicontinuity it is enough to show that $a = 0$ if $a_n = 0$ for all n .

To show upper semicontinuity, consider the C^*-algebra B_0 generated by elements $(f_1, f_2, \ldots, f)w(g)$ where $g \in G$ and $f_n \to f$ in $C_0(\mathbb{R})$ (resp. $C(\mathbb{R}/\mathbb{Z})$) , with the relations $w(g)^* = w(-g)$, $w(g_1+g_2) = w(g_1)w(g_2)$, and $w(g)(f_1, f_2, \ldots, f) = (f_1^{\alpha_1(g)}, f_2^{\alpha_2(g)}, \ldots, f^g)w(g)$, where $f^g(s) = f(s-g)$. There exists a homomorphism of B_0 onto B , since the sections defining B also satisfy the relations. The quotient of B_0 by the closed two-sided ideal generated by the elements $(f_1, f_2, \ldots, 0)$ where $f_n \to 0$ is the crossed product of $C_0(\mathbb{R})$ (resp. $C(\mathbb{R}/\mathbb{Z})$) by G . This ideal maps into sections in B with zero component in A , thus defining a map of the crossed product onto A . But by uniqueness, A also is the crossed product, so this map is injective. In particular, if $b = (a_1, a_2, \ldots, a) \in B$ and $a = 0$ then b is in the image of the above ideal of B_0 , and therefore a_n is small for large n .

To show lower semicontinuity consider the canonical densely defined faithful lower semicontinuous traces $\tau_1, \tau_2, \ldots, \tau$ on $A_1^+, A_2^+, \ldots, A^+$ extending Lebesgue measure on $C_0(\mathbb{R})$ (resp. $C(\mathbb{R}/\mathbb{Z})$) . Thus, $\tau(fu(g))$ is the integral of f if $g = 0$ and is zero otherwise. Since $\alpha_n(g) \to g$ for all g we have $\tau_n(a_n) \to \tau(a)$ whenever $b = (a_1, a_2, \ldots, a)$ belongs to the dense subalgebra F of B of finite sums of sections $(f_1 u(\alpha_1(g)), f_2 u(\alpha_2(g)), \ldots, fu(g))$ where all f_n have support in some fixed compact set. We must show that if $b = (a_1, a_2, \ldots, a) \in B^+$ and $a_n = 0$ for all n then $a = 0$. Note first that for any $b \in B$ and $c = (d_1, d_2, \ldots, d) \in F$, $\tau_n(d_n a_n d_n^*) \to \tau(dad^*)$. Indeed, if $b \in F$ this convergence holds by direct computation as remarked above. Since also $\tau_n(d_n d_n^*) \to \tau(dd^*)$ the positive functionals $d_1^* \tau_1 d_1, d_2^* \tau_2 d_2, \ldots, d^* \tau d$ are uniformly bounded, and hence the convergence holds for any $b \in B$. In particular, if $b = (0, 0, \ldots, a) \in B^+$ then for any $c = (d_1, d_2, \ldots, d) \in F$, $\tau_n(d_n a_n d_n^*) = 0$ for all n , so $\tau(dad^*) = 0$, $dad^* = 0$; it follows that $a = 0$.

Lemma 3. Let B be a C^*-algebra, let v be a selfadjoint multiplier of B , and let h be a selfadjoint operator affiliated with the enveloping von Neumann algebra of B such that $(h+i)^{-1}$ belongs to B . Then also $h+v$ is selfadjoint and $(h+v+i)^{-1}$ belongs to B .

Proof. It is immediate that $h+v$ is selfadjoint. Hence it is sufficient to consider the case that $\|v\| < 1$; repeated application of this case yields the general case. If $\|v\| < 1$, we have $\|(h+i)^{-1}v\| < 1$ since $\|(h+i)^{-1}\| \leq 1$, and hence, with $b = (h+i)^{-1}v \in B$,

$$(h+v+i)^{-1} = (1+b)^{-1}(h+i)^{-1} = (\Sigma_0^\infty(-b)^n)(h+i)^{-1} \in B \ .$$

<u>Theorem</u>. Let G be a countable subgroup of $\mathbb{R}$ (resp. $\mathbb{R}/\mathbb{Z}$) and let $(\alpha_1, \alpha_2, \ldots)$ be a sequence of homomorphisms of G into $\mathbb{R}$ (resp. $\mathbb{R}/\mathbb{Z}$) converging pointwise to the identity. Let V be a real-valued almost periodic function on $\mathbb{R}$ (resp. $\mathbb{Z}$) with frequencies in G, so that V has the Fourier series $\Sigma_{g\in G}\, a_g \exp 2\pi igt$. Then for each $n = 1,2,\ldots$ the Fourier series $\Sigma_{g\in G}\, a_g \exp 2\pi i\alpha_n(g)t$ represents a real-valued almost periodic function V_n on $\mathbb{R}$ (resp. $\mathbb{Z}$), and if H_n denotes the one-dimensional Schrödinger operator in $L_2(R)$ (resp. $\ell_2(\mathbb{Z})$) with potential V_n, and H that with V, then the spectrum of H_n approaches the spectrum of H.

<u>Proof</u>. Since V is almost periodic and the dual $\hat{G}$ of the discrete group G is compact, there is a function $\tilde{V}$ on $\hat{G}$ such that V is equal to $\tilde{V}$ composed with the map $\mathbb{R} \to \hat{G}$ (resp. $\mathbb{Z} \to \hat{G}$) dual to the given map $G \to \mathbb{R}$ (resp. $G \to \mathbb{R}/\mathbb{Z}$) (see [2], page 428). Then V_n is just $\tilde{V}$ composed with the map $\mathbb{R} \to \hat{G}$ (resp. $\mathbb{Z} \to \hat{G}$) dual to the map α_n.

Denoting H by H_0 in the case that $V = 0$, we may view $(H_0+i)^{-1}$ as an element of $C_0(\mathbb{R})$ (resp. $C(\mathbb{R}/\mathbb{Z})$), if at the same time we view V as belonging to the C^*-algebra generated by $\{u(g)\,|\,g\in G\}$, and V_n to the C^*-algebra generated by $\{u(\alpha_n(g))\,|\,g\in G\}$. Then $v = (V_1, V_2, \ldots, V)$ is a self-adjoint multiplier of the C^*-algebra B of continuous sections of the C^*-algebra bundle of Lemma 2, and with h denoting $(H_0, H_0, \ldots, H_0)$, $(h+i)^{-1}$ belongs to B. Applying Lemma 3 with this B, v, and h yields

$$((H_1+i)^{-1}, (H_2+i)^{-1}, \ldots, (H+i)^{-1}) = (h+v+i)^{-1} \in B \ .$$

The conclusion of the Theorem follows by Lemma 1.

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ON A GENERALIZATION OF GAMMA FUNCTIONAL EQUATION.

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Presented by J. Aczél, F.R.S.C.

ABSTRACT- *The general holomorphic solution is given of the functional equation* $f(z+1)=Q(z)f(z)$, *if* $Q(z)=P(z)/R(z)$, *where* P *and* R *are entire functions of genus 1.*

1.- Several generalizations of the Euler functional equation $f(z+1)=zf(z)$ have been studied (see the book of M. Kuczma [3]).

In the present paper we consider the functional equation

$$f(z+1)=Q(z)f(z) \tag{1}$$

where

$$Q(z)=\prod_{k=1}^{\infty} \frac{z-c_k}{z-d_k} e^{\gamma_k z} \quad , \quad \gamma_k=\frac{1}{c_k}-\frac{1}{d_k}$$

and we determine, under suitable hypotheses on c_k and d_k, its holomorphic solutions such that $f(z_o)=1$ for some z_o.This can be done by using the Laplace transformation and specifying the procedure described in [2].

2.- Let $\Omega_a=\{z\in C\colon \operatorname{Re} z>a\}$, $a\in R$, and let $G\in H(\Omega_a)$ be a given function without zeros in Ω_a, where $H(\Omega_a)$ is the set of all functions holomorphic in Ω_a.
Differentiating equation (1) ($z\in\Omega_a$) we see that if $f\in H(\Omega_a)$ is a solution of (1) without zeros in Ω_a, then f is also a solution of the functional equation

$$\frac{f'(z+1)}{f(z+1)}=\frac{G'(z)}{G(z)}+\frac{f'(z)}{f(z)}, \quad z\in\Omega_a. \tag{2}$$

THEOREM 1- *Given* $z_o\in\Omega_a$, *let* $f\in H(\Omega_a)$ *be a solution of (2) such that* $f(z_o)=1$. *Any other holomorphic solution of (2) taking the value 1 at* z_o, *has the form*

$$(3)\qquad h(z)=\Big(\exp\int_{z_0}^{z} F(\zeta)\,d\zeta\Big)\cdot f(z)\,,$$

where F is an arbitrary function in $H(\Omega_a)$, periodic of period 1.

PROOF- Obviously if h has the form (3) then it is a solution of (2) and $h(z_0)=1$. Conversely, let $h\in H(\Omega_a)$, $h(z_0)=1$, be a solution of (2). Then we get

$$(4)\qquad \left(\frac{h'(z+1)}{h(z+1)}-\frac{f'(z+1)}{f(z+1)}\right)-\left(\frac{h'(z)}{h(z)}-\frac{f'(z)}{f(z)}\right)=0,$$

and, putting $K(z)=\frac{h(z)}{f(z)}$, (4) becomes

$$\frac{K'(z+1)}{K(z+1)}-\frac{K'(z)}{K(z)}=0.$$

Hence $F(z)=\frac{K'(z)}{K(z)}$ is holomorphic in Ω_a and $F(z+1)=F(z)$. Since $K(z_0)=1$, we obtain $K(z)=\exp\int_{z_0}^{z} F(\zeta)\,d\zeta$, i.e. $h(z)=f(z)\exp\int_{z_0}^{z} F(\zeta)\,d\zeta$.

If $f\in H(\Omega_a)$ is a solution of equation (2), by integrating and taking the exponentials, we have

$$f(z+1)=\alpha G(z)f(z),\quad z\in\Omega_a\,,$$

for some constant $\alpha\in C-\{0\}$. So, in general, f is not a solution of (1); nevertheless we can get a solution of (1) by taking

$$h(z)=f(z)\exp\int_{z_0}^{z} F(\zeta)\,d\zeta\,,\ z_0\in\Omega_a\,,$$

where $F\in H(\Omega_a)$, periodic of period 1, is such that

$$\exp\int_{z_0}^{z_0+1} F(\zeta)\,d\zeta=\frac{1}{\alpha}\,.$$

Thus the knowledge of a particular solution of (2) sufficies to find the solutions of (1).

3.- In this section we shall find a particular solution of (2), when the given function is

$$Q(z)=\prod_{k=1}^{\infty}\frac{z-c_k}{z-d_k}e^{\gamma_k z}\ ,\ \gamma_k=\frac{1}{c_k}-\frac{1}{d_k}\ . \tag{5}$$

By section 2, this solves equation (1) considering Q instead of G.

We assume that the complex numbers c_k and d_k satisfy the following conditions:

(i) for every k, $Re\ c_k<0$, $Re\ d_k<0$;

(ii) $\sum\limits_{k=1}^{\infty}\frac{1}{|c_k|^2}<\infty$, $\sum\limits_{k=1}^{\infty}\frac{1}{|d_k|^2}<\infty$;

(iii) there exists θ, $0\leq\theta<\pi/2$, such that $|arg(-c_k)|\leq\theta$, $|arg(-d_k)|\leq\theta$, for every k;

(iv) $\sum\limits_{k=1}^{\infty}|\gamma_k|<\infty$.

We set $M=max\ (\ \sup\limits_k Re\ c_k\ ,\ \sup\limits_k Re\ d_k\ )$, so $Q\in H(\Omega_M)$.

In the sequel, if $\psi(t,z)$, $t\in R$, $z\in C$, is a function of two variables, we shall denote with $\mathcal{L}_t(\psi(t,z))(s)$ the Laplace transform of ψ as function of t, calculated in s.

THEOREM 2- *Consider the functional equation (2) with Q defined by (5) instead of G; assume that conditions (i)-(iv) hold and fix* $z_0\in\Omega_M$ *.The function*

$$f(z)=exp\int_{z_0}^{z}\Phi(\zeta)\ d\zeta\ ,\quad z\in\Omega_M\ ,$$

where

$$\Phi(z)=(z-\frac{3}{2})\sum_{k=1}^{\infty}\gamma_k+\sum_{k=1}^{\infty}\{\mathcal{L}_t(\psi(t,z))(-c_k)-\mathcal{L}_t(\psi(t,z))(-d_k)\}$$

and

$$\psi(t,z)=\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\ ,\ t\neq0,\ \psi(0,z)=z-\frac{3}{2}\ ,$$

is a solution of (2), holomorphic in Ω_M *, for which* $f(z_0)=1$ *holds.*

PROOF- We must only prove that $\Phi(z+1)-\Phi(z)=\frac{Q'(z)}{Q(z)}$.

Let $g_k(z)=\dfrac{z-c_k}{z-d_k}\,e^{\gamma_k z}$, then

$$\frac{g_k'(z)}{g_k(z)}=\gamma_k+\frac{1}{z-c_k}-\frac{1}{z-d_k}\,. \tag{6}$$

If we take the inverse Laplace transform of (6), we have

$$H_k(t)=\mathcal{L}^{-1}\left(\frac{g_k'(z)}{g_k(z)}\right)=\gamma_k\cdot\delta+e^{c_k t}-e^{d_k t},$$

where δ is the Dirac distribution. We now set

$$\Phi_k(z)=\int_0^\infty \psi(t,z)H_k(t)\,dt=(z-\tfrac{3}{2})\gamma_k+\int_0^\infty \psi(t,z)(e^{c_k t}-e^{d_k t})\,dt. \tag{7}$$

Since $Re\ z>M$, $Re\ c_k<0$, $Re\ d_k<0$, the last integral in (7) exists and

$$\int_0^\infty \psi(t,z)(e^{c_k t}-e^{d_k t})\,dt=\mathcal{L}_t(\psi(t,z))(-c_k)-\mathcal{L}_t(\psi(t,z))(-d_k).$$

Obviously $\Phi_k\in H(\Omega_M)$, and

$$\Phi_k(z+1)-\Phi_k(z)=\int_0^\infty\{\psi(t,z+1)-\psi(t,z)\}H_k(t)\,dt=\int_0^\infty e^{-zt}H_k(t)\,dt=\mathcal{L}(H_k(t))(z)=\frac{g_k'(z)}{g_k(z)}\,.$$

Now we define

$$\Phi(z)=\sum_{k=1}^{\infty}\Phi_k(z)=\sum_{k=1}^{\infty}\{(z-\tfrac{3}{2})\gamma_k+\mathcal{L}_t(\psi(t,z))(-c_k)-\mathcal{L}_t(\psi(t,z))(-d_k)\}. \tag{8}$$

In our hypotheses on $\{c_k\}$ and $\{d_k\}$, we can apply Theorem 33.4, p. 226 in [1], to obtain

$$\lim_{k\to+\infty}[-c_k\cdot\mathcal{L}_t(\psi(t,z))(-c_k)]=\lim_{k\to+\infty}[-d_k\cdot\mathcal{L}_t(\psi(t,z))(-d_k)]=z-\frac{3}{2}\,,$$

so

$$\mathcal{L}_t(\psi(t,z))(-c_k)=-\frac{z-\frac{3}{2}}{c_k}+o(\frac{1}{c_k})\quad\text{and}\quad\mathcal{L}_t(\psi(t,z))(-d_k)=-\frac{z-\frac{3}{2}}{d_k}+o(\frac{1}{d_k})\ ,\ \text{as } k\to+\infty.$$

$$\left|(z-\tfrac{3}{2})\gamma_k+\mathcal{L}_t(\psi(t,z))(-c_k)-\mathcal{L}_t(\psi(t,z))(-d_k)\right|\leq$$

$$\leq\left|z-\tfrac{3}{2}\right|\cdot|\gamma_k|+\left|z-\tfrac{3}{2}\right|\cdot\left|\ \frac{1}{c_k}-\frac{1}{d_k}+o(\frac{1}{c_k})+o(\frac{1}{d_k})\right|\ ,$$

hence, by (iv), the series in (8) is absolutely uniformely convergent on every compact subset of Ω_M, thus $\Phi\in H(\Omega_M)$. Moreover

$$\Phi(z+1)-\Phi(z)=\sum_{k=1}^{\infty}\{\Phi_k(z+1)-\Phi_k(z)\}=\sum_{k=1}^{\infty}\frac{g_k'(z)}{g_k(z)}=\frac{Q'(z)}{Q(z)}\ .$$

This completes the proof.

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THE BERGMAN-GILBERT OPERATOR AS A TRANSMUTATION

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Presented by F.V. Atkinson, F.R.S.C.

Abstract: We give a spectral characterization of the Bergman-Gilbert operator via Mellin transforms which displays it in the context of general transmutation theory.

Introduction. We consider (*) $\Delta u + F(r^2)u = 0$ in R^n and refer to the formula

$$u(\vec{x}) = h(\vec{x}) + \int_0^1 \sigma^{n-1} G(r,1-\sigma^2) h(\vec{x}\sigma^2) d\sigma \tag{1}$$

as the Bergman-Gilbert integral operator. It was obtained by Gilbert as a "method of ascent" (G does not depend on n) and maps harmonic functions h onto solutions of (*) (cf. [3;4;5]). It is not surprising that it should have transmutational features (cf. [1] for transmutation) and in the course of studying certain singular inverse problems (cf. [2]) it became desirable to spell out this connection. The result is indicated here.

Framework. We recall that $B: P \to Q$ is a transmutation if $BP = QB$ acting on suitable objects and we write here $P = r^2D^2 + (n-1)rD$ with $Q = P + r^2F(r^2)$. Since $r^2\Delta = P + \Omega$ with Ω independent of r we see that a transmutation $B: P \to Q$ in the radial variables automatically induces a transmutation $B: r^2\Delta \to r^2\Delta + r^2F$. Transmutations can be characterized in various ways and in general will be given as integral operators with distribution kernels in the form $Bf(r) = \langle \beta(r,\rho), f(\rho) \rangle$. The kernels β can be represented as spectral integrals using eigenfunctions of P and Q and for radial operators $\tilde{P} = (1/r^2)P$ for example such expressions for β have a particularly nice form in terms of "spherical functions". The present situation is somewhat more complicated and we bypass the spectral analysis as such be representing the kernels as inverse Mellin type transforms directly (in which the spectral theory is implicit).

Results. One changes variables in (1) and writes $u = h + \langle \rho^{n-3}\check{K}(r,\rho), h(\rho,\cdot)\rangle$ (distribution bracket) where $\check{K}(r,\rho) = K(r,\rho)Y(r-\rho)$ with Y the Heavyside function. The differential equation satisfied by G in (1) leads to a differential equation for K when $\rho < r$ with a jump discontinuity $K(r,r) = -\frac{1}{2}\int_0^r F(\rho^2)\rho\, d\rho / r^{n-2}$ at $\rho = r$. This is all embodied in the distribution context (from which the desired jump arises naturally) and one shows that $(Q_r - P_\rho^*)\{\rho^{n-3}\check{K}(r,\rho)\} = -r^2F(r^2)\delta(r-\rho)$ where P_ρ^* is the formal adjoint defined by $P^*\varphi = (r^2\varphi)'' - (n-1)(r\varphi)'$. One determines then that (1) represents a transmutation which we write as $u = \check{B}\{h\} = \langle \check{\beta}(r,\rho), h(\rho,\cdot)\rangle$. Then writing $\check{B}\{\rho^\sigma\} = \langle \rho\check{\beta}(r,\rho), \rho^{\sigma-1}\rangle = M\{\rho\check{\beta}(r,\rho)\}(\sigma)$ (Mellin transform $\rho \to \sigma$) we will have $Q_r\check{\beta} = P_\rho^*\check{\beta}$ and choosing $\check{\psi}_\sigma^Q(r)$ ($= \check{\psi}_\tau^Q(r)$ with $\tau = -\sigma-n+2$) to be the solution of the eigenvalue equation $Q\psi = \sigma(\sigma+n-2)\psi$ with $\check{\psi}_\sigma^Q(r)r^{-\sigma} \to 1$ as $r \to 0$ we have (note $P\rho^\tau = \tau(\tau+n-2)\rho^\tau = \sigma(\sigma+n-2)\rho^\tau$)

Theorem. The "extended" Bergman-Gilbert kernel $\check{\beta}(r,\rho)$ is characterized by the spectral formula

$$(2) \qquad \check{\beta}(r,\rho) = \frac{1}{2\pi i}\int \rho^{-\sigma-1}\check{\psi}_\sigma^Q(r)\, d\sigma = \frac{1}{2\pi i}\int \rho^{n-3}\rho^\tau\, \check{\psi}_\tau^Q(r)\, d\tau$$

where the integral involves suitable contours $c-i\infty \to c+i\infty$.

Examples. The various formulas indicated are calculated explicitly for $n = 3$ and $F(r^2) = k^2$ (which is a typical example from scattering theory) and comparison to other transmutations is made via spectral integrals. Thus in particular in (2) $\check{\psi}_\sigma^Q(r) = \check{\psi}(k,\sigma)z^{-\frac{1}{2}}J_{\sigma+\frac{1}{2}}(z)$ $(z = kr)$ where $\check{\psi} = 2^{\sigma+\frac{1}{2}}k^{-\sigma}\Gamma(\sigma+3/2)$. One knows also from [3;4;5] that $K(r,\rho) = -\frac{1}{2}k(\rho/r)^{\frac{1}{2}}J_1\{kr(1-\rho/r)^{\frac{1}{2}}\}/(1-\rho/r)^{\frac{1}{2}}$ and thus $M\{\rho K(r,\rho)\} = -k^{-\sigma}(kr)^{-\frac{1}{2}}\ell_{\sigma+3/2,\sigma+\frac{1}{2}}(kr)$ where ℓ ($\sim$ s) represents the Lommel function.

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LOCALLY CARTESIAN CLOSED CATEGORIES AND TYPE THEORY

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Presented by J. Lambek, F.R.S.C.

Abstract A modified version ML of Martin-Löf's type theory is presented which characterises the categorical structure of a locally cartesian closed category.

0. Introduction

It is known that for much of the mathematics of topos theory, it is in fact sufficient to use a category $\underline{C}$ whose slice categories $\underline{C}/A$ are cartesian closed, where the notion of "A-indexed family" is represented by morphisms $B \to A$ of $\underline{C}$. It has also been conjectured that the logic of such categories is given by Martin-Löf's type theory. The purpose of this note is to make this precise.

1. Definitions

The type theory sketched here is based on Martin-Löf's, as given in [1], [2]. We present the definition informally for the sake of brevity.

Definition 11: An ML theory is given by type-function constants and term-function constants, which induce sets of types and terms (of given types), satisfying the following conditions. We write $t \in T$ to mean t is a term of type T, b[x] to represent all free occurrences of x in b, and b[a] for the substitution of a for x in b, where appropriate.

Type formation rules: (i) 1 is a type .

(ii) If $a,b \in A$, then $I(a,b)$ is a type .

(iii) If A is a type, and if for $x \in A$, $B[x]$ is a type, x not occurring in the type of any free variable of B, then $\Sigma x{\in}A\ B[x]$ and $\Pi x{\in}A\ B[x]$ are types. (If B does not depend on x, these are written $A{\times}B$, $A{\supset}B$ respectively.)

Term formation rules: $* \in 1$. For each type A, there are (countably many) free variables $x \in A$.

(ΠI) If $t[x] \in B[x]$, where $x \in A$ does not occur in the type of any free variable of t, then $\lambda x{\in}A t[x] \in \Pi x{\in}A\ B[x]$.

(ΠE) If $f \in \Pi x{\in}A\ B[x]$, $a \in A$, then $f(a) \in B[a]$.

(ΣI) If $a \in A$, $b \in B[a]$, then $\langle a,b\rangle \in \Sigma x{\in}A\ B[x]$.

(ΣE) If $c \in \Sigma x{\in}A\ B[x]$, then $\pi(c) \in A$, $\pi'(c) \in B[\pi(c)]$.

(=I) If $a \in A$, then $r(a) \in I(a,a)$.

(=E) If $a,b \in A$, $c \in I(a,b)$, $d \in C[a,a,r(a)]$, then $s(d)[a,b,c] \in C[a,b,c]$.

Substitution instances of type (term)-function constants are types (terms respectively).

Equality rules (notation as above):

(Π red) $\lambda x{\in}A t[x](a) = t[a]$. ($\Pi$ exp) $f = \lambda x{\in}A f(x)$.

(Σ red) $\pi(\langle a,b\rangle) = a$, $\pi'(\langle a,b\rangle) = b$. ($\Sigma$ exp) $c = \langle \pi(c), \pi'(c)\rangle$.

(=red) $s(d)[a,a,r(a)] = d$ (=exp) If $f[a,b,c] \in C[a,b,c]$ then $f = s(f[a,a,r(a)])[a,b,c]$.

(1rule) If $t \in 1$ then $t = *$.

(Irule) If $a[x]$, $b[x] \in A$, $t[x] \in I(a[x],b[x])$, for $x \in X$ not occurring in the type of any free variable of a, b, A, t, then $a[x] = b[x]$ and $t[x] = r(a[x])$.

Definition 12: A locally cartesian closed category (lcc) is a category $\underline{C}$ with finite limits such that for any object A of $\underline{C}$, the slice category $\underline{C}/A$ is cartesian closed.

2. <u>From ML to lcc</u> Given an ML theory $\underline{M}$, we define a category $\underline{C}(\underline{M})$, whose objects are closed types of $\underline{M}$, and morphisms $A \to B$ are closed terms of type $A \supset B$.

<u>Theorem 21</u>: $\underline{C}(\underline{M})$ is locally cartesian closed.

Sketch of the proof: Finite limits are constructed in the evident way: for example, the pullback of $t \in A \supset B$ along $s \in C \supset B$ is given by $\Sigma\, x \in A\, \Sigma\, y \in C\ I(t(x), s(y))$ with projections to A and C. Similarly B^A is just $A \supset B$. Finally, to see $\underline{C}(\underline{M})$ is lcc, we define two $\underline{C}(\underline{M})$-indexed categories, $\underline{P}_1$ and $\underline{P}_2$. A $\underline{C}$-indexed category $\underline{P}$ consists of, for each object A of $\underline{C}$, a category $\underline{P}(A)$, and for each morphism $f: A \to B$ of $\underline{C}$, a functor $f^*: \underline{P}(B) \to \underline{P}(A)$. Here, $\underline{P}_1(A)$ is the category of types $B[x]$, $x \in A$, and terms $t[x] \in B[x] \supset C[x]$; f^* is defined by substitution. $\underline{P}_2(A) = \underline{C}(\underline{M})/A$, with f^* defined by pullback. Two $\underline{C}$-indexed categories are equivalent, $\underline{P}_1 \simeq \underline{P}_2$, if there are equivalences $\underline{P}_1(A) \simeq \underline{P}_2(A)$ which commute with the f^*'s.

<u>Lemma 22</u>: $\underline{P}_1 \simeq \underline{P}_2$; furthermore, $\underline{P}_1$ is a hyperdoctrine.

The main idea of the proof of the lemma is to associate with a term $f \in B \supset A$ the type $f^{-1}(x) =_{df} \Sigma\, y \in B\ I(x, f(y))$ in $\underline{P}_1(A)$, and inversely, with a type $B[x]$ of $\underline{P}_1(A)$, the projection term $\pi \in \Sigma\, x \in A B[x] \supset A$.
A $\underline{C}$-indexed category $\underline{P}$ is a hyperdoctrine if each $\underline{P}(A)$ is cartesian closed, if each f^* has adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$, and if f^* preserves exponents. In the case of $\underline{C}$ with finite limits, $\underline{P}$ given by $\underline{P}(A) = \underline{C}/A$, $\underline{C}$ is lcc iff $\underline{P}$ is a hyperdoctrine. In [3], [4] it is shown

that the category of hyperdoctrines is equivalent to the category of theories in first order logic. Since $\underline{M}$, and so $\underline{P}_1$, is essentially a theory in first order logic, $\underline{P}_1$ is a hyperdoctrine. Lemma 22 completes the proof of Theorem 21, since $\underline{P}_2$ is a hyperdoctrine.

3. *From lcc to ML* Given an lcc $\underline{C}$, we define an ML theory $\underline{M}(\underline{C})$ using the fact that $\underline{C}$ defines a hyperdoctrine (with $\underline{P}(A) = \underline{C}/A$) and hence a theory $\underline{T}(\underline{C})$ in first order logic: briefly, we take predicates of $\underline{T}(\underline{C})$ (*i.e.* morphisms of $\underline{C}$) as the types of $\underline{M}(\underline{C})$, and proofs in $\underline{T}(\underline{C})$ (*i.e.* morphisms of slice categories $\underline{C}/A$) as the terms of $\underline{M}(\underline{C})$. The proof that this defines an ML theory follows closely the proof that $\underline{T}(\underline{C})$ is a first order theory, as in [3] or [4]. Note that the closed types and terms of $\underline{M}(\underline{C})$ are the objects and morphisms of $\underline{P}(1) \cong \underline{C}$.

4. *Equivalences* From the preceeding remark, $\underline{C} \simeq \underline{C}(\underline{M}(\underline{C}))$. It is not difficult to show that $\underline{M} \simeq \underline{M}(\underline{C}(\underline{M}))$, in a suitable sense. If $\underline{ML}$ is the category of ML theories, and $\underline{LCC}$ the category of lcc categories, then:

Theorem: $\underline{ML} \simeq \underline{LCC}$.

5. *Remark* The details of these results were presented in the McGill Categorical Logic Seminar, and are available (from the author) as a publication (with the same title as this note) in the McGill lecture notes series.

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SCALES

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Presented by J.H.H. Chalk, F.R.S.C.

1) Abstract

It is a well known fact [1] that a "discrepancy" divides the musicians capable of pure intonation from those tied to tempered instruments. The aim of this work is to develop arithmetical concepts for a common understanding.

M will denote the set of "musical notes" produced by a certain instrument. With each M will be given a group T isomorphic to $\mathbb{Z}$ operating faithfully and transitively on M, which we will call a chromatic scale.

Each time T will be considered to be imbedded in $\mathbb{R}^+$ as a set, and we will always assume that $2 \in T$. The index $[T:<2>]$ will be called the number of degrees of T, and the orbits of M under the action of $<2>$ will be called the names of the musical notes.

We will now proceed to give an abstract definition of T, which will produce historical scales [2] like a rabbit from a hat.

2) Construction of scales

Let us consider the sequence $\mathcal{S}$ of the subgroups of the multiplicative group $\mathbb{Q}^+$ which are generated by the successive prime numbers :

$$\mathcal{S} \quad : <2,3>, <2,3,5>, <2,3,5,7>, \text{etc} \ldots$$

All the groups in $\mathcal{S}$ are dense in $\mathbb{R}^+$ (Kronecker's theorem) in the usual topology.

2.1) Commas associated to a subgroup $G \in \mathcal{S}$

To avoid cumbersome notations, we will take $G = <2,3,5>$ as a fairly general specimen of a group in $\mathcal{S}$.

<u>Definition</u> .- $2^x 3^y 5^z$ is a "comma" in $<2,3,5>$ iff :

1) $(x,y,z) \neq (0,0,0)$

2) $|\log 2^{x'} 3^{y'} 5^{z'}| < |\log 2^x 3^y 5^z|$ implies

$$|x'|\log 2 + |y'|\log 3 + |z'|\log 5 > |x|\log 2 + |y|\log 3 + |z|\log 5$$

The following criterion is often useful.

<u>Criterion</u> .- Consider $\frac{a+1}{a}$ with $a \in \mathbb{N}$. If $\frac{a+1}{a} \in G$ and $G \in \mathcal{S}$, then $\frac{a+1}{a}$ is a comma in G.

<u>Results</u> .-

a) The commas in $<2,3>$ are :

$$\frac{2}{1}, \frac{3}{2}, \frac{2^2}{3}, \frac{3^2}{2^3}, \frac{2^8}{3^5}, \frac{3^{12}}{2^{19}}, \frac{2^{65}}{3^{41}}, \frac{3^{53}}{2^{84}}, \ldots$$

and their inverses.

b) Those in $<2,3,5>$ are :

$$\frac{2}{1}, \frac{3}{2}, \frac{2^2}{3}, \frac{5}{2^2}, \frac{3^2}{2^3}, \frac{2.5}{3^2}, \frac{2^4}{3.5}, \frac{5^2}{2^3.3}, \frac{3^4}{2^4.5}, \ldots$$

and their inverses.

c) In the case of $<2,3,5,7>$ we can say that the following numbers and their inverses are commas :

$$\frac{2}{1}, \frac{2^2}{3}, \frac{2.3}{5}, \frac{7}{2.3}, \frac{2.5}{3^2}, \frac{3.5}{2.7}, \frac{3.7}{2^2.5}, \frac{2^2.3^2}{5.7}, \frac{7^2}{2^4.3}, \frac{2^6}{3^2.7}, \frac{3^4}{2^4.5}, \frac{3^2.5^2}{2^5.7}, \text{etc} \ldots$$

2.2) Quotient groups

To fix ideas, take again $G = <2,3,5>$ and consider two elements r and r' in G ; we will use the following elementary result to study the quotient group $G/_{<r,r'>}$.

<u>Theorem</u> .-

1) Let us write $r = 2^\alpha 3^\beta 5^\gamma$, $r' = 2^{\alpha'} 3^{\beta'} 5^{\gamma'}$ and $\begin{vmatrix} x & \alpha & \alpha' \\ y & \beta & \beta' \\ z & \gamma & \gamma' \end{vmatrix} = \alpha''x + \beta''y + \gamma''z$

Then ${}^{G}/_{<r,r'>}$ is isomorphic to $\mathbb{Z}$ iff the greatest common divisor of α'', β'', γ'' is equal to 1.

2) Let us consider $s = 2^a\,3^b\,5^c \in G$.

Then the coset of s is a generator of ${}^{G}/_{<r,r'>}$ iff

$$\alpha''a + \beta''b + \gamma''c = 1$$

2.3) Euler's principle

We must now imbed the group ${}^{G}/_{<r,r'>}$ in $\mathbb{R}^+$ and our problem is to choose for each coset of G modulo $<r,r'>$ a representative in $\mathbb{Q}^+$. As these cosets are (generally) dense subsets in $\mathbb{R}^+$, this choice is by no means an obvious one. To make this choice we will use the following remark from L. Euler (free translation) : "hearing is used to perceive as a simple fraction all the fractions which are near to it".

Let $\frac{r}{s}$ be an irreducible fraction, $\frac{r}{s} \in \mathbb{Q}^+$; we recall that the "height" of $\frac{r}{s}$ is the number $h(\frac{r}{s}) = \sup(r,s)$, then we will make Euler's remark precise in choosing in each coset an element of minimal height (which is unique in practice).

3) Results

The following table gives the three scales corresponding to the quotient groups :

$$<2,3>/_{<r>},\quad <2,3,5>/_{<r,r'>},\quad <2,3,5,7>/_{<r,r',r''>}$$

with $r = \frac{3^{12}}{2^{19}}$, $r' = \frac{3^4}{2^4.5}$, $r'' = \frac{3^2.5^2}{2^5.7}$ which are commas in $<2,3>$, $<2,3,5>$ and $<2,3,5,7>$ respectively.

The first scale is similar (if not identical) to Pythagoras', the second to Zarlino's and I will call the last one the "Zarlino A" scale.

Other scales are described in [3] and [4].

First sixty degrees of the scales of Pythagoras, Zarlino and Zarlino A

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	$\frac{2^8}{3^5}$	$\frac{3^2}{2^3}$	$\frac{2^5}{3^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$	$\frac{3^6}{2^9}$	$\frac{3}{2}$	$\frac{2^7}{3^4}$	$\frac{3^3}{2^4}$	$\frac{2^4}{3^2}$	$\frac{3^5}{2^7}$	2	$\frac{2^9}{3^5}$	$\frac{3^2}{2^2}$	$\frac{2^6}{3^3}$	$\frac{3^4}{2^5}$	$\frac{2^3}{3}$	$\frac{3^6}{2^8}$	3	$\frac{2^8}{3^4}$
1	$\frac{2^4}{3.5}$	$\frac{3^2}{2^3}$	$\frac{2.3}{5}$	$\frac{5}{2^2}$	$\frac{2^2}{3}$	$\frac{5.3^2}{2^5}$	$\frac{3}{2}$	$\frac{2^3}{5}$	$\frac{5}{3}$	$\frac{2^4}{3^2}$	$\frac{3.5}{2^3}$	2	$\frac{2^5}{3.5}$	$\frac{3^2}{2^2}$	$\frac{2^2.3}{5}$	$\frac{5}{2}$	$\frac{2^3}{3}$	$\frac{5.3^2}{2^4}$	3	$\frac{2^4}{5}$
1	$\frac{3.5}{2.7}$	$\frac{3^2}{2^3}$	$\frac{2.3}{5}$	$\frac{5}{2^2}$	$\frac{2^2}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{2^3}{5}$	$\frac{5}{3}$	$\frac{2^4}{3^2}$	$\frac{3.5}{2^3}$	2	$\frac{3.5}{7}$	$\frac{3^2}{2^2}$	$\frac{2^2.3}{5}$	$\frac{5}{2}$	$\frac{2^3}{3}$	$\frac{2.7}{5}$	3	$\frac{2^4}{5}$

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	$\frac{3^3}{2^3}$	$\frac{2^5}{3^2}$	$\frac{3^5}{2^6}$	2^2	$\frac{2^{10}}{3^5}$	$\frac{3^2}{2}$	$\frac{2^7}{3^3}$	$\frac{3^4}{2^4}$	$\frac{2^4}{3}$	$\frac{3^6}{2^7}$	2.3	$\frac{2^9}{3^4}$	$\frac{3^3}{2^2}$	$\frac{2^6}{3^2}$	$\frac{3^5}{2^5}$	2^3	$\frac{2^{11}}{3^5}$	3^2	$\frac{2^8}{3^3}$	2.5
1	$\frac{2.5}{3}$	$\frac{2.3^2}{5}$	$\frac{3.5}{2^2}$	2^2	$\frac{2^6}{3.5}$	$\frac{3^2}{2}$	$\frac{2^3.3}{5}$	5	$\frac{2^4}{3}$	$\frac{5.3^2}{2^3}$	2.3	$\frac{2^5}{5}$	$\frac{2^2.5}{5}$	$\frac{2^2.3^2}{5}$	$\frac{3.5}{2}$	2^3	$\frac{5^2}{3}$	3^2	$\frac{2^4.3}{5}$	2.5
1	$\frac{2.5}{3}$	$\frac{2.3^2}{5}$	$\frac{3.5}{2^2}$	2^2	$\frac{2.3.5}{7}$	$\frac{3^2}{2}$	$\frac{2^3.3}{5}$	5	$\frac{2^4}{3}$	$\frac{2^2.7}{5}$	2.3	$\frac{2^5}{5}$	$\frac{2^2.5}{5}$	$\frac{2^2.3^2}{5}$	$\frac{3.5}{2}$	2^3	$\frac{5^2}{3}$	3^2	$\frac{2^4.3}{5}$	2.5

	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
1	$\frac{2^5}{3}$	$\frac{3^6}{2^6}$	$2^2.3$	$\frac{2^{10}}{3^4}$	$\frac{3^3}{2}$	$\frac{2^7}{3^2}$	$\frac{3^5}{2^4}$	2^4	$\frac{2^{12}}{3^5}$	2.3^2	$\frac{2^9}{3^3}$	$\frac{3^4}{2^2}$	$\frac{2^6}{3}$	$\frac{3^6}{2^5}$	$2^3.3$	$\frac{2^{11}}{3^4}$	3^3	$\frac{2^8}{3^2}$	$\frac{3^5}{2^3}$	2^5
1	$\frac{2^5}{3}$	$\frac{5.3^2}{2^2}$	$2^2.3$	$\frac{2^6}{5}$	$\frac{3^3}{2}$	$\frac{2^3.3^2}{5}$	3.5	2^4	$\frac{5.3^3}{2^3}$	2.3^2	$\frac{2^5.3}{5}$	$2^2.5$	$\frac{2^6}{3}$	$\frac{5.3^2}{2}$	$2^3.3$	5^2	3^3	$\frac{2^4.3^2}{5}$	2.3.5.	2^5
1	$\frac{2^5}{3}$	$\frac{5.3^2}{2^2}$	$2^2.3$	$\frac{2^6}{5}$	$\frac{3^3}{2}$	$\frac{2^3.3^2}{5}$	3.5	2^4	$\frac{2^2.7}{5}$	2.3^2	$\frac{2^5.3}{5}$	$2^2.5$	$\frac{2^6}{3}$	$\frac{5.3^2}{2}$	$2^3.3$	5^2	3^3	$\frac{2^4.3^2}{5}$	2.3.5.	2^5

4) Some consequences

Let us mention only two developments.

The first one is to give a meaning to the common ground on which tempered and pure instrumentalists agree, and this can be done using the concept of isomorphism between two scales having the same number of degrees.

The second one is to define, in a natural way, the dissonance between two notes in a scale.

The application $d : (x,y) \longmapsto \log h(\frac{x}{y})$ is a distance on $\mathbb{Q}^+$ [5]. So one is led to define the dissonance of the interval $\frac{x}{y}$ in the scale $T = {}^{G}/_{H}$ by :

$$\delta_T(x,y) = d(xH, yH)$$

and the psychological signifiance of this notion remains to be explored ...

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SOME RADIUS OF CONVEXITY PROBLEMS

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Presented by N.S. Mendelsohn, F.R.S.C.

ABSTRACT.

Let S be the class of functions which are analytic and univalent in the unit disc $E = \{z:|z|<1\}$ and which are normalized by the conditions $f(0)=0$, $f'(0)=1$. Let K be the class of close-to-convex functions. The relationship of K with its variuos subclasses and some radius of convexity problems are studied.

Let K denote the class of functions analytic and univalent in $E = \{z:|z|<1\}$, satisfying $f(0)=0$, $f'(0)=1$ and which are close-to-convex in E[4]. Let C and S^* denote the subclasses of K, consisting of functions which are convex and starlike in E respectively. The condition

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\}>0, \qquad \text{for } |z|<1$$

is necessary and sufficient for f to be univalent and convex in E, whereas f is starlike in E if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0 \qquad \text{for } |z|<1 .$$

It is well-known that S^* and C are connected by the basic property

$$f\varepsilon C \text{ if and only if } zf'\varepsilon S^* \tag{1}$$

Noor and Thomas[2] and Noor[3] have introduced and studied a new class C^* of close-to-convex functions which is a natural analogue of the class C in terms of the property defined in (1), that is

$$f\varepsilon C^* \text{ if and only if } zf'\varepsilon K \tag{2}$$

A function $f\varepsilon C^*$ if f is analytic in E with $f(0)=0$, $f'(0)=1$, and there exists a function $g\varepsilon C$ such that for $z\varepsilon E$,

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\}>0, \qquad \text{for } |z|<1. \tag{3}$$

$f\varepsilon C^*$ is called a quasi-convex function. Form (2) and (1), we can write

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr] & S^* \arrow[d] \\ C^* \arrow[r] & K \end{tikzcd}$$

where the arrows indicate the set inclusion. Previously no relationship between the classes C^* and S^* was known, but Aloboudi[1] has shown recently that there is a function which is in C^* but not in S^*, and also Koebe function εS^* are shown not belonging to C^*.

We shall discuss the relationship of the class K of close-to-convex functions with its various subclasses. The class K is a subclass of S of univalent functions[4] and Kryzyż[5] has shown that radius of close-to-convexity of every function in S is greater than or equal to r_0, $.80 < r_0 < .81$. In this paper, we prove the following results.

THEOREM 1.

Let $f\varepsilon C^*$ in E. Then f maps $|z| < 4\sqrt{2}-5$ onto a convex domain and this result is sharp.

PROOF:

It has shown by Lewandowski[6] that the exact radius R such that the image of $|z| < R$ by $f\varepsilon K$ in E is a starshaped domain (with respect to the origin) is $R = 4\sqrt{2} - 5$. So we see that from this result and the fact that $f\varepsilon C^*$ if and only if $zf'\varepsilon K$ implies that the exact radius for which zf' maps the unit disc onto a starshaped domain is $R = 4\sqrt{2} - 5$. Hence f maps $|z| < 4\sqrt{2}-5$ onto a convex domain.

Furthermore, Lewandowski's method yields the existence of an extremal function which maps E onto the w-plane cut along a half-line not passing through the origin. Consequently we deduce that the result is sharp.

It is shown[2] that if $f\varepsilon C^*$, then $\operatorname{Re} \frac{zf'(z)}{g(z)} > 0$, $g\varepsilon C$ and $z\varepsilon E$. We now prove the following result.

THEOREM 2

Let $f: f(z) = z + a_2z^2 + \cdots\cdots\cdots$ be analytic in E and $g: g(z) = z + b_2z^2 + \cdots\cdots\cdots$ be convex in E. If $\operatorname{Re} \frac{zf'(z)}{g(z)} > 0$, for $z\varepsilon E$, then $\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0$ for $|z| < 1/3$.

Also, if $\operatorname{Re}\frac{zf'(z)}{g(z)} > 0$, $g\varepsilon C$, $z\varepsilon E$, then f maps $|z| < 1/3$ onto a convex domain.

PROOF:

We can write, for $z\varepsilon E$ and $g\varepsilon C$,

$$zf'(z) = g(z)h(z), \qquad \operatorname{Re} h(z) > 0.$$

Hence

$$\frac{(zf'(z))'}{g'(z)} = h(z) + \frac{g(z)}{g'(z)}h'(z),$$

from which it follows that

$$\operatorname{Re}\frac{(zf'(z))'}{g'(z)} \geq \operatorname{Re}h(z) - \left|\frac{g(z)}{g'(z)}h'(z)\right| \tag{4}$$

Since g is convex in E, so we have $\operatorname{Re}\frac{zg'(z)}{g(z)} > 1/2$ for $z\varepsilon E$ and consequently

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geq (1+|z|)^{-1}, \qquad \text{see[8]}.$$

This implies that

$$\left|\frac{g(z)}{g'(z)}\right| \leq \frac{|z|}{1+|z|} \tag{5}$$

Also it is known[7] that $|h'(z)| \leq (2\operatorname{Re}h(z))/(1-|z|^2)$ (6)

From (4), (5) and (6), we get

$$\operatorname{Re}\frac{(zf'(z))'}{g'(z)} \geq \operatorname{Re}h(z) - \frac{|z|}{(1+|z|)}\ \frac{2\operatorname{Re}h(z)}{1-|z|^2}$$

$$= \operatorname{Re}h(z)[(1-3|z|)/(1-|z|)]$$

Hence $\operatorname{Re}\frac{(zf'(z))'}{g'(z)} > 0$ for $|z| < 1/3$.

We now prove that if $\operatorname{Re}\frac{zf'(z)}{g(z)} > 0$, $g\varepsilon C$ and $z\varepsilon E$, then f maps $|z| < 1/3$ onto a convex domain. For $z\varepsilon E$, we have

$$(zf'(z))' = g'(z)h(z) + g(z)h'(z), \qquad \operatorname{Re} h(z) > 0,$$

from which it follows that

$$\operatorname{Re}\frac{(zf'(z))'}{f'(z)} \geq \operatorname{Re}\frac{zg'(z)}{g(z)} - \left|\frac{zh'(z)}{h(z)}\right| \geq \frac{1}{1-|z|} - \frac{2|z|}{1-|z|^2}$$

$$= (1-3|z|)/(1-|z|^2)$$

Hence $\operatorname{Re}\frac{(zf'(z))'}{f'(z)} > 0$ for $|z| < 1/3$, and this gives the required result.

THEOREM 3.

Let $g \in S^*$ and $\operatorname{Re} \frac{zf'(z)}{g(z)} > 0$, for $z \in E$. Then $\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0$ for $|z| < 2-\sqrt{3}$. This result is sharp.

PROOF:

For $z \in E$, we have $zf'(z) = g(z)h(z)$, $\operatorname{Re} h(z) > 0$, from which it follows that

$$\frac{(zf'(z))'}{g'(z)} = h(z) + \frac{g(z)}{g'(z)}h'(z)$$

Thus

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} \geq \operatorname{Re} h(z)\left[1 - \frac{1+|z|}{1-|z|} \, \frac{2|z|}{1-|z|^2}\right].$$

Hence $\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0$ for $|z| < 2-\sqrt{3}$.

The function $f(z) = g(z) = z/(1-z)^2$ shows that this result is sharp.

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RADAR RECEPTION AND NILPOTENT HARMONIC ANALYSIS IV

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Presented by G.F.D. Duff, F.R.S.C.

The prototype among the infinite dimensional irreducible unitary linear representations $(\tilde{U}_\lambda)_{\lambda \in R, \lambda \neq 0}$ of the real Heisenberg nilpotent group $\tilde{A}(R^n)$, $n \geq 1$, which may be constructed for instance by Kirillov theory is the linear Schrödinger representation $\tilde{U}$. It can be singled out by normalizing Planck's constant, that is to say by putting $\lambda = 1$. One of the most important properties of the irreducible unitary representation $\tilde{U}$ is that it belongs to the discrete series of $\tilde{A}(R^n)$. Let Z denote the (one-dimensional) center of $\tilde{A}(R^n)$. According to the Stone-von Neumann-Segal theorem, $\tilde{U}$ forms an isomorphism of the complex convolution algebra $\mathcal{S}(\tilde{A}(R^n)/Z)$ to the algebra of all kernel operators on the complex Schwartz space $\mathcal{S}(R^n)$ with kernel in $\mathcal{S}(R^n \times R^n)$. By transposition, a vector subspace $\mathcal{S}'_2(\tilde{A}(R^n)/Z)$ of the complex vector space $\mathcal{S}'(\tilde{A}(R^n)/Z)$ of all tempered distributions on $\tilde{A}(R^n)$ mod Z acts on $\mathcal{S}(\tilde{A}(R^n)/Z)$ by right and left convolution, i.e., $\mathcal{S}(\tilde{A}(R^n)/Z)$ is equipped with the structure of a $\mathcal{S}'_2(\tilde{A}(R^n)/Z)$-bimodule. In particular, the unitary symplectic Fourier transform of the functions in $\mathcal{S}(\tilde{A}(R^n)/Z)$ is given by right convolution with the constant function $\frac{1}{2^n} \in \mathcal{S}'_2(\tilde{A}(R^n)/Z)$.

It is the purpose of the present part of this series of papers [4-6] to establish Titlebaum's identity for the symplectic Fourier transform of radar crossambiguity functions within the framework of nilpotent harmonic analysis. As a special case, an analogue of Moyal's identity for the mixed Wigner distribution of quantum mechanics obtains.

1. $\mathscr{S}$-Multipliers on $\tilde{A}(R^n)$

Let $\mathscr{S}(\tilde{A}(R^n))$, the Schwartz space of the real Heisenberg nilpotent group $\tilde{A}(R^n)$, to be the usual vector space over the field C of all complex-valued infinitely differentiable functions on R^{2n+1} rapidly decreasing at infinity under its canonical locally convex vector space topology. As is well known, its topological antidual $\mathscr{S}'(\tilde{A}(R^n))$, the complex vector space of all tempered distributions on $\tilde{A}(R^n)$, is too large to be an algebra under convolution. Following Corwin [1], we say that a tempered distribution $T \in \mathscr{S}'(\tilde{A}(R^n))$ is a right $\mathscr{S}$-multiplier on $\tilde{A}(R^n)$ if $f*T \in \mathscr{S}(\tilde{A}(R^n))$ whenever $f \in \mathscr{S}(\tilde{A}(R^n))$ and that T is a left $\mathscr{S}$-multiplier if $T*f \in \mathscr{S}(\tilde{A}(R^n))$ whenever $f \in \mathscr{S}(\tilde{A}(R^n))$. Of course, $T \in \mathscr{S}'(\tilde{A}(R^n))$ is called an $\mathscr{S}$-multiplier on $\tilde{A}(R^n)$ if it is both a left and a right $\mathscr{S}$-multiplier.

The set $\mathscr{S}'_2(\tilde{A}(R^n))$ of all $\mathscr{S}$-multipliers on $\tilde{A}(R^n)$ forms a complex vector space. More precisely, we have the linear embeddings

$$\mathscr{S}(\tilde{A}(R^n)) \longrightarrow \mathscr{S}'_2(\tilde{A}(R^n)) \hookrightarrow \mathscr{S}'(\tilde{A}(R^n)).$$

The elements $T \in \mathscr{S}'_2(\tilde{A}(R^n))$ are exactly the finite sums of terms of the form $D\mu$ where for an arbitrary number $k \in \mathbf{N}$, μ denotes a measure on $\tilde{A}(R^n)$ so that the measure $(1+|(x,y,z)|^2)^k\mu$ is finite on $\tilde{A}(R^n)$ and D denotes a bi-invariant linear differential operator. This characterization implies that $\mathscr{S}'_2(\tilde{A}(R^n))$ forms a complex algebra under convolution. In particular, $\mathscr{S}(\tilde{A}(R^n))$ is a $\mathscr{S}'_2(\tilde{A}(R^n))$-bimodule.

2. The Symplectic Fourier Transform

Let $\mathfrak{n}$ denote the Lie algebra of $\tilde{A}(R^n)$. Then we have the direct sum decomposition

$$\mathfrak{n} = V \oplus Z$$

where $V = \log R^n \oplus \log R^n$ is a vector subspace of $\mathfrak{n}$ which is isomorphic to the space $R^n \oplus R^n$. If we put

$X = (\log x_1, \log x_2) \in V$ and $Y = (\log y_1, \log y_2) \in V$ then the bracket operation of $\mathfrak{n}$ induces on V a symplectic (= nondegenerate alternating bilinear) form $B: V \times V \longrightarrow R$ given by

$$B(X,Y) = [X,Y] = \langle x_2|y_1\rangle - \langle x_1|y_2\rangle.$$

Following a recent paper by Howe [3] we shall call exp V the isotropic cross-section to Z in $\tilde{A}(R^n)$. Since the linear mapping between complex Schwartz spaces

$$s: \mathcal{S}(\tilde{A}(R^n)/Z) \longrightarrow \mathcal{S}(V)$$

which is given by

$$s: f \rightsquigarrow (V \ni X \rightsquigarrow f(\exp X))$$

is an isomorphism, by transport of structure we may consider $\mathcal{S}(V)$ to be a complex algebra under the convolution product on $\mathcal{S}(\tilde{A}(R^n)/Z)$. Indeed, define for functions g_1 and g_2 in $\mathcal{S}(V)$ the convolution product

$$g_1 \underset{B}{*} g_2 = s(s^{-1}(g_1) * s^{-1}(g_2))$$

and observe that any element of $\tilde{A}(R^n)$ can be written in the form

$$(\exp X, z) = (\exp X, 0).(0,0,z)$$

where $X \in V$, $z \in Z$. Then the group law in $\tilde{A}(R^n)$ takes the form

$$(\exp X_1, z_1).(\exp X_2, z_2) = (\exp(X_1+X_2), z_1+z_2+\tfrac{1}{2}B(X_1,X_2))$$

(basic presentation of the real Heisenberg nilpotent group $\tilde{A}(R^n)$). Let $\mathcal{S}(\tilde{A}(R^n)/Z)$ be identified with the complex vector space

$$\{f \in \mathcal{S}(\tilde{A}(R^n)) \mid f((\exp X, z)) = e^{2\pi i z} f((\exp X, 0)),\ X \in V,\ z \in Z\}.$$

Then we obtain explicitly

$$g_1 \underset{B}{*} g_2(X)=\int_V g_1(Y)g_2(X-Y)e^{\pi iB(Y,X)}dY$$
$$=\int_V g_1(X-Y)g_2(Y)e^{\pi iB(X,Y)}dY$$

for $g_j \in \mathcal{S}(V)$, $j \in \{1,2\}$. In particular, the unitary symplectic Fourier transform $\hat{g}$ of $g \in \mathcal{S}(V)$ is given by right convolution with the constant function $\frac{1}{2^n} \in \mathcal{S}'_2(V)$, i.e., we have

$$\hat{g}\colon V \ni X \rightsquigarrow 2^{-n}\int_V g(Y)e^{\pi iB(Y,X)}dY = g \underset{B}{*} 2^{-n}(X)=2^{-n} \underset{B}{*} g(-X)$$

by the skew symmetry of the bilinear form B. Similarly, the unitary symplectic Fourier cotransform is given by left convolution with $\frac{1}{2^n}$ and both transforms may be extended uniquely to involutive automorphisms of the complex Hilbert space $L^2(V)$.

3. Titlebaum's Identity

Let $(f_j)_{1\leq j\leq 4}$ be any four functions belonging to the complex Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and observe that the corresponding radar crossambiguity functions are related to the coefficient functions of the linear Schrödinger representation $\tilde{U}$ of $\tilde{A}(\mathbb{R}^n)$ according to the identity

$$H(f_j,f_k;.) = c_{\tilde{U}}(f_j,f_k;(.,0)),$$

j, $k \in \{1,\ldots,4\}$; cf. [5]. We shall consider the functions $H(f_j,f_k;.)$ as elements of the complex Hilbert space $L^2(V)$. A few exercises in the yoga of computation rules for the coefficient functions of linear representations which are square integrable modulo the center yields for all $X \in V$, $Y \in V$:

$$(H(f_1,f_2,.\ -Y).\bar{H}(f_3,f_4;.\ +Y)) \underset{B}{*} \frac{1}{2^n}(-2X) =$$
$$\frac{1}{2^n}H(f_1,f_3;X-Y).\bar{H}(f_2,f_4;X+Y).$$

The preceding identity expresses the unitary symplectic Fourier transform at the point -2X (or the unitary symplectic Fourier cotransform at the point 2X) of the "twisted" product of radar crossambiguity functions as a twisted product of such ambiguity functions. In the case n = 1, Titlebaum's identity arises (cf. Titlebaum [8] and Helstrom [2]).

Titlebaum's identity as displayed above includes, for Y = 0, Sussman's result [7] for the symplectic Fourier transform of products of radar crossambiguity functions, and for X = Y = 0, an analogue of Moyal's identity for the mixed Wigner phase-space quasiprobability distribution functions which are the Euclidean Fourier transforms of the radar crossambiguity functions (cf. Theorem 1 of [5]).

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ISOMETRY GROUPS, FIXED POINTS AND CONFORMAL TRANSFORMATIONS

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Presented by H.S.M. Coxeter, F.R.S.C.

Abstract

Every finite group of Möbius transformations is conjugate to a group of orthogonal transformations.

Let $S^n = \{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$ denote the n-sphere in $\mathbb{R}^{n+1}$; $\mathcal{M}_n$, the group of conformal (≡Möbius) transformations of S^n; and $\mathcal{O}_{n+1}$, the group of isometries (≡orthogonal transformations) of S^n. These groups are closely related: the group $\mathcal{M}_n$ is generated by inversions in arbitrary (n-1)-spheres and any Möbius transformation can be written as the product of at most n+2 such inversions; the subgroup $\mathcal{O}_{n+1} \subset \mathcal{M}_n$ is generated by inversions in great (n-1)-spheres and any orthogonal transformation can be written as the product of at most n+1 inversions in great (n-1)-spheres. In [5] it is shown, among other results, that if a Möbius transformation $g: S^n \to S^n$ has finite period then there is an inversion $\gamma_o: S^n \to S^n$ such that the conjugate transformation $\gamma_o g \gamma_o$ is an orthogonal transformation. The main purpose of this note is to generalize this result to the following

Theorem. Let $G \subset \mathcal{M}_n$ be a finite group of Möbius transformations of the n-sphere. Then there is an inversion γ_o which conjugates G to a group $\gamma_o G \gamma_o \subset \mathcal{O}_{n+1}$ of orthogonal transformations.

In ([3] pp. 22-24) there is a proof of this theorem for the case $n=2$ where complex numbers can be used as coordinates. The present proof involves the natural extension of the action of G to the ball $H^{n+1}=\{\vec{x}\varepsilon R^{n+1}:\|\vec{x}\|<1\}$. This ball serves as a Poincaré model of hyperbolic (n+1)-space and G acts on it by isometries of the hyperbolic geometry ([2],[4]). Accordingly we are interested in the action of a finite group of isometries on a metric space. The key lemma, which may be of some independent interest, applies to groups which possess a bounded orbit and not just to finite groups.

Lemma: Let X be a metric space in which closed balls are compact and Property M, which is defined below, is satisfied. Then every group of isometries of X which possesses a bounded orbit also possesses an orbit of length 1 i.e. a point which is fixed by all the transformations in the group.

The sphere S^n has the property that closed balls are compact and every group of isometries possesses bounded orbits. Nevertheless the full group O_{n+1} or the group of order 2 generated by the antipodal map serve to show that there are isometry groups which do not possess an orbit of length 1. Some additional hypothesis on the space X is required for the validity of the Lemma and our Property M is convenient because it is easy to verify in Euclidean and hyperbolic space.

Property M: Let $r>0$ and let P, Q, Q' be points of X satisfying $d(P,Q)\leq r$ and $d(P,Q')\leq r$. Then there is a point $M=M(Q,Q',r)$ and a number $\alpha=\alpha(Q,Q',r)$ with $0\leq\alpha<1$ such that $d(P,M)\leq\alpha r$.

In Euclidean and hyperbolic space the point M can be taken to be the midpoint of QQ' and the value of α depends only on r and the number $d=\frac{1}{2}d(Q,Q')\leq r$. To verify Property M we note that d(P,M) is maximal when QQ' is a chord of the sphere of radius r centred at P. In this situation we can apply the appropriate version of "Pythagoris' Theorem" to the right angle triangle MPQ in order to compute α. The Euclidean value is $\alpha=\sqrt{1-(d/r)^2}$ and the hyperbolic value is $\alpha=r^{-1}\cosh^{-1}(\cosh r/\cosh d)$.

The proof of the Lemma is motivated by the proof of an analogous result. If a group of affine transformations G acting on an affine space X has a finite orbit then the centre of gravity of that orbit provides a common fixed point for the transformations in G([1] p. 44).

<u>Proof of the Lemma</u>. Let $G=\{g_i : i\varepsilon I\}$ be a group of isometries acting on the metric space X and let $\Omega_P=\{P_i=P^{g_i} : i\varepsilon I\}$ be a bounded orbit. Let $B(P_i,r)$ be the closed ball with centre P_i and radius r. By taking $r>\text{diameter } \Omega_P$ we can guarantee that

$$B(r) \equiv \bigcap_{i\varepsilon I} B(P_i,r)\neq\emptyset.$$

The set B(r) is invariant under the action of G because if $g\varepsilon G$

$$B(r)^g = \bigcap_{i\varepsilon I} B(P_i,r)^g = \bigcap_{i\varepsilon I} B(P_i{}^g,r) = \bigcap_{j\varepsilon I} B(P_j,r) = B(r).$$

Let $r_o=\inf\{r:B(r)\neq\emptyset\}$. We show that $B(r_o)\neq\emptyset$. If r_j is a sequence of numbers which strictly decreases to r_o then for any $j\varepsilon\{1,2,3,\ldots\}$ $B(r_j)\neq\emptyset$. If $Q_j\varepsilon B(r_j)$ then $Q_j\varepsilon B(P_1,r_j)$ hence $Q_j\varepsilon B(P_1,r_1)$. Since the closed balls in X are compact there is a point Q_o and a subsequence, which we take to be Q_j itself, such that $Q_j\rightarrow Q_o$. For each $i\varepsilon I$, $d(Q_j,P_i)\leq r_j$ hence $d(Q_o,P_i)\leq r_o$ and therefore $Q_o\varepsilon B(r_o)$.

To complete the proof of the Lemma we use Property M of X to show that $B(r_o)=\{Q_o\}$ so that Q_o serves as a common fixed point for the elements in G. If there is a point $Q_o'\neq Q_o$ in $B(r_o)$ then for each $i\varepsilon I$, $d(P_i,Q_o)\leq r_o$ and $d(P_i,Q_o')\leq r_o$. It follows that there is a point $M_o=M(Q_o,Q_o',r_o)$ and a number $\alpha_o=\alpha(Q_o,Q_o',r_o)<1$ such that $d(P_i,M_o)\leq\alpha_o r_o$. This means that $M_o\varepsilon B(\alpha_o r_o)$ and since $\alpha_o r_o<r_o$ we have a contradiction to the definition $r_o=\inf\{r:B(r)\neq\emptyset\}$.

<u>Proof of the Theorem</u>. Each inversion $\gamma:S^n\rightarrow S^n$ extends to an inversion or reflection $\Gamma:\mathbb{R}^{n+1}\cup\{\infty\}\rightarrow\mathbb{R}^{n+1}\cup\{\infty\}$ which acts as a reflection on the hyperbolic (n+1)-space H^{n+1}. If G is a group of Möbius transformations acting on S^n then each transformation in G can be factored into inversions and thus extended to an isometry of H^{n+1}. If the group G is finite then the Lemma applies to its action on H^{n+1} to produce a fixed point $P_o\varepsilon H^{n+1}$. Let Γ_o be the hyperbolic reflection which interchanges P_o with the point $0=\vec{0}$ at the "centre" of the model. The hyperbolic isometries in the conjugate group $\Gamma_o G\Gamma_o$ fix 0 and therefore act as Euclidean isometries on $\mathbb{R}^{n+1}$. If γ_o is the restriction of Γ_o to S^n then by restricting all the transformations to S^n we see that γ_o conjugates the Möbius group G to a group of orthogonal transformations $\gamma_o G\gamma_o$.

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THE CHARACTER GENERATORS OF SO(2k) AND SO(2k+1)

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Presented by G.de B. Robinson, F.R.S.C.

Abstract A combinatorial method of writing down the character generators of SO(2k) and SO(2k+1) is described.

1. Introduction

Character generators of the groups SU(n) [6] and Sp(2k) [3] have been established through the application of certain general results concerning partially ordered sets to the problem of enumerating arrays in the form of Young tableaux which specify the characters of these groups. The combinatorial description of group characters is here extended to the remaining classical groups SO(2k) and SO(2k+1) and thereby the character generators of these groups are obtained.

2. Irreducible characters of SO(2k)

Each irreducible representation $[\lambda]$ of SO(2k) is labelled by means of its highest weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{k-1} \geqslant |\lambda_k|$ where λ_i for $i = 1, 2, \ldots, k$ are either all integers or all half odd integers [4]. The character in the representation λ of an element of SO(2k) in the class labelled by $\underline{\Phi} = (\phi_1, \phi_2, \ldots, \phi_k)$ takes the form

$$\chi^{[\lambda]}(\underline{\Phi}) = \sum_{\underline{w}} M_{\underline{w}}^{[\lambda]} \exp(i\,\underline{w}\cdot\underline{\Phi}) \quad . \qquad (1)$$

The multiplicities $M_{\underline{w}}^{[\lambda]}$ of the weights $\underline{w}$ may be determined through the use of Schur function methods [7] or the use of Young tableaux and their generalisations [2].

The generalised Young diagram, F^{λ}, corresponding to λ consists, if the components of λ are integers, of boxes arranged in left-adjusted rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, |\lambda_k|$ and, if the components of λ are half odd integers, of boxes in rows of lengths $\lambda-\frac{1}{2}, \lambda-\frac{1}{2}, \ldots, \lambda_{k-1}-\frac{1}{2}, |\lambda_k|-\frac{1}{2}$ left-adjusted to a single column of half boxes of length k. Numberings or arrays, N^{λ}, are obtained by inserting positive and negative integers chosen from the set $S = \{\pm1, \pm2, \ldots, \pm k\}$ into the boxes and half boxes of F^{λ} in such a way that : (i) the entries are non-decreasing from left to right across each row, (ii) the entries are strictly increasing in magnitude down a column of half boxes, (iii) the entries are strictly increasing down each column of boxes, (iv) no entry i or -i (denoted by i and $\bar{i}$ respectively) may appear in any row lower than the ith row, (v) no entry i may appear to the right of an entry $\bar{i}$ in the ith row unless it also lies immediately below an entry $\bar{i}$, (vi) if $\lambda_k \neq 0$ and no two entries in the first column are of the same magnitude then the number of negative entries in that column is even or odd according as λ_k is positive or negative respectively. The ordering to be used in applying these rules is defined by $\bar{1}<1<\bar{2}<2< \ldots <\bar{k}<k$.

Typical arrays associated with the representations $[332\bar{1}]$ and $[7/2\ 3/2\ 3/2\ \overline{1/2}]$ of SO(8) are given by

$$\begin{array}{lll} 1 & \bar{2} & 2 \\ \bar{2} & 2 & \bar{3} \\ \bar{4} & 4 & \\ 4 & & \end{array} \qquad \text{and} \qquad \begin{array}{llll} \bar{1}/ & \bar{1} & \bar{1} & \bar{2} \\ \bar{2}/ & \bar{3} & & \\ 3/ & 3 & & \\ \bar{4}/ & & & \end{array}$$

where the symbol / separates entries in half boxes from those in boxes.

The multiplicity $M^{[\lambda]}_{\underline{w}}$ of the weight $\underline{w} = (w_1, w_2, \ldots, w_k)$ is the sum of contributions 2^p arising from each distinct array N^{λ} such that

$$w_i = n_i - n_{\bar{i}} \qquad \text{for } i = 1, 2, \ldots, k \tag{2}$$

where n_i is half the number of entries i in half boxes plus the number of entries i in boxes, and $n_{\bar{i}}$ is the corresponding number calculated for $\bar{i}$. The factor p is the number of pairs of entries $\bar{i}$ and i appearing in the (i-1)th and ith rows, respectively, of the first column of the array with i< k. The arrays illustrated thus each contribute 1 to $M^{[332\bar{1}]}_{(10\bar{1}1)}$ and

$M\binom{7/2\ 3/2\ 3/2\ 1/2}{5/2\ 3/2\ 1/2\ \overline{1/2}}$ respectively.

3. The character generator of SO(2k)

Despite the complex nature of the above combinatorial description of the characters of SO(2k) they may be generated in a manner analagous to that already used for characters of SU(k) and Sp(2k). In this case the relevant shifted Young diagrams, $Z^{\underline{m}}$, are those for which the row lengths, left-adjusted to a diagonal line, are given by $\underline{m} = (2k-1,2k-3, \ldots ,3,1)$. To each $Z^{\underline{m}}$ there corresponds a set of generalised standard shifted Young tableaux (GSSYT). For example in the case $k = 2$, $\underline{m} = (3,1)$ and the set of GSSYT is

$$\begin{array}{cccc} \begin{matrix}1&2&3\\&4&\end{matrix} & \begin{matrix}1&2&\bar{3}\\&4&\end{matrix} & \begin{matrix}1&2&3\\&\bar{4}&\end{matrix} & \begin{matrix}1&2&\bar{3}\\&\bar{4}&\end{matrix} \\ \begin{matrix}1&2&4\\&3&\end{matrix} & \begin{matrix}1&2&\bar{4}\\&3&\end{matrix} & \begin{matrix}1&2&4\\&\bar{3}&\end{matrix} & \begin{matrix}1&2&\bar{4}\\&\bar{3}&\end{matrix} \end{array} \tag{3}$$

The generalisation exemplified here is the inclusion of negative as well as positive entries, but only at the extreme right hand end of each row. It is convenient to specify the signature of the entry at the end of the ath row of the GSSYT π, by $\eta_a(\pi) = \pm 1$ and to define the parity of π by $\eta(\pi) = \eta_1(\pi)\,\eta_2(\pi) \ldots \eta_k(\pi)$.

The character generating formula of SO(2k) is then

$$F_{\underline{m}}(\underline{A},\underline{X}) = \sum_{\pi} \left\{ \prod_{j \in K_\pi} \gamma(\pi^{(j)}) \Big/ \prod_{i=1}^{m} \left[1 - \gamma(\pi^{(i)})\right] \right\} \tag{4}$$

where: π ranges over all GSSYT of shape $\underline{m} = (2k-1,2k-3, \ldots ,3,1)$; $m = k^2$; K_π is the set of those j for which $j+1$ appears in π in a row above one containing j or for which $\bar{j}$ appears in a row of π whilst $\overline{j+1}$ does not appear in a lower row; $\pi^{(i)}$ is obtained from π by deleting all entries greater than i in magnitude. Whilst $\underline{m}$ [3,6] specifies the row lengths of π and $\underline{m}^{(i)}$ those of $\pi^{(i)}$, it is convenient to introduce $\tilde{\underline{m}}^{(i)}$ where $\tilde{m}_a^{(i)}$ is $m_a^{(i)}+1$ if $m_a^{(i)} = 2k-2a+1$, $m_{a-1}^{(i)} = 2k-2a+2$ (for $a>1$) and $\eta_a(\pi) = -1$ and is $m_a^{(i)}$ otherwise, for $a = 1,2, \ldots, k(i)$ where $k(i)$ is the number of parts of $\pi^{(i)}$. Finally

$$\tilde{\gamma}(\pi^{(i)}) = \left[\tilde{A}_{k(i)} X_{\tilde{\underline{m}}_1}(i) X_{\tilde{\underline{m}}_2}(i) \cdots X_{\tilde{\underline{m}}_{k(i)}}(i) \right]^{\varepsilon(i)} \tag{5}$$

where $\varepsilon(i)$ is $1/2$ if $\pi^{(i)} = \pi$ and is 1 if $\pi^{(i)} \neq \pi$, whilst $\tilde{A}_{k(i)}$ is $A_{k(i)}$ if $k(i)<k$ and is $A_{\eta(\pi)k}$ if $k(i)=k$.

The required SO(2k) character generator is then given by

$$F_{\underline{m}}(\underline{A},\underline{X}) = \sum_{\lambda} c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_k^{\lambda_k} \chi^{[\lambda]}(\underline{\phi}) \tag{6}$$

with $\underline{m} = (2k-1, 2k-3, \ldots, 3, 1)$ and $\underline{A} = (c_1, c_1c_2, \ldots, c_1c_2\cdots c_k)$. It is to be understood that, as indicated, $A_k = c_1c_2\cdots c_k$ but that $A_{-k} = c_1c_2\cdots c_{k-1}c_k^{-1}$. The dependence on $\underline{\phi}$ is determined by $\underline{X} = (e^{i\phi_k}, e^{-i\phi_k}, \ldots, e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1})$. The denominators in (4) are products of $m = k^2$ factors, whilst the number of terms in the sum is

$$2^k g^{(2k-1,2k-3,\ldots,1)} = \frac{2^k (k^2)!\,(k-1)!\,(k-2)!\,\ldots\,1!}{(2k-1)!\,(2k-2)!\,\ldots\,(k+1)!\,k!} . \tag{7}$$

4. Example: SO(4)

In the case of the group SO(4) $k = 2$, $\underline{m} = (3,1)$ and through the use of (3) the expression (4) takes the form :

$$\begin{aligned} F_{(3,1)}(\underline{A},\underline{X}) = \; & \left\{ 1/(1 - A_1X_1)(1 - A_1X_2) \right\} \cdot \\ \cdot \Big\{ & 1/(1 - A_1X_3)(1 - A_2^{\frac{1}{2}}X_3^{\frac{1}{2}}X_1^{\frac{1}{2}}) + A_1X_4/(1 - A_1X_4)(1 - A_{\bar{2}}^{\frac{1}{2}}X_4^{\frac{1}{2}}X_1^{\frac{1}{2}}) \\ & + A_{\bar{2}}^{\frac{1}{2}}X_3^{\frac{1}{2}}X_2^{\frac{1}{2}}/(1 - A_1X_3)(1 - A_{\bar{2}}^{\frac{1}{2}}X_3^{\frac{1}{2}}X_2^{\frac{1}{2}}) + A_2^{\frac{1}{2}}X_4^{\frac{1}{2}}X_2^{\frac{1}{2}}/(1 - A_1X_4)(1 - A_2^{\frac{1}{2}}X_4^{\frac{1}{2}}X_2^{\frac{1}{2}}) \\ & + A_2X_2X_1/(1 - A_2X_2X_1)(1 - A_2^{\frac{1}{2}}X_3^{\frac{1}{2}}X_1^{\frac{1}{2}}) + A_{\bar{2}}^{\frac{1}{2}}X_4^{\frac{1}{2}}X_1^{\frac{1}{2}}/(1 - A_{\bar{2}}X_2X_1)(1 - A_{\bar{2}}^{\frac{1}{2}}X_4^{\frac{1}{2}}X_1^{\frac{1}{2}}) \\ & + A_{\bar{2}}X_2X_1/(1 - A_{\bar{2}}X_2X_1)(1 - A_{\bar{2}}^{\frac{1}{2}}X_3^{\frac{1}{2}}X_2^{\frac{1}{2}}) + A_2X_2X_1A_2^{\frac{1}{2}}X_4^{\frac{1}{2}}X_2^{\frac{1}{2}}/(1 - A_2X_2X_1)(1 - A_2^{\frac{1}{2}}X_4^{\frac{1}{2}}X_2^{\frac{1}{2}}) \Big\} . \end{aligned} \tag{8}$$

The character generator is then obtained in the required form,(6), by setting $A_1=c_1$, $A_2=c_1c_2$, $A_{\bar{2}}=c_1c_2^{-1}$ and $\underline{X} = (e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1})$.

This particular result may be considerably simplified. Setting $A_1=BC$, $A_2=B^2$ and $A_{\bar{2}}=C^2$ along with $\underline{X} = (YZ^{-1}, Y^{-1}Z, YZ, Y^{-1}Z^{-1})$ yields

$$F_{(3,1)}(\underline{A},\underline{X}) = 1/(1 - BY)(1 - BY^{-1})(1 - CZ)(1 - CZ^{-1}) \quad . \qquad (9)$$

Since the character generator of SU(2) takes the form [5,6]

$$F_2(\underline{A},\underline{X}) = 1/(1 - AX)(1 - AX^{-1}) \quad , \qquad (10)$$

the result (9) merely serves to indicate the well known isomorphism between SO(4) and SU(2)XSU(2). A similar simplification of (4) occurs in the case of SO(6) which is isomorphic to SU(4), for which the character generator has been given explicitly [6].

5. <u>The character generator of SO(2k+1)</u>

It is straightforward to extend the results to SO(2k+1). Each irreducible representation is labelled by $[\lambda]$, as for SO(2k), but now with no negative parts so that $\lambda_k \geq 0$ [4]. As before, arrays serve to specify contributions to the weight multiplicities but this time with the entries chosen from the set S extended to $S_0 = \{\pm 1, \pm 2, \ldots, \pm k, 0\}$ by the inclusion of 0. This entry, 0, may only appear in a box (not a half box) at the foot of a column. This rule replaces rule (vi) given for SO(2k), and the ordering used now is $\bar{1}<1<\bar{2}<2< \ldots <\bar{k}<k<0$. The entries 0 make no contribution to the components of the weight vector which are still defined by (2).

The resulting change in the character generator is such that the relevant shifted Young diagrams, $Z^{\underline{m}}$, are those for which $\underline{m} = (2k, 2k-2, \ldots, 4, 2)$, precisely as for Sp(2k)[3]. Now however GSSYT are required with negative as well as positive entries allowed at the end of each row. The generating formula is still (4) with $\tilde{Y}(\pi^{(i)})$ defined by (5) but now $\tilde{A}_{k(i)}$ is $A_{k(i)}$ for all k(i), whilst $\tilde{m}_a^{(i)}$ is $m_a^{(i)}+1$ if $m_a^{(i)} = 2k-2a+2$, $m_{a-1}^{(i)} = 2k-2a+4$ (for a>1) and $\eta_a(\pi) = -1$, and is $m_a^{(i)}$ otherwise. The required SO(2k+1) character is then given by (6) with $\underline{m} = (2k, 2k-2, \ldots, 4, 2)$, $\underline{A} = (c_1, c_1c_2, \ldots$

..., $c_1c_2\ldots c_k$) and $\underline{X} = (1, e^{i\phi_k}, e^{-i\phi_k}, \ldots, e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1})$.

For k = 1, the simplest case, the GSSYT are just 1 2 and 1 $\bar{2}$, so that

$$F_{(2)}(\underline{A},\underline{X}) = 1/(1 - A_1X_1)(1 - A_1^{\frac{1}{2}}X_2^{\frac{1}{2}}) + A_1^{\frac{1}{2}}X_3^{\frac{1}{2}}/(1 - A_1X_1)(1 - A_1^{\frac{1}{2}}X_3^{\frac{1}{2}})$$

$$= 1/(1 - A_1^{\frac{1}{2}}X_2^{\frac{1}{2}})(1 - A_1^{\frac{1}{2}}X_2^{-\frac{1}{2}}) \qquad (11)$$

in agreement with (10) when regard is taken of the isomorphism between SO(3) and SU(2). More generally the denominators in (4) are products of $m = k(k+1)$ factors and the number of terms in the sum is

$$2^k g^{(2k,2k-2,\ldots,2)} = 2^k \frac{[k(k+1)]!\,(k-1)!\,(k-2)!\,\ldots,\,1!}{(2k)!\,(2k-1)!\,(2k-2)!\,\ldots\,(k+1)!}\,. \qquad (12)$$

6. Conclusion

The results presented here complete the task of constructing the character generators of all the classical groups. The procedure used to pass from the arrays specifying irreducible characters to SSYT and GSSYT via certain partially ordered sets will be discussed in more detail elsewhere, along the lines suggested by Baclawski [1].

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A RELATION BETWEEN AN AFFINE KILLING VECTOR AND THE STRAIN TENSOR OF A PSEUDO-RIEMANNIAN MANIFOLD

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Presented by P.G. Rooney, F.R.S.C.

In [2] Yano has shown that a Killing vector field in a Riemannian manifold is an affine Killing vector-field. Moreover, he has shown that an affine Killing vector-field in a compact and orientable Riemannian manifold without boundary is Killing. THE purpose of this paper is to characterize an affine Killing vector-field of any pseudo-Riemannian manifold(not necessarily compact) in terms of its strain tensor. The characterization is given by

Theorem: <u>The strain tensor corresponding to an affine Killing vector-field in a pseudo-Riemannian manifold</u>(not necessarily compact) <u>is parallel with respect to the Levi-Civita connexion of the pseudo-Riemannian metric</u>.

Proof: Let M be a pseudo-Riemannian manifold with metric g. Then g induces the Levi-Civita connexion ∇ on M. Let X be an affine Killing vector-field on M so that $L_X\nabla=0$. Now if we set $\nabla_X Y=\nabla(X,Y)$ then the commutation of the operator of Lie-derivation L_X and Covariant derivation ∇_Y is given by

$$L_X \nabla_Y Z - \nabla_Y L_X Z - \nabla_{[X,Y]} Z$$

$$= L_X(\nabla(Y,Z)) - \nabla(Y,L_X Z) - \nabla([X,Y],Z)$$

$$= (L_X\nabla)(Y,Z) + \nabla(L_XY,Z) + \nabla(Y,L_XZ) - \nabla(Y,L_XZ) - \nabla(L_XY,Z)$$

$$= (L_X\nabla)(Y,Z).$$

Thus we obtained the formula

(1) $$L_X\nabla_YZ - \nabla_YL_XZ = (L_X\nabla)(Y,Z) + \nabla_{[X,Y]}Z.$$

Next, we have

(2) $$\begin{aligned}(\nabla_YL_Xg)(U,V) &= Y((L_Xg)(U,V)) - (L_Xg)(\nabla_YU,V) - (L_Xg)(U,\nabla_YV)\\ &= Y(X(g(U,V))) - g(L_XU,V) - g(U,L_XV))\\ &\quad - Xg(\nabla_YU,V) + g(L_X\nabla_YU,V) + g(\nabla_YU,L_XV)\\ &\quad - X\,g(U,\nabla_YV) + g(L_XU,\nabla_YV) + g(U,L_X\nabla_YV)\\ &= -Y(g(U,L_XV)) + g(\nabla_YU,L_XV) + g(U,L_X\nabla_YV)\\ &\quad + (YX)\,g(U,V) - Y(g(V,L_XU)) + g(L_XU,\nabla_YV)\\ &\quad + g(V,L_X\nabla_YU) - X\,g(\nabla_YU,V) - X\,g(\nabla_YV,U)\\ &= -g(U,\nabla_YL_XV - L_X\nabla_YV) - g(V,\nabla_YL_XU - L_X\nabla_YU)\\ &\quad -(XY - YX)\,g(U,V),\end{aligned}$$

where we used the fact that g is covariant constant. Using (1) in (2) we find

$$\begin{aligned}(\nabla_YL_Xg)(U,V) &= g(U,(L_X\nabla)(Y,V)) + g(U,\nabla_{[X,Y]}V)\\ &\quad + g(V,(L_X\nabla)(Y,U) + \nabla_{[X,Y]}U) - [X,Y]\,g(U,V)\end{aligned}$$

$$= g\,(U,(L_X\nabla)(Y,V)) + g\,(V,(L_X\nabla)(Y,U)).$$

But $L_X\nabla = 0$.THerefore it follows that $\nabla_Y L_X g = 0$ which implies that $L_X g$ is parallel with respect to ∇. That is, the strain tensor $L_X g$ of (M,) along X [1] is parallel with respect to ∇.

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IDENTIFICATION OF RATIONAL FUNCTIONS: LOST AND REGAINED

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Presented by P. Ribenboim, F.R.S.C.

Abstract. We discuss conjectures and results concerning the Taylor coefficients of rational functions thereby correcting allegations made in this Journal 3 (1981) 279-84.

1. Apology.

In "Identifying a rational function" [1] I induced two colleagues to join me in purporting that up to obvious ambiguities one may recognise a rational function from any infinite subset of its Taylor coefficients. This assertion is sufficiently vague to be difficult to refute. Unfortunately we did hazard more explicit claims. One is false, though 'only just'. But worse, we sketched a 'proof' and it, so to speak, does succeed in 'proving' all our claims. Thus the proof must be dangerously defective.

In the event, it turns out that the growth theorems for recurrence sequences [4], to which we had anyhow made an appeal in passing, in themselves already suffice to prove as much as is true in the matter of identifying a rational function.

2. Introduction.

Accordingly, this note is organised as follows: In section 3 I mention a fine conjecture generated by our original 'proof' and point to a typical counterexample to our explicit claims. In section 4 I explain the delusion that caused our principal error. In section 5 I show that the results of [4] do after all allow us to recover all that we should have wanted to claim.

3. Common values of recurrence sequences. If $\sum_{h\geq 0} f_h X^h$ is a rational function then the sequence (f_h) satisfies a linear homogeneous recurrence relation with constant coefficients, and the terms of the sequence are generalised power sums, or equivalently, an exponential polynomial evaluated at the non-negative integers:

$$f_h =: f(h) = \sum_{i=1}^{m} a_i(h)\alpha_i^{\,h}$$

with the a_i polynomials. Suppose that $\sum g_h X^h$ is rational; so also (g_h) is a recurrence sequence.

We are interested in the intersection of the *sets* $\{f_h\}$ and $\{g_h\}$. In [1] we falsely imply that if this intersection is infinite then there are positive integers d, d' so that all but finitely many of the common elements are complete subrecurrences

$$(f_{dh+r}) = (g_{d'h+r'}), \quad \text{some } r, r' \text{ with } 0\leq r<d,\ 0\leq r'<d'.$$

Were this so we would have had a wonderful generalisation of the Lech-Mahler theorem (see, for example [2]), which asserts that any constant sequence $(c, c, \ldots.)$ intersects (f_h) along complete subrecurrences.

But our claim is false as is shown by

$$f(x) = 2^X \qquad\qquad g(Y) = Y2^Y$$

which take their common values at $x = h + 2^h$, $y = 2^h$.

We had argued as follows: If $f(X) = g(Y)$ for infinitely many $(X,Y) \in \mathbb{Z}^2$ then a p-adic argument leads us to write (in effect)

$$x(h) = f^{-1} \circ g \circ y(h) .$$

Because f,g are exponential polynomials, if one of x,y is an exponential polynomial then both may be supposed to be exponential polynomials. In [1] we somehow came to claim that both x,y could be supposed linear which would have been justified if one of x,y had been known to be linear. I decided that at any rate both x,y would have to be exponential polynomials, an error that dealt handily with the threatening counterexample, and which committed me to what I now realise to be only a speculation: *If* f,g *are exponential polynomials and* $f(X) = g(Y)$ *for infinitely many* $(X,Y) \in \mathbb{Z}^2$ *then there are exponential polynomials* $x(t)$, $y(t)$ *so that identically* $f \circ x(t) = g \circ y(t)$.

<u>4. Determining a rational function.</u> Suppose f,g are exponential polynomials and that y_h is in $\mathbb{Z}$ (or, more generally, in a finitely generated subring of a field of characteristic zero) for all $h \in \mathbb{N} \cup \{0\}$. But say we are given only that

$$f(h) = g \circ y(h) , \qquad h \in N$$

for some suitable subset N of $\mathbb{N} \cup \{0\}$. In [1] we explained that y may be analytically continued to all of $\mathbb{N} \cup \{0\}$ and suggested that then also y is an exponential polynomial on the grounds that y_h is in $\mathbb{Z}$ for all h. This is a nonsense, though only because there is no reason, in general, why the values $\bar{y}_h$ obtained by p-adic continuation should relate to the given y_h, except of course for h in N. For $\mathbb{N} \cup \{0\} \setminus N$ finite, our argument is valid.

On the other hand Perelli and Zannier [3] show that when N meets every arithmetic progression and hence meets every arithmetic progression infinitely many times then, for $g : t \longmapsto t^k$, y_h an integer for $h \in N$ implies y_h an integer for $h \in \mathbb{N} \cup \{0\}$. I believe this to hold for all exponential polynomials g.

Indeed this raises the following question: Suppose that $\sum f_h X^h$ is a rational function regular at ∞; so (f_h) is a recurrence sequence; but we know the f_h only for h in N. *Which subsets* N *completely determine* (f_h)? It is easy to see that N must meet every arithmetic progression for if N does not meet $(hd + r)$ then the rational function

$$\sum f_{hd+r} X^{hd+r}$$

is entirely undetermined by N. Our remarks below go close to implying that the condition is also sufficient: the data f_h for h in N, where N meets every arithmetic progression, determine at most one rational function $\sum f_h X^h$ regular at ∞.

5. <u>Identifying a rational function.</u> A recurrence sequence (f_h) given by

$$f_h = \sum_{i=1}^{m} a_i(h)\alpha_i^{\,h}$$

is said to be nondegenerate if no pairwise quotient α_i/α_j, $i \neq j$, nor any of its characteristic roots α_i, is a root of unity.

Suppose f and g are nondegenerate and consider the diophantine equation

$$f(X) = \sum_{i=1}^{m} a_i(X)\alpha_i^{X} = \sum_{i=1}^{n} b_i(Y)\beta_i^{Y} = g(Y) .$$

If $f(X) = g(Y)$ for infinitely many $(X,Y) \in \mathbb{Z}^2$ then Theorem 2 of [4] implies that $m = n$ and, after a reordering of the terms we have for infinitely many h :

$$\alpha_i^{d} = \beta_i^{d'} \quad \text{and} \quad a_i(hd + r)\alpha_i^{r} = b_i(hd' + r')\beta_i^{r'}, \qquad (1 \le i \le m)$$

for given integers d, d' and integers $r(h)$, $r'(h)$. Thus if two nondegenerate recurrence sequences have infinitely many common values then this is accounted for by a virtual identity of their characteristic roots up to the freedom of choosing d/d', *and* by relationships satisfied by the corresponding coefficients. These relations can yield infinitely many solutions h only if, at the least, each quotient $a_i(hd + r)/b_i(hd' + r')$ is a power of a polynomial linear in h. Certainly *if the sets* $\{f_h\}$ *and* $\{g_h\}$ *have infinite intersection then there are recurrence sequences* (x_h) *and* (y_h) *so that*

$$f_{x(h)} = g_{y(h)}, \qquad h \in \mathbb{N} \cup \{0\} .$$

The 'counterexample' of section 3 is in fact a typical instance of x, y nonlinear. In the degenerate case the argument goes through with only notational change unless both f and g collapse to polynomials on some arithmetic progressions. The results of [4] do not deal directly with an equation $a(X) = b(Y)$ in polynomials a, b .

In summary: excepting degeneracy, a rational function may indeed be recognised, up to acceptable ambiguities, from an infinite subset of its Taylor coefficients presented in any order. The situation is less straightforward than is implied in [1] and the proof depends on quite different and considerably deeper principles.

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