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RADAR RECEPTION AND NILPOTENT HARMONIC ANALYSIS II

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Presented by G. de B. Robinson, F.R.S.C.

In the present paper we adopt the same point of view as in its first part [3]: Motivated by the analogy of the radar uncertainty principle with the Heisenberg uncertainty principle of quantum mechanics, we shall use harmonic analysis on the real Heisenberg nilpotent group $\tilde{A}(\mathbb{R})$ for studying the radar cross- and autoambiguity functions. An important rôle will be played by the property of the linear Schrödinger representation of $\tilde{A}(\mathbb{R})$ of being square-integrable modulo the center $\mathfrak{Z}$ of $\tilde{A}(\mathbb{R})$, i.e., by the Stone-von Neumann-Segal theorem. Among the specific waveforms dealt with in radar synthesis, one of the most extensively treated single signal forms is the monochromatic Gaussian pulse. One reason for this fact is that the area of the "uncertainty ellipse" which plays an important rôle in the discussion of the resolution of signals and the measurement of signal parameters, takes its largest value for the single Gaussian pulse. Since the radar autoambiguity function of the Gaussian signal has likewise a Gaussian shape, another important advantage is the absence of subsidiary peaks from its Wigner-Woodward relief (= ambiguity surface). This property is important because it much reduces the risk of large errors in measuring the epoch and the frequency of the Gaussian signal. However, Klauder [1] and Wilcox [8] among other authors have emphasized that there exists an instructive generalization of the Gaussian signal, to wit, the Hermite-Weber waveforms. These signals have an oscillatory amplitude modulation of their radar carrier frequency, i.e., the real envelope of the Hermite-Weber signals changes signs before finally decreasing exponentially to zero. The radar autoambiguity function of the Hermite-Weber signals may be expressed in terms of the Laguerre-Weber functions. In this case, the Wigner-Woodward reliefs have circular ridges around the central peak whose heights decrease from the center (0,0) of the time-frequency plane. Between them there are circular contour lines on which the autoambiguity function vanishes. For further details, see the paper by Klauder [1] cited above. - Our approach to the radar cross- and autoambiguity functions of the Hermite-Weber waveforms via nilpotent harmonic analysis is based on the complex wave model $\tilde{V}$ or Bargmann-Fock-Segal model by which $\tilde{A}(\mathbb{R})$ is made to act on a certain complex Hilbert space $F(\mathbb{C})$ of entire holomorphic functions on the plane $\mathbb{C}$. This alternate realization of the linear Schrödinger representation of $\tilde{A}(\mathbb{R})$ reveals itself to be more convenient for our purposes than the analog model $\tilde{U}$ (= microparticle model) on the complex Hilbert space $L^2(\mathbb{R};\lambda^1)$ (λ^n= Lebesgue measure of $\mathbb{R}^n$) since the complex wave model of $\tilde{A}(\mathbb{R})$ provides a very effective operational approach to the Hermite-Weber functions.

1. The Radar Crossambiguity Function

Whenever it is necessary to resolve two narrowband signals in the presence of white Gaussian noise, the performance of the receiver depends upon the crosscorrelation of the two signals involved. Therefore, the notion of crosscorrelation function is of great importance in information theory. - Given any pair $(f,g) \in L^2(\mathbb{R};\lambda^1) \times L^2(\mathbb{R};\lambda^1)$ of envelopes, the radar crossambiguity function $H(f,g;.)$ on $\mathbb{R}^2$ is defined according to the prescription

$$H(f,g;v) = H(f,g;x,y) = \int_{\mathbb{R}} f(t+{}^1/_2x)\bar{g}(t-{}^1/_2x)e^{2\pi i ty}d\lambda^1(t)$$

where $v=(x,y) \in \mathbb{R}^2$. The standardization $\|f\| = \|g\| = 1$ is customary. In the case $f=g$, the radar autoambiguity function $H(f,f;.)$ on $\mathbb{R}^2$ arises which will be denoted by $H(f;.)$ as in the first part [3].

Recall that the real Heisenberg nilpotent group $\tilde{A}(\mathbb{R})$ may be realized as the real manifold $\mathbb{R}^2 \oplus \mathbb{R}$ with group multiplication law

$$(v_1,z_1)(v_2,z_2) = (v_1+v_2,z_1+z_2+{}^1/_2B(v_1,v_2))$$

where the symplectic form (= nondegenerate, skew symmetric, $\mathbb{R}$-bilinear form) B on the plane $\mathbb{R}^2$ is defined by the exterior product

$$B(v_1,v_2) = B((x_1,y_1),(x_2,y_2)) = v_1\wedge v_2 = x_1y_2-y_1x_2.$$

(basic presentation of $\tilde{A}(\mathbb{R})$). Observe that Haar measure of $\tilde{A}(\mathbb{R})$ may be identified with Lebesgue measure λ^3 and that the center $\mathfrak{Z}$ of $\tilde{A}(\mathbb{R})$ is isomorphic to $\mathbb{R}$; see [4] and [5] . The analog model $(\tilde{U};L^2(\mathbb{R};\lambda^1))$ of the linear Schrödinger representation which is a mod $\mathfrak{Z}$ square-integrable irreducible unitary linear representation of $\tilde{A}(\mathbb{R})$ is defined for $(v,z)\in\mathbb{R}^2\oplus\mathbb{R}$ as follows:

$$\tilde{U}(v,z)f(t)=\tilde{U}(x,y,z)f(t)=\exp 2\pi i(z+ty+{}^1/_2xy)f(t+x) \quad (t\in\mathbb{R})$$

Let $(v,z)\rightsquigarrow c_{\tilde{U}}(f,g;v,z)=\langle\tilde{U}(v,z)f|g\rangle\in\mathbb{C}$ denote the coefficient function of $\tilde{U}$ relative to the pair $(f,g)\in L^2(\mathbb{R};\lambda^1)\times L^2(\mathbb{R};\lambda^1)$. In the case f=g, let $c_{\tilde{U}}(f,f;.,.)=c_{\tilde{U}}(f;.,.)$ be as in part I. Observe that $c_{\tilde{U}}(f,g;.,.)$ is a continuous and bounded complex-valued function on $\tilde{A}(\mathbb{R})$ which belongs to $L^2(\tilde{A}(\mathbb{R})/\mathfrak{Z};\lambda^2)$ and satisfies

$$H(f,g;v) = c_{\tilde{U}}(f,g;v,0)$$

for all pairs $v=(x,y)\in\mathbb{R}^2$; cf. Theorem 1 of part I [3]. Since $\tilde{U}$ admits formal degree 1, the well-known properties of the coefficient functions of square-integrable representations modulo the center (cf. Moore-Wolf [2]) combined with some standard facts about the unitary dual of the two-step nilpotent Lie group $\tilde{A}(\mathbb{R})$ imply the following result:

Theorem 1. Let arbitrary elements f,f',g,g' of $L^2(\mathbb{R};\lambda^1)$ be given. Then the identity

$$\int_{\mathbb{R}^2} H(f,g;v)\overline{H}(f',g';v)\,d\lambda^2(v) = \langle f|f'\rangle\langle g|g'\rangle$$

holds in $L^2(\mathbb{R}^2;\lambda^2)$. If $(f_m)_{m\geq 0}$ is a Hilbert basis of the space $L^2(\mathbb{R};\lambda^1)$ then the double sequence $(H(f_m,f_n;.))_{m\geq 0,n\geq 0}$ is a Hilbert basis of $L^2(\mathbb{R}^2;\lambda^2)$.

The preceding formula includes as a special case the identity

$$\int_{\mathbb{R}^2} |H(f;v)|^2 d\lambda^2(v) = \|f\|^2$$

(cf. Corollary 1 of Theorem 1 of part I), which can be geometrically interpreted as radar uncertainty principle or "law of conservation of ambiguity". For additional properties of the radar crossambiguity functions, the reader is referred to the paper by Titlebaum [7].

2. The Complex Wave Model

Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by assigning the complex numbers $v = x+iy$ and $\bar{v} = x-iy$ to the pair $(x,y) \in \mathbb{R}^2$. Actually, $\mathbb{C}$ may be viewed in the light of Kirillov theory as the complexified orbit associated with the linear Schrödinger representation of $\tilde{A}(\mathbb{R})$ in the dual of the Lie algebra of $\tilde{A}(\mathbb{R})$ under the coadjoint action. Denote by μ the measure on $\mathbb{C}$ which admits density $v \rightsquigarrow e^{-\pi|v|^2}$ with respect to λ^2. Then the monomials $M_m(v) = \sqrt{\frac{\pi}{m!}}\, v^m$ $(m \geqq 0)$ form a Hilbert basis of the closed subspace $F(\mathbb{C})$ of $L^2(\mathbb{C};\mu)$ consisting of the entire holomorphic functions on $\mathbb{C}$ which are square-integrable with respect to the measure μ. For $(v,z) \in \mathbb{C} \times \mathbb{R}$ and $f \in F(\mathbb{C})$ define

$$\tilde{V}(v,z)f(w) = \exp 2\pi(iz - {}^1/_4|v|^2 - {}^1/_2\bar{v}w)f(w+v) \quad (w \in \mathbb{C}).$$

Then $\tilde{V}$ forms a mod $\mathfrak{Z}$ square-integrable irreducible unitary linear representation of $\tilde{A}(\mathbb{R})$ on the complex Hilbert space $F(\mathbb{C})$. The pair $(\tilde{V};F(\mathbb{C}))$ is called the complex wave model, or Bargmann-Fock-Segal model of the linear Schrödinger representation of $\tilde{A}(\mathbb{R})$. Let $(W_m)_{m\geqq 0}$ denote the sequence of Hermite-Weber functions (= harmonic oscillator wave functions) obtained by orthonormalization from the sequence of functions $(x^m e^{-\pi x^2})_{m\geqq 0}$ in the vector subspace $\mathcal{S}(\mathbb{R})$ of the complex Hilbert space $L^2(\mathbb{R};\lambda^1)$; cf. [6]. It is well known that $(W_m)_{m\geqq 0}$ forms a Hilbert basis of $L^2(\mathbb{R};\lambda^1)$. Indeed, the irreducibility of the unitary linear representation $\tilde{U}$ of $\tilde{A}(\mathbb{R})$ in $L^2(\mathbb{R};\lambda^1)$ implies that the sequence $(W_m)_{m\geqq 0}$ forms a total family in $L^2(\mathbb{R};\lambda^1)$. The $\mathbb{C}$-linear isometry T of $L^2(\mathbb{R};\lambda^1)$ onto $F(\mathbb{C})$ which maps $\bar{W}_m$ onto M_m for all $m \in \mathbb{N}$ defines a unitary isomorphy of $\tilde{U}$ onto $\tilde{V}$, i.e., $T \circ \tilde{U} = \tilde{V} \circ T$. Hence we have

Theorem 2. The radar crossambiguity functions relative to the Hermite-Weber functions $(W_m)_{m \geq 0}$ satisfy the identities

$$H(W_m, W_n; v) = c_{\tilde{V}}(M_m, M_n; v, 0) \quad (m \geq 0,\ n \geq 0)$$

for all numbers $v \in \mathbb{C}$.

It is not difficult to evaluate the right hand side. An explicit calculation verifies that the functions $v \rightsquigarrow H(W_m, W_n; v)$ $(m \geq 0, n \geq 0)$ which form a Hilbert basis of $L^2(\mathbb{R}^2; \lambda^2)$ by Theorem 1, can be expressed in terms of the products

$$v \rightsquigarrow M_{m-n}(v) L_n^{(m-n)}(\tfrac{1}{2}|v|^2) \quad (m \geq n)$$

where $(L_n^{(\alpha)})_{n \geq 0}$ denote the Laguerre-Weber functions of order α. The final expression coincides with a formula established by Wilcox [8] . This author, however, uses a totally different method and avoids any group-theoretic reasoning. - In particular, we obtain for m=n as a special case the following result.

Corollary. Let $(L_m)_{m \geq 0}$ denote the sequence of Laguerre-Weber functions (of order 0). The radar autoambiguity functions relative to the Hermite-Weber functions $(W_m)_{m \geq 0}$ admit the form

$$H(W_m; v) = L_m(\tfrac{1}{2}|v|^2) \quad (m \geq 0)$$

for all $v \in \mathbb{C}$.

If we combine our results with Theorem 2 of part I, then we obtain immediately the following

Theorem 3. The Wigner-Woodward relief $H(f; \mathbb{R}^2)$ of a function $f \in L^2(\mathbb{R}; \lambda^1)$ with norm $\|f\| = 1$ is invariant under the action of the orthogonal subgroup $\underline{O}(2,\mathbb{R})$ of $\underline{Sp}(1,\mathbb{R})$ if and only if there exist a phase factor, i.e., a number $\zeta \in \mathbb{C}$ of absolute value $|\zeta| = 1$ and an integer $m \geq 0$ such that the identity

$$f = \zeta W_m$$

holds λ^1-almost everywhere on the real line $\mathbb{R}$.

It follows that the Hermite-Weber waveforms can be characterized by the property that the resolution of the target range and range rate are the same. - Of course, using tensor products, the preceding results can be adapted to the real Heisenberg nilpotent group $\widetilde{A}(\mathbb{R}^n)$, $n > 1$.

References

1. Klauder, J.R.: The design of radar signals having both high range resolution and high velocity resolution. Bell System Tech. J. 39, 809-819 (1960)

2. Moore, C., Wolf, J.: Square integrable representations of nilpotent groups. Trans. Amer. Math. Soc. 185, 445-462 (1973)

3. Schempp, W.: Radar reception and nilpotent harmonic analysis I. C. R. Math. Rep. Acad. Sci. Canada 4 (1982)

4. Schempp, W.: Gruppentheoretische Aspekte der Signalübertragung und der kardinalen Interpolationssplines I. Math. Meth. in the Appl. Sci. (to appear)

5. Schempp, W.: On the rôle of the Heisenberg group in information theory (to appear)

6. Schempp, W., Dreseler, B.: Einführung in die harmonische Analyse. Stuttgart: B. G. Teubner 1980

7. Titlebaum, E. L.: A generalization of a two-dimensional Fourier transform property for ambiguity functions. IEEE Trans. Inform. Theory IT-12, 80-81 (1966)

8. Wilcox, C. H.: The synthesis problem for radar ambiguity functions. MRC Technical Summary Report, no. 157. The University of Wisconsin, Madison, Wisconsin 1960

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THE SCHLÄFLIAN OF A CRYSTALLOGRAPHIC COXETER GROUP

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Abstract

A Coxeter group is crystallographic if and only if the Schläflian of each distinguished subgroup is a rational integer.

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§1. Let G be the Coxeter group with presentation

$$\langle r_1, \ldots, r_n \mid (r_i r_j)^{p_{ij}} = 1,\ 1 \le i,j \le n \rangle$$

where each $p_{ii} = 1$, and each relation with $p_{ij} = \infty$ is omitted. For a basis $\{\vec{d}_1, \ldots, \vec{d}_n\}$ of $\mathbb{R}^n$ we define on $\mathbb{R}^n$ a bilinear form $(\cdot,\cdot)$ by $(\vec{d}_i, \vec{d}_j) = -2 \cos \pi/p_{ij}$. Thus G is faithfully represented in $GL(\mathbb{R}^n)$ where r_i is the reflection $r_i : \vec{x} \to \vec{x} - (\vec{x}, \vec{d}_i)\vec{d}_i$, $\vec{x} \in \mathbb{R}^n$, [1, Chap. v, §4]. The <u>Schläflian</u> of G is the parameter $s(G) = \det[(\vec{d}_i, \vec{d}_j)]$. For any $I \subset I_n = \{1, 2, \ldots, n\}$, $G(I)$ will denote the <u>distinguished subgroup</u> of G generated by $\{r_i \mid i \in I\}$.

Now G is <u>crystallographic</u> if it leaves invariant some lattice L spanning $\mathbb{R}^n$. It follows [4, Prop. 1.3] that L contains a G-invariant root lattice Q(B) generated by a basic system of roots $B = \{\vec{e}_i = t_i\vec{d}_i \mid 1 \le i \le n\}$, for certain $t_i > 0$, where

$$c_{ji} = t_j(\vec{d}_j, \vec{d}_i)/t_i \in \mathbb{Z}, \quad (1 \le i,j \le n) \qquad (1).$$

<u>Proposition</u>. A Coxeter group G is crystallographic if and only if $s(G(I)) \in \mathbb{Z}$ for each distinguished subgroup G(I) of G.

Proof. Suppose G is crystallographic and let $I \subset I_n$. Then $s(G(I)) = \det[(\vec{d}_i, \vec{d}_j)]$ $,(i, j \in I)$,

$= \det[t_j c_{ij}/t_i] = \det[c_{ij}] \in Z$.

Conversely, suppose $s(G(I)) \in Z$ for each $I \subset I_n$. We shall attempt to label each node i of the Coxeter diagram Γ for G by $t_i > 0$ satisfying (1). Now for $I = \{i,j\}$ we have $4 - (2 \cos \pi/p_{ij})^2 = s(G(I)) \in Z$. Thus either $(c_i, c_j) = 0$ and nodes i and j are non-adjacent, or $2 \cos \pi/p_{ij} = 1, \sqrt{2}, \sqrt{3}, 2$ and the branch joining nodes i and j is labelled by $p_{ij} = 3, 4, 6, \infty$, respectively. In the latter case, condition (1) implies that the integer $t_i(2 \cos \pi/p_{ij})/t_j$ divides $4 \cos^2 \pi/p_{ij}$, so that letting $\alpha_{ij} = t_i/t_j$, we obtain α_{ij} or $\alpha_{ji} = 1, \sqrt{2}, \sqrt{3}, 2$ or 1, respectively.

In each connected component of Γ we may freely choose one label, say $t_1 = 1$. In labelling the remaining nodes along paths starting at node 1, an inconsistency is forced only by a circuit, with nodes $i \in I = \{1, \ldots, k\}$, along which $\alpha_{12}\alpha_{23} \cdots \alpha_{k1} \neq 1$. However, by an easy induction, the principal minors of $s(G(I))$ are integral; and by expansion along any row, the integer $s(G(I))$ equals another integer minus $2(2 \cos \pi/p_{12}) \ldots (2 \cos \pi/p_{k1})$. Thus, since an even number of $p_{i,\ i+1}$'s must equal 4 (and likewise 6), it is possible to choose $\alpha_{12}\alpha_{23} \cdots \alpha_{k1} = 1$. Hence, one can indeed choose all t_i satisfying (1), and each r_j leaves invariant the lattice Q(B).//

§2. Algebraic Crystallography and Remarks. The above arguments are easily generalized by replacing Z with any Euclidean subdomain A of $\mathbb{R}$. For instance, if we wish to admit 5-fold symmetry, we may take $A = Z[\tau]$, the ring of integers in $Q(\sqrt{5})$, where $\tau = (1 + \sqrt{5})/2$. Then G fixes a $Z[\tau]$ - lattice spanning $\mathbb{R}^n$ if and only if $s(G(I)) \in Z[\tau]$, for each $I \subset I_n$. In this case, each $p_{ij} \in \{1, 2, 3, 4, 5, 6, 10, \infty\}$, and in each circuit of Γ the numbers of branches labelled 4, 6, and 10 are even.

The application of the Schläflian to spherical and Euclidean Groups is described in [2, pp. 133-141]. Our Proposition is mentioned as an observation in [3], where one can also find the well known graphical description of crystallographic groups which underlies the above discussion: c.f. [1, Chap. v, §4, Exer. 6], [4, p. 17].

REFERENCES.

[1] N. Bourbaki, Groupes et algèbres de Lie, Chap. iv-vi., Hermann, Paris, 1968.

[2] H. S. M. Coxeter, Regular Polytopes (3rd Ed.), Dover, New York, 1973.

[3] N. W. Johnson, The Theory of Uniform Polytopes and Honeycombs, University of Toronto, Ph.D. Thesis, Toronto 1966.

[4] G. Maxwell, On the Crystallography of Infinite Coxeter Groups, Math. Proc. Cambridge Philos. Soc., 82 (1977), pp. 13-24.

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NOTE ON THE TAYLOR SERIES AND GROWTH OF MEANS

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Presented by L.A. Lorch, F.R.S.C.

Abstract:

Let f be analytic on D, the unit disc $|z| < 1$ in the complex plane $\mathbb{C}$. For $1 \leq p \leq \infty$, we denote the corresponding means of f by $M_p(r,f) = \|f(re^{i\theta})\|_{L^p(\mathbb{T})}$, where $\mathbb{T}$ is the unit circle with normalized Lebesgue measure. We describe a method for determining the growth rate of $M_p(r,f')$ in terms of the growth rate in H^p-norm of a certain sequence of "weighted" blocks from the Maclaurin's series of f (for $1 < p < \infty$ the "weightings" can be dispensed with). This leads to a new characterization by Taylor coefficients of Bloch functions and of the analytic functions Lipschitz or smooth on the boundary. The method employed admits a large class of weightings, provides a direct link with trigonometric approximation, and affords a convenient viewpoint for fractional differentiation.

1. Main results on growth rates. We denote by $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}^+$ the real line, the set of integers, and the set of positive integers, respectively. The H^p norm is denoted by $\|\cdot\|_p$.

(1.1) Definition. Let F denote the class of all functions ϕ on $\mathbb{R}$ such that $\phi \geq 0$, $\phi(0) = 1$, ϕ vanishes on $\mathbb{R}\backslash(-\frac{1}{2}, 1)$, and the restriction of ϕ to $[-\frac{1}{2}, 1]$ is absolutely continuous, with $\phi' \in L^2[-\frac{1}{2}, 1]$.

(1.2) Theorem. *Let* $\phi \in F$. *If* $1 \leq p \leq \infty$, $-1 < \gamma < \infty$, *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic on* D, *then the following are equivalent:*

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(i) $M_p(r,f') = O(1/(1-r)^{1+\gamma})$,

(ii) $\sigma \equiv \sup_{N \in \mathbb{Z}^+} N^{-\gamma} \| \sum_{k=1}^{\infty} \phi(\frac{k}{N} - 1) a_k z^k \|_p < \infty$.

For $\phi \in F$, $\gamma \in (-1,+\infty)$ *there are positive constants* c_ϕ *and* $C_{\phi,\gamma}$ *such that whenever* f *and* p *are as in the hypotheses and* (i) *and* (ii) *hold, then*

$$c_\phi \sigma \leq \sup_{0 \leq r < 1} (1-r)^{1+\gamma} M_p(r,f') \leq C_{\phi,\gamma} \sigma .$$

(1.3) *Theorem. Suppose* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic on* D, $1 < p < \infty$, $\phi \in F$, *and* $\eta \in \mathbb{R}$. *Then*

$$s_1 \equiv \sup_{N \in \mathbb{Z}^+} N^{-\eta} \| \sum_{k=1}^{\infty} \phi(kN^{-1}-1) a_k z^k \|_p < \infty$$

if and only if

$$s_2 \equiv \sup_{N \in \mathbb{Z}^+} N^{-\eta} \| \sum_{k=N}^{2N} a_k z^k \|_p < \infty .$$

If this is the case, then $\omega_{\phi,\eta} s_1 \leq s_2 \leq \Omega_{\phi,\eta,p} s_1$, *where* $\omega_{\phi,\eta}$ *and* $\Omega_{\phi,\eta,p}$ *are positive constants depending only on their subscripts.*

(1.4) *Theorem. Let* $\phi \in F$. *If* $1 \leq p \leq \infty$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic on* D, *then the following are equivalent*:

(i) $\| \sum_{k=1}^{\infty} \phi(kN^{-1}-1) a_k z^k \|_p = O(N^{\gamma})$,

(ii) $\| \sum_{k=1}^{\infty} \phi(kN^{-1}-1) k^{\beta} a_k z^k \|_p = O(N^{\gamma+\beta})$.

(1.5) *Corollary. If* $\phi \in F$, $1 \leq p \leq \infty$, $\alpha > 0$, *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic on* D, *then* $M_p(r,f) = O((1-r)^{-\alpha})$ *if and only if*

$$\| \sum_{k=1}^{\infty} \phi(kN^{-1}-1) a_k z^k \|_p = O(N^{\alpha}) .$$

2. *Comments on the proofs*. In obtaining the results of §1, repeated use is made of the following lemma due to Robert Kaufman (private communication), based in part on the Poisson summation formula.

(2.1) *Lemma*. *Let* $f\colon \mathbb{R} \to \mathbb{C}$ *be absolutely continuous on each compact interval*, $f \in L^2(\mathbb{R})$, *and* $f' \in L^2(\mathbb{R})$. *Then* $\sum_{k=-\infty}^{\infty} f(k)e^{ik\theta}$ *is the Fourier series of a function* $G \in L^1(\mathbb{T})$ *and*

$$\|G\|_{L^1(\mathbb{T})} \le 2\pi^{-1/2}[\|f\|_{L^2(\mathbb{R})} \|f'\|_{L^2(\mathbb{R})}]^{1/2} .$$

The proofs of Theorems (1.2) and (1.3) require extensive, technically involved estimates. In this brief note, we confine ourselves to highly condensed accounts of the main ideas.

Proof of Theorem (1.2). To see that (ii) implies (i) a positive integer M sufficiently large for certain explicit requirements is first chosen. One then defines $g(x) = \sum_{n=-\infty}^{\infty} \phi(x + nM^{-1})$ for $x \in \mathbb{R}$, and $h_j(x) = \phi(M \log x - jM^{-1})/g(M \log x)$ for $x > 0$, $j \in \mathbb{Z}$. In particular, $\sum_{j=-\infty}^{\infty} h_j(x) \equiv 1$, $h_j \ge 0$ for all j, and $h_j(x) = 0$ unless $|\log x - jM^{-2}| < M^{-1}$. Since $\sum_{k=1}^{\infty} ka_k r^k e^{ik\theta} = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} ka_k r^k h_j(k)e^{ik\theta}$, the demonstration reduces to estimating, for fixed j, $\|\sum_{k=1}^{\infty} ka_k r^k h_j(k)e^{ik\theta}\|_{L^p(\mathbb{T})}$. Use of Bernstein's inequality for trigonometric polynomials gives a further reduction to consideration of $\|\sum_{k=1}^{\infty} a_k r^k h_j(k)e^{ik\theta}\|_{L^p(\mathbb{T})}$. The terminating series in this last expression can be treated as the convolution of $\sum_{k=1}^{\infty} \phi(kN_j^{-1}-1)a_k e^{ik\theta}$ and another trigonometric polynomial P_j with suitable Fourier coefficients, the positive integer N_j being chosen so that $|\log N_j - jM^{-2}| < M^{-1}$. Lengthy calculations in conjunction with Lemma (2.1) provide a suitable estimate for

$\|P_j\|_{L^1(\mathbb{T})}$ to complete this half of the proof. To show that (i) implies (ii) let $r_N = 1 - N^{-1}$, and write $\sum_{k=1}^{\infty} \phi(kN^{-1}-1)a_k e^{ik\theta}$ as the convolution of $\sum_{k=1}^{\infty} k a_k r_N^k e^{ik\theta}$ and a suitable trigonometric polynomial. The norm of the latter in $L^1(\mathbb{T})$ can then be estimated by Lemma (2.1).

Proof of Theorem (1.3). The proof that $s_1 < \infty$ implies $s_2 < \infty$ proceeds from an application of Stečkin's theorem [5_{II}, Chapter XV, Theorem (4.14)]. To show that $s_2 < \infty$ implies $s_1 < \infty$, one first notes that by Lemma (2.1) $\tau_\phi \equiv \sup_{N \in \mathbb{Z}^+} \|\sum_{k=1}^{\infty} \phi(kN^{-1}-1)e^{ik\theta}\|_{L^1(\mathbb{T})} < \infty$. Representation of $\sum_{k=M}^{2M} \phi(kN^{-1}-1)a_k e^{ik\theta}$ (M,N positive integers) as the convolution of $\sum_{k=M}^{2M} a_k e^{ik\theta}$ with $\sum_{k=1}^{\infty} \phi(kN^{-1}-1)e^{ik\theta}$ shows that $\|\sum_{k=M}^{2M} \phi(kN^{-1}-1)a_k z^k\|_p \le \tau_\phi s_2 M^{\eta}$. Suitable choices for M and N in this last inequality lead to the desired result.

Proof of Theorem (1.4). Take $z = e^{i\theta}$. For the series in (i) (resp., (ii)) Lemma (2.1) provides a corresponding trigonometric polynomial of suitable norm in $L^1(\mathbb{T})$ whose convolution with the series is the series in (ii) (resp., (i)).

3. Applications

(a) Analytic functions on D which are Lipschitz or smooth on $\mathbb{T}$. Such functions are characterized by growth conditions on the means of their first or second derivative, respectively (see [4, Theorem 13] and [3, Chapter 5]). The results of §1 make it possible to characterize such functions in terms of weighted blocks from their Maclaurin's series. For instance, the following theorem results.

(3.1) Theorem. Let $\phi \in F$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D. Then f is continuous on $|z| \le 1$ and $f(e^{i\theta}) \in \Lambda_*$ if and only if $\|\sum_{k=1}^{\infty} \phi(kN^{-1}-1)a_k z^k\|_\infty = O(N^{-1})$.

(b) Best approximation by trigonometric polynomials. If $f \in L^p(\Pi)$, $1 \leq p < \infty$ (resp., $f \in C(\Pi)$), for each positive integer N we write $E_{N,p}(f)$ (resp., $E_{N,\infty}(f)$) for the distance in $L^p(\Pi)$ (resp., $C(\Pi)$) from the subspace of trigonometric polynomials of degree at most N. Let $\{V_N\}_{N=1}^{\infty}$ denote de la Vallée Poussin's kernel. Thus $V_N = 2K_{2N-1} - K_{N-1}$, where $\{K_N\}_{N=0}^{\infty}$ is the Fejér kernel (see, e.g., [5_I, page 115]). Let $T\colon \mathbb{R} \to \mathbb{R}$ be defined as follows: $T(x) = 2x + 1$ for $x \in [-1/2,0]$, $T(x) = 1 - x$ for $x \in [0,1]$, and $T(x) = 0$ for $x \in \mathbb{R}\setminus[-1/2,1]$. Obviously $T \in F$. A straightforward computation shows that for $N \geq 1$, $k \geq 0$, $\hat{V}_{2N}(k) - \hat{V}_N(k) = T(k(2N)^{-1}-1)$, where $\hat{V}_N$ denotes the Fourier transform of V_N. By combining this last fact (and basic properties of de la Vallée Poussin's kernel) with the results in §1 and those mentioned in 3(a) above, one obtains, with comparative ease, the analytic versions of standard facts relating Lipschitz or smooth behavior to best approximation by trigonometric polynomials ([2, Theorem 2.4.1]). For example, the following theorem ensues.

(3.2) Theorem. Let f be continuous on $|z| \leq 1$ and analytic on D. Then $f(e^{i\theta}) \in \Lambda_*$ if and only if $E_{N,\infty}(f(e^{i\theta})) = O(N^{-1})$.

(c) Fractional differentiation. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic on D and $\beta > 0$, let $f^{(\beta)}(z) = \sum_{n=1}^{\infty} n^{\beta} a_n z^n$. Thus $f^{(\beta)}$ is (essentially) the Weyl fractional derivative of f of order β. The preceding results discussed in this note facilitate considerations involving fractional differentiation. For instance the following basic result of Hardy and Littlewood is an obvious consequence of Theorem (1.4) and Corollary (1.5).

(3.3) Theorem. If f is analytic on D, $1 \leq p \leq \infty$, $\alpha > 0$ and $\beta > 0$, then $M_p(r,f) = O((1-r)^{-\alpha})$ if and only if $M_p(r,f^{(\beta)}) = O((1-r)^{-\alpha-\beta})$.

(d) Bloch functions. For the case $\gamma = 0$, $p = \infty$, Theorem (1.2) provides a new characterization of Bloch functions in terms of Taylor coefficients (for a treatment of Bloch functions see [1]). This approach unifies various facts about Bloch functions.

A full account of the circle of ideas discussed in this note will be given in a forthcoming paper by the author.

References

|1| J. M. Anderson, J. Clunie, and Ch. Pommerenke, On Bloch functions and normal functions. J. Reine Angew. Math., 270 (1974), 12-37.

|2| P. Butzer and R. Nessel, Fourier analysis and approximation, Vol. 1. Pure and Applied Mathematics, Vol. 40-I. Academic Press, New York, 1971.

|3| P. Duren, Theory of H^p spaces. Pure and Applied Mathematics, Vol. 38. Academic Press, New York, 1970.

|4| A. Zygmund, Smooth functions. Duke Math. J., 12 (1945), 47-76.

|5| A. Zygmund, Trigonometric series. 2nd ed., Vol. I. Cambridge University Press, New York, 1959.

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SUR UN SYSTEME D'INEQUATIONS FONCTIONNELLES

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Presented by J. Aczél, F.R.S.C.

Résumé. Désignons par N, R les ensembles des nombres naturels ($\geqslant 1$) et réels, respectivement. Le théorème suivant sera démontré.

Théorème. Soit $g: [0,\infty) \longrightarrow [0,\infty)$ une fonction continue, croissante (donc injective mais non nécessairement surjective), de sorte que

(1) $g(0) = 0,\ g(1) = 1.$

Si la fonction $f: R \longrightarrow R$ satisfait aux inéquations

(2) $f(x+y) \geqslant f(x) + f(y)$ $\qquad (x,y \in R),$

(3) $f(g(|x|)) \geqslant g(|f(x)|)$ $\qquad (x \in R)$

et à la condition

(4) $f(1) = 1,$

alors $f(x) \equiv x.$

I. Remarques. 1. Soit $n \in N$, $n \geqslant 2$. Considérons les équations fonctionnelles

(5) $f(x+y) = f(x) + f(y)$ $\qquad (x,y \in R),$

(6) $f(x^n) = f(x)^n$ $\qquad (x \in R).$

Selon J.Aczél [2] (voir aussi P.M.Vasić et R.P.Lučić /4/) la solution de (4),(5),(6) est $f(x) = x$; si $n = 2k$ ($k \in N$), cela résulte déjà d'un théorème classique de Darboux (voir J.Aczél [1], p.45). On peut se demander, si ce résultat subsiste, si l'on remplace les signes d'égalité de (5),(6) par le signe $\geqslant$. Pour $n = 2k$ la réponse est affirmative: Il suffit de choisir $g(t) = t^{2k}$ dans le théorème plus haut. Pour $n = 2k+1$ la réponse est négative: la fonction

$$f(x) = \begin{cases} x & (x > 0) \\ x - 1 & (x \leqslant 0) \end{cases}$$

satisfait aux conditions (2),(4) et aux inéquations

$$f(x^{2k+1}) \geqslant f(x)^{2k+1} \qquad (x \in R, k \in N).$$

2. Si $g(t) = t^2$, l'inéquation (3) est de la forme

$$(7) \quad f(x^2) \geqslant f(x)^2 \qquad (x \in R),$$

et d'après le théorème plus haut, $f(x) = x$ est la solution de (2),(4),(7). Ici la condition (4) est indispensable, car les fonctions

$$f(x) = \begin{cases} ax & (x \geqslant 0) \\ bx & (x \leqslant 0) \end{cases}$$

(où $b^2 \leqslant a \leqslant b \leqslant 1$) satisfont aux inéquations (2),(7). Il est intéressant de comparer ce fait avec un résultat de M.Rădulescu /3/, selon lequel la solution du système des inéquations fonctionnelles (2) et

$$f(xy) \geqslant f(x)f(y) \qquad (x, y \in R)$$

est donnée par les fonctions $f(x) = x$ et $f(x) \equiv 0$ (voir aussi /5/).

<u>II. Démonstration du théorème.</u> 1. Soit $x \geqslant 0$. On peut écrire

$$x = ny,$$

où $n \in N$ et $0 \leqslant y < 1$. De plus,

$$y = g(t)$$

avec un $t \geqslant 0$. En tenant compte de (2),(3), il résulte que

$$f(x) = f(ny) \geqslant nf(y) = nf(g(t)) \geqslant ng(|f(t)|) \geqslant 0 .$$

On a donc $f(x) \geqslant 0$ pour $x \geqslant 0$. De ce fait et de l'inéquation (2) on obtient facilement que $f: R \longrightarrow R$ est une fonction monotone non-décroissante, $f(0) = 0$ et

$$(8) \quad f(-x) \leqslant -f(x) \qquad (x \in R).$$

2. Les formules (1),(3),(4) entraînent

$$g(1) = 1 = f(1) = f(g(1)) = f(g(|-1|)) \geqslant g(|f(-1)|) ,$$

d'où

$$|f(-1)| \leqslant 1 .$$

D'autre part, $f(-1) \leqslant -1$ (d'après (4),(8)), donc

$$f(-1) = -1 .$$

De cette formule et de (2),(8) on obtient pour $n \in N$

$$f(n) \leqslant -f(-n) = -f(n(-1)) \leqslant -nf(-1) = n .$$

D'autre part, $f(n) \geqslant nf(1) = n$ (d'après (2),(4)), donc

$$(9) \quad f(n) = n \qquad (n \in N).$$

3. Soient $p,q \in N$. Les formules (2),(9) entraînent

$$p = f(p) = f(q\tfrac{p}{q}) \geqslant qf(\tfrac{p}{q}) ,$$

donc $f(\frac{p}{q}) \leqslant \frac{p}{q}$. Grâce à la monotonie de la fonction f il en résulte

$$f(x) \leqslant x \qquad (x \geqslant 0).$$

De là on obtient pour x réel, arbitraire (en tenant compte de (3))

$$g(|x|) \geqslant f(g(|x|)) \geqslant g(|f(x)|) ,$$

ce qui entraîne

$$(10) \quad |f(x)| \leqslant |x| \qquad (x \in R).$$

4. Pour établir la relation désirée

$$f(x) = x \qquad (x \in R),$$

considérons d'abord le cas $x = -\frac{p}{q}$, où $p,q \in N$. Les formules

(2),(8),(9) entraînent

$$qf(-\tfrac{p}{q}) \leqslant f(q(-\tfrac{p}{q})) = f(-p) \leqslant -f(p) = -p ,$$

donc $f(-\frac{p}{q}) \leqslant -\frac{p}{q}$, d'où avec (10)

$$f(-\tfrac{p}{q}) = -\tfrac{p}{q} .$$

Grâce à la monotonie de f il en résulte

$$(11)\quad f(x) = x \qquad (x<0).$$

Soit finalement $x \geqslant 0$. Il existe $n \in N$ de sorte que $x - n < 0$. Les formules (2),(9),(11) entraînent

$$f(x) \geqslant f(x-n) + f(n) = (x-n) + n = x ,$$

et à l'aide de (10) on obtient $f(x) = x$.

Littérature

[1] Aczél, J.: Vorlesungen über Funktionalgleichungen und ihre Anwendungen. Birkhäuser Verlag, Bâle - Stuttgart, 1961.

[2] —— : General solution of "isomoment" functional equations. Amer. Math. Monthly 74 (1967), 1068 - 1071.

[3] Rădulescu, Marius: On a supra-additive and supra-multiplicative operator of C(X). Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S.) 24(72) (1980), No. 3, 303 - 305.

[4] Vasić, Petar M., et Lučić, Radomir P.: Note on a system of functional equations. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 544 - 576 (1976), 125 - 126 (1977).

[5] Volkmann, Peter: Sur les fonctions simultanément suradditives et surmultiplicatives. Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S.), à paraître.

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ON LINEARLY COMPACT COMMUTATIVE REGULAR RINGS

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1. Introduction. In this paper we shall consider rings with unity and all modules will be left unitary modules. Furthermore all topological spaces are Hausdorff.

A topological module whose open submodules form a fundamental system of neighbourhoods of zero is called linearly topologized. By a linear variety in a module M, we shall mean a coset of a submodule of M. A linearly topologized R-module M is linearly compact if every collection of closed linear varieties in M with the finite intersection property has a non-void intersection. For the basic properties of linearly topologized modules, the reader is referred to [4] and [7]. LTM denotes the category of linear topologized modules and continuous homomorphisms.

In [5, Theorem 6] we have the following result: A commutative ring is von Neumann regular if and only if every simple R-module is injective. In this note we present a topological version of this result for linearly compact commutative rings (3.2).

We shall require the following technical lemma.

1.1 Lemma. Let R be a linearly topologized ring. If every linearly compact simple (equivalently discrete simple) R-module is projective (injective) in LTM, then for any $a \neq 0$ in R, there exists a clopen maximal ideal M in R not containing a.

Proof: First we assume that every linearly compact simple R-module is projective. Let $0 \neq a$ in R. Then the set of open ideals in R not containing a is not empty and contains a maximal element M. Since every open ideal is closed, we conclude that M is clopen ideal. Now, let (M, a) be the ideal generated by a and M. Then both M and (M, a) are clopen, and $(M, a)/M$ is a simple ring. Thus $(M, a)/M$ is a linearly compact simple submodule of R/M. Also we note that the ring R/M is a linearly topologized discrete module. Since $(M, a)/M$ is a simple R-module, we can put $(M, a)/M = R\hat{a}$ where $\hat{a} = a + M$. Now define $g : R/M \to R\hat{a}$ by $g(\hat{r}) = r\hat{a}$. To show that g is well-defined, let $r_1, r_2 \in R$ with $\hat{r}_1 = \hat{r}_2$. Then $r_1 - r_2 \in M$. Thus $g(\widehat{r_1 - r_2}) = (r_1 - r_2)\hat{a} = 0$. But $(r_1 - r_2)\hat{a} = r_1\hat{a} - r_2\hat{a} = g(\hat{r}_1) - g(\hat{r}_2)$. Hence $g(\hat{r}_1) = g(\hat{r}_2)$. Since R/M is discrete, it follows that the epimorphism g is continuous. $R\hat{a}$ is projective by assumption. Therefore there exists a continuous monomorphism $\kappa : R\hat{a} \to R/M$ such that $R/M = \kappa\big((M,a)/M\big) \oplus \mathrm{Ker}(g)$. Thus there exist ideas L and K in R such that $\kappa\big((M,a)/M\big) = L/M$ and $\mathrm{Ker}(g) = K/M$. Since $\kappa\big((M,a)/M\big) \neq (0)$, we have $M \neq L$. By the maximality of M, $a \in L$; and so $M \subset (M,a) \subset L$.

Since L/M is simple, it follows that $L/M = (M,a)/M$. Hence $R/M = (M,a)/M \oplus K/M$. Note $\hat{a} \notin K/M$; and thus $a \notin K$.

Since $M \subset K$, the maximality of M implies that $K = M$. It follows that R/M is simple, and hence that M is a maximal ideal in R.

Now we assume every linearly compact simple R-module is injective. Then, as in the argument above we can show that there exists an open maximal ideal M in R not containing $a \neq 0$. Choose M as in the previous situation. Then $(M,a)/M$ is a direct summand of R/M and so R/M is simple. This concludes the proof.

<u>2. OM-semisimple rings</u>. A linearly topologized ring is <u>OM-semisimple</u> if the intersection of all its open maximal ideals is (0).

It is known that every semisimple linearly compact ring is OM-semisimple ([6]). As an immediate consequence of 1.1 we obtain

<u>2.1 Proposition</u>. In LTM, a linearly topologized ring R where every linearly compact simple R-module is projective (injective), is OM-semisimple.

<u>2.2 Theorem</u>. Let R be a linearly topologized OM-semisimple ring. Then R is linearly compact iff it is injective in LTM.

<u>Proof</u>: Let R be injective and $\{M_\alpha\}$ be the set of all clopen maximal ideals in R. Then $\bigcap M_\alpha = (0)$. Define a mapping $\kappa : R \to \prod_\alpha R/M_\alpha$ by $\kappa(a) = \hat{a}$, where $\hat{a}$ is defined in such a way

that $\hat{a}(\alpha) = a + M_\alpha$. Since the projection $p_\alpha : \Pi R/M_\alpha \to R/M_\alpha$ is continuous for each α, κ is a continuous monomorphism. The injectivity of R implies that there exists a continuous epimorphism $g : \underset{\alpha}{\Pi} R/M_\alpha \to R$. Note that R/M_α is linearly compact for each α. By [7, propositions 1 and 2] R is linearly compact. The converse follows from [6, Theorem 1].

<u>3. Linearly compact commutative von Neumann regular rings</u>. A commutative ring R is <u>von Neumann regular</u> if $aR = a^2R$ for all a in R ([2]).

<u>3.1 Proposition</u>. A linearly topologized commutative von Neumann regular ring is OM-semisimple.

<u>Proof</u>: As noted earlier a simple module over a regular ring is injective in the algebraic sense. Let $a \in R\setminus\{0\}$. Then the set of all open ideals in R not containing a contains a maximal clopen ideal P not containing a, and $(P,a)/P$ is a direct summand of R/P. It follows that R/P is simple and so R is OM-semisimple.

We may now state our main result.

<u>3.2 Theorem</u>. Let R be a linearly compact commutative ring. Then the following statements are equivalent in LTM.

(a) R is von Neumann regular.

(b) Every linearly compact R-module is injective.

(c) Every linearly compact R-module is projective.

(d) Every linearly compact simple R-module is injective.

(e) Every linearly compact simple R-module is projective.

Proof: Since a regular ring is semisimple ([2]) Theorem 1 of [6] shows that (a) => (b) and (c). By 2.1 each of (d) and (e) implies that R is OM-semisimple and hence semisimple. Thus, (d) <=> (e). To complete the proof it suffices to show that (e) => (a).

Let $a \in R\backslash\{0\}$. Since R is OM-semisimple, there exists a clopen maximal ideal M in R such that $a \notin M$. Moreover (e) implies that the simple module R/M is algebraically flat (see [2], page 83 and page 133).

Hence $a^2R \otimes R/M$ is a subring of $aR \otimes R/M$. But R/M is a field, and hence it follows that $o = (aR \otimes R/M)/(a^2R \otimes R/M) = (aR/a^2R) \otimes R/M$. Thus $aR = a^2R$, and so R is von Neumann regular.

Combining [6, Theorem 1] and 3.1 with the theorem above we obtain

Corollary. A linearly compact commutative ring is von Neumann regular if and only if it is semisimple.

3.1, 2.2 and the theorem above imply

Corollary. A linearly topologized commutative von Neumann regular ring is linearly compact iff it is injective.

REFERENCES

[1] I. Kaplansky, Projective Modules: Annals of Math. 68 (1958), 372-377.

[2] J. Lambek, Lectures on Rings and Modules, Blaisdell, Toronto (1966).

[3] S. Lefschetz, Algebraic Topology, Colloquium Publications, vol. 27, N.Y. (1942).

[4] M. Petrich, Rings and Semigroups, Lecture Notes in Math. 380, Springer-Verlag (1974).

[5] A. Rosenberg and D. Zelinsky, Finiteness of the Injective Hull, Math. Z. 70 (1959), 372-380.

[6] R. Wiegandt, Über halbeinfache linear kompakte Ringe, Studia Sci. Math. Hung 1 (1966), 31-38.

[7] D. Zelinsky, Linearly Compact Modules and Rings, Amer. J. Math. 75 (1953), 79-90.

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L_2-ESTIMATES FOR VARIATIONAL INEQUALITIES

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ABSTRACT.
For the finite element approximation of the solution of variational inequalities, we derive the L_2-error estimates.

1. INTRODUCTION.

Variational inequalities are now fundamental in studying the obstacle and unilateral problems arising in engineering and mathematical sciences. Finite element techniques are being applied for obtaining numerical solutions of variational inequalities. Using piecewise linear elements, Falk[4], Mosco and Strang[8], Brezzi, Hager and Raviart[2]; and Noor[10,11,12], have shown the O(h) convergence in the energy norm.

In this paper, using the unilateral approximation result of Mosco and Strang[8] and the well-known Aubin-Nitsche trick[3], we obtain the error estimates for the finite element approximation of a class of variational inequalities in the L_2-norm, which are of order h^2. Using a different approach, Johnson[7] and Berger and Falk[1] have obtained the rate of convergence for the numerical solution of parabolic variational inequalities in the L_2-norm. The nonlinear obstacle problem and approximation is considered in Section 2. The L_2-error estimates are derived in Section 3.

2. OBSTACLE PROBLEM AND APPROXIMATION.

For simplicity, we consider the nonlinear obstacle problem of finding $u \in M$ such that

$$(1) \qquad (Tu, v-u) \geq 0, \qquad \text{for all } v \in M,$$

where $(Tu,v) = \sum_{i=1}^{2} \frac{1}{q} \int_D \frac{\partial(u+\phi)}{\partial x^i} \frac{\partial v}{\partial x^i}\, dD$, $M = \{v \in H_0^1;\ v \geq \psi \text{ on } D\}$, a closed closed convex subset of H_0^1, see[6], $D \subset R^2$ is a convex polygonal domain with boundary S, $\bar{D} = D \cup S$ its closure, and ψ is a given function in $H_0^1 \cap H^2$, see Glowinski[6].

The problem (1) arises in the study of permanent compressible irrotational flow. Here $(u+\phi)$ is the stream function of the motion. It is assumed that the flow is subsonic . For the complete mathematical formulation, see[5].

The space $W_2^k(D) = H^k$ is taken to be the usual Sobolev space, where for $u \in H^k$, k a positive integer, we have the norm

$$\| u \|_{k,D}^2 = \sum_{|\alpha| \leq k} \| D^\alpha u \|_{L_2(D)}^2$$

The space of functions from H^k, which in generalized sense satisfies the homogeneous boundary conditions on S, is denoted by H_0^k. The spaces H^k and H_0^k are Hilbert spaces. Let $H^2 = W_2^2(D)$ for simplicity.

It has been shown in [5] that the operator T is strongly monotone and Lipschitz continuous from H_0^1 into H^{-1}, being the dual space of H_0^1, so there does exist a unique solution of problem (1), see Noor[9].

We consider $S_h \subset H_0^1$, a subspace of continuous piecewise linear functions on the triangulation of the polygonal domain D vanishing on its boundary S. Let ψ_h be the interpolant of ψ such that ψ_h agrees at all the vertices of the triangulation. For our purpose, it is enough to choose the finite dimensional convex subset $M_h = S_h \cap \{v_h \geq \psi_h$ only at the vertices of the triangulations$\}$, as in Berger and Falk[1]. For other choices of convex subsets, see [2,4, 11,12].

The approximate solution u_h is defined as that function from a given finite dimensional convex subset M_h which satisfies

$$(2) \qquad (Tu_h, v_h - u_h) \geq 0 \qquad \text{for all } v_h \in M_h.$$

Concerning the regularity of the solution $u \in M$ satisfying (1), we assume the following hypothesis, see[11].

(A) : {For $\psi \in H_0^1 \cap H^2$, $u \in M$ satisfying (1) also lies in H^2}.

We also need the following results.

Theorem 1 [8].

Suppose that $U \geq 0$ in the polygon D and that U lies in $H_0^1 \cap H^2$. Then there exists a V_h in S_h such that

$$0 \leq V_h \leq U \quad \text{in } D$$

and

$$\| U - V_h \|_{1,D} \leq C h \| U \|_{2,D}$$

Theorem 2 [10].

If u and u_h are the solutions of (1) and (2) respectively and hypothesis(A) holds, then

$$\|u - u_h\|_{1,D} = O(h)$$

3. L_2-ERROR ESTIMATE.

In this section, using the unilateral approximation result of Mosco and Strang and Aubin-Nitsche trick, we derive the error estimates for the finite element approximation of variational inequalities in the L_2-norm. We would also like to point out that these are only partial results under the assumption that the solution of an auxiliary problem is smooth enough.

We now state and prove the main result of this paper.

Theorem 3.

If u and u_h are solutions of (1) and (3) respectively, and hypothesis (A) holds, then

$$\|(u-u_h)^+\|_{L_2(D)} = O(h^2), \tag{3}$$

and

$$\|(u-u_h)^-\|_{L_2(D)} = O(h^2), \tag{4}$$

where

$$(u-u_h)^+ = \operatorname{Sup}\{u-u_h, 0\}, \quad (u-u_h)^- = \operatorname{Inf}\{u-u_h, 0\}.$$

Proof:

In order to prove (3), we consider for

$$w = (u-u_h)^+ \tag{5}$$

the problem

$$\begin{cases} y \le 0 \quad \text{on } E, \quad y \in H_0^1 \\ (Ty, z-y) \ge (w, z-y), \quad \text{for all } z \le 0 \text{ on } E, \ z \in H_0^1 \end{cases} \tag{6}$$

and E is a set of points in D where $u = \psi$. This set is known as contact set.

The solution y of (6) exists and $y \in H^2_{(D-E)}$, with

$$\|y\|_{2,D-E} \le C\|w\|_{L_2(D)} \tag{7}$$

and by the maximum principle, we have $y \geq 0$ on D, hence $y \equiv 0$ on E. Thus, by theorem 1, there exists $\tilde{y}_h \varepsilon S_h$ satisfying $0 \leq \tilde{y}_h \leq y$ in D, with $\tilde{y}_h = 0$ on E, such that

(8) $$\|y-\tilde{y}_h\|_{1,D-E} \leq Ch\|y\|_{2,D-E}$$

Since $\tilde{y}_h = 0$ on E, we have $(Tu,\tilde{y}_h) = 0$ and $v_h = u_h + \tilde{y}_h \in M_h$ implies that $(Tu_h,\tilde{y}_h) \geq 0$. Thus we have

(9) $$(Tu-Tu_h,\tilde{y}_h) \leq 0.$$

Moreover, we know that $u-u_h \leq 0$ on E, $z = y+u-u_h \leq 0$ on E, from which, we get

(10) $$(w,Tu-Tu_h) \leq (y,Tu-Tu_h)$$

From (9) and (10), we obtain

$$(w,Tu-Tu_h) \leq (y-\tilde{y}_h,Tu-Tu_h)$$

which implies that

(11) $$\begin{aligned} \|(u-u_h)^+\|^2_{L_2(D)} &\leq C_1\|y-\tilde{y}_h\|_{1,D-E}\|u-u_h\|_{1,D-E} \\ &\leq Ch\|(u-u_h)^+\|_{L_2(D)}\|u-u_h\|_{1,D-E} \quad \text{by (7) and (8).} \end{aligned}$$

From (11) and theorem 2, we obtain the required estimate (3).

In order to prove (4), we solve with $w = (u-u_h)^-$, the problem

(12) $$\begin{cases} y \geq 0 \text{ on } E_h, \ y \varepsilon H^1_0 \\ (Ty,z-y) \geq (w,z-y), \quad \text{for all } z \geq 0 \text{ on } E_h, \ z \varepsilon H^1_0, \end{cases}$$

and E_h is a set of points in D where $u_h = \psi$.

The solution of (12) exists and belongs to $H^2_{(D-E_h)}$, with

(13) $$\|y\|_{2,D-E_h} \leq C\|w\|_{L_2(D)}.$$

By the maximum principle, $y \leq 0$ on D, hence $y \equiv$ on E_h. Thus from theorem 1, there exists $\tilde{y}_h \varepsilon S_h$ satisfying $0 \leq \tilde{y}_h \leq y$ in D with $\tilde{y}_h = 0$ on E_h such that

(14) $$\|y-\tilde{y}_h\|_{2,D-E_h} \le Ch\|y\|_{2,D-E_h}$$

Since $\tilde{y}_h = 0$ on E_h, we have $(Tu_h,\tilde{y}_h) = 0$, and from $\tilde{y}_h \le 0$, $v = u-\tilde{y}_h \varepsilon M$. it follows that $(Tu,\tilde{y}_h) \le 0$. Thus we obtain

(15) $$(Tu-Tu_h,\tilde{y}_h) \le 0.$$

We also know that $u-u_h \ge 0$ on E_h, and $z = y+u-u_h \ge 0$ on E_h, from which it follows that

(16) $$(w,Tu-Tu_h) \le (y,Tu-Tu_h).$$

From (15) and (16), we obtain

$$(w,Tu-Tu_h) \le (y-\tilde{y}_h,Tu-Tu_h),$$

which implies that

$$\|(u-u_h)^{-}\|^2_{L_2(D)} \le C\|y-\tilde{y}_h\|_{1,D-E_h}\|u-u_h\|_{1,D-E_h}.$$

Thus from (13), (14), (17) and theorem 2, the required estimate (4) follows.

REFERENCES

1. A. E.Berger and R. Falk;An error estimate for the truncation method for the solution of parabolic obstacle variational inequalities, Math. Comp. 31(1977),619 -628.
2. F. Brezzi, W. Hager and P. Raviart;Error estimate for the finite element solution of variational inequalities, Num. Math. 28(1977),431-443.
3. P. G. Ciarlet;The finite element method for elliptic problems, North-Holland, 1978.
4. R. S. Falk;Error estimate for the approximation of a class of variational inequalities, Math. Compt. 28(1974),963-971.
5. H. Froidevaux;A finite element method for solving nonlinear boundary value problem in The Mathematics of Finite Elements and Applications II, (ed). by J. R. Whiteman, Academic Press, London, New York,(1976),217-224.

6. R. Glowinski; Introduction to the approximation of elliptic variational inequalities, T/R 76006, University of Paris VI, France, 1976.

7. C. Johnson;A convergence estimate for an approximation of a parabolic variational inequality, SIAM Num. Anal. 13(1976),599-606.

8. U. Mosco and G. Strang;One sided approximation and variational approximation, Bull. Amer. Math. Soc. 80(1974),308-312.

9. M. Aslam Noor;On variational inequalities, Ph.D. Thesis, Brunel University, U. K. 1975.

10. M. Aslam Noor:Finite element methods and nonlinear programming, Pak. J. Sc. (1982), to appear.

11. M. Aslam Noor;Finite element error analysis of nonlinear variational inequalities, Zeit. Ang. Math. Mech.(ZAMM), 60(1980), 631-632.

12. M. Aslam Noor; Mildly nonlinear variational inequalities, Queen's Math. pre-print, 1980-21.

13. G. Strang and G. Fix;An analysis of the finite element method, Prentice-Hall, N. J. 1973.

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AN ISOMORPHISM BETWEEN THE HALIMSKIĬ AND PAULOWICH FUNDAMENTAL GROUPS

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Presented by K. Murasugi, F.R.S.C.

For each Hausdorff space (X,x_0), let $\pi R(X,x_0)$ denote the Halimskiĭ fundamental group defined for the family R of all compact segments [2, pp. 75-87]. In this note $\pi R(X,x_0)$ is shown to be isomorphic to the author's fundamental group $\lambda(X,x_0)$, as defined in [6, §4]. This simplifies the computation of $\pi R(X,x_0)$. We note that the homology theory in [2, Theorem II.6, p. 74] is defined for the family of all segments (not just the compact segments, as a misprint in MR57#4118 states).

Let α be a fixed infinite cardinal and let $x_0 \in X$, where X is a fixed Hausdorff space such that, for each compact $Y \subset X$, $\text{weight}(Y) \leq \alpha$. Let $A = [a,b]$ be a compact segment (called an <u>arc</u> in [5], [6]) with non-cutpoints a and b. A function $f\colon A \longrightarrow X$ is called a <u>map</u> if f is continuous and $f(a) = f(b) = x_0$. Maps $f,g\colon A \longrightarrow X$ are said to be <u>homotopic</u> if there exist both a compact segment $C = [c,d]$ and a continuous function $h\colon A \times C \longrightarrow X$ such that, for each $s \in A$ and $t \in C$,

(1) $h(a,t) = x_0 = h(b,t)$

(2) $h(s,c) = f(s)$ and $h(s,d) = g(s)$.

We write $[f]$ for the set of all maps which are homotopic to f.

Maps $f_1\colon A_1 \longrightarrow X$ and $f_2\colon A_2 \longrightarrow X$ are said to be <u>equivalent</u> if there exist a compact segment A and order preserving continuous surjections $g_1\colon A \longrightarrow A_1$ and $g_2\colon A \longrightarrow A_2$ such that $(f_1 \circ g_1)$ is homotopic to $(f_2 \circ g_2)$. This is an equivalence relation [2, pp. 77-80]. We write $\{f_1\}$ for the class of all maps which are equivalent to f_1. Halimskiĭ defines the fundamental group $\pi R(X,x_0)$ on the class of all equivalence classes of maps from compact segments to X. The following lemma proves that $\pi R(X,x_0)$ is a set in the sense of [3, Appendix].

LEMMA. For each compact segment A and each map $f\colon A \longrightarrow X$, there exist a compact segment A_2, with weight$(A_2) \le \alpha$, and a map $f_2\colon A_2 \longrightarrow X$ which is equivalent to f.

PROOF. Consider the monotone-light factorization $f = (f_2 \circ f_1)$ and let $A_2 = f_1(A)$. Then weight$(A_2) \le \alpha$, by [7], and f_2 is equivalent to f.

For any Hausdorff space Z we define the character of Z to be the least cardinal β such that, for each $x \in Z$, the neighborhood system of the point x has a base of cardinality $\le \beta$. If C is compact Hausdorff and Y is a Hausdorff space with infinite weight, then the proof of [1, Proposition 3.4] shows that character$(Y^C) \le$ weight(Y), where Y^C is the space of all continuous functions from C to Y, equipped with the compact-open topology. Thus if K is a compact subspace of X^C, then character$(K) \le \alpha$.

A compact segment B is said to be flexible if each compact segment $C \subset B$ admits an order preserving continuous surjection $g\colon C \longrightarrow B$ and there also exists an order reversing homeomorphism $h\colon B \longrightarrow B$. We choose a flexible compact segment B which admits order preserving continuous surjections onto all compact segments of character at most α. Such a segment B exists and may be chosen to have cardinality at most 2^{2^α}, by [6, §2] and [4, Theorem I.19]. We define $\lambda(X,x_0)$ to be the set of homotopy classes $[f]$, where $f: B \longrightarrow X$ is any map, together with the multiplication defined in [6, §3 and §4].

THEOREM. The function sending each homotopy class $[f]$ to the equivalence class $\{f\}$ is an isomorphism from $\lambda(X,x_0)$ onto $\pi R(X,x_0)$.

PROOF. By the LEMMA, each equivalence class in $\pi R(X,x_0)$ contains a map $f_2: A_2 \longrightarrow X$, where weight$(A_2) \leq \alpha$. We choose an order preserving continuous surjection $h: B \longrightarrow A_2$ and let $f = (f_2 \circ h)$. Then $\{f\} = \{f_2\}$.

Now suppose $f_1, f_2: B \longrightarrow X$ are maps such that $\{f_1\} = \{f_2\}$. Then there exist a compact segment A and order preserving continuous surjections $g_1, g_2: A \longrightarrow B$ such that $[f_1 \circ g_1]$ equals $[f_2 \circ g_2]$. Let $g: A \longrightarrow B$ be any order preserving continuous surjection. One can easily prove that $[f_1 \circ g]$ equals $[f_2 \circ g]$. The proof of [6, Theorem 4.5] shows that $[f_1]$ equals $[f_2]$. Thus we have a bijection between $\lambda(X,x_0)$ and $\pi R(X,x_0)$. The theorem follows upon examining the group structures defined in [6] and [2].

REMARK. The preceding proof shows that $\pi R(X,x_0)$ could also be defined by using the class of all compact segments of cardinality at most 2^α . This answers [2, Problem II.2, p. 86] for the family of all compact segments.

REFERENCES

1. C.R. Borges, Compact-open versus k-compact-open, Proc. Amer. Math. Soc. 73(1979), 129-133. MR 80b:54027.

2. E.D. Halimskiĭ, Ordered topological spaces (Russian), Izdat. "Naukova Dumka", Kiev, 1977, MR 57#4118.

3. J.L. Kelley, General topology, Van Nostrand, New York, N.Y., 1955. MR 16, pp. 1136-1138.

4. M.A. Maurice, Compact ordered spaces, second edition, Mathematisch Centrum, Amsterdam, 1970. MR 36#3318.

5. D.G. Paulowich, Weak contractibility and hyperspaces, Fund. Math. 94(1977), 41-47. MR 55#1278.

6. ____________, Generalized homotopy groups, preprint.

7. V.V. Proizvolov, Decomposition mappings of locally connected spaces, Soviet Math. Dokl. 5(1964), 1540-1542. MR 30#563.

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ALEXANDROV-TYPE TRANSFORMATIONS ON EINSTEIN'S CYLINDER UNIVERSE

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Presented by J. Aczél, F.R.S.C.

A theorem of A.D. Alexandrov [1] states that bijections of Minkowski spacetime which preserve pairs of events connected by unreflected light signals are essentially Lorentz transformations. The purpose of this note is to obtain the corresponding transformations for Einstein's cylinder universe. These may in fact be somewhat pathological.

Einstein's cylinder universe C_4 can be realized as the set of points $(t,\vec{r}):=(t,w,x,y,z)\in\mathbb{R}^5$ on the cylinder $\vec{r}\cdot\vec{r}=1$ ("$\cdot$" denotes the usual dot product on $\mathbb{R}^4$) with the line element $ds^2=-dt^2+d\vec{r}\cdot d\vec{r}$ (see [2]). In terms of a parameter τ, its geodesics have the form

$$\vec{r}(\tau)=\cos(\omega\tau)\vec{a}+\sin(\omega\tau)\vec{b},\quad t=\alpha\tau+\beta \tag{1}$$

for constants ω,α,β $(\alpha^2+\omega^2\neq 0)$ and orthonormal $\vec{a},\vec{b}\in\mathbb{R}^4$. The separation between two points $(t_1,\vec{r}_1),(t_2,\vec{r}_2)\in C_4$ (obtained by integrating ds along a geodesic joining them) is $-(t_1-t_2)^2+$ $+\{\cos^{-1}(\vec{r}_1\cdot\vec{r}_2)\}^2$, and is zero iff the geodesic is null, i.e. iff $\alpha=\pm\omega$ in (1). Null geodesics represent light signals; thus events $(t_1,\vec{r}_1)$ and $(t_2,\vec{r}_2)$ are connected by an unreflected light signal iff

$$\cos(t_1-t_2)=\vec{r}_1\cdot\vec{r}_2 \tag{2}$$

The transformations preserving all separations on C_4 have the form $(t,\vec{r}) \to (\pm t+\alpha, A\vec{r})$ for some constant α and 4×4 orthogonal matrix A. Let $f\colon C_4 \to C_4$ be a bijection preserving (2) on C_4 in both directions. We determine below all such f's.

Lemma: Define the equivalence relation $\equiv$ on C_4 by $(t_1,\vec{r}_1) \equiv (t_2,\vec{r}_2)$ iff for all $(t,\vec{r}) \in C_4$

$$\cos(t-t_1) = \vec{r}\cdot\vec{r}_1 \quad \text{iff} \quad \cos(t-t_2) = \vec{r}\cdot\vec{r}_2.$$

Then the equivalence classes of $\equiv$ are the subsets

$$[(t,\vec{r})] := \{(t+k\pi, (-1)^k\vec{r});\ k \text{ an integer}\}.$$

Proof: If $(t_1,\vec{r}_1),(t_2,\vec{r}_2) \in [(t,\vec{r})]$, then clearly $(t_1,\vec{r}_1) \equiv (t_2,\vec{r}_2)$. Assume that $(t_1,\vec{r}_1) \equiv (t_2,\vec{r}_2)$. For some $\omega \in \mathbb{R}$, $\vec{r}_1\cdot\vec{r}_2 = \cos\omega$, so for $t := t_1 \pm \omega$ and $\vec{r} := \vec{r}_2$, $\cos(t-t_1) = \vec{r}\cdot\vec{r}_1$. Then $\cos(t_1-t_2\pm\omega) = \vec{r}_2\cdot\vec{r}_2 = 1$, so for some integers m, n, $t_1-t_2+\omega = 2m\pi$ and $t_1-t_2-\omega = 2n\pi$. Set $k := m-n$; then $t_1-t_2 = k\pi$ and $\omega = (k+2n)\pi$. Since ω is the angle between $\vec{r}_1$ and $\vec{r}_2$, $\vec{r}_1 = (-1)^k\vec{r}_2$, thus $(t_1,\vec{r}_1) \in [(t_2,\vec{r}_2)]$. $\square$

Although f maps equivalence classes onto each other, arbitrary permutations within equivalence classes also preserve (2). We can thus only determine the effect of f on these classes, not on each individual point.

The space C_4 is, as described below, a covering space for conformal Minkowski space $\bar{M}_4$ (see [3] and [4] for details about $\bar{M}_4$; the relevant points follow). The points of $\bar{M}_4$ may be coordinatized by homogeneous 6-tuples $R\propto(W,X,Y,Z,U,V)^t \in \mathbb{R}^6$ (superscript "t" denotes "transpose") satisfying $R^tGR = 0$, where $G := \mathrm{diag}\{1,1,1,1,-1,-1\}$. Points $R_1, R_2 \in \bar{M}_4$ have zero separation iff $R_1{}^tGR_2 = 0$, and bijections of $\bar{M}_4$ preserving such pairs of points (in both directions) must have the form $R \to TR$ for some 6×6 matrix T satisfying $T^tGT = G$ (a direct corollary of the main result of [4]).

The mapping $C_4 \to \bar{M}_4$ given by

$$(t,\vec{r}) \to (\vec{r}, \cos t, \sin t)^t \tag{3}$$

is surjective, maps each equivalence class onto a point and maps distinct equivalence classes onto distinct points. Moreover, since (2) may be rewritten as

$$(\vec{r}_1, \cos t_1, \sin t_1)G(\vec{r}_2, \cos t_2, \sin t_2)^t = 0,$$

f induces through (3) a bijection on $\bar{M}_4$ satisfying the hypotheses of the above mentioned result from [4]. Thus, up to a permutation within equivalence classes, f has the form

$$(\vec{r}, \cos t, \sin t)^t \to \lambda T(\vec{r}, \cos t, \sin t)^t$$

for some 6×6 matrix T satisfying $T^tGT = G$ and a scalar $\lambda \neq 0$. A tedious calculation verifies that all such f's do indeed preserve (2), so we are finished.

If T has the form

$$T := \mathrm{diag}\left\{A, \begin{pmatrix} \cos\alpha, & \mp\sin\alpha \\ \sin\alpha, & \pm\cos\alpha \end{pmatrix}\right\}$$

for a 4×4 orthogonal matrix A and constant α, then (again, up to permutations within equivalence classes) f has the form $(t,\vec{r}) \to (\pm t+\alpha, A\vec{r})$. For the still unconvinced, we exhibit an example of a transformation not of this form which still preserves (2): for $\lambda^2 := (\sqrt{3}z + 2\cos t)^2 + \sin^2 t$,

$$(\cos t, \sin t) \to \lambda^{-1}(\sqrt{3}z + 2\cos t, \sin t)$$
$$\vec{r} \to \lambda^{-1}(w,x,y,2z+\sqrt{3}\cos t).$$

One final remark: since the above mentioned result from [4] is in fact valid on conformal Minkowski spaces $\bar{M}_n$ of any dimension $n \geq 3$, the above characterization is valid on any Einstein cylinder universe C_n of dimension $n \geq 3$. For C_2, as for $\bar{M}_2$, the group of separation zero preserving transformations is even larger.

References

1. A.D. Alexandrov, A contribution to chronogeometry, Canad. J. Math. 19 (1967) 1110-1128.

2. S.W. Hawking; G.F.R. Ellis, The large scale structure of space-time, Cambridge University Press, 1973.

3. J.A. Lester, Conformal spaces, J. Geom. 14 (1980) 108-117.

4. J.A. Lester, A physical characterization of conformal transformations of Minkowski spacetime. Ann. Discrete Math. (to appear).

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THE IDEAL OF FORMS VANISHING AT A FINITE SET OF POINTS IN P^n

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Presented by P. Ribenboim, F.R.S.C.

Abstract: We study the ideal of forms vanishing at a finite set of distinct points in P^n and seek, in general, the least number of generators for such an ideal.

Introduction:

Let $P_1,\dots,P_s$ be distinct points of $P^n(k)$, with $n \geq 2$ and k an algebraically closed field. Let $I = I_d \oplus I_{d+1} \oplus \dots$, with $I_d \neq (0)$, be the ideal generated by all forms in $k[x_0,\dots,x_n]$ vanishing simultaneously at $P_1,\dots,P_s$ and let $\nu(I)$ denote the minimal number of generators of I.

We are interested in the following problems which are motivated by prior studies in [A], [G], $[G\text{-}O]_1$, $[G\text{-}O]_2$ and [R].

Problem (A): Does there exist, for any given n and s, a non-empty Zariski open set $U_{n,s} \subset (P^n)^s$ such that if $P = (P_1,\dots,P_s) \in U_{n,s}$ and I is the ideal of $P_1,\dots,P_s$ in $k[x_0,\dots,x_n]$, then $\nu(I)$ is a constant independent of P and explicitly computable in terms of n and s ?

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Problem (B): If such an open set $U_{n,s}$ exists, can one find some dense subsets of it which are "easily" identifiable in a concrete geometric or algebraic way?

1. In order to discuss our response to these problems we recall the following definition. (See also $[G\text{-}O]_1$, $[G\text{-}O]_2$).

Definition 1: Let $P_1,\dots,P_s$ be distinct points of $P^n(k)$, I the ideal of these points and $A = k[x_0,\dots,x_n]/I = \sum_{i=0}^{\infty} A_i$. If $\dim_k A_i = \min(s, \binom{i+n}{n})$ for $i \geq 0$ we say that $P_1,\dots,P_s$ are in generic s-position.

Remarks: 1) The sets of s distinct points in P^n, considered as points in $(P^n(k))^s$, which are in generic s-position form a non-empty Zariski open subset of $(P^n(k))^s$. (See $[G\text{-}O]_1$). 2) If $P_1,\dots,P_s$ are s distinct points of $P^n(k)$ in generic s-position and $I = I_d \oplus I_{d+1} \oplus \dots$, with $I_d \neq (0)$, is the ideal generated by the forms in $k[x_0,\dots,x_n]$ vanishing at $P_1,\dots,P_s$ then I may always be generated by forms of degree $\leq d+1$.

An affirmative answer to (A) above is given, in case $n = 2$, by:

Theorem 2: For any integer $s > 0$, there is a non-empty open set, $U_{2,s} \subset (P^2(k))^s$, contained in the open set of all s-tuples of points in generic s-position, with the following property:

If $(P_1,\dots,P_s) \in U_{2,s}$ and $I = I_d \oplus I_{d+1} \oplus \dots$, with $I_d \neq (0)$, is the ideal of the points $P_1,\dots,P_s$ in

$k[x_0,x_1,x_2]$ then

$$(*) \quad \nu(I) = \dim_k I_d + \dim_k I_{d+1} - \min(3\dim_k I_d, \dim_k I_{d+1}) .$$

Proof: We first observe that for s points of P^2 in generic s-position, the numbers d, $\dim_k I_d$ and $\dim_k I_{d+1}$ are independent of the set of s points chosen and easily determined by s. The main point of the proof depends on finding at least one set of s points of P^2 in genric s-position for which the claimed value for $\nu(I)$ is valid.

The proof constructs such a set for every s and is too technical to include here. The details will appear elsewhere.

Remark: This theorem gives an affirmative response in A^3, to the conjecture made by L. Roberts in [R].

In order to discuss our response to (B) we recall the following definition (See also $[G\text{-}O]_1$, $[G\text{-}O]_2$) .

Definition 3: The points $P_1,\ldots,P_s$ of P^n are said to be in uniform position if for every t, $1 \leq t \leq s$, each t element subset of $\{P_1,\ldots,P_s\}$, consists of points in generic t-position.

Theorem 4: If $P_1,\ldots,P_s$ are any s points of P^2 in uniform position and I is the ideal in $k[x_0,x_1,x_2]$ of these points then $\nu(I)$ is given by $(*)$ in each of the following cases:

i) $s = \binom{d+2}{2} - i$, $0 \leq i \leq 2$.
ii) $s = \binom{d+2}{2} + 1$.
iii) $s = 12$. (char $k = 0$)

Remark: Since the sets of s points in P^n in uniform position describe a non-empty Zarski open subset of $(P^n)^s$ for any s we have, for the s of this theorem, a response to Question (B) above in these cases.

We also show that the notion of uniform position alone is not enough to guarantee a fixed value for $\nu(I)$. Thus, we strengthen this notion to that of transversal uniform position and are able to extend Theorem 4, for such points, to the case $s = \binom{d+2}{2}-3$. We also obtain similar results in P^n for $n > 2$.

2 . An important consideration, in obtaining the results above, is the linear system of hypersurfaces of least degree passing through the given points. We have obtained the following result:

Theorem 5: Let $P_1,\dots,P_s$ be points of P^n in uniform position and write $s = \binom{(d-1)+n}{n} + h$, $0 \leq h < \binom{(d-1)+n}{n-1}$, $n \geq 2$, $d > 2$, Then:

i) the set, R , of all reducible hypersurfaces of degree d through $P_1,\dots,P_s$ is an algebraic set of dimension $n-1-h$ in P^N , $N = \binom{d+n}{n}-1$.

ii) if $h \geq n$, every hypersurface of degree d through $P_1,\dots,P_s$ is irreducible.

This theorem, and variants of it to cases when $P_1,\dots,P_s$ are only assumed to be in generic s-position allow us to give a partial answer (in characteristic zero) to the question raised by Abhyankar in [A] about the least degree of a nonsingular curve passing through a given finite set of points in P^2 (as well as to the obvious extension of Abhyankar's question to hypersurfaces in P^n.)

BIBLIOGRAPHY

[A] Abhyankar, S.S., On Macaulay's Examples, Springer Lect. Notes Math. 311 (1973) 1-16.

[B-E] Buchsbaum, D. and D. Eisenbud, Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (1977) 447-485.

[F] Fulton, W., Algebraic Curves, Benjamin, New York, 1969.

$[G\text{-}O]_1$ Geramita, A.V. and F. Orecchia, On the Cohen-Macaulay Type of s Lines in A^{n+1}, (to appear, J. Algebra).

$[G\text{-}O]_2$ Geramita, A.V. and F. Orecchia, Minimally Generating ideals Defining Certain Tangent Cones (preprint).

[G-S] Geramita, A.V. and C. Small, Introduction to Homological Methods in Commutative Rings, Queen's Papers in Pure and Applied Mathematics, Vol. 43 (2nd edition), Kingston, Ontario, Canada.

[G] Geyer, W.-D., On the Number of Equations which are Necessary to Describe an Algebraic Set in n-space, Atas da 3^a Escola de Algebra, Brasilia, 1976; IMPA Lecture Notes, 9, Rio de Janeiro, 1977, 183-317.

[G-H] Griffiths, P. and J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978.

[R] Roberts, L.G., A conjecture on Cohen-Macaulay type, C.R. Math. Rep. Acad. Sci. Canada 3 (1981) 43-48.

[Sa] Sally, J.D., Numbers of Generators of Ideals in Local Rings, Lecture Notes in Pure and Applied Math., Vol. 35, Dekker, New York, 1978.

[S-R] Semple, J.G. and L. Roth, Introduction to Algebraic Geometry, Clarendon Press, Oxford, 1949.

[Sh] Shafarevich, I.R., Basic Algebraic Geometry, Grundlehren 213, Springer, Heidelberg, 1974.

[St] Stanley, R. Hilbert Functions of Graded Algebras, Advances in Math., 28 (1978) 57-83.

[Z-S] Zariski, O. and P. Samuel, Commutative Algebra, Vol. 2, Van Nostrand, Princeton, 1960.

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