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K_0-REGULARITY OF UNIONS OF PLANES

Barry H. Dayton

Presented by P. Ribenboim, F.R.S.C.

Let k be a field of characteristic 0, A the coordinate ring of n (two dimensional) planes through the origin of affine p-space $\mathbb{A}_k^p$ over k. Upper and lower bounds are given for $K_0(A)$, and necessary and sufficient conditions are given for A to be K_0-regular.

K_0-regularity is a stronger condition on A than is seminormality (see [DR] for a discussion of seminormality of unions of planes). For instance, the number of planes possible in a K_0-regular configuration in $\mathbb{A}_k^p$ is bounded, unlike the case for seminormality. Like seminormality, however, there is no simple geometric condition for K_0-regularity. The conditions give here involve checking seminormality and calculating a Hilbert function and can, in theory, be checked for any explicit example. For p = 3,4 there are no K_0-regular configurations of planes other than those already described in [DR], [DW]. For $p \geq 5$ it appears likely that there will be some "new" examples of K_0-regular configurations.

1. $K_0(A)$

Let $R = k[x_1,\ldots,x_p]$, $\mathbb{A}_k^p = \operatorname{Spec} R$. For $i = 1,\ldots,n$ let I_i be an ideal of R generated by p-2 independent linear forms of R. Then $\pi_i = \operatorname{Spec} R/I_i \approx \operatorname{Spec} k[x,y]$ is a plane through the origin of $\mathbb{A}_k^p$, and $A = R/(\cap I_i)$ is the coordinate ring of $\cup \pi_i$. Let $B = \prod R/I_i$ be the normalization of A, $\mathfrak{C}$ the conductor of A in B and $C = (A/\mathfrak{C})_{red}$, $D = (B/\mathfrak{C})_{red}$. Note that C is the coordinate ring of the union of lines in $\mathbb{A}_k^p$ which are intersections of two or more of the planes (see [DR] Lemma 2.5). Let ${}^+A$, ${}^+C$, ${}^+D$ be the seminormalizations of A,C,D respectively (see [S]). By universality of seminormalization there is a map ${}^+C \longrightarrow {}^+D$ over the inclusion $C \longrightarrow D$. Let $\delta: {}^+C/C \longrightarrow {}^+D/D$ be the induced map and $H = \text{coker}\ \delta$.

<u>Theorem 1</u>: With notation as above, there is an exact sequence

$$\text{nil}(A/\mathfrak{C}) \longrightarrow \text{nil}(B/\mathfrak{C}) \longrightarrow {}^+A/A \longrightarrow {}^+C/C \xrightarrow{\delta} {}^+D/D$$

and a non-canonical isomorphism

$$\text{Pic}(A) \approx {}^+A/A$$

Let k' be the smallest subfield of k containing the coefficients of the generators of the I_i's, $\Omega_{k/k'}$, $\Omega_k = \Omega_{k/\mathbb{Z}}$ the groups of relative Kähler differentials and $\rho: \Omega_k \longrightarrow \Omega_{k/k'}$ the projection.

<u>Theorem 2</u>: With notation as above, there is a non-canonical splitting

$$K_0(A) = \mathbb{Z} \oplus \text{Pic}(A) \oplus SK_0(A)$$

and surjections

$$H \otimes_k \Omega_k \longrightarrow\!\!\!\!\rightarrow SK_0(A) \longrightarrow\!\!\!\!\rightarrow H \otimes_k \Omega_{k/k'}$$

whose composition is $1 \otimes \rho$. In particular, if $\Omega_{k'} = 0$ (eg. $k' = \mathbb{Q}$) then

$$K_0(A) = \mathbb{Z} \oplus {}^+A/A \oplus (H \otimes_k \Omega_k)$$

Theorem 3: With notation as above, there is an isomorphism

$$K_{-1}(A) = H.$$

Further, $K_i(A) = 0$ for $i < -1$.

2. K_0-regularity

Theorem 4: With notation as above, the following are equivalent:

1) A is seminormal
2) $Pic(A) = 0$
3) the conductor C is radical in B and ∂ is injective.

Theorem 5: With notation as above, the following are equivalent:

1) A is K_0-regular
2) $NK_0(A) = 0$
3) A is seminormal and K_{-1}-regular
4) A is seminormal and ∂ is surjective
5) C is radical in B and ∂ is bijective
6) A is seminormal and $\dim_k {}^+C/C = \dim_k {}^+D/D$.

If Spec C consists of r lines and if the plane π_i contains δ_i of these lines then $\dim_k {}^+C/C = \Sigma_{d=1}^{\infty}(r - f(d))$ where f(d) is the dimension of the degree d part of the graded ring C, and $\dim_k {}^+D/D = \Sigma_{i=1}^{n}\binom{\delta_i - 1}{2}$. f is the Hilbert function of C and is fairly well understood (eg. see [GO]), thus, if A is known to be seminormal, it is relatively easy to ascertain if A is K_0-regular.

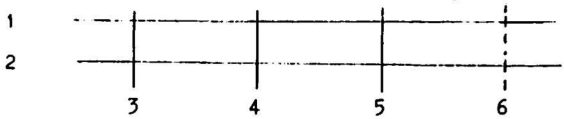
3. Examples

In the following examples we may take $k' = Q$ and k to be any field containing Q.

Example 1: $A = k[X,Y,Z]/(XYZ(X+Y-Z))$ is a well known example of a union of planes in $\mathbb{A}^3_k$ which is seminormal but not K_0-regular. Here $\dim_k {}^+C/C = 3$, $\dim_k {}^+D/D = 4$ so H is a k-vector space of dimension 1. Hence from Theorem 2, we recover the computation $K_0(A) = \mathbb{Z} \oplus \Omega_k$ of [D].

We illustrate our next examples using the classical correspondence between planes through the origin of $\mathbb{A}^p_k$ and lines in $\mathbb{P}^{p-1}$ as in [DR]. Planes intersecting only at the origin correspond to skew lines, planes intersecting in lines correspond to intersecting lines.

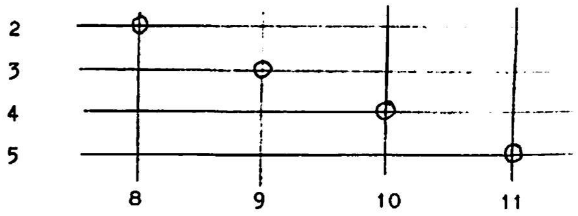
Example 2: Let $A = k[X,Y,Z,W]/(\cap I_i)$ where $I_1 = (X,Z)$, $I_2 = (Y,W)$, $I_3 = (X,W)$, $I_4 = (Y,Z)$ and $I_5 = (X+Z,Y+W)$. Note $\cup \pi_i$ is contained in the quadric hypersurface $XY-ZW = 0$. A is K_0-regular



(see [DR, Examples 7,11]) but $\dim_k {}^+C/C = \dim_k {}^+D/D = 2$. All the examples of K_0-regular configurations of planes in [DW] had ${}^+D/D = 0$. This example, which indicates that K_0-regularity is a more subtle notion than I had previously thought, was the motivation for this paper.

Example 3: Adding the plane π_6 with ideal $I_6 = (X-Z,Y-W)$ to the configuration of Example 2 gives a configuration for which C is not radical in B (see Remark 9 of [DR]) but for which ∂ is still an isomorphism.

Example 4: Planes 2,3,4,5,8,9,10,11 of the explicit Example 16 of [DR] give a "double 4" with radical conductor but which is not seminormal. Here ∂ is surjective but not injective. $K_0(A) = Z \oplus k^2$ for this example.



From Theorem 5, part 5, there are three conditions necessary for K_0-regularity: a) C is radical in B, b) ∂ is injective and c) ∂ is surjective. Examples 1,3,4 show that any one of these may fail with the others holding. In general, however, one expects none of these to hold. Thus, K_0-regularity is a fairly strong condition.

Theorem 6: Let A be the coordinate ring of n planes through the origin of $\mathbb{A}_k^p$ as in §1. If A is K_0-regular then $n \leq 2\binom{p+1}{p-1}$.

At least for small p, there is a much lower bound on n:

Theorem 7: Let A be the coordinate ring of n planes through the origin of $\mathbb{A}_k^p$ as in §1, p = 3,4. If A is K_0-regular then the configuration of planes can be built up, one plane at a time, so that at each stage the coordinate ring is K_0-regular and each plane added intersects the previous union in at most two lines. In particular $\delta_i \leq 2$ for at least one plane. If p = 3 there are at most 3 planes and if p = 4 there are at most 6 planes in the configuration.

Theorem 7 says that all examples of K_0-regular unions of planes, p = 3,4, have already been described; for p = 3 in [DW, Example 2.4] and for p = 4 in [DR, Example 11]. I do not know of an example of a K_0-regular configuration with $\delta_i \geq 3$ for all i. Example 4 shows that there are such examples which are K_{-1}-regular and there certainly are seminormal configurations like this. It thus seems possible that in $\mathbb{A}_k^p$, $p \geq 5$ such a K_0-regular configuration might exist.

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Northeastern Illinois University
Chicago, Illinois 60625, USA.

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PLONGEMENTS DE PG(n,q) ET AG(n,q) DANS PG(m,q'), m < n.

Monique Limbos

Presented by H.S.M. Coxeter, F.R.S.C.

On montre que PG(n,q) et AG(n,q) (sauf AG(2,3)) sont plongeables dans PG(m,q'), $m < n$, ssi $q'=q^r$ pour certaines valeurs de r suffisament grandes.
Une caractérisation de ces plongements est obtenue, ainsi que des bornes pour la valeur de r.

§1. Introduction.

Un espace linéaire E est dit plongeable dans un espace projectif P ssi il existe un sous-ensemble $S \subset P$, tel que S muni des restrictions à S des droites de P est un espace linéaire isomorphe à E. S est appelé un plongement de E.
Nous supposerons P arguésien et fini.
Il est bien connu que AG(2,3) se plonge dans PG(2,q) $q \equiv 1 \pmod 3$ [2].
D'autre part Thas [6] a construit un plongement de AG(n,q) (en tant qu'espace linéaire) dans $AG(2,q^n)$.
Nous nous proposons d'étudier le problème de l'existence de plongements des espaces linéaires PG(n,q) et AG(n,q) dans PG(m,q'), $m < n$, et de caractériser ces plongements.

§2. Projections de "plongements naturels".

Il est bien connu que PG(2,q) ne peut être plongé dans PG(2,q') que si $q'=q^r$, pour r entier positif [1]. De même Rigby [5] a montré que AG(2,q) ne peut être plongé que dans $PG(2,q^r)$ dès que $q > 3$.
D'autre part si $n \geq 3$, nous avons pu montrer que AG(n,3) ne peut être plongé que dans $PG(m,3^r)$ [3]. Ainsi en toute généralité PG(n,q) et AG(n,q) (sauf AG(2,3)) ne peuvent être plongés que dans $PG(m,q^r)$ pour certaines valeurs de r.
PG(n,q) est évidement plongeable dans $PG(n,q^r)$ pour tout r entier positif.
Dans l'espace $PG(n,q^r)$ coordonné l'ensemble S des points à coordonnées sur GF(q) constitue un plongement de PG(n,q) dans $PG(n,q^r)$. Tout plongement S' équivalent à S par une collinéation est appelé plongement naturel de PG(n,q)

dans $PG(n,q^r)$. Si H est un hyperplan de S' (considéré comme espace projectif), S' \ H est appelé plongement naturel de AG(n,q) dans $PG(n,q^r)$. L'idée de base pour construire des plongements de PG(n,q) et AG(n,q) dans $PG(m,q^r)$, $m < n$, est la suivante: soit S un plongement naturel de PG(n,q) (resp.AG(n,q)) dans $PG(n,q^r)$; supposons qu'il existe une variété Vn-m-1 de $PG(n,q^r)$, de dimension n-m-1, et une variété Vm $\cong PG(m,q^r)$ de $PG(n,q^r)$ telles que Vm $\cap$ Vn-m-1=$\emptyset$ et Vn-m-1 ne rencontre aucun plan engendré par trois points de S, alors en projetant S sur Vm à partir de Vn-m-1 on obtient un plongement de PG(n,q) (resp AG(n,q)) dans $PG(m,q^r)$.
La propriété remarquable est la suivante.

Théorème 1: Tout plongement S de PG(n,q) (resp AG(n,q) $\neq$ AG(2,3)) dans $PG(m,q^r)$, $m < n$, est obtenu comme projection d'un plongement naturel de PG(n,q) (resp AG(n,q)) dans $PG(n,q^r)$, à partir d'une variété Vn-m-1 de dimension n-m-1, sur une variété complémentaire Vm de dimension m. La variété Vn-m-1 ne rencontre aucun plan de S.

Il suffit dès lors d'étudier les valeurs de r pour lesquelles la variété Vn-m-1 décrite existe pour obtenir une classification complète des plongements de PG(n,q) et AG(n,q) dans $PG(m,q^r)$, $m < n$. Nous pouvons démontrer les résultats suivants

Théorème 2: PG(n,q) est plongeable dans $PG(n-1,q^r)$ $(n \geq 3)$ ssi $r \geq 4$

la démonstration repose sur le

lemme: Soit S un plongement naturel de PG(n,q) dans $PG(n,q^r)$. Tout point $p \in PG(n,q^r) \setminus S$ appartient à au moins un sous-espace de dimension r-1 de S, et il existe des points qui n'appartiennent à aucun sous-espace de dimension r-2 de S.

Par descente on déduit que PG(n,q) est plongeable dans $PG(2,q^{4^{n-2}})$. Cette borne peut être améliorée si $n \geq 4$. Supposons qu'il existe H un hyperplan de $PG(n,q^r)$ qui ne coupe pas S un plongement naturel de PG(n,q) dans $PG(n,q^r)$. Alors S est en fait un plongement de PG(n,q) dans $AG(n,q^r)$. On montre qu'un tel hyperplan H existe dès que $r \geq n+1$. De plus il existe alors dans H un point p qui n'appartient à aucun plan de S. En projetant S à partir de p sur un hyperplan H' $\not\ni$ p, on obtient un plongement S' de PG(n,q) dans $AG(n-1,q^{n+1})$. Thas a montré que $AG(n-1,q^{n+1})$ est plongeable dans $AG(2,q^{n^2-1})$. Nous avons ainsi le

<u>Théorème 3</u>: PG(n,q) est plongeable dans $PG(2,q^{n^2-1})$ si $n \geq 4$, dans $PG(2,q^4)$ si n=3.

§3. Conclusion.

Les résultats précédents caractérisent les plongements de PG(n,q) dans $PG(m,q^r)$. Lors de l'étude des plongements d'espaces linéaires E quelconques dans les espaces projectifs arguésiens finis on peut se ramener à l'étude des plongements de E dans les plans PG(2,q). En effet tout plongement de E dans PG(n,q) donne lieu à un plongement de E dans certains plans $PG(2,q^r)$ et le plongement initial peut être "reconstruit" à partir de celui-ci. (La manière dont cette "reconstruction" peut se faire apparaît dans la démonstration du théorème 1 [2]).

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Université Libre de Bruxelles
Département de Mathématique
Campus Plaine C.P.215
Bd du Triomphe
B-1050 BRUXELLES

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SUR UNE QUESTION DE L. LESIEUR

Marc CHAMARIE
UNIVERSITE CLAUDE-BERNARD - LYON I
Département de Mathématiques
43, boulevard du 11 Novembre 1918
69622 VILLEURBANNE CEDEX (France)

Presented by G. de B. Robinson, F.R.S.C.

Soient A un anneau (unitaire) principal à gauche et intègre, Z(A) son centre, K le corps des fractions de Z(A) et L = Z(Frac(A)) le centre du corps des fractions à gauche de A.

L. LESIEUR a montré dans [4], que Z(A) est un domaine de Krull (ceci résulte aussi directement du lemme 1.3 de [2]) et que K est algébriquement fermé dans L. Il pose alors la question suivante : a-t-on toujours L = K ? (G. CAUCHON a montré dans [1] que ceci est vrai si A est une extension de Ore : $D[X,\sigma,\delta]$, où D est un corps gauche, σ un endomorphisme et δ une σ-dérivation de D).

Le résultat suivant montre que la réponse est négative.

PROPOSITION 1. - *Soient* R *un domaine de Krull commutatif,* K *son corps des fractions et* L *une extension transcendante pure de* K. *Si* carac(K) = 0, *il existe un anneau principal intègre* A, *dont tous les idéaux sont bilatères, tel que :*

Z(A) = R *et* Z(Frac(A)) = L .

PREUVE. - Soit $(X_i)_{i \in I}$ une base pure de L sur K, et soit Y une indéterminée. Considérons le domaine de Krull $S = R[X_iY,Y]_{i \in I}$ de corps des fractions $K(X_i, Y)_{i \in I} = L(Y)$. Sur L(Y), notons δ la L-dérivation définie par : $\delta(Y) = Y$. Puisque δ laisse stable S, on peut considérer l'extension de Ore : $B = S[Z,\delta]$; on montre alors comme en 3.2 et 3.3 de [3] que B est un anneau de Krull non commutatif.

Soit $\mathcal{P}$ l'ensemble de tous les idéaux premiers de hauteur 1 de S, δ-stables. Pour tout $P \in \mathcal{P}$, BP = PB est un c-idéal complètement premier ; la partie multiplicative B - PB vérifie les conditions de Ore à droite et à gauche et le localisé correspondant B_{PB} est un anneau principal quasi-local (proposition 2.5 de [3]). Notons $\Sigma = \bigcap_{\mathcal{P}} (B - PB)$ et remarquons qu'un idéal à droite I de B coupe Σ si et seulement si $I \not\subseteq PB$, $\forall P \in \mathcal{P}$; en effet, supposons cette dernière condition vérifiée ; soient $f \neq 0$ dans I et $n = \deg_Z f$; le coefficient dominant de f n'appartient pas à P, pour tout $P \in \mathcal{P}$, sauf un nombre fini : $P_1, P_2, \ldots, P_q$; $\forall i$, $1 \leq i \leq q$, soit $f_i \in I - PB$ et soit $n_i = \deg_Z f_i$; alors :

$$f + f_1 Z^{n+1} + f_2 Z^{n+n_1+2} + \ldots + f_q Z^{n+n_1+\ldots+n_q+q}$$

appartient à $I \cap \Sigma$. On en déduit que Σ vérifie les conditions de Ore à droite et à gauche. Notons $A = \bigcap_{\mathcal{P}} B_{PB}$ le localisé de B par rapport à Σ.

Puisque PB est complètement premier, tous les idéaux de B_{PB} sont bilatères ; par suite tous les idéaux de A sont bilatères.

Montrons que A est principal. On sait que A est un anneau de Krull non commutatif (Lemme 2.6 de [3]) ; pour tout idéal I de A, il existe donc $f_1, f_2, \ldots, f_n$ dans $I \cap B$ tels que : $\overline{I} = \overline{\sum_i f_i A}$ (où $\overline{I}$ désigne le plus petit idéal réflexif contenant I). Il suffit donc de montrer que si $f \neq 0$ et g sont dans B, alors fA + gA est principal. Soit $n = \deg_Z f$ et considérons : $h = f + gZ^{n+1}$; pour tout $P \in \mathcal{P}$, il existe un entier $m \geq 0$ tel que : $h \in P^{(m)}B - P^{(m+1)}B$;

on en déduit : $(P^{-m}h)(h^{-1}f) \subseteq B$ et : $P^{-m}h \not\subseteq PB$; par suite : $f \in hA$; de même on montre : $g \in hA$ et donc : $fA + gA = hA$.

Il est clair que $\mathrm{Frac}(A) = \mathrm{Frac}(L(Y)[Z,\delta])$; comme L est de caractéristique nulle, on en déduit facilement : $Z(\mathrm{Frac}(A)) = L$.

Calculons : $Z(A) = L \cap A$; il est clair que $R \subseteq Z(A)$. Réciproquement soit $x \neq 0$ dans $L \cap A$; alors $I = S\,x^{-1} \cap S$ est non nul et δ-stable ; tout idéal premier de hauteur 1, P contenant I est donc δ-stable ; en effet, il existe $n > 1$ tel que : $IS_p = P^nS_p$; soit $y \in P$ tel que : $PS_p = yS_p$; il vient : $n\,\delta(y) \in yS_p \cap S = P$; par suite soit $\delta(y) \in P$ et P est δ-stable, soit $n \in P$, auquel cas, la caractéristique de K étant nulle, $P = S(P \cap R)$ et P est aussi δ-stable. Puisque $x \in A$, on en déduit : $I = S$ et donc $x \in L \cap S = R$.

Dans le même ordre d'idées, soit A un anneau de Bezout à gauche (tout idéal à gauche de type fini de A est principal) et intègre ; on vérifie aisément qu'un tel anneau a un corps des fractions à gauche.

PROPOSITION 2. - *Soient* L *un corps commutatif et* R *un sous-anneau de* L. *Les conditions suivantes sont équivalentes :*

1) *Il existe un anneau de Bezout (à gauche) intègre* A *tel que :*

$$Z(A) = R \quad et \quad Z(\mathrm{Frac}(A)) = L .$$

2) R *est intégralement fermé dans* L.

PREUVE. - 1) $\Rightarrow$ 2). Soit x dans L, entier sur R ; comme x est central, il existe $\lambda \neq 0$ dans A tel que $A[x]\lambda$ soit un idéal à gauche de type fini, donc principal ; puisque on a : $x\,A[x]\lambda \subset A[x]\lambda$, on en déduit que x est dans A, donc dans R.

2) $\Rightarrow$ 1). Soient K le corps des fractions de R et $(X_i)_{i \in I}$ une base de transcendance de L sur K. Soit Y une indéterminée ; notons $S = R[X_i Y, Y]_{i \in I}$; L(Y) est algébrique sur $K(X_i, Y)_{i \in I}$, le corps des fractions de S ; notons $\overline{S}$ la fermeture intégrale de S dans L(Y).

Soient $(U_n)_{n\in\mathbb{Z}}$ des indéterminées ; sur le corps $L(Y,U_n)_{n\in\mathbb{Z}}$, considérons σ le L-automorphisme défini par : $\sigma(U_n) = U_{n+1}$ et $\sigma(Y) = YU_o$; il est clair que σ laisse stable $S[U_n]$ et donc aussi $\overline{S}[U_n]$.

A toute valuation v du corps L(Y), positive sur S, on va associer une valuation w sur le corps $L(Y,U_n)$, positive sur $\overline{S}[U_n]$ et vérifiant : $w|_L = v|_L$ et $\forall x \in L(Y,U_n)$, $w(\sigma(x)) = w(x)$. Il suffit de définir w sur $L[Y,U_n]$; si $x \in L[Y]$, $x = \sum_i \lambda_i Y^i$, on pose : $w(x) = \inf_i(v(\lambda_i Y^i))$; si $x \in L[Y][U_n]$, $x = \sum_\nu x_\nu U^\nu$, on pose : $w(x) = \inf_\nu(w(x_\nu))$. Comme $S = \bigoplus_{n\geqslant o}(S \cap LY^n)$, il est clair que w est positive sur S, donc sur $\overline{S}$ et donc aussi sur $\overline{S}[U_n]$. On vérifie facilement que pour tout $x \in L[Y,U_n]$, $w(\sigma(x)) = w(x)$. Considérons maintenant l'extension de Ore : $B = \overline{S}[U_n][Z,\sigma]$; les valuations w se prolongent à B et donc à Frac(B) en posant : $w(\sum_n x_n Z^n) = \inf_n(w(x_n))$. Notons $A = \{x \in \mathrm{Frac}(B)\ ;\ \forall w,\ w(x) \geqslant 0\}$. Il est clair que tous les idéaux de A sont bilatères. Montrons que A est de Bezout ; soient x, y non nuls dans A ; il existe $u \neq 0$ dans B tel que : $ux \in B$ et $uy \in B$; si $n = \deg_Z(ux)$, posons : $z = ux + uy\,Z^{n+1}$; puisque $w(z) \leqslant w(ux)$ et $w(z) \leqslant w(uy)$, on en déduit :

$$xA + yA = u^{-1}zA.$$

Il est clair que : $\mathrm{Frac}(A) = \mathrm{Frac}(B) = \mathrm{Frac}(L(Y,U_n)[Z,\sigma])$; on vérifie aisément que : $Z(L(Y,U_n)[Z,\sigma]) = \{x \in L(Y,U_n)\ ;\ \sigma(x) = x\} = L$. Donc $Z(\mathrm{Frac}(A)) = L$. Calculons $Z(A) = L \cap A$; évidemment : $R \subseteq Z(A)$; réciproquement si $x \in L \cap A$, pour toute valuation v de L(Y), positive sur S, $v(x) = w(x)$ est positif ; donc x est entier sur S ; il existe une relation de la forme :

$$x^n + s_{n-1}(X_iY,Y)x^{n-1} + \ldots + s_o(X_iY,Y) = 0\ ;$$

on en déduit : $x^n + s_{n-1}(0,0)x^{n-1} + \ldots + s_o(0,0) = 0$, ce qui prouve que x est entier sur R et donc x est dans R, par hypothèse.

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POLYNOMIALS OF GIVEN DISCRIMINANT AND INTEGRAL ELEMENTS OF GIVEN DISCRIMINANT OVER INTEGRAL DOMAINS

K. Győry

Presented by P. Ribenboim, F.R.S.C.

Abstract. Let K be an algebraic number field with ring of integers R, L a finite extension of K, and T the integral closure of R in L (i.e. the ring of integers of L). In [2] we proved that up to the translation by elements of R, there are only finitely many elements in T with a given non-zero discriminant over K and a full set of representatives of such elements of T can be effectively determined. In this paper we present a generalization of the first part of this theorem to the case when K is an arbitrary field of finite type over $\mathbb{Q}$, and R is an integrally closed subring of K of finite type over $\mathbb{Z}$ with quotient field K. Our result is the consequence of a theorem concerning polynomials of given discriminant. Some applications are given to integral elements of given discriminant and to discriminant form and index form equations.

A detailed and generalized version of this work will appear in [4].

1. Polynomials with given discriminant

Let R be a finitely generated integral domain over $\mathbb{Z}$, and K its quotient field. Suppose that R is integrally closed. If $f \in R[X]$ is a monic polynomial and $f^*(X) = f(X + a)$ with some $a \in R$, then for their discriminants $D(f) = D(f^*)$ holds. Such polynomials $f, f^* \in R[X]$ will be called R-equivalent. For linear f let $D(f) = 1$.

Let G be a finite extension field of K, and D_0 a non-zero element in R. Under the above assumptions we have the following

THEOREM 1. *There are only finitely many pairwise R-inequivalent monic polynomials* $f \in R[X]$ *with roots in* G *and discriminant* D_0.

COROLLARY. *Given a non-zero element* f_0 *in* R, *there are only finitely many monic polynomials* $f \in R[X]$ *with roots in* G *such that* $D(f) = D_0$ *and* $f(0) = f_0$.

In Sections 2 and 3 some further consequences of Theorem 1 will be presented.

2. Integral elements of given discriminant over integral domains

Let R, K and D_0 be as above. Let L be a finite extension of K of degree ≥ 2, and T the integral closure of R in L. If α is an element of T then its discriminant $D_{L/K}(\alpha)$ over K lies in R. Further, if $\alpha^* \in T$ and $\alpha - \alpha^* \in R$ then α and α^* have the same discriminant over K. Such elements of T will be called

R-equivalent.

The following theorem is an easy consequence of Theorem 1.

<u>THEOREM 2</u>. <u>There are only finitely many pairwise R-inequivalent elements</u> α <u>in</u> T <u>with</u> $D_{L/K}(\alpha) = D_o$.

We note that Theorems 1 and 2 do not remain valid in general if R is not integrally closed. If R is an arbitrary integral domain of finite type over $\mathbb{Z}$ with quotient field K and if $\bar{R}$ denotes the integral closure of R in K , then R and $\bar{R}$ have the same integral closure T in L and, by Theorem 2, there are only finitely many $\bar{R}$-inequivalent $\alpha \in T$ with $D_{L/K}(\alpha) = D_o$.

We present now some consequences of Theorem 2 . Let R and T be as in Theorem 2 , and let $N_{L/K}$ be the norm from L to K .

<u>COROLLARY 1</u>. <u>Given a non-zero</u> $N_o \in R$, <u>there are only finitely many elements</u> α <u>in</u> T <u>with</u> $D_{L/K}(\alpha) = D_o$ <u>and</u> $N_{L/K}(\alpha) = N_o$.

<u>COROLLARY 2</u>. <u>There are only finitely many invertible elements</u> α <u>in</u> T <u>with</u> $D_{L/K}(\alpha) = D_o$.

Denote by R^* the multiplicative group of invertible elements of R . Let S be a subring of L containing R . Suppose that S is integral over R . If $S = R[\alpha]$ with some $\alpha \in S$ and $\alpha' = a + b\alpha$ with some $a \in R$ and $b \in R^*$, then $S = R[\alpha']$.

COROLLARY 3. *Up to the multiplication by elements of* R^* *and translation by elements of* R, *there are only finitely many* $\alpha \in S$ *with* $S = R[\alpha]$.

3. Discriminant form and index form equations

Let K and L be as above, and let $1, \alpha_1, \ldots, \alpha_m$ be linearly independent elements of L over K such that $L = K(\alpha_1, \ldots, \alpha_m)$. Let $n = [L : K]$. We recall that by hypothesis $n \geq 2$. There are n K-isomorphisms of L into the normal closure of L/K; denote the images of α_j under these isomorphisms by $\alpha_j^{(1)}, \ldots, \alpha_j^{(n)}$. Consider the linear forms

$$\ell^{(i)}(X) = X_0 + \alpha_1^{(i)} X_1 + \ldots + \alpha_m^{(i)} X_m, \quad i = 1, \ldots, n.$$

Then

$$D(\alpha_1 X_1 + \ldots + \alpha_m X_m) = \prod_{1 \leq i < j \leq n} (\ell^{(j)}(X) - \ell^{(i)}(X))^2$$

is a decomposable form of degree $n(n-1)$ in $X_1, \ldots, X_m$ with coefficients in K. Such a form is said to be a discriminant form.

Let R' be a finitely generated subring of K over $\mathbb{Z}$. Our Theorem 2 is equivalent to the following statement.

THEOREM 3. *For given non-zero* $D_0 \in K$, *the discrim-*

inant form equation

$$D(\alpha_1 x_1 + \ldots + \alpha_m x_m) = D_o$$

has only finitely many solutions $x_1, \ldots, x_m$ *in* R' .

Finally, we present an important consequence of Theorem 3 . Let A be an integral domain with quotient field K , and B a subring of L containing A . Suppose that B is a free module over A having a basis of the form $\{1, \omega_2, \ldots, \omega_n\}$. It is easily verified that

$$D(\omega_2 X_2 + \ldots + \omega_n X_n) = [F(X_2, \ldots, X_n)]^2 . D(1, \omega_2, \ldots, \omega_n)$$

where $D(1, \omega_2, \ldots, \omega_n)$ denotes the discriminant of the basis $\{1, \omega_2, \ldots, \omega_n\}$ over K , and $F(X_2, \ldots, X_n)$ is a decomposable form of degree $n(n-1)/2$ with coefficients in A. The form F is called the index form of the basis $\{1, \omega_2, \ldots, \omega_n\}$ of B over A .

COROLLARY. For given non-zero $F_o \in K$, *the index form equation*

$$F(x_2, \ldots, x_n) = F_o$$

has only finitely many solutions $x_2, \ldots, x_n$ *in* R' .

Remark 1. We note that Theorems 1,2 and 3 cannot be extended to arbitrary integrally closed integral domains R, R' of characteristic 0 .

Remark 2. There exist several finiteness theorems for polynomials with algebraic integer coefficients and given discriminant, for algebraic integers with given discriminant and for discriminant form and index form equations considered over number fields (cf. [1-3], [6-8] and the references given there). Our theorems and their corollaries generalize (in an ineffective form) some of these results. Certain effective versions of our results have been established in [5].

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Mathematical Institute, Kossuth Lajos University
4010 Debrecen, Hungary

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SOME SPECTRAL PROPERTIES OF BOUNDED OPERATOR COSINE FUNCTIONS

Dieter Lutz

Presented by J. Aczél, F.R.S.C.

Abstract: Using Fourier transform methods we show that the infinitesimal generator of a bounded strongly continuous operator cosine function is bounded iff its spectrum is bounded.

Let X denote a Banach space and let $B(X)$ be the algebra of bounded linear operators on X.

An operator cosine function on X is a strongly continuous function C defined on $\mathbb{R}$ with values in $B(X)$ satisfying d'Alembert's functional equation

$$C(t+s) + C(t-s) = 2\,C(t)C(s), \qquad s,t \in \mathbb{R}$$
$$C(0) = I \quad .$$

The infinitesimal generator A of C is a closed operator defined on a dense subset of X given by

$$Ax := C''(0)x \quad ,$$

whenever the right-hand side makes sense.

For general information concerning operator cosine functions and their spectral theory see [5], [3], [1].

The operator cosine function C on X is called bounded if there is a constant $M \geq 0$ with

$$\|C(t)\| \leq M$$

for every $t \in \mathbb{R}$.

By $\sigma(A)$ we denote the spectrum of A. If A is the infinitesimal generator of a bounded operator cosine function, then we have

$$\sigma(A) \subset \{x \in \mathbb{R} \mid x \leq 0\}$$

([5], for another proof see [1], where Satz 1.2.2 contains a misprint: read $\sigma(A) \subseteq \mathbb{R}^-$ instead of $\sigma(A) \subseteq \mathbb{R}$). Furthermore, denoting the resolvent operator of A by $R(z,A)$, we have in this case

$$z\,R(z^2,A) = \int_0^\infty e^{-zt}\,C(t)dt$$

for $z > 0$.

There are cosine operator functions C whose (unbounded) infinitesimal generator A has a void and hence bounded spectrum. For operator semigroups (compare e.g. [4]) results have been published relating boundedness properties of $\sigma(A)$ to spectral and other structural properties of $C(t)$ and A . Nothing of this kind seems to be known for operator cosine functions. It is the aim of the present note to prove the following partial result.

Theorem: Let C denote a bounded operator cosine function on the Banach space X with infinitesimal generator A .
Then

i) $\sigma(A) \neq \emptyset$, if $X \neq \{0\}$.
ii) $\sigma(A)$ is bounded iff A is bounded (that is equivalent to the continuity of C with respect to the operator norm topology).

Remark: For periodic (and hence bounded) operator cosine functions these results are well-known, e.g. [2]).

Proof: If C is a bounded operator cosine function on X for every $f \in L^1(\mathbb{R})$ and every $x \in X$ the function

$$t \to f(t)C(t)x$$

is Bochner integrable on $\mathbb{R}$.

We put

$$C(f)x := \int_{-\infty}^{\infty} f(t)C(t)x\,dt \quad .$$

Then

$$\begin{aligned}\|C(f)x\| &\leq \int_{-\infty}^{\infty} |f(t)|\,\|C(t)x\|dt\\ &\leq M\,\|f\|_1\|x\|, \quad x \in X\end{aligned} \quad ,$$

if the norm bound of C is denoted by M .

Hence $C(f) \in B(X)$

and $\|C(f)\| \leq M\cdot\|f\|_1$.

The mapping
$$L^1(\mathbb{R}) \to B(X) \quad ,$$
$$f \to C(f) \quad ,$$
is linear and continuous.

For $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ define $h \in L^1(\mathbb{R})$ by
$$h(t) := e^{-\varepsilon|t|} e^{i\alpha t} , \quad t \in \mathbb{R} .$$

Then, omitting the argument $x \in X$, we get
$$\begin{aligned} C(h) &= \int_{-\infty}^{\infty} e^{-\varepsilon|t|} e^{i\alpha t} C(t)dt \\ &= \int_0^{\infty} e^{-(\varepsilon-i\alpha)t} C(t)dt \\ &+ \int_0^{\infty} e^{-(\varepsilon+i\alpha)t} C(t)dt \\ &= (\varepsilon-i\alpha)R((\varepsilon-i\alpha)^2,A) + (\varepsilon+i\alpha)R((\varepsilon+i\alpha)^2,A) \\ &=: g(\varepsilon,\alpha) . \end{aligned}$$

Now let $f,\hat{f} \in L^1(\mathbb{R})$, $\hat{f}$ being the Fourier transform of f . Then, for every $\varepsilon > 0$, we have
$$\begin{aligned} \int_{-\infty}^{\infty} f(t)e^{-\varepsilon|t|} C(t)dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha)\Big(\int_{-\infty}^{\infty} e^{i\alpha t} e^{-\varepsilon|t|} C(t)dt\Big)d\alpha \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha)g(\varepsilon,\alpha)d\alpha \end{aligned}$$

and thus

$$\text{(1)} \qquad C(f) = \lim_{\varepsilon\to 0^+} C(f\, e^{-\varepsilon|.|}) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha)g(\varepsilon,\alpha)d\alpha .$$

Now we assume that $\sigma(A) = \emptyset$. We show that $C(f) = 0$ for every $f \in L^1(\mathbb{R})$: This is true for all $f \in L^1(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$ and supp $\hat{f}$ compact since
$$\lim_{\varepsilon\to 0^+} g(\varepsilon,\alpha) = -i\alpha\, R(-\alpha^2,A) + i\alpha\, R(-\alpha^2,A) = 0 .$$

But these f form a dense subset of $L^1(\mathbb{R})$ and C is continuous on $L^1(\mathbb{R})$, so $C(f) = 0$ for every $f \in L^1(\mathbb{R})$.

Putting
$$f(t) := e^{-|t|}$$

we get

$$0 = C(f) = \int_{-\infty}^{\infty} e^{-|t|}\, C(t)dt$$

$$= 2 \int_{0}^{\infty} e^{-t}\, C(t)dt$$

$$= 2\, R(1,A) \quad .$$

$R(1,A) = 0$ can only be true for $X = \{0\}$.

ii) We prove only the nontrivial part. So let $\sigma(A)$ be bounded and let K be a circle in $\mathbb{C}$ containing $\sigma(A)$ in its interior. Then

$$P := \frac{1}{2\pi i} \int_K R(z,A)dz$$

is a projection reducing $C(t)$ for each $t \in \mathbb{R}$.
The restricted mapping $t \to C(t) \mid \mathrm{Ker}\, P$ is a bounded operator cosine function with infinitesimal generator $A_1 = A \mid \mathrm{Ker}\, P \cap D(A)$. Thus $\sigma(A_1) = \emptyset$ from which we can conclude

$$\mathrm{Ker}\, P = \{0\}, \quad P = \mathrm{id}_X \quad .$$

Thus, for every $x \in D(A)$, we have

$$Ax = \frac{1}{2\pi i} \int_K R(z,A)Ax\, dz$$

$$= \frac{1}{2\pi i} \int_K (z\, R(z,A)x - x)dz$$

$$= \frac{1}{2\pi i} \int_K z\, R(z,A)x\, dz \quad ,$$

from which the boundedness of A follows.

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FB 6 - Mathematik - Universität Essen
P.O.B. 6843, D-4300 Essen 1, Germany(Fed.Rep.)

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PEAUCELLIER AND THE TORUS

Neville Reed
Presented by H.S.M. Coxeter, F.R.S.C.

We use a Peaucellier inverting linkage as the key to a new proof that certain oblique sections of a torus are circles. Other proofs of this surprising result can be found in [1] pp. 132-133 and [2] pp. 154-155.

Inversion in the circle with centre (0,0) and radius $\sqrt{a^2-b^2}$ fixes the circle with centre (a,0) and radius b shown in Figure 1. By rotating this figure about the z-axis we see that inversion in the sphere Σ with centre (0,0,0) and radius $\sqrt{a^2-b^2}$ leaves invariant the torus τ consisting of all points at distance b from the circle of radius a centred at (0,0,0) and lying in the plane z=0. The intersection of Σ with a plane π of the form z=cy is a circle of radius $\sqrt{a^2-b^2}$ and since inversion in this circle is induced by inversion in Σ it must leave invariant the cross section $\pi\cap\tau$.

Figure 2 shows that inversion in a circle with centre (0,0) and radius $\sqrt{a^2-b^2}$ interchanges the two circles of radius a with centres at (b,0) and (-b,0). Our object is to prove that circles like these constitute the cross section $\pi_0\cap\tau$ which occurs when $c=\frac{b}{\sqrt{a^2-b^2}}$ so that π_0 is a bitangent plane inclined at $\psi=\arcsin\frac{b}{a}$ to the horizontal. In effect we show that when Figure 2 is imbedded in 3-space as the plane π_0 then mates like P_1 and P_2 both lie on τ.

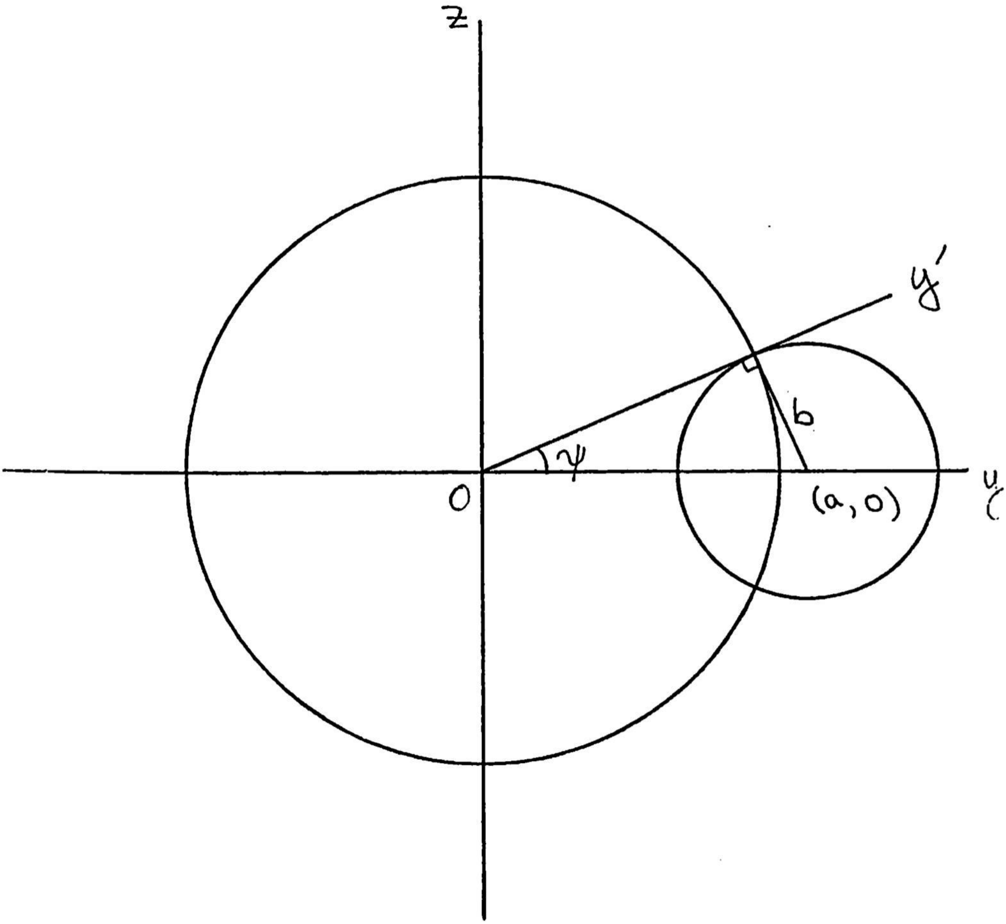

Figure 1

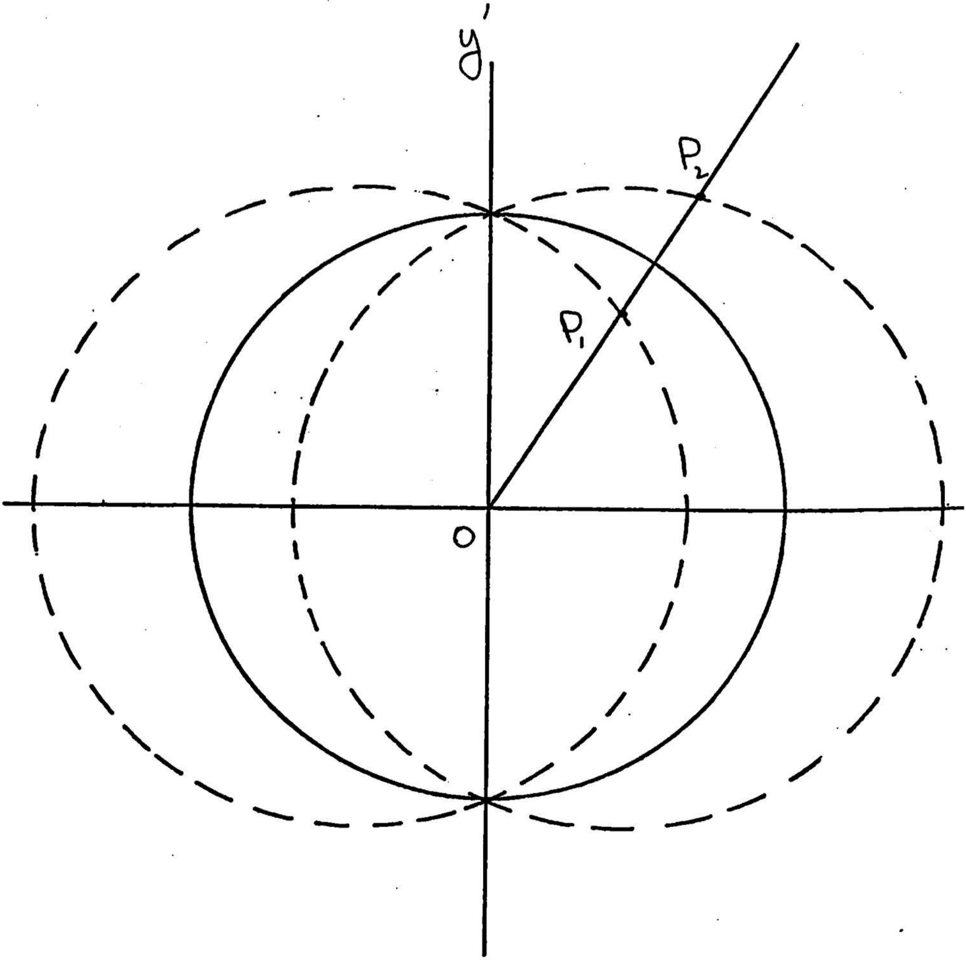

Figure 2

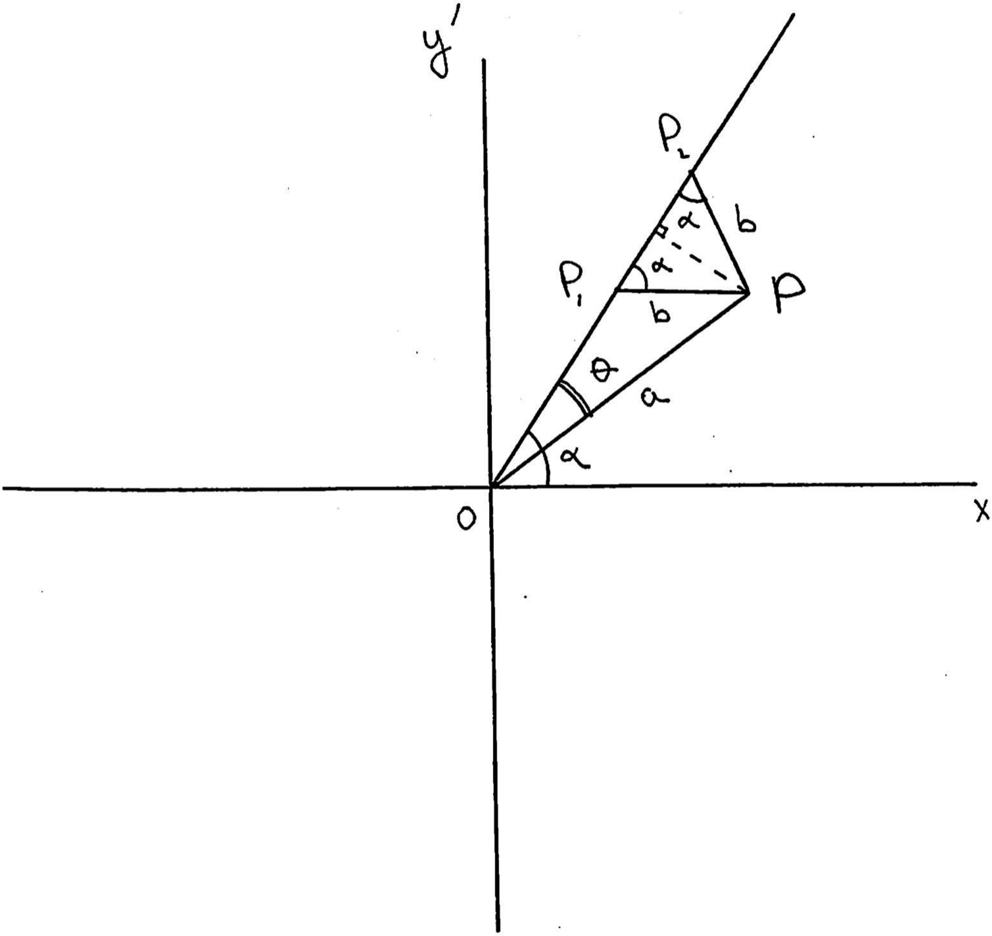

Figure 3

We imagine the inversion of Figure 2 performed with the Peaucellier linkage shown in Figure 3. It follows that

$$a \sin\theta = b \sin\alpha \text{ and therefore}$$

$$\sin\psi = b/a = \sin\theta/\sin\alpha.$$

This implies that if ΔP_1P_2P in π_0 is rotated to $\Delta P_1P_2P'$ in the vertical plane then P' must lie in the plane $z=0$. Since P' obviously lies on a circle of radius a centred at (0,0,0), this proves that P_1 and P_2 lie on τ as required.

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4 Menin Way,
Farnham,
Surrey GU98DY,
England.

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ON SIMPLE GROUPS WHICH ARE HOMOMORPHIC IMAGES OF MULTIPLICATIVE SUBGROUPS OF SIMPLE ALGEBRAS OF DEGREE 2

Michitaka Hikari

Presented by P. Ribenboim, F.R.S.C.

Let $M_2(D)$ be the full matrix algebra of degree 2 over a division algebra D of characteristic 0. In [G] we proved: If G is a finite multiplicative subgroup of $M_2(D)$ with abelian Sylow 2-subgroups, then G is a solvable group. In this paper we will determine the simple groups which are homomorphic images of multiplicative subgroups of $M_2(D)$. Our main result is as follows.

Theorem. Let S be a simple group. If there exist a division algebra D of characteristic 0, a finite multiplicative subgroup G of $M_2(D)$ and a normal subgroup N of G satisfying $G/N \cong S$, then $N \neq 1$ and S is isomorphic to $PSL(2,5)$ or $PSL(2,7)$.

In the theorem $N \neq 1$ means the following.

Corollary. Let G be a finite group and let K be a field of characteristic 0. If one of the simple components of the group ring KG is the full matrix algebra of degree 2 over a division algebra, then G is not simple.

The corollary can not be generalized to the full matrix algebra of degree ≥ 3. In fact,

$$\mathbb{C}A_5 \cong \mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus M_5(\mathbb{C}) \quad \text{and}$$

$$\mathbb{C}A_n \cong \mathbb{C} \oplus M_{n-1}(\mathbb{C}) \oplus \dots , \quad n \geq 5 ,$$

where $\mathbb{C}$ is the complex number field.

Let S be a simple group. We define $m(S) = \{(D,G,N) | D$ is a division algebra of characteristic 0, G is a finite multiplicative subgroup of $M_2(D)$ and N is a normal subgroup of G such that $G/N \cong S\}$.

We assume $m(S) \neq \emptyset$. Let (D,G,N) be an element of $m(S)$. In [5] we proved: The 2-rank of G (the maximal rank of an abelian 2-subgroup) is ≤ 2. In particular, the Sylow 2-subgroups of G possess no abelian normal subgroups of rank 3, which implies these 2-groups are generated by at most 4 elements (See MacWilliams [8]). Therefore S is a known simple group (See Gorenstein - Harada [3]). Since $\mathbb{Q}$ (the rational number field) is a subfield of the center of D, we can define

$$V_{\mathbb{Q}}(G) = \{ \Sigma a_g g | \ a_g \in \mathbb{Q} , \ g \in G\}$$

as a $\mathbb{Q}$-subalgebra of $M_2(D)$. If $S \ncong PSL(2,5)$, then there exists an element (D,G,N) in $m(S)$ such that $M_2(D) = V_{\mathbb{Q}}(G)$ and $|G| \leq |G'|$ for any element $(D',G',N') \in m(S)$. For (D,G,N) we have the following.

<u>Proposition 1</u>. (1) $[G,G] = G$.

(2) N is a 2-group.

(3) If $S \ncong PSL(2,5)$, $PSL(2,7)$, $PSL(2,9)$, A_7 and A_8, then N is cyclic and $N \subsetneq Z(G)$.

The quasi-simple groups G (i.e. $G = [G,G]$ and $G/Z(G)$ is a simple group) of 2 rank ≤ 2 with $O(G)=1$ have been determined by Alperin-Brauer-Gorenstein [1] and Gorenstein-Harada [3]. Using their theorem we have

<u>Proposition 2</u>. If $m(S) \neq \emptyset$, then $S \cong PSL(2,q)$, q:odd , $PSU(3,4^2)$, A_7 , A_8 or M_{11} .

Since a central extension G of PSL(2,q) with [G,G] = G is a homomorphic image of SL(2,q) , $q \neq 4$, 9 , the Janusz's results [7] on $\mathbb{Q}$[SL(2,q)] show the following.

<u>Proposition 3</u>. If $m(S) \neq \emptyset$, then $S \cong PSL(2,5)$, PSL(2,7) , PSL(2,9) , $PSU(3,4^2)$, A_7 , A_8 or M_{11} .

Finally, using the results of Gow [4] on Schur indices of $PSU(3,4^2)$, we have our main result.

<u>Theorem 4</u>. Let S be a simple group.

(1) $m(S) \neq \emptyset$ if and only if $S \cong PSL(2,5)$ or PSL(2,9).

(2) If $(D,G,N) \in m(S)$, then $N \neq 1$.

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Department of Mathematics
Keio University
Hiyoshi, Yokohama, Japan

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EQUIVALENCES LIEES A UNE THEORIE DE TORSION

Claude LEMAIRE *

Presented by P. Ribenboim, F.R.S.C.

Résumé

Nous étudions deux relations d'équivalence déterminées par une théorie de torsion et inspirées de la théorie du genre des groupes et des espaces topologiques.

Introduction

L'étude du genre peut être définie comme la recherche des propriétés communes à deux objets qui ont les mêmes localisés. Pourtant, une analyse plus poussée montre qu'une équivalence plus forte est fréquemment utilisée, qu'elle soit explicitement introduite ([4],[5]) ou qu'elle soit, dans le cas considéré, équivalente à avoir les mêmes localisés ([2]).

Ce travail commence l'étude de ce type de relations pour les modules et une localisation définie par une théorie de torsion, et donne quelques applications générales. Des applications plus spécifiques lorsque l'anneau est un anneau de groupe, feront l'objet d'un travail ultérieur. (Notons que dans ce cas, la nilpotence d'une action (cf [6] par exemple) est un cas particulier de "τ-borné" ci-dessous).

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1- Préliminaires et résultats généraux

Λ est un anneau unitaire, τ une théorie de torsion héréditaire sur la catégorie des Λ-modules à gauche. τ est défini par sa topologie de Gabriel F_τ , donc un module est de τ-torsion ssi tout élément est annulé par un $\alpha \in F_\tau$. ([1],[8]). $T_\tau A$ désigne le sous-module de τ-torsion de A et L_τ la localisation associée à τ . A est <u>τ-borné</u> si $\alpha A = 0$ pour un $\alpha \in F_\tau$.

τ' désigne une théorie de torsion disjointe de τ ([1], p. 80), donc $\alpha+\beta = \Lambda$ si $\alpha \in F_\tau$, $\beta \in F_{\tau'}$.

<u>Dans la suite, nous nous limitons aux modules A pour lesquels on peut trouver un $\alpha \in F_\tau$ avec $\alpha A \cap T_\tau A = 0$</u> .

<u>Définitions</u>: Un τ-isomorphisme ϕ de A dans B (en symboles: $A \overset{\phi}{\underset{\tau}{\rightarrow}} B$) est un homomorphisme de A dans B dont le noyau et le conoyau sont τ-bornés.

A est <u>τ-commensurable</u> à B ($A \underset{\tau}{\sim} B$) s'il existe un X avec $X \underset{\tau}{\rightarrow} A$ et $X \underset{\tau}{\rightarrow} B$.

<u>Théorème 1</u>: <u>Les trois propriétés suivantes sont équivalentes:</u>

(1) <u>$A \underset{\tau}{\sim} B$</u>

(2) <u>il existe $S \overset{\subseteq}{\underset{\tau}{\rightarrow}} A$, $T \overset{\subseteq}{\underset{\tau}{\rightarrow}} B$ avec $S \simeq T$</u>

(3) <u>il existe Y avec $A \underset{\tau}{\rightarrow} Y$ et $B \underset{\tau}{\rightarrow} Y$</u>

$A \underset{\tau}{\sim} B$ entraîne $L_\tau A \simeq L_\tau B$. La réciproque est loin d'être vraie. Cependant:

Théorème 2: Si F_τ est engendré par des idéaux bilatères et si A et B sont de type fini, alors $L_\tau A \simeq L_\tau B$ entraîne $A \underset{\tau}{\sim} B$.

2. Universalité

Définition: A est τ-équivalent à B ($A \underset{\tau}{\approx} B$) si $A \underset{\tau}{\dotplus} B$ et $B \underset{\tau}{\dotplus} A$. Cette relation est plus naturelle et plus efficace que la τ-commensurabilité mais son étude se heurte à un obstacle traditionnel (cf. [3],[5],[7]): $B \underset{\tau}{\dotplus} A$ n'entraîne pas $A \underset{\tau}{\dotplus} B$. La symétrie est rétablie pour les modules que nous appelerons τ-universels (cf. [3],[5],[7]):

Définition: A est τ-universel si $S \overset{\subset}{\underset{\tau}{\dotplus}} A$ entraîne $A \underset{\tau}{\dotplus} S$.

Théorème 3: Si A et B sont τ-universels, $A \underset{\tau}{\approx} B$ ssi $A \underset{\tau}{\sim} B$. En particulier, $B \underset{\tau}{\dotplus} A$ entraîne $A \underset{\tau}{\dotplus} B$.

Exemples: -Tous les modules sont τ-universels, pour n'importe quel τ, si Λ est semi-simple, ou commutatif principal (cas de [5]), ou un anneau d'entiers algébriques.

- Si Λ est commutatif et P une famille d'idéaux premiers, $F_P = \{\alpha \mid \alpha \not\subset \pi, \forall \pi \in P\}$ est une topologie de Gabriel. Tous les modules sont τ-universels pour le τ correspondant si P a un ou deux éléments ou si Λ est de Dedekind et P fini.
- Si $\Lambda = \mathbb{R}[x,y]$ et F_τ est engendré par les $(x,y)^n$, $\mathbb{R}[x,y]$ n'est pas τ-universel mais un module A tel que $yA = 0$ l'est.

- Les modules τ-divisibles ($\alpha A = A$ pour tout $\alpha \in F_\tau$) et notamment les modules de τ-torsion sont toujours τ-universels.

3. Applications

Théorème 4: Si $\psi : A \to B$ et $\phi : B \to A$ sont tels que $\psi\phi$ soit un τ-automorphisme de B, alors $A \underset{\tau}{\sim} B' \oplus C$ où $B' \underset{\tau}{\sim} B$. Si, de plus, A et B sont τ-universels, alors $A \underset{\tau}{\approx} B' \oplus C$ et $B' \underset{\tau}{\approx} B$.

Théorème 5: Si A ou B est noethérien et $A \underset{\tau}{\approx} B$, alors $A/\alpha A \simeq B/\alpha B$ pour tout $\alpha \in F_\tau$, et $\hat{A} \simeq \hat{B}$ (où $\hat{X}$ est le complété de X pour la topologie des αX, $\alpha \in F_\tau$,).

Théorème 6: Si $A \underset{\tau}{\approx} B$ et $A \underset{\tau'}{\approx} B$, alors A est isomorphe à un quotient et à un sous-module de $B \oplus B$.

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Département de Mathématiques
Université Laval
Québec. Canada, G1R 7P4

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ON THE ARGABRIGHT CONJECTURE FOR r^*-INVARIANT MEASURES

Samuel Bourne

Presented by H.J. Zassenhaus, F.R.S.C.

Abstract: We prove that the support of an r^*-invariant measure on a locally compact semigroup is a left group.

0. Introduction. A regular Borel measure μ on a locally compact semigroup S is r^*-invariant if $\mu(Ba^{-1}) = \mu(B)$ for all Borel sets B and every point a of S, where $Ba^{-1} = \{x \in S, xa \in B\}$. In [1] Argabright conjectured that the support F of a r^*-invariant measure μ on a locally compact semigroup S is a left group. He succeeded in proving the case when S is a discrete semigroup [1]. Furthermore he established that F is a closed right ideal and that the cl(Fa) = F for every $a \in S$. It is clear that $\mu(Ka) \geq \mu(K)$ if K is compact and $a \in S$.

Mukherjea and Tserpes [6] have proved this conjecture in the case where the measure μ is finite. However, their method of proof does not apply when the measure μ is infinite. Furthermore, they proved in [6] that aF is left regular, for every $a \in S$. In this paper we establish the Argabright conjecture in general. The importance of the theorem resides in the fact that a characterization is obtained of the support F of the r^*-invariant measure as $F = G \times E$, where G is a locally compact group and E is a left zero semigroup and $\mu = \lambda * \nu$, where λ is a right Haar measure on G and ν is positive regular Borel measure on E.

1. <u>The Argabright Conjecture</u>. S is a locally compact semigroup continuous in both of its variables. Since F is a right ideal in S then aF is a right ideal and a subsemigroup of S. The measure μ induces a r^*-invariant measure on aF. As Tserpes and Mukherjea [6] have established the conjecture in the case μ is finite, we suppose that the measure induced on aF is infinite. According to [6] and [7], aF is left regular. Hence, we may assume, without loss of generality, that F is left regular. Here follows a quick proof in the discrete case: F is left regular and $F = cl(Fa) = Fa$, $a \in F$, then F is right simple and F is a left group.

LEMMA. Let F be the support of a r^*-invariant measure μ on locally compact semigroup S. Then the equation $tu = u$ can be solved in F for some $u \in F$.

<u>Proof</u>. A suitable modification of part of the proof of Mukherjea and Tserpes [7, pp. 974-5] is applicable to the proof of this lemma. For the sake of completeness a transcription of this proof follows now. Let $a,b \in F$ and let K' be compact such that $\mu(K') > 0$ If $K = aK'b$, then $\mu(a^{-1}K) \geq \mu(K'bb^{-1}) \geq \mu(K') > 0$. Let K_0 be a compact set selected so that $\mu(a^{-1}K \cap K_0) > 0$ and $K_1 = K_0K \cup K_0$. Define $\mu_0(B) = \mu(a^{-1}B \cap K_1)$. Whence it follows that μ_0 is a non-zero finite regular measure on F. The product measure $\mu_0 \times \mu_0$ is constructed as in [5, pp.152-3]. Let $\theta(x,y) = (x,yx)$ and $\beta(x,y) = (y,x)$. Therefore θ is continuous and β is measure preserving. By Fubini's theorem [5, p.153]

$$\mu_0 \times \mu_0[\theta(K\times K)] = \int_K \mu_0(Kx)\mu_0(dx) = \int_K (a^{-1}(Kx) \cap K_0x)\mu_0 dx$$
$$\geq \int_K \mu(a^{-1}K \cap K_0)\mu_0(dx) > 0 ,$$

since $a^{-1}(Kx \cap K_0x) \supset (a^{-1}K \cap K_0)x$ and $\mu\{a^{-1}(Kx \cap K_0x)\} \geq \mu\{(a^{-1}K \cap K_0)x\} \geq \mu(a^{-1}K \cap K_0)$. $a^{-1}K$ is compact, $(x,y) \longrightarrow xy$ is left continuous and $\mu(Ax) \geq \mu(A)$ for compact A. The set $B = aFb \times aFb - \theta(aFb \times aFb)$ cannot contain any compact set with positive $\mu_0 \times \mu_0$ measure. Otherwise, if C is compact and $C \subset B$ then for $x \in aFb$, $C_x = \{z\colon (x,z) \in C\}$ is contained $aFb - aFbx$ and $\mu_0(C_x) \leq \mu(a^{-1}C_x) = \mu(a^{-1}C_x x^{-1}b^{-1}) = 0$. Consequently, $a^{-1}C_x x^{-1}b^{-1} \cap F$ is empty and $\mu_0(B) = 0$ if $B \cap aFb$ is empty.

Since the mapping is measure preserving, then $\beta(K \times K) \cap \theta(aFb \times aFb)$ is non-empty and there exist u, v, w and z in aFb such that $(u, vu) = (zw, w)$ which implies $u = zw$ and $vu = w$. It follows that $(zv)u = u$ and $tu = u$ is soluble.

THEOREM. Let F be the support of a r^*-invariant measure μ on a locally compact semigroup S. Then F is a left group.

<u>Proof</u>. The lemma supplies a solution of the equation $tu = u$. Hence, $(uu^{-1}) = \{x \in F \mid xu = u\}$ is not empty, for some $(u^{-1}u)u = u$. Following Bourne [4], then $F(uu^{-1})u = Fu \subset F$ and $F(uu^{-1}) \subset Fu^{-1}$. The right representation $\rho_\mu\colon (Fu)u^{-1} \longrightarrow Fu$ defined by $\rho_\mu[(fu)u^{-1}] = [(fu)u^{-1}]u$ is bijective. Hence $[Fu(uu^{-1})u]u^{-1} = Fu(uu^{-1})[(Fu)u]u^{-1} = Fu(uu^{-1})$ by [3]. Since $[fu](uu^{-1}) \subset [(fu)u]u^{-1}$ and $[Fu](uu^{-1}) = [(Fu)u]u^{-1}$, there must exist an f_1 such that $(f_1u) \subset (f_1u)(uu^{-1})$. Therefore, $(f_1u) = (f_1u)e$, $e \in (uu^{-1})$ and $(f_1u)e = (f_1u)e^2$. Left cancellability yields $e = e^2$ an idempotent in F. In other words, the support F of the measure μ is a left group.

2. Remark. The fact that the support of every r^*-invariant measure is a left group is equivalent to the statement that $\pi(F)$ is contained in the group of unitary operators $L_2(F,\pi)$ [2, p.99].

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Department of Mathematics
University of California
Berkeley, California 94720

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LINEAR GROUPS WITH BOUNDED TRACE VALUES

John D. Dixon

Presented by H. Schwerdtfeger, F.R.S.C.

In their paper [1] Greenleaf, Moskowitz and Rothschild consider subgroups G of $GL(n,C)$ with the property:

(*) there exists $\beta > 0$ such that $|\text{tr } x| \leqq \beta$ for all $x \in G$.

As they note, this condition is equivalent to assuming that the eigenvalues of all elements in G have absolute value 1. For each $x \in GL(n,C)$ write $x = s(x)u(x) = u(x)s(x)$ where $s(x)$ and $u(x)$ are the semisimple and unipotent parts, respectively. The main result of [1] is that, for each group G satisfying (*), the subgroup U generated by all $u(x)$ $(x \in G)$ consists entirely of unipotent matrices. The object of the present note is to prove a stronger property of groups satisfying (*) which in particular implies this latter result.

Theorem. If G is a subgroup of $GL(n,C)$ satisfying (*) and G is completely reducible, then G is conjugate in $GL(n,C)$ to a group of unitary matrices. In particular, each element of G is semisimple.

Proof. It is evidently enough to prove that the conclusion holds when G is irreducible. Then, by Burnside's Theorem (see [5] Corollary 1.17), there exists a basis of the vector space $Mat(n,C)$ of all $n \times n$ matrices, say x_i ($i = 1,\dots, n^2$), consisting of elements of G. Define y_j $(j = 1,\dots, n^2)$ to be the dual basis in the sense that $\text{tr } (x_i y_j) = \delta_{ij}$ for all i and j. Then each element $x \in G$ can be written $x = \sum \xi_j y_j$ where $\xi_j = \text{tr } (x_j x)$. The condition

(*) shows that $|\xi_j| \leq \beta$ for each j, and so we conclude that the entries of x are uniformly bounded for all elements $x \in G$. Thus G is bounded in Mat(n,C) with the usual topology, and so its closure $\bar{G}$ is a compact subgroup of GL(n,C). Hence a well-known theorem (see, for example, [3], Section 28) shows that $\bar{G}$ (and hence G) is conjugate in GL(n,C) to a group of unitary matrices.

Corollary. If G is a subgroup of GL(n,C) satisfying (*), then the subgroup U generated by all u(x) $(x \in G)$ is unipotent.

Proof. Put G into reduced form. Specifically, choose $a \in GL(n,C)$ and irreducible representations $r_i : G \to GL(n_i,C)$ $(i = 1, \ldots, m)$ such that for each $x \in G$, $a^{-1}xa$ has the block upper triangular form:

$$a^{-1}xa = \begin{bmatrix} r_1(x) & & * \\ & \ddots & \\ 0 & & r_m(x) \end{bmatrix}$$

For any $y \in GL(n,C)$, s(y) and u(y) lie in the Zariski closure of the group generated by y (see [5] Theorem 7.3). Hence $a^{-1}u(x)a$ and $a^{-1}s(x)a$ have block triangular form similar to that of $a^{-1}xa$:

$$a^{-1}u(x)a = \begin{bmatrix} u_1(x) & & \\ 0 & \ddots & * \\ & & u_m(x) \end{bmatrix} \quad \text{and} \quad a^{-1}s(x)a = \begin{bmatrix} s_1(x) & & \\ 0 & \ddots & * \\ & & s_m(x) \end{bmatrix}$$

Since a matrix is unipotent [respectively, semisimple] if and only if its minimal polynomial has all roots equal to 1 [all roots distinct], it is clear that all $u_i(x)$ are unipotent and all $s_i(x)$

are semisimple. Since $a^{-1}u(x)a$ and $a^{-1}s(x)a$ commute, the same is true for all pairs $u_i(x)$, $s_i(x)$; thus $u_i(x)$ and $s_i(x)$ are the unipotent and semisimple parts of $r_i(x)$. However, as we noted above, the hypothesis (*) is equivalent to assuming all eigenvalues of all elements of G have absolute value 1. Thus, for each i, $r_i(G)$ is an irreducible group satisfying (*); hence the Theorem implies that $s_i(x) = r_i(x)$ and $u_i(x) = 1$. Thus all matrices $a^{-1}u(x)a$ are upper triangular matrices with diagonal entries 1, and so the group $a^{-1}Ua$ which they generate is unipotent. Hence U is unipotent as asserted.

<u>Note 1.</u> The idea behind the proof of our theorem goes back at least to [2] (see also [4]). In those papers the authors are studying subgroups G of GL(n,C) in which each cyclic subgroup is bounded. It is easily verified that this latter property holds exactly when (*) holds and each element of G is semisimple, and our theorem is proved (in essence) in [4] under this stronger hypothesis.

<u>Note 2.</u> It is <u>not</u> true that (*) implies that the subgroup generated by all s(x) $(x \in G)$ consists entirely of semisimple elements. For example, if G is generated by

$$\begin{bmatrix} \xi & 1 \\ 0 & \xi^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \xi^{-1} & 1 \\ 0 & \xi \end{bmatrix}$$

for any $\xi \in C$ with $|\xi| = 1$ and $\xi^2 \neq 1$, then G clearly satisfies (*), and the two generators are semisimple but their product is not.

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Department of Mathematics and Statistics
Carleton University
Ottawa, Canada, K1S 5B6

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BASIC PROPERTIES OF NONEXPANSIVE MAPPINGS

Michael Edelstein

Presented by E.E. Granirer, F.R.S.C.

Introduction.

Let C be a weakly compact convex set in a Banach space and $f: C \to C$ a nonexpansive mapping. It has recently been shown that in certain Banach spaces such maps may fail to have fixed points (cf. [1], [4], [5]). Thus minimal invariant convex sets under such mappings need not be singletons. The geometry of such minimal sets K has some interesting special properties. Since the existence of a nondiametral point x in K is known to be incompatible with minimality, each $z \in K$ must be diametral; i.e. $\sup\{||z-y||: y \in K\} = \operatorname{diam} K$.

In this note we discuss several additional features of the geometry of such sets. Some of these flow directly from the above mentioned fact; others involve explicitly the mapping f .

Proposition 1. Let K be a closed bounded set in a normed linear space such that each of its points is diametral. Suppose that $d = \sup\{||x-y||: x,y \in K\} > 0$ and $0 < r' < r < d$. Then $B_K(x,r) \not\subset B_K(x',r')$ for any $x,x' \in K$. (Here $B_K(y,t) = \{z \in K: ||z-y|| \le t\}$.)

Proof. Let $z \in K \setminus B_K(x,r)$. Let y be a point on the line segment joining x and z such that $||y-x|| = r$. Suppose, for a contradiction, that $B_K(x',r') \supset B_K(x,r)$. Then $||z-x'|| \le ||z-y|| + ||y-x'|| \le ||z-x|| - r + ||y-x'|| \le ||z-x|| - r + r'$. Hence $d = \sup\{||z-x'||: z \in K\} \le \sup\{||z-x||: z \in K\} - (r-r') < d$, which is absurd.

Corollary. The Chebyshev radius r_S of $\phi \neq S \subset K$, with respect to K, is defined as $\inf\{r_S(x): x \in K\}$, where $r_S(x) = \sup\{||x-y||: y \in S\}$. The preceding result, then, states that for any set $B_K = B_K(x,\rho)$, $r_{B_K} = \rho$.

Proposition 2. Let K be as in Proposition 1. Then (i), K is not symmetric about any of its points, that is $K-x$ is not symmetric with respect to the origin for any $x \in K$; and (ii) K is nowhere dense.

Proof. The verifications of both (i) and (ii) are straightforward and omitted.

Proposition 3. Let f be a nonexpansive selfmapping of a weakly compact convex set in a Banach space. Let K be a minimal invariant convex subset. If $0 < r' < r < d$ where d is the diameter of K then, for any $x \in K$, $f[B_K(x,r)] \not\subset B_K(f(x),r')$.

Proof. Suppose not and let z be a point in $K \backslash B_K(x,r)$. Let y be the point of intersection of the line segment joining z with x, so that $||z-x|| = ||z-y|| + r$. Then

$$\begin{aligned} ||f(z)-f(x)|| &\leq ||f(z)-f(y)+f(y)-f(x)|| \\ &\leq ||z-y|| + r' = ||z-x|| - r + r' . \end{aligned}$$

Choose a sequence $\{z_n\} \subset K$ such that $||f(z_n)-z_n|| \to 0$. Then, by a result of Karlowitz [3], $||u-z_n|| \to d$, for each $u \in K$. Hence

$$||z_n-f(x)|| \le ||z_n-f(z_n)|| + ||f(z_n)-f(x)|| \le \varepsilon + ||z_n-x|| - (r-r')$$

which is impossible since $\lim||z_n-f(x)|| = d = \lim||z_n-x||$, and $\varepsilon \to 0$ as $z \to \infty$.

Clearly the Chebyshev radii of sets $\phi \neq S \subset K$ cannot increase under f. However, an argument similar to that of the proof of Proposition 3, shows that as stated in the following proposition, these radii of balls (in K) and their images must be equal.

Proposition 3a. With the notation and assumptions of Proposition 3, the Chebyshev radii of $B_K(x,r)$ and its image are equal.

Proof. Suppose not. Then there is a $u \in B_K(x,r)$ such that

$$||f(u)-f(v)|| \le r' < r$$

for all $v \in B_K(x,r)$. As in the preceding proof choose $\{z_n\} \subset K \backslash B_K(x,r)$ such that $||f(z_n)-z_n|| \to 0$ and let y_n be the point on the line segment joining z_n with x. Now $\{||u-y_n||\}$ contains a convergent subsequence. Assume that $||u-y_n|| \to \rho \ge 0$. We have

$$||f(z_n)-f(u)|| \le ||f(z_n)-f(y_n)|| + ||f(y_n)-f(u)||$$

$$\le ||z_n-y_n|| + r' \le ||z_n-x|| - r$$

leading to a contradiction; (cf. proof of Proposition 3).

Proposition 4. Let f and K be as in the preceding proposition. Then (i), $f^{-1}[B_K(x,r)] \neq \phi$ whenever x is a strongly exposed point of K and $r > 0$; and (ii) no sequence of iterates $\{f^n(y): n=1,2,\ldots\}$ in K is precompact.

Proof. (i) Suppose not, and let x be strongly exposed and $f^{-1}[B_K(x,r)] = \phi$. Then there is a closed hyperplane strictly separating x and $K/B_K(x,r)$. It follows that $\overline{co}(K\backslash B_K(x,r))$ is mapped into itself by f , against the minimality of K .

(ii) Here $\overline{co}\{f^n(y)\}$ is compact; hence, as is well known, has a nondiametral point. It readily follows that the Chebyshev center of this set (with respect to K) is a proper subset of K . Let z be a cluster point of $\{f^n_{(y)}\}$. Then the Chebyshev center of $\overline{co}\{f^n(y)\}$ is invariant under f and properly contained in K , which is impossible.

Remark. If K is contained in a strictly convex Banach space X then no subsequence $\{f^{n_i}(y)\} \subset \{f^n(y)\} \subset K$ can converge. This, because, as is well known, the restriction of f to $\overline{co}\{f^n(z)\}$, where $z = \lim f^{n_i}(y)$, if such a-limit exists, is an affine isometry; then $\text{diam } K = 0$, i.e. f has a fixed point.

It would be interesting to prove or disprove that the same is true even where X is not strictly convex.

Proposition 5. Let f and K be as in the preceding proposition. Suppose in addition that f is uniformly asymptotically regular; that is, for any $\varepsilon > 0$ there is a positive integer N such that $||f^N(x)-f^{N+1}(x)|| < \varepsilon$ for all $x \in K$. Let $0 < r < d = \text{diam } K$. Then, if $x \in K$,
(i) $f^{-n}[B_K(x,r)] = \phi$, (ii) $f^n[B_K(x,r)] \cap B_K(x,r) = \phi$.

Proof. We observe that

$$\varepsilon = \inf\{||y-f(y)||: y \in B_K(x,r)\} > 0$$

since no sequence $\{z_n\}$ is in $B_K(x,r)$ if $||z_n-f(z_n)|| \to d = \text{diam } K$; (cf. proof of Proposition 3). Let N be such that $||f^N(y)-f^{N+1}(y)|| < \varepsilon$ for all $y \in K$. Then, clearly, $f^{-N}(y) =$ for all $y \in B_K(x,r)$, proving (i).

To prove (ii) let N be as above. Then, clearly,

$$y \in B_K(x,r) \Rightarrow f^N(y) \notin B_K(x,r)$$

(as, otherwise $f^{-N}(f^N(y)) \neq \phi$ against (i)).

Remark. If $f: K \to K$ is nonexpansive then $F_\lambda = \lambda I + (1-\lambda)f$ where I is the identity map and $0 < \lambda < 1$ is nonexpansive and as shown in [2] uniformly asymptotically regular. Furthermore f and F_λ have exactly the same fixed point sets. If then f is fixed point free then so is F_λ . However minimal invariant convex sets under F_λ have properties which are not shared by similar sets of general nonexpansive maps.

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Dalhousie University
Halifax, N.S.
B3H 4H8

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ON THE LEVI-CIVITA CONNECTION OF A GAUGED LEVI-CIVITA CONNECTION

E. Binz

Presented by K. Ranger, F.R.S.C.

Abstract: Given a C^∞-manifold M and a fixed Riemannian metric. Any other metric G can be represented via a symmetric positive definite endomorphism f^2 of the tangent bundle TM of M. We answer the following question: How depends the Levi-Civita connection of G on f in the dimensions 2 and 3 in case M is oriented.

1) The Levi-Civita connection of a gauged Levi-Civita connection

$\mathfrak{M}(M)$ denotes the collection of all C^∞-Riemannian metrics on a C^∞-manifold M. Fix $G^o \in \mathfrak{M}(M)$. Given any $G \in \mathfrak{M}(M)$ and $p \in M$ there is a fibrewise positive definite C^∞-bundle map $A : TM \longrightarrow TM$, satisfying

1) $$G(p)(v_p, w_p) = G^o(p)(A(p)v_p, w_p)$$

for any couple $v_p, w_p \in T_pM$. A is uniquely determined by the fibrewise formed positive definite square root f of A and vice versa. Call therefore G by $G(f)$. Denote by ΓTM the collection of all C^∞-vector fields of M. Given $X, Y \in \Gamma TM$ define $\nabla(f)_X Y := f^{-1}\nabla^o_X fY$. The following is immediate:

Lemma 1: $G(f)$ is parallel with respect to $\nabla(f)$.

$\nabla(f)$ however is in general not the Levi-Civita connection of $G(f)$ as one immediately verifies in the conformal case. Thus the Levi-Civita connection $\nabla(G(f))$ is determined by

$$\nabla(G(f))_X Y = \nabla(f)_X Y + \Phi(X,Y) \quad , \tag{3}$$

for a well defined two tensor Φ . Write $\Psi(X)\cdot Y$ instead of $\Phi(X,Y)$. Clearly $G(f)(\Psi(X)\cdot Y,Z) + G(f)(Y,\Psi(X)\cdot Z)=o$ for any triple $X,Y,Z \in \Gamma TM$. $\Psi(X)$ can be described more specifically by using the theorem of Nash. Associated with G^o and G are two embeddings $i_o, j : M \longrightarrow \mathbb{R}^m$ for some m , large enough to guarantee that $G^o = i_o^* <,>$ and $G = j^*<,>$. Here $<,>$ denotes a fixed scalar product on $\mathbb{R}^n$. Denote by $\nu(i)$ and $\nu(j)$ the respective normal bundles of i and j . By [Bi] there is a C^∞-map $g : M \longrightarrow O(m)$ such that for all $p \in M$

$$g(p)(di\cdot f(p)(v)) = dj(v) \quad \text{and} \quad g(p)(\nu_p(i)) = \nu_p(j)$$

for any $v \in T_pM$ and which in addition satisfies

$$\Psi(X)\cdot Y = f^{-1}\cdot P(i)\cdot g^{-1}\cdot dg(X)\cdot di\cdot f\cdot Y \quad . \tag{4}$$

The dot denotes (also in forthcoming formulas) the pointwise formed composition. By di and dg we denote the linear parts of Ti and Tg respectively. Moreover $P(i) : M \times \mathbb{R}^m \longrightarrow TM$ is the composition of the (fibrewise) orthogonal projection onto $Ti(TM)$ followed by $(Ti)^{-1}$. Equation (4) allows the following interpretation. If M is simply connected, $\bar{f} : M \longrightarrow GL(n)$ defined by $\bar{f}(p)|\nu_p(i) = id$ for all $p \in M$ and $\bar{f}\cdot di = di\cdot f$ on $diTM$, yields the form $\bar{f}\cdot di$, which is in general not the differential of an immersion j of M into $\mathbb{R}^n$. An "integrating factor" g is needed to write $dj = g\cdot\bar{f}\cdot di$ for some j. The following is immediate. The torsion $T(f)$ of $\nabla(f)$ is given by

5) $$T(f)(X,Y) = f^{-1}(\nabla^o_X(f)\cdot Y - \nabla^o_Y(f)\cdot X) \quad .$$

Hence for any $X,Y \in \Gamma TM$ the following holds

6) $$\Psi(X)\cdot Y - \Psi(Y)\cdot X = - T(f)(X,Y) \quad .$$

Since $\Psi(X)\cdot Y = \nabla(G(f))_X Y - \nabla(f)_X Y$ we have in addition to (4) the well known expression

$$G(f)(\Psi(X)\cdot Y,Z) = -\frac{1}{2}[G(f)(T(f)(X,Y),Z) + G(f)(T(f)(Z,X),Y) - G(f)(T(f)(Y,Z),X)] \quad .$$

2) $\Psi(X)$ in the conformal case

2a) dim $M = 2$

Let M be oriented. We work locally in $U \subset M$. Since $\Psi(X)$ is with respect to $G(f)$ skew symmetric, we have for any pair $Z_o,Z_1 \in \Gamma TM$

7) $$G(f)(\Psi(X)\cdot Z_o,Z_1) = \alpha(X)\cdot\mu(G(f))(Z_o,Z_1)$$

where α is a one form on U of class C^∞ and $\mu(G(f))$ denotes the Riemannian volume form of $G(f)$. There is a unique C^∞-bundle map $V : TU \longrightarrow TU$ satisfies both:

8) $$\mu(G(f))(Z_o,Z_1) = G((f))(V\cdot Z_o,Z_1) \quad .$$

9) $$\Psi(X) = \alpha(X)\cdot V \quad .$$

<u>Proposition 2:</u> Let $f = \gamma \cdot \iota$ for some real valued C^∞-function γ on U. Then for any $X \in \Gamma TU$

10) $$\Psi(X) = \mu(G(f))(\text{grad} \log\gamma, X) \cdot V \quad ,$$

where grad denotes the gradient with respect to $G(f)$.

<u>Proof:</u> Let $\lambda = \log\gamma$ and $Z_o = \text{grad}\,\lambda$. By (6) we have

11) $$G(f)(\Psi(X) \cdot \text{grad}\,\lambda, X) = G(f)(\text{grad}\,\lambda, \text{grad}\,\lambda) \cdot G(f)(X,X) - (G(f)(\text{grad}\,\lambda, X))^2$$

or

12) $$\alpha(X) \cdot \mu(G(f))(\text{grad}\,\lambda, X) = (\mu(G(f))(\text{grad}\,\lambda, X))^2$$

from which (10) follows.

<u>2b) dim M = 3</u>

Let M be oriented. Given a real-valued C^∞-map γ on M, we again denote by grad the gradient with respect to the metric $G(\gamma \cdot \iota)$. The symbol $\times$ marks the cross product ([Gr]) with respect to $G(\gamma \cdot \iota)$ and its Riemannian volume.

<u>Proposition 3:</u> Let M be oriented and of dimension three. If $f = \gamma \cdot \iota$, where γ is a real-valued C^∞-function on M, then

13) $$\Psi(X) \cdot Y = (\text{grad}(\log\gamma) \times X) \times Y$$

and the torsion of $\nabla(\gamma \cdot \iota)$ is for each $X, Y \in \Gamma TM$ given by

14) $$T(\gamma \cdot \iota)(X,Y) = (X \times Y) \times \text{grad}(\log\gamma) \, .$$

<u>3) The general situation for dim M = 3 and M oriented</u>

Again assume that M is oriented. Since $\Psi(X) \cdot Y$ is skew symmetric in Y, we find a C^∞-bundle map $u : TM \longrightarrow TM$

such that

15) $$u(X) \times Y = \Psi(X)\cdot Y \ .$$

Here $\times$ marks the cross product with respect to $G(f)$ and its Riemannian volume $\mu(G(f))$. One immediately observes: For each triple $X,Y,Z \in \Gamma TM$ holds

16) $$tr\,u\ \mu(G(f))(X,Y,Z) = G(f)(\Psi(Z)\cdot X,Y) - G(f)(T(f)(X,Y),Z) \ .$$

Next let us split u into $u = u_o + (tr\,u)\cdot\iota + a$, where u_o is (with respect to $G(f)$) fibrewise selfadjoint and a is skew symmetric. ι denotes the identity on TM. Then an easy calculation shows: $T(f)(X,Y) = (u_o - a)(X \times Y)$ for each touple $X,Y \in \Gamma TM$. Due to $\dim M = 3$ there is on the other hand a well defined C^∞-bundle map $C : TM \longrightarrow TM$ such that both equations are satisfied:

17) $$T(f)(X,Y) = 2C(X \times Y) \quad \text{and} \quad C = \frac{1}{2}\cdot(u_o - a) \ .$$

Hence $a = C^* - C$, where C^* denotes the adjoint of C with respect to $G(f)$. Then by (17) we obtain:

<u>Proposition 4:</u> Let M be oriented. Then the following holds:

18) $$\Psi(X)\cdot Y = u(X)\times Y \quad \text{with} \quad u = 2C^* - (tr\ C)\cdot\iota \ .$$

This proposition immediately yields proposition 3.

If X_1, X_2, X_3 denotes a local orthonormed eigenframe of f with respective eigenvalues $\lambda_1, \lambda_2, \lambda_3$ then

$$2C(X_i \times X_j) = (X_i \times X_j) \times (X_i(\log\lambda_j)\cdot X_i + X_j(\log\lambda_i)X_j) + \\ + \lambda_j f^{-1}(\nabla^o_{X_i} X_j) - \lambda_i f^{-1}(\nabla^o_{X_j} X_i) - [X_i, X_j] \ .$$

Let $C(X \times Y) = V \times (X \times Y)$ for some $V \in \Gamma TM$ and each pair $X, Y \in \Gamma TM$. If

19) $$G(f)(\nabla(G)_Y V, X) = G(f)(\nabla(G)_X V, Y)$$

then locally $V = \text{grad}\lambda$ for some real-valued C^∞-map λ.
Moreover λ is harmonic if V is constant.
Finally one easily verifies the identities

$$- [G(f)(T(f)(X,Y),Z) + G(f)(T(f)(Y,Z),X) + G(f)(T(f)(Z,X),Y)]$$
$$= 2[G(f)(\Psi(X)\cdot Y,Z) + G(f)(\Psi(Y)\cdot Z,X) + G(f)(\Psi(Z)\cdot X,Y)] =$$
$$= 2\cdot \mathcal{t}\kern-1pt\mathcal{r}u\cdot\mu(G(f))(X,Y,Z) \quad .$$

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Department of Mathematics
University of Mannheim
D 68 Mannheim/Germany

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INVARIANT TEMPERED DISTRIBUTIONS ON THE REDUCTIVE p-ADIC GROUP $GL_n(F_p)$

Peter A. Mischenko

Presented by J.G. Arthur, F.R.S.C.

INTRODUCTION. Professor Harish-Chandra has defined the Schwartz Space $\underline{C}(G)$ for reductive p-adic groups G [H-Ch. I] ; $\underline{C}(G)$ is a topological vector space, and algebra under convolution product of functions. A tempered distribution on G is a continuous linear functional Λ on $\underline{C}(G)$; Λ is said to be invariant if $\Lambda(f_1 \cdot f_2) = \Lambda(f_2 \cdot f_1)$, $\forall f_1, f_2 \in \underline{C}(G)$. A representation π of G is said to be tempered if it is unitary, irreducible, and its character ch_π (given by $f \longmapsto \mathrm{trace}(\int_G f(x)\pi(x)dx)$, $f \in C_c^\infty(G) \subset \underline{C}(G)$) extends to a tempered distribution. We have obtained the following result for $G = GL_n(F_p)$, the general linear group over a p-adic field F_p: The space of invariant tempered distributions on G is, in fact, the weak*-closure of the characters ch_π of the tempered representations π of G.

FOURIER TRANSFORM IMAGE OF THE SCHWARTZ SPACE

To attack the problem of invariance, some useful harmonic analysis for arbitrary reductive p-adic groups G is developed. We obtain a complete description of the image $\underline{C}(\hat{G})$ of the Schwartz Space $\underline{C}(G) = \bigcup_{K_0} \underline{C}_{K_0}(G)$ (the union being over all compact open subgroups K_0 of G) under the Fourier Transform $\mathcal{F}$. In notation based on that of [Arthur I] and [H-Ch. II] , $\mathcal{F}$ is an isomorphism from $\underline{C}_{K_0}(G)$ to

$$\underline{C}_{K_0}(\hat{G}) = \Big\{ \underline{a} \in \bigoplus_{\underline{P} \in Cl(G)} \bigoplus_{P \in \underline{P}} \bigoplus_{\sigma_{\sigma_0} \subset \underline{E}_2(M : K_0 \cap M)} \underline{C}\big(\sigma_{\sigma_0} ; H^2(\underline{H}_{F(K_0)}(\sigma_0))\big) :$$

$$\underline{a}(s \cdot \sigma_{0,\nu}) = R(s : \sigma_0, \nu)\, \underline{a}(\sigma_{0,\nu})\, R(s : \sigma_{0,\nu})^{-1}, \forall s \in w(A|A'), \forall \nu \in i\sigma^* \Big\}.$$

Here the $R(s:\sigma_0,\nu)$ are well-known intertwining operators (c.f. [Shahidi]). $\underline{C}(\sigma; H)$ denotes the space of C^∞-maps, from the compact orbit space σ into a finite dimensional Hilbert space H, equipped with the Schwartz

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topology for a compact manifold. This space $\underline{C}_{K_o}(\hat{G})$ is also realizable as $\{ \underline{x} \in \bigoplus_{\underline{P}} \bigoplus_{P} \bigoplus_{\sigma_{\sigma_o}} \underline{C}(\sigma_{\sigma_o}; \underline{C}_{\sigma_o}(M, \rho_{F(K_o),M})):$

$$\underline{x}(s\cdot\sigma_o,\nu) = {}^{o}c(s:\sigma_o,\nu)\,\underline{x}(\sigma_o,\nu), \ \forall s \in w(A|A'), \ \forall \nu \in i\mathfrak{a}^*\}.$$

To prove this, we demonstrate by direct calculation that $\mathcal{F}$ maps $\underline{C}_{K_o}(G)$ continuously into $\underline{C}_{K_o}(\hat{G})$. Then a theorem of Harish-Chandra, involving his theory of Eisenstein integrals and wave packets, is employed to show that $\mathcal{F}^{-1}$ maps $\underline{C}_{K_o}(\hat{G})$ continuously into $\underline{C}_{K_o}(G)$. The stated result answers a question posed in [Silberger], §5.5.4, pp. 351-352, as to the decomposition of the Schwartz Space into algebras of wave packets, modulo certain equivalence relations.

<u>THE FIRST MAIN THEOREM</u> (G= arbitrary reductive p-adic group)

DEF. $\underline{D}_{T,K_o}(G) = (\underline{C}_{K_o}(G))^*$ equipped with the weak*-topology.

DEF. $\underline{D}^{inv}_{T,K_o}(G)$ is the subspace in $\underline{D}_{T,K_o}(G)$ of invariant tempered distributions (with subspace topology).

The spaces $\underline{D}_{T,K_o}(\hat{G})$ and $\underline{D}^{inv}_{T,K_o}(\hat{G})$ are defined likewise (with $\underline{C}_{K_o}(\hat{G})$ playing the role of $\underline{C}_{K_o}(G)$). We let $\underline{D}^{inv}_{T}(G) = \bigcup_{K_o} \underline{D}^{inv}_{T,K_o}(G)$.

Let $\underline{E}_T(G)$ be the set of equivalence classes $[\pi]$ of tempered representations π of G, and let $\underline{E}_{T,K_o}(G) = \{[\pi] : \pi$ is an irreducible subrepresentation of some induced representation $\pi(\sigma_o,\nu)$, for some $\sigma_o \in \underline{E}_2(M:K_o \cap M)$, some M, and some $\nu \in i\mathfrak{a}^*\}$. $\underline{E}_T(G)$ is the union, over all K_o, of the $\underline{E}_{T,K_o}(G)$.

Let S denote the set of distinguished tori A in G. For any complex-valued function t on $\underline{E}_{T,K_o}(G)$, define, for $A \in S$ and $D \in \mathcal{D}(i\sigma^*)$,

$$\|t\|_D^{(A)} = \sup_{\substack{\sigma_o,\nu \in \underline{E}_2(M:K_o \cap M) \\ Q \in \mathcal{P}(A)}} |D_\nu t(\pi(\sigma_{o,\nu}))| .$$

Here $\pi(\sigma_{o,\nu})$ is the induced representation $\mathrm{Ind}_Q^G(\sigma_{o,\nu})$ acting on $\underline{H}(\sigma_o)$; the derivative of any non-differentiable function is taken to be ∞.

DEF. $\underline{I}_{K_o}(\hat{G}) = \{t: \underline{E}_{T,K_o}(G) \longrightarrow \mathbb{C} \mid \|t\|_D^{(A)} < \infty, \forall A \in S, \forall D \in \mathcal{D}(i\sigma^*)\}$, with the semi-norm topology. Let $\underline{I}(\hat{G}) = \bigcup_{K_o} \underline{I}_{K_o}(\hat{G})$.

For $\phi \in \underline{C}_{K_o}(G)$, we define $\mathcal{J}_{K_o}\phi: \underline{E}_{T,K_o}(G) \longrightarrow \mathbb{C}$ by $(\mathcal{J}_{K_o}\phi)([\pi]) = ch_\pi(\phi)$, $[\pi] \in \underline{E}_{T,K_o}(G)$. For $\underline{a} \in \underline{C}_{K_o}(\hat{G})$, we define $\hat{\mathcal{J}}_{K_o}\underline{a}: \underline{E}_{T,K_o}(G) \longrightarrow \mathbb{C}$ by $(\hat{\mathcal{J}}_{K_o}\underline{a})([\pi]) = \mathrm{trace}|_{H_\pi} \underline{a}(\sigma_{o,\nu})$, if π is a subrepresentation of $\pi(\sigma_{o,\nu})$, $[\pi] \in \underline{E}_{T,K_o}(G)$. These maps are related by $\mathcal{J}_{K_o} = \hat{\mathcal{J}}_{K_o} \circ \mathcal{F}|_{\underline{C}_{K_o}(G)}$.

Let $\mathcal{J}$ denote the extension of the $\mathcal{J}_{K_o}$ to all of $\underline{C}(G)$. Let primes denote the adjoint map. Our <u>first main theorem</u> is:

$\mathcal{J}'$ (resp. $\mathcal{J}'_{K_o}$) injects $\underline{I}(\hat{G})^*$ (resp. $\underline{I}_{K_o}(\hat{G})^*$) into $\underline{D}_T^{\mathrm{inv}}(G)$ (resp. $\underline{D}_{T,K_o}^{\mathrm{inv}}(G)$).

This theorem is proved by explicitly showing that $\hat{\mathcal{J}}_{K_o}$ is a surjective map from $\underline{C}_{K_o}(\hat{G})$ onto $\underline{E}_{T,K_o}(G)$.

<u>FACTORIZATION THROUGH THE TRACE</u>

DEF. $\Lambda \in \underline{D}_{T,K_o}(G)$ (resp. $\mathcal{A} \in \underline{D}_{T,K_o}(\hat{G})$) factors through the trace if Λ (resp. $\mathcal{A}$) vanishes on $\ker(\mathcal{J}_{K_o})$ (resp. on $\ker(\hat{\mathcal{J}}_{K_o})$).

If Λ does factor through the trace, there is a linear functional $\tilde{\Lambda}$ on $\underline{I}_{K_o}(\hat{G})$ such that $\Lambda\phi = \tilde{\Lambda}([\pi] \longmapsto ch_\pi(\phi), [\pi] \in \underline{E}_{T,K_o}(G))$, for all $\phi \in \underline{C}_{K_o}(G)$; this justifies the terminology. Clearly factorization through the trace implies invariance.

From now on, we suppose G is $GL_n(F_p)$. In this case the induced representations $\pi(\sigma)=\mathrm{Ind}_{P=NM}^{G}(\sigma)$ ($\sigma\in \underline{E}_2(M)$) are all irreducible and tempered ([Jacquet]).

THEOREM. If $\mathcal{A}\in \underline{D}_{T,K_o}(\hat{G})$ is invariant, then $\mathcal{A}$ factors through the trace.

For the proof, we show that $\mathcal{A}$ vanishes on $\ker(\hat{\mathcal{J}}_{K_o})$ by a localization method (rather as in the real case, [Arthur II]). It turns out that it is enough to demonstrate that if $\mathcal{A}$ is an invariant linear functional on the algebra $C_c^\infty(U)\otimes E \cong C_c^\infty(U;E)$ ($E=H^2(\underline{H}_{F(K_o)}(\sigma_o)$, U=small open neighbourhood of 0 in a certain Euclidean space), then $\mathcal{A}$ vanishes on ${}^{(o)}C_c^\infty = \{\alpha\in C_c^\infty(U;E) : \mathrm{trace}(\alpha(\nu))=0, \forall\nu\in U\}$. This is easy to see, either by a direct matrix entry calculation, or by noting that ${}^{(o)}C_c^\infty$ equals its commutator bracket $[{}^{(o)}C_c^\infty, {}^{(o)}C_c^\infty]$.

Combining the first main theorem with the previous one yields

THEOREM. $\mathcal{J}'_{K_o}:\underline{I}_{K_o}(\hat{G})^* \longrightarrow \underline{D}_{T,K_o}^{inv}(G)$ is an isomorphism, for each K_o.

<u>THE SECOND MAIN THEOREM</u> is:

For $G = GL_n(F_p)$, the space $\underline{D}_T^{inv}(G)$ (resp. $\underline{D}_{T,K_o}^{inv}(G)$) is the weak*-closure of $\{ ch_\pi : [\pi]\in\underline{E}_T(G)\}$ (resp. $\{ ch_\pi : [\pi]\in\underline{E}_{T,K_o}(G)\}$).

To prove this we note that any element Λ of $\underline{D}_{T,K_o}^{inv}(G)$ corresponds to some $\tilde{\Lambda}\in \underline{I}_{K_o}(\hat{G})^*$, by the preceding theorem. But $\underline{I}_{K_o}(\hat{G})$ is identifiable with $C^\infty(X_{K_o};\text{symmetric})$, the space of C^∞-functions f on the compact real manifold $X_{K_o} = \bigcup_{A\in S}\bigcup_{\boldsymbol{\sigma}\subset \underline{E}_2(M:K_o\cap M)} \boldsymbol{\sigma}$, which are symmetric in that $f(s\cdot\sigma) = f(\sigma)$, $\forall s\in w(A|A'), \sigma\in\boldsymbol{\sigma}$. Any element $\tilde{\Lambda}$ in the dual space can therefore be obtained by integration (over X_{K_o}) against a symmetric complex Borel measure $\mu = \mu_{K_o, \tilde{\Lambda}}$. Standard integration theory then shows Λ to be in the weak*-closure of the $ch_{\pi}, [\pi] \in \underline{E}_{T,K_o}(G)$.

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These results form part of the author's doctoral thesis. The author expresses thanks to his supervisor, Professor James Arthur, for advice and generous encouragement.

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Dept. of Mathematics
University of Toronto
Toronto, Ontario, Canada
M5S 1A1

ON THE CLASSIFICATION OF KNOTS

C.H. Dowker and M.B. Thistlethwaite

Presented by I. Halperin, F.R.S.C.

We use the following modification of one of Tait's notations [2]. For a knot projection with n crossings we count the crossing points, 1, 2, ... , 2n, as we go continuously around the knot curve. Each crossing point is counted twice, by an odd number and an even number. Thus we have a matching of the odd numbers with the even numbers up to 2n . We write successively the even numbers matched, respectively, with 1, 3, ... , 2n-1 . If we count from different initial points, or in a different direction, we may get different arrangements of the even numbers for the same projection. We choose the arrangement least in lexicographic order as the standard symbol for the knot projection. The only knots with 3 or 4 crossings, and the two knots with 5 crossings, are denoted:

4 6 2, 4 6 8 2, 4 8 10 2 6, 6 8 10 2 4

As usual, we reject composite knots. Hence, (Rule 1) we do not accept any matching of odd with even numbers such that, for some proper subinterval of the numbers 1, 2, ..., 2n , the odd numbers in the subinterval are matched with the even numbers in the same subinterval. In particular, no number may be matched with an adjacent number modulo 2n . It can be shown that the matching, or the arrangement of even numbers, corresponding to a knot projection, determines the projection in the plane uniquely up to homeomorphism of the extended complex plane.

An arrangement of the first n even numbers, satisfying Rule 1, need not correspond to a knot projection. For example, there is no knot projection giving the arrangement: 4 8 2 10 6 . But we can write down a condition (Rule 2) on a matching of the odd with the even numbers up to 2n satisfying Rule 1, which is necessary and sufficient for the matching to represent a knot projection. Rule 2 is a formal requirement on the number matching which we shall state in full in a paper now in preparation. It expresses some consequences of the fact that the projection of each arc of a knot, going from a crossing point to the matching crossing point, divides the plane into regions which can be coloured black and white alternately.

When we have a matching that determines a knot projection, we can easily compute all the other matchings, including the standard one, that determine the same knot projection. This enables us to solve numerically Kirkman's problem of finding all projections of non-composite knots with n crossings.

Since, as is usual for tabulations of knots, we identify a knot with its mirror image, we may assume that 1 denotes an undercrossing. If the knot is alternating, the even numbers denote overcrossings. For non-alternating knots, we write a minus sign before each even number which denotes an undercrossing. Thus we have symbols for all knots with n crossings. Unlike Conway's notation [1], this notation gives little indication of the structure of the knot. But it is a simple uniform notation which is very suitable for computer use. Entering the arrangement of signed even numbers allows the computer to find the Alexander polynomial, any of the invariants based on group representations, or even the Conway symbol for the knot.

As was pointed out by Tait, the same knot in 3-space may have different projections in the plane. These projections of the same knot are regarded as equivalent projections. Tait conjectured that his flyping operation [2] is sufficient for finding all equivalences of projections of alternating knots with n crossings. The flyping operation can be carried out numerically. Thus, if Tait's conjecture is true, we have an algorithm for listing, without duplication, all alternating knots with n crossings. We have verified Tait's conjecture for $n \leq 12$.

For non-alternating knots, it may be necessary to use operations which change a projection with n crossings to one with more than n crossings, and then eventually to change back to n crossings. We do not even have a conjecture about how many such operations may be needed to find all equivalences, and hence, we have to be content with an empirical approach.

We have found 1288 alternating knots and 888 non-alternating knots with 12 crossings. Apart from possible computing errors, this list is complete. Each of these knots is distinguished by its invariants from all other knots with up to 12 crossings, so there are no duplicates.

References

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Birkbeck College and
Polytechnic of the South Bank
London, England.

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MAILING ADDRESSES

1.	E. Binz	Dept. of Mathematics, University of Manheim D68 Manheim, W. Germany
2.	Samuel Bourne	Dept. of Mathematics, University of California, Berkeley, CA 94720, U.S.A.
3.	Marc Chamarie	Université Claude-Bernard - Lyon I Département de Mathematiques, 43, Boulevard du 11 Novembre 1918, 69622 Villeurbanne Cedex (France)
4.	John D. Dixon	Dept. of Mathematics and Statistics, Carleton University, Ottawa, Ontario Canada, K1S 5B6
5.	C. H. Dowker	Birkbeck College and Polytechnic of the South Bank, London, England
6.	Michael Edelstein	Dept. of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada, B3H 4H8
7.	K. Győry	Mathematical Institute, Kossuth Lajos University, 4010 Debrecen, Hungary
8.	Michitaka Hikari	Dept. of Mathematics, Keio University, Hiyoshi, Yokohama, Japan
9.	Claude Lemaire	Département de Mathématiques, Universite Laval, Cité Quebec, Québec, Canada, G1R 7P4
10.	Monique Limbos	Université Libre de Bruxelles, Département de Mathematiques, Campus Plaine C.P. 215, Bd. du Triomphe, B-1050, Bruxelles, Belgium
11.	Dieter Lutz	F.B. 6 - Mathematik - Universität Essen, P.O.B. 6843, D-4300 Essen 1, Germany (Fed. Rep.)
12.	Peter A. Mischenko	Dept. of Mathematics, University of Toronto, Toronto, Ontario, Canada, M5S 1A1
13.	Neville Reed	4 Menin Way, Farnham, Surrey GU98DY, England
14.	Morwen B. Thistlethwaite	Birkbeck College and Polytechnic of the South Bank, London, England