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ON THE DUAL OF A GENERALIZED B^*-ALGEBRA

Subhash J. Bhatt

Presented by I. Halperin, F.R.S.C.

Abstract: A pseudo-complete symmetric locally convex *-algebra each continuous hermitian functional on which is a difference of two positive functionals is a GB^*-algebra. Conversely, a locally convex GB^*-algebra admits a largest locally convex GB^*-topology with respect to which the positive cone is normal, and hence such a decomposition is available.

DEFINITION ([2], [7]): Let A be a locally convex algebra with identity 1 and with a continuous involution $x \longrightarrow x^*$. Let t be its topology. Let $\mathcal{B}^*(t)$ be the collection of all $B \subset A$ such that B is absolutely convex, $B^2 \subset B$, $B^* = B$, $1 \in B$ and closed and bounded. Then A is called a GB*-algebra if

(i) A is symmetric in the sense that for each $x \in A$, $(1 + x^*x)^{-1}$ exists and is bounded. (An element $a \in A$ is bounded if for some $\lambda > 0$, $\{(\lambda^{-1}a)^n | n = 1,2,\ldots\}$ is bounded. The set of all bounded elements of A is denoted by $(A, t)_0$ or A_0.)

(ii) the collection $\mathcal{B}^*(t)$ has a greatest member B_0 under inclusion, called the unit ball of A,

(iii) the *subalgebra $A(B_0) = \{\lambda x | \lambda \in \mathbb{C}, x \in B_0\}$ is complete with the Minkowski functional $||\cdot||_{B_0}$ of B_0 as a norm. (It turns out [2, Theorem 2.6] that $(A(B_o), ||\cdot||_{B_o})$ is a B^*-algebra.)

The results in [2], [7] and [12] indicate that a GB^*-algebra is a well-behaved topological analogue of a B^*-algebra. We aim to discuss the GB^*-analogues of Grothendieck's dual characterization of B^*-algebras [8] and its converse [6, Corollary 2.6.4]. The paper [11] contains results about the decomposition of hermitian functionals as differences of positive functionals in the context of topological algebras of unbounded operators.

THEOREM 1. Let (A, t) be a pseudo-complete locally convex *-algebra with 1. Assume that

(a) A is symmetric

and (b) every continuous hermitian functional on A is a difference of two positive functionals.

Then A is a GB^*-algebra. If A is complete, then in place of (a), it suffices to assume

(a') A is hermitian (i.e. for each $h = h^*$ in A, $(1 + h^2)^{-1}$ is bounded).

Proof: Let A^P be the complex linear span of the set $P(A)$ of all positive functionals on A. By (b), the dual $A' = (A,t)' \subset A^P$; and so the weak topology $\sigma = \sigma(A,A') \leq \sigma(A,A^P) = \sigma_P$ (say). Let $P_1(A) = \{f \in P(A) | f(1) = 1\}$. Let $B_1 = \{x \in A | f(x^*x) \leq 1$ for all $f \in P_1(A)\}$, $B_o = B_1 \cap B_1^*$. Then exactly as in [2, Lemma 5.4], each of B_1 and B_1^* is absolutely convex, bounded and satisfies $B_1^2 \subset B$, $B_1^{*2} \subset B_1^*$. Hence so is B_o and $B_o = B_o^*$. Let $B \in \mathcal{B}^*(t)$. Pseudo-completeness of A implies that the *-subalgebra $A(B) = \{\lambda x \mid \lambda \in \mathbb{C}, x \in B\}$ containing 1 is complete under $||x||_B = \inf\{\lambda > 0 \mid x \in \lambda B\}$. As $B = B^*$, $||x||_B = ||x^*||_B$ $(x \in A(B))$. The remark following [4, p 198] gives the norm continuity on A(B) of each $f \in P_1(A)$ with $||f|| = f(1) = 1$. Let $x \in B$. Then $x^*x \in B$, $xx^* \in B$. Hence $f(x^*x) \leq 1$, $f(xx^*) \leq 1 (f \in P_1(A))$. Thus $x \in B_o$, $B \subset B_o$. In particular, the t-closure $\overline{B}_o$ of B_o is in $\mathcal{B}^*(t)$ and so $\overline{B}_o \subset B_o$. It follows that $B_o \in \mathcal{B}^*(t)$ and is the greatest member of $\mathcal{B}^*(t)$. As A is pseudo-complete, $(A(B_o), ||\cdot||_{B_o})$ is complete. Thus A is a GB^*-algebra. If A is complete, then under the greatest member condition, the conditions (a) and (a') are known [3] to be equivalent.

Now we take the converse problem. Given a locally convex GB^*-algebra A, we construct a largest locally convex GB^*-topology on A in which the positive cone A^+ of A is normal. Then we apply [10, Corollary 3, p. 220].

Let (A,t) be a locally convex GB^*-algebra. Let $P(A,t) = \{f \in P(A) \mid f$ is t-continuous$\}$. For each $f \in P(A,t)$, the GNS construction yields (in general, an unbounded) representation π_f of A as: Let $N_f = \{x \in A \mid f(x^*x) = 0\}$, $X_f = A/N_f$, a pre-Hilbert space with inner product $\langle a + N_f, b + N_f\rangle = f(b^*a)$. Let H_f be its completion. Let $\pi_f(a)(b + N_f) = ab + N_f$ $(a, b \in A)$. The direct sum of these representations is the representation π_t with dense domain $\mathcal{D}_t$ in a Hilbert space H_t defined as: $H_t = \Sigma^{\oplus} H_f$, a Hilbert space direct sum, $D_t = \{ z = (z_f) \in H_t \mid \Sigma \|\pi_f(x)z_f\|^2 < \infty$ for all $x \in A \}$ and $\pi_t(x)(z_f) = (\pi_f(x)z_f)$ for $x \in A$, $(z_f) \in D_t$. Then $\pi_t(A)$ is an Op^*-algebra [9, § 2] on D_t. The induced topology t_A on D_t is the topology defined by the semi-norms $z \longrightarrow \|Sz\|$ ($S \in \pi_t(A)$). Let G be the collection of all bounded subsets of (D_t, t_A). Let $\tau_t(t)$ be the topology on A defined by the seminorms $p_M(a) = \sup \{ |\langle \pi_t(a)z, w\rangle| \ z, w \in M \}$ where M varies over G. If T denote the Dixon topology [7, § 6] on A, we denote $\tau_T(T)$ by $\pi(T)$.

<u>THEOREM 2</u>. The topology $\tau(T)$ on a locally convex GB^*-algebra (A, t) is the largest locally convex GB^*-topology on A in which A^+ is normal.

<u>Proof</u>: By the first paragraph of [7, § 8], $P(A) = P(A,T)$. As in the proof of theorem 7.6 of [7], the representation π_T (briefly π) is faithful. Hence $\tau(T)$ is Hausdorff; and it is immediate from [9, Theorem 3.1 (ii)] that $(A, \pi(T))$ is a locally convex *-algebra. By [11, Proposition 4.1], A^+ is $\pi(T)$-normal. Further since (A, t) is barrelled [7, Lemma 6.2], an adaptation of [9, Theorem 4.1] shows that $\pi:(A,T) \longrightarrow (\pi(A), \tau_{D_T})$ is continuous. (Here τ_{D_T} is the uniform topology [9, § 3] on $\pi(A)$ as an Op^*-algebra on D_T). Hence $\tau(T) \leq T$. This gives $(A,T)_0 \subset (A, \pi(T))_0$ showing that $(A, \pi(T))$ is symmetric.

Since A^+ is $\pi(T)$-normal, [10, Corollary 3, p. 220] implies that $(A, \pi(T))' \subset A^P$. Conversely, let $f \in P(A)$. Let (x_α) be a net in A such that $x_\alpha \xrightarrow{\pi(T)} 0$. Then for each $z \in D_T$, $\langle \pi(x_\alpha)z,z\rangle \to 0$. Choose $z = (z_g | g \in P(A,T) = P(A))$ as $z_g = 1 + N_f$ if $g = f$, $z_g = 0$ if $g \neq f$. Then $f(x_\alpha) = \langle \pi_f(x_\alpha)z_f, z_f\rangle = \langle \pi(x_\alpha)z, z\rangle \to 0$ giving the $\pi(T)$ continuity of f. Thus $(A, \pi(T))' = A^P$. Now let $\tau = \tau(A,A^P)$, the Mackey topology of the duality $\langle A,A^P\rangle$. As (A,σ_P) is a GB^*-algebra [7, § 8, paragraph 1 and Corrolary 7.8] with the same unit ball B_0, it can be shown as in [2, § 5] that (A,τ) too is a GB^*-algebra whose unit ball is B_0 [7, Corollary 7.8]. Since $(A,\pi(T))' = A^P$ and T is the largest locally convex GB^*-topology, $\sigma_P \leq \pi(T) \leq T$. Now let $B \in \beta^*(\pi(T))$. By [10, § 3.1 p.130], B is σ_P-closed and σ_P-bounded. Then $B \in \beta^*(\sigma_P)$, and so $B \subset B_0 \in \beta^*(\pi(T))$. Thus $(A, \pi(T))$ is a GB^*-algebra.

Finally, if t' is any locally convex GB^*-topology on A, then $t' \leq T$. This gives $\pi_{t'}(t') \leq \pi_T(T) = \pi(T)$. If A^+ is t'-normal, again by remark 1 in [11, p. 224], $t' \leq \pi_{t'}(t')$ giving $t' \leq \pi(T)$. Hence the theorem.

<u>COROLLARY 3</u>. Let A be a locally convex GB^*-algebra.

(a) The following are equivalent

(i) A^+ is T-normal, (ii) $\pi(T) = T$, (iii) (A,T) is an $A\sigma^*$-algebra. In this case, $\pi(T) = \tau(A,A^P) = T$.

(b) Every $\pi(T)$ continuous hermitian functional on A is a difference of two positive functionals.

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Dept. of Mathematics
Sardur Patel University
Vallabh Vidyanagar 388120
Gujarat, India

BASES OF HYPERIDENTITIES OF LATTICES AND SEMILATTICES

R. Padmanabhan and P. Penner*

Presented by G. Grätzer, F.R.S.C.

Abstract. W. Taylor investigated hyperidentities of various well known varieties. In this paper we shall summarize some results which state whether or not the bases of certain subsets of semilattice or lattice hyperidentities are finitely based. We also state some structure theorems for certain types of algebras satisfying all semilattice hyperidentities.

1. Introduction. Following W. Taylor [5] we define a hyperidentity ϵ to be formally the same as an identity. However, a variety V is said to satisfy a hyperidentity ϵ, if whenever the operation symbols of ϵ are replaced by arbitrary polynomials (of appropriate arity) in the operations of V, then the resulting identity holds in V in the usual sense. We shall follow the convention that hyperidentities have no nullary operations. By way of an example, a proof that

$$\epsilon: \quad F(G(x,y),x) = G(x,F(y,x))$$

holds in L, the variety of all lattices, consists of a case by case examination of $(F,G) \in \{x, y, x \vee y, x \wedge y\}^2$. For example, if $(F,G) = (x, x \vee y)$, we obtain the lattice identity $x \vee y = x \vee y$.

The set of all hyperidentities satisfied by a variety V, will be denoted by H(V). We shall let $H^m(V)$ be the set of all hyperidentities holding in V with operation symbols of arity at most m, and $H_n(V)$ will denote

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the set of all hyperidentities of V with at most n distinct variables. The hyperidentities of $H^m(V)$ will be said to be m-ary. The set of hyperidentities of V which are of type $\langle m,m,\ldots,m\rangle$ will be denoted by $H(V)\langle m,m,..,m\rangle$. We thus have $H(V)\langle m,m,\ldots,m\rangle \subseteq H^m(V)$. To avoid confusing hyperidentites with identities, whenever ε is being viewed as a hyperidentity we shall underline the function symbols of ε. For example, if ε is the lattice hyperidentity discussed above we shall write

$$\varepsilon: \quad \underline{F}(\underline{G}(x,y),x) = \underline{G}(x,\underline{F}(y,x)).$$

In the binary case we often use the infix notation $x\underline{\lambda}y$ rather than $\underline{F}(x,y)$. By a hyperpolynomial we shall mean an expression of the type $\underline{P}(\underline{F},\underline{G}) = \underline{F}(\underline{G}(x,y),x)$. When $\underline{F}$ and $\underline{G}$ are replaced by polynomials p and q, we shall write $\underline{P}(p,q)$ for the resulting polynomial. For a more comprehensive introduction to hyperidentities we refer the interested reader to W. Taylor [5]. Proofs for the results of Sections 2,3 and 4 shall appear in [1], [2] and [3] respectively.

2. Binary semilattice hyperidentities. Let $\underline{\lambda}$ be a distinguished binary operation symbol, and let SL denote the variety of all semilattices. We define inductively two syntactical transforms for hyperpolynomials of type $\langle 2,2,..\rangle$.

$$L_\lambda(x) = R_\lambda(x) = x \text{ for all variables } x,$$
$$L_\lambda(\underline{P}\underline{\lambda}\underline{Q}) = L_\lambda(\underline{P}),$$
$$L_\lambda(\underline{P}\underline{\mu}\underline{Q}) = (L_\lambda(\underline{P}))\underline{\mu}(L_\lambda(\underline{Q})),\ \underline{\mu} \neq \underline{\lambda},$$
$$R_\lambda(\underline{P}\underline{\lambda}\underline{Q}) = R_\lambda(\underline{Q}),$$
$$R_\lambda(\underline{P}\underline{\mu}\underline{Q}) = (R_\lambda(\underline{P}))\underline{\mu}(R_\lambda(\underline{Q})),\ \underline{\mu} \neq \underline{\lambda}.$$

Thus the transform L_λ (R_λ) replaces each occurrence of the operation $\underline{\lambda}$ by the left (right) projection. Thus if $\underline{P} = \underline{Q}$ is a hyperidentity of type <2>, $\underline{P} = \underline{Q}$ is satified by SL iff $L_\lambda(\underline{P}) = L_\lambda(\underline{Q})$ (i.e., the first variables of $\underline{P}$ and $\underline{Q}$ are equal), $R_\lambda(\underline{P}) = R_\lambda(\underline{Q})$, and $\underline{P} = \underline{Q}$ is regular (i.e., $\underline{P}$ and $\underline{Q}$ have the same variables). If Σ<2> denotes the set of three hyperidentities $\{x\underline{\lambda}x = x;\ x\underline{\lambda}(y\underline{\lambda}z) = (x\underline{\lambda}y)\underline{\lambda}z;\ (x\underline{\lambda}y)\underline{\lambda}(z\underline{\lambda}t) = (x\underline{\lambda}z)\underline{\lambda}(y\underline{\lambda}t)\}$ we have the following lemma:

<u>Lemma 2.1</u>. The set Σ<2> is a basis for H(SL)<2>.

<u>Theorem 2.2</u>. For an algebra A = <A;λ> of type <2> the following are equivalent:

(1) A satisfies Σ<2> as a set of identities, with λ substituted for $\underline{\lambda}$;

(2) A is a Plonka sum of diagonal semigroups (see [4] for the definition of this concept);

(3) A ∈ SL ∨ D, where D is the variety of diagonal semigroups;

(4) A satisfies all semilattice hyperidentities.

<u>Proof</u>. The proof utilizes the result of J. Plonka [4] and the fact that $p(x,y) = x\lambda y\lambda x$ is a partition function in the sense of J. Plonka [4].

We now let Σ^2 be the following set of hyperidentities:

$$x\underline{\lambda}x = x,$$
$$x\underline{\lambda}(y\underline{\lambda}z) = (x\underline{\lambda}y)\underline{\lambda}z,$$
$$(x\underline{\lambda}y)\underline{\mu}(z\underline{\lambda}t) = (x\underline{\mu}z)\underline{\lambda}(y\underline{\mu}t).$$

$\underline{P} = \underline{Q}$ is a consequence of Σ^2 will be denoted by $\underline{P} \approx \underline{Q}$. If we set $\underline{\lambda} = \underline{\mu}$ we get $\Sigma\langle 2\rangle$ as a consequence of Σ^2. In Lemma 2.3 we shall use the binary operation symbols $\underline{\lambda}$ and $\underline{\cdot}$ instead of $\underline{\lambda}$ and $\underline{\mu}$.

Lemma 2.3. If $\underline{P}$ is a hyperpolynomial consisting of two binary operation symbols $\underline{\lambda}$ and $\underline{\cdot}$, and if

$$L_\lambda(\underline{P}) = \underline{P}_1,$$
$$R_\lambda(\underline{P}) = \underline{P}_k,$$
$$\mathrm{Var}(\underline{P}) = \{x_1, x_2, \dots, x_n\},$$
$$\underline{X} = x_1 \underline{\cdot} x_2 \underline{\cdot} \dots \underline{\cdot} x_n,$$
$$\mathrm{Var}(L.(\underline{P})) \times \mathrm{Var}(R.(\underline{P})) = \{(a_1,b_1),(a_2,b_2),\dots,(a_h,b_h)\},$$

then $\underline{P} \approx \underline{P}_1 \;\underline{\lambda}\; a_1 \underline{\cdot} \underline{X} \underline{\cdot} b_1 \;\underline{\lambda}\; a_2 \underline{\cdot} \underline{X} \underline{\cdot} b_2 \;\underline{\lambda}\; \dots \;\underline{\lambda}\; a_h \underline{\cdot} \underline{X} \underline{\cdot} b_h \;\underline{\lambda}\; \underline{P}_k$.

Theorem 2.4. Σ^2 is a basis for $H^2(SL)$.

Proof. Since for any semilattice hyperidentity $\underline{P} = \underline{Q}$ in the binary operations $\underline{\lambda}$ and $\underline{\cdot}$ we have $L_\lambda(\underline{P}) \approx L_\lambda(\underline{Q})$, $R.(\underline{P}) \approx R.(\underline{Q})$ and $\mathrm{Var}(\underline{P}) = \mathrm{Var}(\underline{Q})$, Lemma 2.3 implies that Σ^2 is a basis for $H(SL)\langle 2,2\rangle$. Theorem 2.4 can now be proved by inducting on the number of operation symbols $\underline{\lambda}_1, \underline{\lambda}_2, \dots, \underline{\lambda}_n$, and noting that $H^2(SL)$ is a consequence of $H(SL)\langle 2,2,\dots\rangle$.

In order to state another structure theorem we define $D_{\langle 2,2\rangle}$ to be the smallest equational class generated by all algebras $A = \langle A; \lambda, \cdot\rangle$ of type $\langle 2,2\rangle$, where $\langle A; \vee\rangle$ is a semilattice and $\{\lambda, \cdot\} \subsetneq \{x, y, x \vee y\}$, the set of binary semilattice polynomials, and we let $\mathrm{Id}(\Sigma^2)$ be the set of six identities obtained by replacing $\underline{\lambda}$ and $\underline{\mu}$ of Σ^2 by all possible choices of $\cdot$ and λ.

Theorem 2.5. For an algebra A of type <2,2> the following are equivalent:

(i) A satisfies all semilattice hyperidentities;

(ii) A satisfies $\mathrm{Id}(\Sigma^2)$;

(iii) $A \in D_{\langle 2,2\rangle}$.

3. n-ary semilattice hyperidentities. If we let $\Sigma\langle n\rangle$ be the following set of n-ary hyperidentities:

(i) $F(x,\ldots,x) = x$,

(ii) $F(F(x_1,\ldots,x_n),F(y_1,\ldots,y_n),z_1,\ldots,z_{n-2})$
$= F(F(x_1,y_1,x_3,x_4,\ldots,x_n),F(x_2,y_2,y_3,\ldots,y_n),z_1,\ldots,z_{n-2})$

(iii) $F(F(x_1,\ldots,x_n),F(u,u,y_1,\ldots,y_{n-2}),z_1,\ldots,z_{n-2})$
$= F(F(x_1,\ldots,x_n),F(x_2,u,y_1,\ldots,y_{n-2}),z_1,\ldots,z_{n-2})$

and $\Sigma^n = \Sigma\langle n\rangle \cup \{F(G(x_{11},\ldots,x_{1n}),\ldots,G(x_{n1},\ldots,x_{nn})) =$
$G(F(x_{11},\ldots,x_{n1}),\ldots,F(x_{1n},\ldots,x_{nn}))\}$,

we obtain results analogous to Lemma 2.1, Theorem 2.2 and Theorem 2.4.

4. H(V) for lattices and semilatttices.

Theorem 4.1. Let V be the variety of all semilattices or a nontrivial variety of lattices. Then for any two integers m and n, there exists a hyperidentity ϵ, such that ϵ holds in V but ϵ is not a consequence of $H^m(V) \cup H_n(V)$.

Thus, if V is the variety of all semilattices or a nontrivial variety of lattices, then H(V) is not finitely based. This is a partial solution to a problem of W. Taylor [5, Problem 3].

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Dept. of Mathematics
University of Manitoba
Winnipeg
Manitoba, Canada
R3T 2N2

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EXTENSIONS OF AF-ALGEBRAS ARE DETERMINED BY K_0

Lawrence G. Brown and George A. Elliott

Presented by P. Ribenboim, F.R.S.C.

Abstract. An extension of one AF-algebra by another is shown to be determined completely (i.e., up to strong equivalence) by the corresponding extension of dimension groups with interval.

Recall that a separable approximately finite-dimensional C*-algebra (AF-algebra) is the direct limit of a sequence of finite-dimensional C*-algebras ([1; 2.2]).

Recall from [4] that AF-algebras are determined by K_0, in the following sense. For an AF-algebra A, the dimension range $D(A)$ is defined to be the set of Murray - von Neumann equivalence classes of projections in A, with addition induced by addition of orthogonal projections.

<u>Theorem 0</u> ([4], Theorem 4.3). Let A and A' be AF-algebras, and let

$$D(A) \overset{\varphi}{\to} D(A')$$

be an isomorphism of their dimension ranges. Then φ is induced by an isomorphism of C*-algebras

$$A \overset{\varphi}{\to} A' .$$

From Theorem 0, using the lifting and extending theorems of [3] (stated in terms of dimension-preserving automorphisms instead of derivations; see Theorem 2 below), we deduce the following analogous result for extensions.

Recall that if A and B are AF-algebras and E is a C*-algebra extension of B by A, that is, there is a short exact sequence of C*-algebra morphisms $0 \to B \to E \to A \to 0$, then by [5; 3.3] and [2], E is AF.

<u>Theorem 1</u>. Let A and B be AF-algebras, and let E and E' be extensions of B by A. Suppose that the corresponding extensions of dimension ranges are equivalent, that is, there is an isomorphism $D(E) \overset{\varphi}{\to} D(E')$ such that the diagram

$$\begin{array}{ccccc} D(B) & \to & D(E) & \to & D(A) \\ 1\downarrow & & \varphi\downarrow & & 1\downarrow \\ D(B) & \to & D(E') & \to & D(A) \end{array}$$

is commutative. Then there exists an isomorphism $E \overset{\varphi}{\to} E'$ inducing the isomorphism $D(E) \overset{\varphi}{\to} D(E')$, such that for some unitary multiplier u of B the diagram

$$\begin{array}{ccccc} B & \to & E & \to & A \\ \mathrm{Ad}\,u\downarrow & & \varphi\downarrow & & 1\downarrow \\ B & \to & E' & \to & A \end{array}$$

is commutative.

<u>Theorem 2</u> (cf. [3]). Let A and B be AF-algebras and let E be an extension of B by A. Then

(i) any dimension-preserving automorphism of A lifts to a dimension-preserving automorphism of E;

(ii) any dimension-preserving automorphism of B can be expressed as the product of two automorphisms of B, one determined by a unitary multiplier of B, and the other extending to a dimension-preserving automorphism of E which induces the identity in A.

Proof. Let $E_1 \subset E_2 \subset \ldots$ be an increasing sequence of finite-dimensional sub-C*-algebras of E with union $\cup E_n$ dense in E, so that by [1; 3.1], $\cup E_n \cap B$ is dense in B. We may assume that E is unital and $1 \in E_1$. Denote the image of E_n in A by $\dot{E}_n$; $\cup \dot{E}_n$ is dense in A.

Ad (i). Let α be a dimension-preserving automorphism of A. (In other words, α induces the identity in $D(A)$.) By [1; proof of 2.6], there is a unitary in A close to 1 transforming $\alpha \cup \dot{E}_n$ onto $\cup \dot{E}_n$. Such a unitary lifts to E, so to lift α to E we may assume that α takes $\cup \dot{E}_n$ onto itself, or, in particular, passing to a subsequence, that $\alpha \dot{E}_n \subset \dot{E}_{n+1}$.

One may now argue very much as for derivations in [3; 2.5]. Since α preserves dimension, there is a unitary $\dot{u}_{n+1} \in \dot{E}_{n+1}$ such that $\alpha | \dot{E}_n = \mathrm{Ad}\, \dot{u}_{n+1} | \dot{E}_n$. Choose u_{n+1} unitary in the preimage of $\dot{u}_{n+1}$ in E_{n+1}. Write $E_n = E_n e_n \dot{+} E_n g_n$ where $E_n e_n = E_n \cap B$ and $e_n + g_n = 1$, and denote the difference $e_n - e_{n-1}$ by f_n, where $e_0 = 0$. Set

$$f_1 + g_1 u_1 = v_1 \,, \; f_1 + f_2 u_1 + g_2 u_2 = v_2, \ldots \,,$$
$$f_1 + f_2 u_1 + \ldots + f_n u_{n-1} + g_n u_n = v_n \,.$$

Then a short computation yields, for $p = 1,2,\ldots$,

$$\mathrm{Ad}\, v_{n+1+p} | E_n = \mathrm{Ad}\, v_{n+1} | E_n \,.$$

Therefore the sequence $(\mathrm{Ad}\, v_n)$ converges simply to an endomorphism of E. Since, moreover, the sequence $(v_n e_n)$ converges simply to a unitary multiplier of B, the range of this endomorphism contains all of B. Since $v_n - u_n \in B$, the endomorphism lifts α. In particular it is surjective modulo B and hence surjective. Thus we have a dimension-preserving automorphism of E lifting α.

Ad (ii). Let β be a dimension-preserving automorphism of B. As above, we may modify β by an inner automorphism of $B + \mathbb{C}1$ in such a way that β takes $\cup E_n \cap B$ onto itself, and then assume that $\beta(E_n \cap B) \subset E_{n+1} \cap B$. Let e_n, f_n, and g_n be as defined above.

Since (e_n) is an approximate unit for B, and βf_n is equivalent to f_n, there is a unitary multiplier v of B transforming βf_n into f_n for all n. Replacing β by $(\mathrm{Ad}\, v)\beta$, we may suppose that β fixes each e_n. Then as in [3; 3.2], since β preserves dimension, there is a unitary multiplier u of B such that for all n, $\beta | f_n E_n f_n = \mathrm{Ad}\, u | f_n E_n f_n$. Replacing β by $(\mathrm{Ad}\, u^{-1})\beta$, we may suppose that β acts trivially on $f_n E_n f_n$.

Now, as in [3; 3.3], define morphisms $\beta_n : E_n \to E$ by

$$\beta_n | E_n e_n = \beta | E_n e_n , \quad \beta_n | E_n g_n = 1 | E_n g_n .$$

One computes easily that β_{n+1} extends β_n, using that β acts trivially on $f_{n+1}E_{n+1}f_{n+1}$ (and that $E_n e_n \subset E_{n+1}e_{n+1}$, $E_n f_{n+1} \subset f_{n+1}E_{n+1}f_{n+1}$, and $E_n g_{n+1} \subset E_{n+1}g_{n+1}$). Hence the sequence (β_n) converges simply to an endomorphism of E, say $\bar{\beta}$, agreeing with β on $\cup E_n e_n$ and so on all of B. Since the image of $E_n g_n$ in A is equal to $\dot{E}_n$, the endomorphism $\dot{\beta}$ of A induced by $\bar{\beta}$ is equal to the identity on $\cup \dot{E}_n$ and so on all of A. In particular $\dot{\beta}$ is surjective, and hence as in the proof of (i) above, $\bar{\beta}$ is surjective. Thus we have a dimension-preserving automorphism of E extending β and inducing the identity in A.

<u>Proof of Theorem 1</u>. By Theorem 0, there is an isomorphism

$E \overset{\varphi}{\to} E'$ inducing $D(E) \overset{\varphi}{\to} D(E')$. By hypothesis, $\varphi|B$ and $\varphi^{-1}|B$ preserve dimension, and in particular take the ideal B into itself. Thus there exist automorphisms $B \overset{\beta}{\to} B$ and $A \overset{\alpha}{\to} A$ such that the diagram

$$\begin{array}{ccccc} B & \to & E & \to & A \\ \beta\downarrow & & \varphi\downarrow & & \alpha\downarrow \\ B & \to & E' & \to & A \end{array}$$

is commutative. By hypothesis, both β and α preserve dimension.

By Theorem 2 (i), α lifts to a dimension-preserving automorphism γ of E . Hence, replacing φ by $\varphi\gamma^{-1}$, we may suppose that α is the identity.

By Theorem 2 (ii), $\beta = (\mathrm{Ad}\ u)\beta_1$, where u is a unitary multiplier of B and β_1 extends to a dimension-preserving automorphism δ of E inducing the identity in A . Replacing φ by $\varphi\delta^{-1}$, we have that φ restricts to Ad u in B and induces the identity in A , as desired.

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Department of Mathematics
Purdue University
West Lafayette
Indiana 47907

Mathematics Institute
Universitetsparken 5
2100 Copenhagen Ø
Denmark

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FREQUENCY ANALYSIS BY THE EAR

J. Malzan

Presented by G.F.D. Duff, F.R.S.C.

We consider below the problem of determining which frequencies are perceived when a periodic vibration of the air is detected by the ear. It is commonly believed that the ear, in its analysis of complex vibrations, performs a Fourier analysis into sinusoidal components and that, at least with practise, these are the perceived frequencies. This view of the function of the ear, called Ohm's Law for Hearing, was first advanced by Helmholtz in [2], and has enjoyed great, if not universal, acceptance since then. Helmholtz carried out a large number of experiments (of a necessarily crude kind) which yielded the following anomalies:

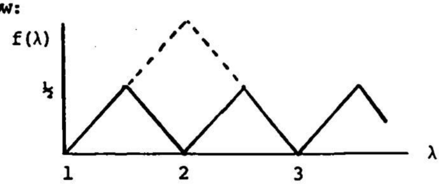
(i) There are many complex vibrations for which significant overtones are inaudible.

(ii) It is often the case that frequencies are perceived which do not arise in the Fourier spectrum.

As an example of the second anomaly we might consider the vibration represented by the function

$$g(t) = \sin(2\pi t) + R \sin\,(2\pi\lambda t + \delta),\ \lambda \geq 1.$$

For R not greatly different from 1 this vibration has a "difference tone", whose frequency, $f(\lambda)$, is given by $f(\lambda) = |\lambda - n|$, where n is the integer such that $|\lambda - n| \leq \frac{1}{2}$. We have the graph below:



(The dotted line represents a more weakly discerned pitch.) Helmholtz believed that perceived pitches which were not present in the Fourier spectrum could be explained by distortion or psychology (although the diagram above is difficult to explain on this basis). Fourier components which, by Ohm's Law, should have been audible but were not he ascribed to insufficient training, a device which led him to distinguish the "analytic" ear (trained as a Fourier analyser by intensifying the otherwise inaudible Fourier components) from the "synthetic" ear (the kind the rest of us have).

The commonly held modern view is still of the ear as a Fourier analyser, but with two differences from Helmholtz's time: First, the failure to "hear out" overtones is called masking, and is thought to be related to frequency discrimination (in a critical bandwidth). Also, while Helmholtz postulated resonating hairs as the basis for frequency analysis, the modern view, called place theory, assumes that frequency analysis is based on the localization of vibrational activity on the basilar membrane of the inner ear. See [3] for a discussion of this view.

Another modern, but minority, view (see [1]) has it that perceived frequencies are those whose periods arise as the distance between major peaks of the vibration. This is sometimes called "peak-picker" theory. The approach below represents a fusion of these two points of view, with frequency analysis being performed mainly by the neural network near the (physical) ear.

A Model for the Synthetic Ear

At the heart of any theory of hearing lies the question: Which vibrations are perceived as a single pitch? Let us call such vibrations simple. Helmholtz's basic claim was that only sinusoids are simple. We have, however, the following experimental data:

(i) A vibration $g(t) = \sin(ft) + R\sin(nft + \gamma)$, $n \in Z^+$, $n > 1$, R, f and γ constants, is simple according to the approximate rule that $R < 1/n$. Otherwise the frequency $nf/2\pi$ can be distinguished.

(ii) If $g(t)$ is periodic and is such that it is monotonic between its (unique) maximum and minimum in a period, then $g(t)$ is simple.

(iii) If $g(t)$ is simple, then so too is $Ag(\lambda t + \gamma)$ so long as frequencies and amplitudes are in the "middle" range of hearing.

(iv) There are vibrations from which the fundamental (first Fourier component) is absent, but are simple nonetheless.

(v) No vibration can be simple at two different frequencies. (This, of course, is a logical requirement rather than an experimental datum.)

It does not seem to be possible to find a criterion for simple vibrations in terms of Fourier coefficients (ignoring phase) which satisfies (i) - (v). One solution involves instead the following construct:

<u>Definition</u>. For a function $g(t)$, periodic of period 1, define, for each positive integer n, the <u>nth part of</u> $g(t)$ as

$$g_n(t) = \frac{1}{n}\sum_{k=1}^{n} g(t+\frac{k}{n}).$$

Note that if $g(t) = \sum_{k=1}^{\infty} R_k \sin(2\pi kt + \gamma_k)$ then

$$g_n(t) = \sum_{k=1}^{\infty} R_{nk} \sin(2\pi knt + \gamma_{kn}).$$

That is, $g_n(t)$ is the "part" of $g(t)$ which is periodic of period $1/n$. Similarly if $g(t)$ has period other than 1. We can now present the criterion for simplicity.

<u>Criterion</u>. A periodic function $g(t)$, representing a vibration, is simple if, and only if,

$$n(\max - \min)\, g_n(t) \leq (\max - \min)\, g(t) \text{ for all } n \in Z^+.$$

This criterion can be shown to satisfy (i) - (v) above. The nth part analysis can be seen as a crude Fourier analysis, while the dependence on peak values in the taking of maxima and minima above is suggestive of the "peak-picker" analysis mentioned above.

Now let $g(t)$ be a periodic function which is not necessarily simple, of period 1. The rth part, $g_r(t)$, of $g(t)$ is defined by averaging, in units of $1/r$ over, if necessary, several periods of $g(t)$. (Here we take r rational.) We shall say that $g_s(t)$ <u>masks</u> $g_r(t)$ if

$$(\max - \min)\, g_r(t) \leq \min\left(\frac{s}{r}, 1\right)(\max - \min)\, g_s(t).$$

This theory predicts that the frequency r is audible if and only if $g_r(t)$ is not masked by some $g_s(t)$, $s \neq t$. This definition is compatible with our definition of simple tone. It implies that if $g(t)$ has minimum period l then the frequency $1/l$ is discerned, but no frequency smaller than $1/l$ is discerned. This definition also explains the "difference" tone perceived with $g(t) = \sin(2\pi t) + R \sin(2\pi mt + \gamma)$ mentioned above, without recourse to distortion.

It is worth remarking, at this point, on an experiment that is a particular favorite of the "peak-picker" school. It nicely illustrates difficulties encountered by the Fourier analysts, the peak-pickers and the theory presented here.

Consider the vibration represented by

$$g(t) = \sum_{n=a}^{b} \cos((n+\varepsilon)2\pi ft)$$

where ε is small, and $b < 12$ (say), and $b-a > 3$ and the frequency f is in the middle of the audible range. With $\varepsilon = 0$ all theories have a plausible case to make: What must be explained is simply that the frequency f is strongly perceived, though not in the Fourier spectrum. Disciples of Helmholtz can point to distortion, and the Fourier components of $g^k(t)$, $k \in Z^+$, while "peak-pickers" can point to the obvious peaks that $g(t)$ possesses at times $1/f$. From the point of view presented on the paper, it can be mentioned that the "fth part" of $g(t)$ is large enough not to be masked and the perception of this frequency is again predicted.

Problems arise, however, when $\varepsilon \neq 0$. The experimental evidence is that, for ε small, the frequency $f(1 + \frac{2\varepsilon}{a+b-1})$ is strongly perceived. There is no obvious explanation for this in terms of Fourier analysis, since this frequency does not arise even in distorted Fourier spectra, which containing only frequencies of the form $f(m + \varepsilon k)$, $m, k \in Z$.

Peak-pickers are vindicated by this experiment since the identity

$$g(t) = \cos\left(\left(\frac{b+a-1}{2} + \varepsilon\right)t\right) \left[\frac{\sin(b-a)t}{\sin(\frac{t}{2})}\right]$$

shows that g(t) has maxima and minima appropriately spaced for the perception of the frequency $f(1+\frac{2\varepsilon}{a+b-1})$, provided that successive positive peaks are compared only to positive peaks (when they exist) and similarly for negative peaks. The peak-pickers are, however, weak on the subject of other perceived frequencies (and there are others in this experiment), and fail to explain them properly.

In using the theory presented here care must be taken in computing the part of g(t) which produces the desired frequency. Calculated strictly (that is, over time $1/\varepsilon$) it is zero. However, it should be remembered that the sensory system is limited to time spans of a very few periods and responds accordingly. Thus the appropriate part of g(t) is significant and predictably audible.

This theory is also suggestive of a physical model of hearing: While we might suppose that localization on the basilar membrane contributes a certain focussing effect in the discrimination of pitch, it seems likely that the real work of pitch discrimination takes place in the neural pathways close to the physical ear. For while neural networks that perform Fourier analysis are difficult to imagine, neural networks that extract nth parts are very simple indeed, relying only on time delay. In fact, it seems as if neurons and their associated dendrites have exactly the properties required for this role, since the velocity of pulses along dentrites is very nearly constant, while the neurons themselves are capable of the addition and subtraction required to for nth parts, and to provide masking for improved frequency discrimination.

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Dept. of Mathematics
University of Toronto
Toronto, Ontario, Canada
M5S 1A1

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REGULAR SOLUTIONS OF NON LINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

EUGENIO SINESTRARI

Presented by G. de B. Robinson, F.R.S.C.

1. Abstract results.

Let E and F be Banach spaces with norm $\|\cdot\|$ and $\|\cdot\|_F$ respectively such that $F \hookrightarrow E$. We want to study the abstract non linear Volterra integrodifferential equation:

$$(1)\quad \begin{cases} u'(t) = f(t,u(t)) + \int_0^t g(t,s,u(t),u(s))ds, & t \geq 0 \\ u(0) = x \end{cases}$$

where f and g satisfy the following assumptions:

(F1) $(t,x) \to f(t,x)$ is continuous from $\mathbb{R}_+ \times F$ to E and its partial Fréchet derivative $f_x(t,x) = A(t,x)$ is continuous from $\mathbb{R}_+ \times F$ to $L(F,E)$. In addition for each (t,x):

(F2) $A(t,x)$ generates an analytic semigroup $s \to e^{A(t,x)s}$ in E with domain $D_{A(t,x)} = F$ and the graph norm of $D_{A(t,x)}$ is equivalent to $\|\cdot\|_F$.

(F3) for each $T > 0$ and $\varphi \in C(0,T;E)$ the function $u(s) = \int_0^s e^{A(t,x)(s-s')}\varphi(s')ds'$, $0 \leq s \leq T$ belongs to $C(0,T;F)$; hence it is the unique function $u \in C(0,T;F) \cap C^1(0,T;E)$ which verifies: $u'(s) = A(t,x)u(s) + \varphi(s)$ for $0 \leq s \leq T$ and $u(0) = 0$.

(G) g is continuous from $\{(t,s,x,x'),\ 0 \leq s \leq t < \infty;\ x;x' \in F\}$ to E and given $x_0 \in F$ and $T > 0$ there exist $\rho(x_0,T)$ and $K(x_0,T)$ such that $\|g(t,s,x_1,x_1') - g(t,s,x_2,x_2')\| \leq K(x_0,T)(\|x_1 - x_2\|_F + \|x_1' - x_2'\|_F)$ for $0 \leq s \leq t \leq T$ and $x_1,x_1',x_2,x_2' \in B_F(x_0,\rho(x_0,T))$, the closed ball in F with center in x_0 and radius $\rho(x_0,T)$.

This problem in the completely non linear case and in a general Banach space[1] does not seem to have been previously studied,

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[1] It is known that property (F3) for an (unbounded) generator A of an analytic semigroup in a general Banach space does not hold: but as we shall see it is valid when the operator A is considered in a suitable intermediate space between D_A and E.

although non linear integrodifferential equations of parabolic type (see (F2)) in a general Banach space are investigated in [2,3,5,6]. In [4] we studied this problem when f is linear with respect to u and when g does not depend on u(t). Our main result is the following:

Theorem 1.1. Let (F1)-(F3) and (G) hold. Given $x \in F$ there is a unique solution $u \in C(0,T;F) \cap C^1(0,T;E)$ of (1) with suitable $T > 0$. Moreover u depends continuously on x; more precisely: given $x_o \in F$ there exist $\delta = \delta(x_o)$ and $T = T(x_o)$ such that if $x \in B_F(x_o,\delta)$ then there is a function $u(t,x)$ from $[0,T] \times B_F(x_o,\delta)$ such that $u(\cdot,x)$ verifies (1) in $[0,T]$ and $x \to u(\cdot,x)$ is continuous from $B_F(x_o,\delta)$ to $C(0,T;F)$. If in addition f and g verify the condition:

(H) if $0 \le t_o < t_1$ and $w \in C(t_o,t_1;F)$, where the function $\tilde{f}: [t_1,\infty[\times F \to E$ defined as

$$\tilde{f}(t,x) = f(t,x) + \int_{t_o}^{t_1} g(t,s,x,w(s))ds$$

verifies (F1)-(F3) for each $(t,x) \in [t_1,\infty[\times F$, then for each $x \in F$ there exist a unique maximally defined solution of (1).

These results are obtained by using a linearization technique and the maximal regularity property stated in (F3). We will show how the assumptions (F1)-(F3), (G) and (H) can be verified in some examples.

2. Applications.

We will study the non linear parabolic integrodifferential equation in $\mathbb{R}^2$:

$$(2.1)\begin{cases} u_t(t,x) = \varphi(t,u_{xx}(t,x)) + \displaystyle\int^t K(t,s,u_{xx}(t,x),u_{xx}(s,x))ds \\ u(0,x) = u_o(x), \quad u(t,0) = u(t,1) = 0; \quad 0 \le x \le 1, \; 0 \le t \le T. \end{cases}$$

To apply the preceding results, let us introduce the Banach space of little hölder continuous functions on [0,1] with parameter $\theta \in]0,1[$:

$$h^{\theta}(0,1) = \{u: [0,1] \to \mathbb{R}, \lim_{\delta\to 0} \sup_{|t-s|\leq\delta} \frac{|u(t)-u(s)|}{\delta^{\theta}} = 0\},$$

$$\|u\|_{h^{\theta}(0,1)} = \sup_{0\leq t\leq 1} |u(t)| + \sup\{\frac{|u(t)-u(s)|}{|t-s|^{\theta}};\ t,s \in [0,1],\ t\neq s\}.$$

If we choose $\theta \in]0,\frac{1}{2}[$ and define:

$E = \{u \in h^{2\theta}(0,1),\ u(0) = u(1) = 0\},\ \|u\|_E = \|u\|_{h^{2\theta}(0,1)}$

$F = \{u \in C^2(0,1),\ u'' \in h^{2\theta}(0,1),\ u(0) = u''(0) = u(1) = u''(1) = 0\},$

$\|u\|_F = \|u\|_{C^0(0,1)} + \|u''\|_{h^{2\theta}(0,1)}$

$u(t) = u(t,\cdot),\ f(t,v)(x) = \varphi(t,v''(x)),\ g(t,s,v,w)(x) = K(t,s,v''(x),w''(x))$

then the abstract version of (2.1) in the space E is given by (1.1). An application of theorem 1.1 gives the following result:

<u>Theorem 2.1</u>. Let us suppose that:

(i) $\varphi(t,x)$, $\varphi_x(t,x)$ and $\varphi_{xx}(t,x)$ are continuous from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$,

(ii) $\varphi(t,0) = 0$ and $\varphi_x(t,x) > 0$,

(iii) $K(t,s,x,x')$ and its first and second partial derivatives with respect to x and x' are continuous from $\{(t,s,x,x'), 0\leq s\leq t<\infty;\ x,x' \in \mathbb{R}\}$ to $\mathbb{R}$,

(iv) $K(t,s,0,0) = 0$ and $K_x(t,s,x,x') \geq 0$.

Then given $u_0 \in F$, there is $T > 0$ and a unique real function $u(t,x)$ defined in $[0,T] \times [0,1]$ such that $t \to u(t,\cdot)$ belongs to $C(0,T;F) \cap C^1(0,T;E)$ and satisfies (2.1). In addition $t \to u(t,\cdot)$ can be uniquely extended as a solution in a maximal time-interval.

The proof is based on the fact that $Au = u''$ generates an analytic semigroup in the space $X = \{u \in C^0(0,1),\ u(0) = u(1) = 0\}$ with sup-norm and if $\theta \in]0,\frac{1}{2}[$, E is isomorphic to $D_A(\theta)$, the continuous interpolation space between D_A and E (see [1]) and in this space the maximal règularity property (F3) holds.

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Istituto Matematico "Castelnuovo"
Università di Roma
00185, Roma, Italy.

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THE CURVATURE TENSOR OF LORENTZ MANIFOLDS WITH SPIN STRUCTURE - PART I

W.H. Greub and H.R. Petry

Presented by H.S.M. Coxeter, F.R.S.C.

Abstract: Let M be a Lorentz manifold which admits a spin structure and denote the spin-bundle by ξ (see sec. 6). Then there is a unique linear connection ∇ in ξ which induces the Levi-Civita connection in the tangent bundle of M and the curvature tensor of M can be expressed explicitly in terms of the curvature for the connection in ξ.

1. Antilinear transformations. Let $\mathbb{C}^2$ be a 2-dimensional complex vector space. An antilinear transformation of $\mathbb{C}^2$ is a linear transformation of the underlying real vector space such that

$$\alpha(\lambda x) = \bar{\lambda}\alpha(x), \qquad \lambda \in \mathbb{C},\ x \in \mathbb{C}^2.$$

The antilinear transformations of $\mathbb{C}^2$ form a complex vector space of dimension 4, denoted by A.

Now fix a nonzero skew symmetric complex bilinear function ε. Then every antilinear transformation α determines a second anti-linear transformation α^* by the relation

$$\varepsilon(\alpha x, y) = \bar{\varepsilon}(x, \alpha^* y), \qquad x, y \in \mathbb{C}^2.$$

It will be called the adjoint of α. It follows from the definition that

1) $(\lambda\alpha)^* = \bar{\lambda}\,\alpha^*$

2) $\alpha^{**} = \alpha$

3) $\alpha\circ\alpha^* = \frac{1}{2}\,tr\,(\alpha\circ\alpha^*)\cdot\iota$

4) $tr\,(\alpha^*\circ\beta) = \overline{tr}\,(\alpha\circ\beta^*)$

Next we define the adjoint of a complex linear transformation ϕ by the equation

$$\varepsilon(\Phi x, y) = \varepsilon(x, \Phi^* y), \qquad x, y \in \mathbb{C}^2,$$

or equivalently, by

$$\Phi^* = tr\,\Phi\cdot\iota - \Phi.$$

Thus $\phi^* = -\phi$ if and only if $tr\,\phi = 0$.

Proposition: Suppose that ϕ is complex linear and that α , β are antilinear. Then

(1) $$tr\,(\Phi\circ\alpha\circ\beta) = \overline{tr}\,(\alpha^*\circ\Phi^*\circ\beta^*).$$

Introduce a (complex) inner product in the space A by setting

$$(\alpha,\beta) = \frac{1}{2}\,tr(\alpha\circ\beta^*).$$

It satisfies the relation

$$(\alpha^*, \beta^*) = \overline{(\alpha,\beta)}$$

2. Selfadjoint antilinear transformations. An antilinear transformation α will be called selfadjoint, if $\alpha^* = \alpha$.

These transformation form a real vector space of dimension 4 denoted by S . The inner product in A restricts to a (real) relations hold, there is a unique connection ∇ in ξ such that $\nabla\varepsilon = 0$ and such that the induced connection coincides with D.

<u>4. The curvature forms</u>. Again, let ∇ be a linear connection in ξ such that $\nabla\varepsilon = 0$ and denote its curvature form by R_ξ. It is a 2-form on M with values in the bundle whose fibre at x consists of the complex linear transformations of F_x. Since $\nabla\varepsilon = 0$ it follows that for any two vector fields X, Y

$$\operatorname{tr} R_\xi(X,Y) = 0$$

or equivalently

$$R_\xi(X,Y)^* = -R_\xi(X,Y). \tag{3}$$

Let $\hat{R}$ denote the curvature form of the induced connection in ξ_A,

$$\hat{R}(X,Y)\alpha = R_\xi(X,Y)\circ\alpha - \alpha\circ R_\xi(X,Y), \quad \alpha \in \operatorname{Sec}\xi_A.$$

$\hat{R}$ determines via the metric g_A the <u>covariant curvature</u> form

$$R_A(X,Y,\alpha,\beta) = g_A(\hat{R}(X,Y)\alpha,\beta), \quad \alpha,\beta \in \operatorname{Sec}\xi_A.$$

Using relations (2), (1) and (3) we obtain the formula

$$R_A(X,Y,\alpha,\beta) = \frac{1}{2}\left[\operatorname{tr}(R_\xi(X,Y)\circ\alpha\circ\overset{*}{\beta}) + \overline{\operatorname{tr}}(R_\xi(X,Y)\circ\overset{*}{\alpha}\circ\beta)\right], \tag{4}$$
$$\alpha,\beta \in \operatorname{Sec}\xi_A.$$

which expresses R_A in terms of R_ξ.

<u>5. The bundle ξ_S</u>. Next consider the vector bundle ξ_S whose fibre at x is the space of <u>selfadjoint</u> antilinear transformations of F_x. It is a real vector bundle of rank 4 and the metric g_A restricts to a Lorentz metric g_S in ξ_S (cf. sec. 2). Let $\hat{\nabla}$ be the connection defined in sec. 3. Since

inner product in S which is of type (+,-,-,-). Thus it makes S unto a Minkowski space.

Finally observe that every antilinear map α can be uniquely decomposed in the form

$$\alpha = \alpha_1 + i\alpha_2, \qquad \alpha_1, \alpha_2 \in S .$$

Hence A can be regarded as the complexification of the real vector space S .

<u>3. The bundle ξ_A</u>. Let ξ be an orientable complex vector bundle of rank 2 over a smooth manifold M . Denote its fibre at x by F_x. Since ξ is orientable, there exists a smooth function ε which assigns to every point $x \in M$ a skew symmetric complex bilinear function $\varepsilon(x)$ in F_x such that $\varepsilon(x) \neq 0$, $x \in M$. We shall call ε a (complex) <u>orientation of ξ</u>.

Now consider the bundle ξ_A whose fibre at x is the space of antilinear transformations of F_x. It is a complex vector bundle of rank 4 over M . The inner product defined in sec. 1 defines a (complex) inner product g_A in ξ_A,

(2) $$g_A(\sigma, \tau) = \tfrac{1}{2}\operatorname{tr}(\sigma \circ \tau^*), \qquad \sigma, \tau \in \operatorname{Sec} \xi_A .$$

Next, let ∇ be a linear connection in ξ such that ε is parallel, $\nabla\varepsilon = 0$ (such connections always exist).

The induced connection $\hat{\nabla}$ in ξ_A is given by

$$\hat{\nabla}_X \alpha = \nabla_X \circ \alpha - \alpha \circ \nabla_X, \qquad \alpha \in \operatorname{Sec} \xi_A .$$

<u>Theorem I</u>: The induced connection satisfies the relations

$$\hat{\nabla}_X g_A = 0 \quad \text{and} \quad \hat{\nabla}_X(\alpha^*) = (\hat{\nabla}_X \alpha)^* \qquad \alpha \in \operatorname{Sec} \xi_A .$$

Conversely, if D is a linear connection in ξ_A such that these

$$(\hat{\nabla}_X \alpha)^* = \hat{\nabla}_X (\alpha^*), \qquad \alpha \in \mathrm{Sec}\, \xi_A$$

it restricts to a linear connection in ξ_S. Its covariant curvature form R_S is just the restriction of R_A to ξ_S. Thus, (4) yields the formula

$$\boxed{R_\xi(X, Y, \alpha, \beta) = \mathrm{Re}\,\mathrm{tr}(R_\xi(X, Y) \circ \alpha \circ \beta)}$$

6. Lorentz manifolds with a spin structure. Let M be a Lorentz manifold with tangent bundle τ_M. A spin structure on M consists of an orientable complex rank 2 bundle ξ over M (called the spin-bundle) and a strong bundle isomorphism Γ from the complexified tangent bundle $\tilde{\tau}_M$ onto the bundle ξ_A which preserves the complex inner products (the inner product in $\tilde{\tau}_M$ is defined by $\tilde{g}(\lambda \otimes X, \mu \otimes Y) = \lambda\mu g(X, Y)$ where g is the Lorentz inner product in τ_M).

Now consider the Levi-Civita connection ∇_M in τ_M and let $\tilde{\nabla}_M$ be the induced connection in $\tilde{\tau}_M$. It determines via Γ a linear connection D in ξ_A which satisfies the conditions in Theorem I. Hence there is a unique connection ∇ in ξ inducing D and such that $\nabla\varepsilon = 0$. Let R_ξ denote its curvature form.

Theorem II: The covariant curvature forms $\tilde{R}_M$ and R_M for the connections $\tilde{\nabla}_M$ and ∇_M are given respectively by

$$\tilde{R}_M(X, Y, Z, W) = \frac{1}{2}\Big[\mathrm{tr}\,(R_\xi(X, Y) \circ \Gamma(Z) \circ \Gamma(W)^*) + \overline{\mathrm{tr}}\,(R_\xi(X, Y) \circ \Gamma(Z)^* \circ \Gamma(W))\Big], \quad Z, W \in \mathrm{Sec}\,\tilde{\tau}_M$$

and

$$R_M(X, Y, X_1, Y_1) = \mathrm{Re}\,\mathrm{tr}\,(R_3(X, Y) \circ \Gamma(X_1) \circ \Gamma(Y_1)),$$
$$X_1, Y_1 \in \mathrm{Sec}\, T_M .$$

The ultimate intention is to give new solutions of Einstein's vacuum field equations.

Dept. of Mathematics
University of Toronto
Toronto, Ontario
M5S 1A1

Institut fuer theoretische
Kernphysik
Nussallee 14
Bonn, W. Germany

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BOUNDS ON PARTICLE MOTIONS FOR THE NAVIER - STOKES EQUATIONS IN THREE SPACE DIMENSIONS

G. F. D. Duff, F.R.S.C.

Abstract: Bounds are given for particle distances travelled in three dimensional fluid motion.

1. Introduction. The highly developed mathematical theory of the Navier-Stokes equations in three space dimensions serves as a foundation for much of classical hydrodynamics as well as a starting point for more general theories of the Boltzman equation in statistical mechanics. The theory is also noted for the problem of the existence of singular, "turbulent" or non-unique solutions first noted by Leray (8) in the case of R^3. To establish more direct links among these theories it is useful to emphasize wherever possible a physical interpretation of the analytical properties of solutions, especially of singular solutions whenever they may exist. Here we present two results on boundedness of particle or fluid element motions: the total distance travelled is bounded for all time in regions with a positive lowest Dirichlet eigenvalue. For other regions satisfying a weak cone condition (2), the distance travelled in time T is bounded by $CT^{\frac{1}{4}}(\log T)^{\frac{1}{2}}$. An initial velocity distribution with finite energy is assumed, with zero body forces subsequently. The bounds are valid in the presence of singular solutions, and have application to problems involving strange attractors. The case where certain body forces are present in a periodic rectangular solid has been treated in (4).

2. The initial value problem. Let $\Omega \subset R^3$ be a region in which solutions are defined for the Navier-Stokes equations.

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = - \frac{\partial p}{\partial x_i} + \nu \Delta u_i, \quad i = 1,2,3$$

where $u_i = u_i(x,t)$ and $x = (x_1, x_2, x_3)$. We assume Ω satisfies a cone condition so that certain embedding theorems apply (1,2). The boundary conditions are $u_i = u_i(x,t) = 0$, $x \in \partial\Omega$. Under fairly general conditions, solutions for $0 \leq t < \infty$ have been shown to exist by Leray (7), E. Hopf (5) and Ladyzhenskaya (6). Such solutions are smooth except on a singular set confined to a countable set of time instants ending before a fixed finite time, while its space dimension is at most two (8).

The initial condition is $u_i(x,0) = u_{i0}(x)$, where $||u_0||_2 < \infty$. Here we use the Lebesgue p-norm for 3-vectors, given by

$$||u||_p^p = \int \sum_{i=1}^{3} |u_i(x)|^p \, dV, \quad p > 1.$$

For $p = \infty$ the maximum norm is used.

3. <u>Motions of fluid elements or "particles"</u>. Let $x_i = x_i(x_{j0},t)$ be the position at time t of the fluid element which at time $t = 0$ was situated at $\{x_{j0}\} \in \Omega$. Thus $\frac{dx_i}{dt} = u_i(x_j,t)$, while if s denotes distance moved, we have

$$s = s(x_{j0},t) = \int_0^t |u(x_m(\tau),\tau)|\, d\tau .$$

Hence

$$S = \max_{x_{j0}} s \leq \int_0^t \max|u(\tau)|\, d\tau = \int_0^t ||u(\ ,\tau)||_\infty \, d\tau .$$

To estimate the maximum norm we use an inequality of the type (1,2,4) ess. max $|u| = ||u||_\infty \leq C\,||u||_6^{\frac{1}{2}}\,||\nabla u||_6^{\frac{1}{2}} \leq C_1||\nabla u||_2^{\frac{1}{2}}\,||\Delta u||_2^{\frac{1}{2}}$. Here ∇u denotes the gradient and Δu the Laplacian of the vector function u. Such inequalities hold for vector functions u satisfying the boundary condition, or being in the closure in a suitable space of functions of compact support in Ω.

Then we have

<u>Theorem I</u>. Particle distances s travelled for $0 \leq t \leq T$ satisfy

$$s \leq C T^{\frac{1}{4}} (\log(1+T))^{\frac{1}{2}}, \quad T \to \infty$$

and are essentially uniformly bounded for every finite time interval.

Theorem II. If Ω has positive lowest eigenvalue $\lambda_1 > 0$ all particle distances travelled s are essentially uniformly bounded for all time.

4. Bounds for the derivative norms. We employ the known inequality (4,7,9,10)

$$\frac{d}{dt}\,||\nabla u||_2^2 + \nu||\Delta u||_2^2 \le K||\nabla u||_2^6$$

and let f(t) denote a positive non-increasing function on $0 \le t < \infty$. Then

$$\frac{d}{dt}(f(t)+||\nabla u(t)||_2^2) + \nu||\Delta u||_2^2$$

$$\le \frac{d}{dt}||\nabla u||_2^2 + \nu||\Delta u||_2^2 \le K||\nabla u||_2^6$$

$$\le K||\nabla u||_2^2\,(f(t) + ||\nabla u||_2^2)^2$$

Dividing by the factors containing f(t), we find

$$-\frac{d}{dt}\,(f(t) + ||\nabla u||_2^2)^{-1} + \frac{\nu\,||\Delta u||_2^2}{(f(t) + ||\nabla u||_2^2)^2} \le K||\nabla u||_2^2$$

and upon integration we obtain

$$(f(0) + ||\nabla u(0)||_2^2)^{-1} + \nu\int_0^T \frac{||\Delta u(\tau)||_2^2\,d\tau}{(f(\tau) + ||\nabla u(\tau)||_2^2)^2}$$

$$\le (f(T) + ||\nabla u(T)||_2^2)^{-1} + K\int_0^T ||\nabla u(\tau)||_2^2\,d\tau\ .$$

Set $f(t) = (1+t)^{-1}$ so that

$$\nu\int_0^T \frac{||\Delta u(\tau)||_2^2\,d\tau}{((1+\tau)^{-1} + ||\nabla u(\tau)||_2^2)^2}$$

$$\le \frac{1}{((1+T)^{-1} + ||\nabla u(\tau)||_2^2)} + K\int_0^T ||\nabla u(T)||_2^2\,d\tau$$

$$\le \frac{1+T}{1 + (1+T)||\nabla u(T)||_2^2} + K\int_0^\infty ||\nabla u(T)||_2^2\,d\tau$$

$$\le K_1 + T$$

since the improper integral on the right side is known to converge.

Following (4) we next observe that by Hölder's inequality

$$\int_0^T ||\Delta u||_2^{2/3}\,dt = \int_0^T \left(\frac{1}{1+t} + ||\nabla u(t)||_2^2\right)^{2/3} \frac{||\Delta u(t)||_2^{2/3}\,dt}{\left(\frac{1}{1+t} + ||\nabla u(t)||_2^2\right)^{2/3}}$$

$$\le \left[\int_0^T \left(\frac{1}{1+t} + ||\nabla u(t)||_2^2\right)dt\right]^{2/3} \left[\int_0^T \frac{||\Delta u(t)||_2^2\,dt}{\left(\frac{1}{1+t} + ||\nabla u(t)||_2^2\right)^2}\right]^{1/3}$$

The first factor is less than $(\log(1+T) + K_2)^{2/3}$ while the second factor is already known. Thus we obtain

$$\int_0^T ||\Delta u(t)||_2^{2/3}\,dt \le (\log(1+T) + K_2)^{2/3} (K_1+T)^{1/2}$$

$$\le AT^{1/3}(\log(1+T))^{2/3}$$

for T sufficiently large.

We may now quote the special case $m=1$, $p=q=6$, $n=3$, $\theta=\frac{1}{2}$ of (2,p718, Theorem 4), see also (4), namely

$$||u||_\infty \le K||u||_6^{1/2}\ ||\Delta u||_6^{1/2} \le K_1||\nabla u||_2^{1/2}\ ||\Delta u||_2^{1/2}$$

by Sobolev's inequality. Hence

$$S_{max} \le \int_0^T ||u||_\infty\,dt \le K\int_0^T ||\nabla u||_2^{1/2}\ ||\Delta u||_2^{1/2}\,dt$$

$$\le K\left[\int_0^T ||\nabla u||_2^2\,dt\right]^{1/4} \left[\int_0^T ||\Delta u||_2^{2/3}\,dt\right]^{3/4}$$

$$\le K_3\ T^{\frac{1}{4}}\ (\log(1+T))^{\frac{1}{2}}$$

This completes the proof of Theorem 1.

<u>5. Proof of Theorem 2.</u> The preceding basic estimate with $f(t) \equiv 0$ can be written

$$-\frac{d}{dt}\,||\nabla u(t)||_2^{-2} + \nu\,\frac{||\Delta u(t)||_2^2}{||\nabla u(t)||_2^4} \le K||\nabla u(t)||_2^2$$

Since by (6, p21) we have the Rayleigh - Ritz type estimate (3)

$$0 < \lambda_1 \leq \frac{||\Delta u(t)||_2^2}{||\nabla u(t)||_2^2} ,$$

we may conclude that $w(t) \equiv ||\nabla u(t)||_2^{-2}$ satisfies $\frac{dw(t)}{dt} \geq \nu\lambda_1 w(t) - \frac{K}{w(t)}$.

Since the right side is an increasing function of $w(t) > 0$, it follows that $w(t)$ dominates the solution $y(t)$ of

$$\frac{dy(t)}{dt} = \nu\lambda_1 y(t) - \frac{K}{y(t)}$$

which has the same initial value say at $t = t_0$. With $z(t) = y(t)^2$ we have

$\frac{dz}{dt} = w\nu\lambda_1 z(t) - 2K$ so that

$$z(t) = z(t_0)e^{2\nu\lambda_1(t-t_0)} - \frac{K}{\nu\lambda_1} (e^{2\nu\lambda_1(t-t_0)}-1) .$$

Thus if $z(t_0) > K/\nu\lambda_1$ we have

$$z(t) > (z(t_0) - \frac{K}{\nu\lambda_1}) e^{2\nu\lambda_1 (t-t_0)}, \qquad t > t_0$$

We must therefore show that $z(t_0) = y(t_0)^2 = ||\nabla u(t_0)||_2^{-4}$ can be made arbitrarily large by the choice of t_0. Since however the integral $\int_0^\infty ||\nabla u(t_0)||_2^2 dt$ converges, it follows that $||\nabla u(t_0)||_2^2$ takes arbitrarily small values. We may therefore choose t_0 so that $z(t_0) > \frac{K}{\nu\lambda_1}$.

Then we find $||\nabla u(t)||_2^{-2} = w(t) > y(t) > z^{\frac{1}{2}}(t) > Ae^{\nu\lambda_1(t-t_0)}, \qquad t > t_0$,

so that, finally, $||\nabla u(t)||_2 \leq Be^{-\nu\lambda_1(t-t_0)/2} , \qquad t > t_0$.

While it is probable that the second derivative norm $||\Delta u(t)||_2$ also decreases exponentially, the polynomial bound found earlier for the integral of its 2/3 power is now enough to yield our result, by the following

__Lemma.__ Let $g(t) > 0$, $g(t) \in L(0,T)$ for $T > 0$ and let $G(T) = \int_0^T g(t)dt \leq KT^a$, $a > 0$ Then the integral $\int_0^\infty e^{-\mu t} g(t)\, dt$ is absolutely convergent for $\mu > 0$.

The proof uses only integration by parts and is omitted.

Now

$$\int_{t_0}^{T} ||u(t)||_\infty \, dt \le K \int_{t_0}^{T} ||\nabla u(t)||_2^{1/2} ||\Delta u(t)||_2^{1/2} \, dt$$

$$\le K_3 \int_{t_0}^{T} e^{-\nu\lambda_1 (t-t_0)/2} ||\Delta u(t)||_2^{1/2} \, dt$$

$$\le K_3 \left(\int_{t_0}^{T} e^{-\nu\lambda_1 (t-t_0)} \, dt \right)^{1/4} \left(\int_{t_0}^{T} e^{-\frac{1}{3}\nu\lambda_1 (t-t_0)} ||\Delta u(t)||_2^{2/3} \, dt \right)^{3/4}$$

by Hölder's inequality and the preceding estimate for $||\nabla u(t)||_2$. We can now apply the Lemma with $g(t) = ||\Delta u(t)||_2^{2/3}$ in view of the result in Section 4 above. The absolute convergence as $T \to \infty$ follows, and completes the proof of Theorem II.

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University of Queensland and University of Toronto
Toronto, Ontario, Canada M5S 1A1

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RADAR RECEPTION AND NILPOTENT HARMONIC ANALYSIS I

Walter Schempp

Presented by G.F.D. Duff, F.R.S.C.

One of the fundamental tenets of quantum mechanics which has far reaching consequences is the Heisenberg uncertainty principle. The term "quantum mechanics" stands here for the quantum-mechanical description, at a given instant of time, of a (finite) number of non-relativistic particles moving in the configuration space $\mathbb{R}^n$ and having $\mathbb{R}^n \oplus \mathbb{R}^n$ as their phase space. According to this principle, not all the physical quantities observed in any realizable experiment can be determined simultaneously with an arbitrarily high accuracy. Even under ideal experimental conditions, an increase in the measurement accuracy of one variable can be achieved only at the expense of decreasing the measurement accuracy of the complementary ("conjugate") non-commutating variable.

It is fascinating to see that also in the macro-world there is a similar phenomenon, to wit, the uncertainty principle for radar measurements. Radar observations can be interpreted as an experiment for determining the target range and range rate at a given time. The improvement of the range accuracy results in the worsening of the range rate accuracy, and vice versa. A rigorous statement of the radar uncertainty principle can be formulated by means of the signal autocorrelation function H on $\mathbb{R}^n \oplus \mathbb{R}^n$ for combined time delays and (Doppler) frequency shifts associated with the transmitted and reflected signal waveforms. The autocorrelation function H is more commonly known as the radar autoambiguity function. Its "sharpness" in a given direction is of great importance to the measurement accuracy of the target parameters. - It is the purpose of the present paper and of its forthcoming second part to indicate that the (reduced) Heisenberg nilpotent group $A(\mathbb{R}^n)$ and its linear representation theory which are important ingredients of quantum mechanics govern also the properties of the radar autoambiguity function H. A central rôle in these investigations will be played by the Stone-von Neumann-Segal theorem which states that the Schrödinger representation is up to unitary isomorphy the unique irreducible unitary linear representation U of $A(\mathbb{R}^n)$ with the identity as central character, and that the representation U is strictly square-integrable.

1. The Analog Model

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the compact circle group. The underlying manifold of the reduced Heisenberg group $A(\mathbb{R}^n)$ is given by the product manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$ and the multiplication law on $A(\mathbb{R}^n)$ is defined by

$$(x_1,y_1,\zeta_1).(x_2,y_2,\zeta_2) = (x_1+x_2,y_1+y_2,e^{2\pi i(x_1|y_2)}\zeta_1\zeta_2).$$

The universal covering group $\tilde{A}(\mathbb{R}^n)$ of the connected, two-step nilpotent Lie group $A(\mathbb{R}^n)$ is called the real Heisenberg nilpotent group. The center $\tilde{\mathfrak{Z}}$ of $\tilde{A}(\mathbb{R}^n)$ which lies over the compact center $\mathbb{T}$ of $\tilde{A}(\mathbb{R}^n)$ is one-dimensional over $\mathbb{R}$ and the quotient group $\tilde{A}(\mathbb{R}^n)/\tilde{\mathfrak{Z}}$ is isomorphic to the phase space $\mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^{2n}$. If $\tilde{\mathfrak{Z}}$ is identified to $\mathbb{R}$, then the commutator operation induces a symplectic form B on $\tilde{A}(\mathbb{R}^n)/\tilde{\mathfrak{Z}}$. Using this form, the real Heisenberg nilpotent group $\tilde{A}(\mathbb{R}^n)$ may be realized as $\mathbb{R}^{2n} \oplus \mathbb{R}$ with group multiplication law

$$(v_1,z_1).(v_2,z_2) = (v_1+v_2,z_1+z_2+\tfrac{1}{2}B(v_1,v_2)),$$

where B is given by

$$B(v_1,v_2) = B((x_1,y_1),(x_2,y_2)) = (x_1|y_2)-(y_1|x_2).$$

From this presentation of $\tilde{A}(\mathbb{R}^n)$ it becomes obvious that the symplectic group of the space $\mathbb{R}^{2n}$ defined by

$$Sp(n,\mathbb{R}) = \{g \in GL(2n,\mathbb{R}) \mid B(gv,gw) = B(v,w) \text{ for } (v,w) \in \mathbb{R}^{2n}\times\mathbb{R}^{2n}\}$$

acts on $\tilde{A}(\mathbb{R}^n)$ as a group of automorphisms, preserving $\mathbb{R}^{2n}$ in the direct sum decomposition $\mathbb{R}^{2n} \oplus \mathbb{R}$ and leaving the center $\tilde{\mathfrak{Z}}$ pointwise fixed; cf. Rallis-Schiffmann [1] or the monograph [4]. According to the Stone-von Neumann-Segal theorem there is (up to unitary isomorphy) a unique irreducible faithful unitary linear representation U of the reduced Heisenberg group $A(\mathbb{R}^n)$ with the identity of $\mathbb{T}$ as central character, and this representation is strictly square-integrable. Lifting U to the real Heisenberg group $\tilde{A}(\mathbb{R}^n)$ yields an irreducible unitary linear representation $\tilde{U}$ of $\tilde{A}(\mathbb{R}^n)$ which is square-integrable mod $\mathfrak{Z}$ and admits the Schwartz-Bruhat subspace $\mathcal{S}(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$ as its space of $\mathcal{C}^\infty$ vectors. The realization of $\tilde{U}$ on $\mathcal{S}(\mathbb{R}^n)$ takes the form

$$\tilde{U}(x,y,z)f(t) = e^{2\pi i(z+(t|y)+\frac{1}{2}(x|y))} f(t+x) \qquad (t \in \mathbb{R}^n).$$

In information theory, $(\tilde{U};L^2(\mathbb{R}^n))$ will be called the analog model of the Schrödinger representation of $\tilde{A}(\mathbb{R}^n)$. For the digital model of the Schrödinger representation, see [4] and [5].

2. The Radar Autoambiguity Function

To see the applicabilities of nilpotent harmonic analysis to the mathematical theory of radar reception, consider a transmitted signal waveform s of total input signal energy $\|s\| = 1$. Specifically we shall suppose that s has the form of a pulsed signal

$$s: \mathbb{R}^n \ni t \rightsquigarrow f(t)e^{2\pi i(\omega|t)}$$

admitting a (standardized) $\mathcal{C}^\infty$ envelope f on $\mathbb{R}^n$ that is rapidly decreasing at infinity, i.e., $f \in \mathcal{S}(\mathbb{R}^n)$, $\|f\| = 1$, and a carrier frequency $\omega \in \mathbb{R}^n$. For instance, the pulse with Gaussian envelope is a frequently encountered signal in radar analysis. The signal autocorrelation function for combined target ranges (time delays) and range rates (Doppler frequency shifts) associated with the waveform s is defined on $\mathbb{R}^n \oplus \mathbb{R}^n$ according to the prescription

$$H(f;x,y) = \int_{\mathbb{R}^n} f(t+\tfrac{1}{2}x)f(t-\tfrac{1}{2}x)e^{2\pi i(t|y)}dt.$$

Since in radar analysis $H(f;.,.)$ is an efficient tool to discriminate moving targets, a more commonly known term is radar autoambiguity function (cf. Woodward [6]).

<u>Theorem 1.</u> Let $\tilde{A}(\mathbb{R}^n) \ni (x,y,z) \rightsquigarrow c_{\tilde{U}}(f;x,y,z) \in \mathbb{C}$ denote the

coefficient function of the analog model $(\tilde{U};L^2(\mathbb{R}^n))$ relative to the standardized pulse envelope $f \in \mathcal{S}(\mathbb{R}^n)$. Then the signal autocorrelation function $H(f;.,.)$ satisfies the identity

$$H(f;x,y) = c_{\tilde{U}}(f;x,y,0)$$

for all target ranges $x \in \mathbb{R}^n$ and range rates $y \in \mathbb{R}^n$. In particular, $H(f;.,.)$ is a mod $\mathfrak{Z}$ square-integrable function of positive type on the real Heisenberg nilpotent group $\tilde{A}(\mathbb{R}^n)$.

Notice that the function $H(f;.,.)$ does not need to be of positive type on $\mathbb{R}^n \oplus \mathbb{R}^n$.

<u>Corollary 1.</u> For all standardized pulse envelopes $f \in \mathcal{S}(\mathbb{R}^n)$ the associated signal autocorrelation function $H(f;.,.)$ satisfies the conditions

(i) $$\sup_{(x,y) \in \mathbb{R}^n \oplus \mathbb{R}^n} |H(f;x,y)| = H(f;0,0) = 1,$$

and

(ii) $$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |H(f;x,y)|^2 dxdy = 1.$$

If we observe that the automorphism

$$\tau : (x,y,z) \rightsquigarrow (y,-x,z)$$

of $\tilde{A}(\mathbb{R}^n)$ which leaves the center $\mathfrak{Z}$ pointwise fixed gives rise to the mod $\mathfrak{Z}$ square-integrable irreducible unitary linear representation $\tilde{U}^\tau = \tilde{U} \circ \tau$ of $\tilde{A}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and that the Fourier cotransform $\bar{\mathcal{F}}_{\mathbb{R}^n}$ defines a unitary isomorphism of $\tilde{U}$ onto $\tilde{U}^\tau$, i.e.,

$$\bar{\mathcal{F}}_{\mathbb{R}^n} \circ \tilde{U} = \tilde{U}^\tau \circ \bar{\mathcal{F}}_{\mathbb{R}^n},$$

then Theorem 1 implies the following

<u>Corollary 2.</u> Let $f \in \mathcal{S}(\mathbb{R}^n)$ be given. Then the identity

$$H(f;x,y) = H(\overline{\mathcal{F}}_{\mathbb{R}^n} f;y,-x)$$

holds for all pairs $(x,y) \in \mathbb{R}^n \oplus \mathbb{R}^n$.

It should be noticed that the radar autoambiguity function H has an important counterpart in the statistical theory of quantum mechanics. Indeed, its Fourier transform $\mathcal{F}_{\mathbb{R}^{2n}} H$ is the Wigner phase-space quasi-probability distribution function (cf. Wigner [7]).

3. Invariants of the Wigner-Woodward Relief

In view of the importance of the signal autocorrelation function $H(f;.,.)$ for the discrimination of targets which are moving relative to the radar and separated by target range and range rate, the following problem in radar analysis arises: Given the Wigner-Woodward relief $H(f;\mathbb{R}^n,\mathbb{R}^n)$ of a standardized pulse envelope $f \in \mathcal{S}(\mathbb{R}^n)$, determine its energy preserving linear invariants, i.e., the energy preserving linear mappings u that transform the set $H(f;\mathbb{R}^n,\mathbb{R}^n)$ onto itself. A solution of this problem is given by the following theorem.

Theorem 2. Let the functions $f \in \mathcal{S}(\mathbb{R}^n)$ and $f' \in \mathcal{S}(\mathbb{R}^n)$ satisfy the standardizations $\| f \| = \| f' \| = 1$. There exists for any pair of vectors $(x,y) \in \mathbb{R}^n \oplus \mathbb{R}^n$ a pair $(x',y') \in \mathbb{R}^n \oplus \mathbb{R}^n$ such that

$$H(f;x,y) = H(f';x',y')$$

if and only if there are a complex number $\zeta \in \mathbb{T}$ and a (unique) automorphism $u \in Sp(n,\mathbb{R})$ of the real Heisenberg nilpotent group $\tilde{A}(\mathbb{R}^n)$ such that the identities

$$f = \zeta f', \quad (x,y) = u(x',y')$$

hold.

The proof of the preceding theorem depends upon an extension of the covering map $Mp(n,\mathbb{R}) \longrightarrow Sp(n,\mathbb{R})$. Theorem 2 includes as a special case the uniqueness results established by Stutt [6] and the transformation rule for the radar ambiguity functions established by Reis [2]. For a detailed exposition the reader is referred to [5].

This paper includes also an approach to the Whittaker-Shannon sampling theorem for band-limited signal functions which is based on the digital model for the Schrödinger representation of the real Heisenberg nilpotent group $\tilde{A}(\mathbb{R}^n)$ (cf. [3]).

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Lehrstuhl für Mathematik I,
Universität Siegen,
Hölderlinstraße 3,
D-5900 Siegen 21, W. Germany

SÁNDOR'S THEOREM ON POLYNOMIAL CONGRUENCES AND HENSEL'S LEMMA

J.H.H. Chalk, F.R.S.C. and R.A. Smith

1. Introduction. Let

(1) $$f(X) = a_0(X-\xi_1)^{e_1}(X-\xi_2)^{e_2}\ \dots\ (X-\xi_m)^{e_m} \in Z_p[X],$$

where $a_0 \neq 0$, $\deg f(X) \geq 2$ and ξ_i $(1 \leq i \leq m)$ are the distinct zeros of f in some algebraic closure of the p-adic field Q_p . Then $K = Q_p(\xi_1,\xi_2,\dots,\xi_m)$ is a finite separable extension of Q_p and we denote by ord_p the unique extension of the p-adic valuation on Q_p to K , normalized with

(2) $$\mathrm{ord}_p\, p = 1 \quad .$$

We denote by $\mathcal{O} = \{\alpha \in K : \mathrm{ord}_p \alpha \geq 0\} \supset Z_p$ the extended ring of integers. If

(3) $$V(f,p^\alpha) = \{x \bmod p^\alpha : f(x) \equiv 0 \pmod{p^\alpha}\}$$

and $N(f,p^\alpha) = \mathrm{card}\ V(f,p^\alpha)$, the best known upper bound for $N(f,p^\alpha)$, up to 1980 when Loxton and Smith [2] secured an improvement, was due to Sándor ([3], pp.15-16) who proved that

(4) $$N(f,p^\alpha) \leq m\, p^{\frac{1}{2}\mathrm{ord}_p D} \quad , \quad (D \neq 0 \ , \ \alpha > \mathrm{ord}_p D) \quad ,$$

where D is the discriminant of $f(X)$. Both results are stated for polynomials $f(X)$ defined over Z but are readily adapted to work over Z_p. In [2], the role of the discriminant is replaced by that of a suitably defined global different of $f(X)$. For our purpose, it is convenient to state and prove the results in terms of a local different* $\mathcal{D}(f)$ of $f(X)$; namely $\mathcal{D}(f)$ is a generator of the fractional ideal

(5) $$I = \bigcap_{1\leq i\leq m} f_i\, \mathcal{O} \ ,$$

where f_i is the Taylor coefficient $f^{(e_i)}(\xi_i)/e_i!$. Since $\mathcal{O}$ is a principal ideal domain, we have $I = \mathcal{D}(f)\mathcal{O}$, where $\mathcal{D}(f) \in \mathcal{O}$ (cf. Lemma 1, Corollary),

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* Note that $\mathrm{ord}_p \mathcal{D}(f) \leq \frac{1}{2}\ \mathrm{ord}_p D$ (cf. [2], (4)).

and so

$$(6) \qquad \operatorname{ord}_p \mathfrak{D}(f) = \operatorname{ord}_p \mathfrak{D}(f)\mathcal{O} = \max_{1\le i\le m} \operatorname{ord}_p f_i\mathcal{O} = \max_{1\le i\le m} \operatorname{ord}_p f_i .$$

For conciseness, we shall use the following notation:

$$\delta_i = \operatorname{ord}_p f_i , \quad \delta = \max_{1\le i\le m} \delta_i = \operatorname{ord}_p \mathfrak{D}(f) , \quad e = \max_{1\le i\le m} e_i .$$

Our principal aim is to give a simple direct proof of the following:

THEOREM

$$(7) \qquad N(f,p^{\alpha}) \le m\, p^{\alpha-(\alpha-\delta)/e} .$$

This is the estimate in [2], with δ replaced by the p-adic order of the global different and with their restriction "$\alpha \ge \delta$" deleted. The key idea for the proof is a new version (lemma 1) of Hensel's lemma. If the latter is stated in the form

$$(8) \qquad a \in V(f,p^{\alpha}) , \quad \alpha > 2\operatorname{ord}_p f'(a) \Longrightarrow$$

$$\exists \; i \; (1\le i\le m) \text{ such that } \operatorname{ord}_p(\xi_i - a) \ge \alpha - \operatorname{ord}_p f'(a)$$

then, for the case $e = 1$, the restriction $\alpha > 2\operatorname{ord}_p f'(a)$ is dropped and $f'(a)$ is replaced by $\mathfrak{D}(f)$, in the new version. Moreover, it implies Hensel's lemma since $\operatorname{ord}_p f'(a) \le \operatorname{ord}_p \mathfrak{D}(f)$, (cf. Lemma 2). In the course of this work, we examined Sándor's proof of (4) and found that his argument could be expressed in terms of $\mathfrak{D}(f)$ rather than the discriminant. In fact, if $e_i = 1$ $(1\le i\le m)$ it gave the same bound as in the Theorem, but as it depended upon an application of Hensel's lemma there was a restriction on α of the type in (8). For completeness, we state his result and indicate the modification to his proof in §3. The usual proof of Hensel's lemma utilizes Newton's process of successive approximation and requires some condition on α of the type in (8) (see e.g. [1], pp.72-74, for a weaker restriction on α). These combine to produce an element θ in Z_p with $\operatorname{ord}_p(\theta - a) \ge \alpha - \operatorname{ord}_p f'(a)$, which, by the construction satisfies $f(\theta) = 0$. Thus $\theta = \xi_i$ for some i $(1\le i\le m)$ and it follows that $a \in V(f,p^{\alpha})$, $\alpha > 2\operatorname{ord}_p f'(a)$ implies that $f(X)$ is reducible in $Q_p[X]$, or equivalently, $N(f,p^{\alpha}) = 0$ if $f(X)$

is irreducible over Q_p .

2. LEMMA 1. Let $f(X) = a_0(X-\xi_1)^{e_1}(X-\xi_2)^{e_2} \ldots (X-\xi_m)^{e_m} \in Z_p[X]$ and let $x \in K$. By a suitable permutation of $\xi_1,\ldots,\xi_m$,

$$\mathrm{ord}_p(x-\xi_1) = \max_{1\le i\le m} \mathrm{ord}_p(x-\xi_j) . \tag{9}$$

Then

$$\mathrm{ord}_p f(x) - e_1\, \mathrm{ord}_p(x-\xi_1) \le \mathrm{ord}_p\, f_1 . \tag{10}$$

Proof. We prove firstly that if

$$\mathrm{ord}_p(x-\xi_1) = \mathrm{ord}_p(\xi_1-\xi_j) \quad \text{for some } j \text{ with } 2 \le j \le m \tag{11}$$

then $\mathrm{ord}_p(x-\xi_1) = \mathrm{ord}_p(x-\xi_j)$ and $x-\xi_1 = u_j(\xi_1-\xi_j)$, where $\mathrm{ord}_p u_j = 0$ and

$$\mathrm{ord}_p(u_j+ 1) = 0. \tag{12}$$

This is immediate, since

$$\begin{aligned}\mathrm{ord}_p(x-\xi_j) &= \mathrm{ord}_p[(u_j+1)(\xi_1-\xi_j)]\\ &= \mathrm{ord}_p(u_j+1) + \mathrm{ord}_p(\xi_1-\xi_j)\\ &\le \mathrm{ord}_p(x-\xi_1) = \mathrm{ord}_p(\xi_1-\xi_j) ,\end{aligned}$$

by (9). Now, if (11) holds for some j with $2 \le j \le m$, then

$$\mathrm{ord}_p(x-\xi_j) = \mathrm{ord}_p(u_j+1)(\xi_1-\xi_j) = \mathrm{ord}_p(\xi_1-\xi_j)$$

by (12); but if it is false for that value of j , then

$$\begin{aligned}\mathrm{ord}_p(x-\xi_j) &= \mathrm{ord}_p[(x-\xi_1) + (\xi_1-\xi_j)]\\ &= \min[\mathrm{ord}_p(x-\xi_1),\mathrm{ord}_p(\xi_1-\xi_j)]\\ &\le \mathrm{ord}_p(\xi_1-\xi_j) .\end{aligned}$$

Hence, in either case,

$$\operatorname{ord}_p f(x) - e_1 \operatorname{ord}_p(x-\xi_1) = \sum_{2\le j\le m} e_j \operatorname{ord}_p(x-\xi_j) + \operatorname{ord}_p a_0$$

$$\le \sum_{2\le j\le m} e_j \operatorname{ord}_p(\xi_1-\xi_j) + \operatorname{ord}_p a_0$$

$$= \operatorname{ord}_p f_1 \ .$$

Corollary

(13) $$\operatorname{ord}_p \mathcal{D}(f) \ge 0 \ .$$

For this, it suffices to note that all the coefficients of the polynomial $f(X)/(X-\xi_1)^{e_1}$ belong to $\mathcal{O}$, by Gauss' lemma, and that, consequently,

$$\operatorname{ord}_p f(x)/(x-\xi_1)^{e_1} = \operatorname{ord}_p f(x) - e_1 \operatorname{ord}_p(x-\xi_1) \ge 0$$

for all $x \in \mathcal{O}$. Now pick $x = 1$ say and permute $\xi_1,\ldots,\xi_m$ so that $(1-\xi_1)$ satisfies (9) and the result follows from (10).

To effect the modification to Sándor's proof without disturbing his application of Hensel's lemma it suffices to prove that

$$\operatorname{ord}_p f'(a) \le \operatorname{ord}_p \mathcal{D}(f) \ , \text{ for all } a \in V(f,p^\alpha)$$

in the special case $e = 1$. For $e > 1$, his argument seems to fail for lack of a suitable modification to Hensel's lemma itself.

LEMMA 2. Let $f(X) = a_0(X-\xi_1)(X-\xi_2)\ldots(X-\xi_m) \in Z_p[X]$ and suppose that $f(a) \equiv 0 \pmod{p^\alpha}$, where

(14) $$\alpha > 2 \operatorname{ord}_p \mathcal{D}(f) \ .$$

If $\xi_1,\xi_2,\ldots,\xi_m$ are distinct and permuted so that

$$\operatorname{ord}_p(\xi_1-a) = \max_{1\le j\le m} \operatorname{ord}_p(\xi_j-a) \ , \text{ then}$$

(15) $$0 \le \operatorname{ord}_p f'(a) = \operatorname{ord}_p f'(\xi_1) \le \operatorname{ord}_p \mathcal{D}(f) \ .$$

Proof. On writing $f(X) = (X-\xi_1)h(X)$ say, we have, on differentiating,

(16) $$f'(X) = (X-\xi_1)h'(X) + h(X) \ .$$

Since $f(X) \in Z_p[X] \subset \mathcal{O}[X]$, it follows by a lemma of Gauss on the content of polynomials, that $h(X) \in \mathcal{O}[X]$ and so also $h'(X) \in \mathcal{O}[X]$. Applying Lemma 1, with $x = a$ and $e = 1$, we have firstly

$$\mathrm{ord}_p\, h(a) \le \mathrm{ord}_p\, f'(\xi_1)$$

by definition of $h(X)$ and secondly

$$\begin{aligned} \mathrm{ord}_p[(a-\xi_1)h'(a)] &\ge \mathrm{ord}_p(a-\xi_1) \\ &\ge \mathrm{ord}_p f(a) - \mathrm{ord}_p f'(\xi_1) \\ &> \mathrm{ord}_p f'(\xi_1) \end{aligned}$$

by (14) and (16). Hence

$$\mathrm{ord}_p h(a) = \mathrm{ord}_p f'(a) \le \mathrm{ord}_p f'(\xi_1) ,$$

which gives (15).

3. Sándor's Result. This takes the following form upon introducing the different $\mathfrak{D}(f)$ with $e = 1$:

(17) $$\alpha > \mathrm{ord}_p\, \mathfrak{D}(f) \Longrightarrow N(f,p^\alpha) \le m\, p^{\mathrm{ord}_p \mathfrak{D}(f)} .$$

For, on fixing some $a \in V(f,p^\alpha)$, we may arrange, by a suitable permutation of $\xi_1,\xi_2,\ldots,\xi_m$ (where now $m = \deg f(X)$) that $\mathrm{ord}_p(\xi_1-a) = \max_{1\le j\le m} \mathrm{ord}_p(\xi_j-a)$.

Then, by Lemma 2,

(18) $$\mathrm{ord}_p f'(a) = \mathrm{ord}_p f'(\xi_1) \le \mathrm{ord}_p\, \mathfrak{D}(f) .$$

Since $\mathrm{ord}_p f(a) \ge \alpha > 2\, \mathrm{ord}_p\, \mathfrak{D}(f)$, Hensel's lemma (cf. (8)) assures the existence of an element $\theta \in Z_p$ for which $f(\theta) = 0$ and $\mathrm{ord}_p(\theta-a) \ge \alpha - \mathrm{ord}_p f'(a)$.

If we define the sets

$$A_\alpha(\theta) = \{a \bmod p^\alpha : \mathrm{ord}_p(a-\theta) \ge \alpha - \mathrm{ord}_p f'(a)\}$$

$$B_\alpha(\theta) = \{a \bmod p^\alpha : \mathrm{ord}_p(a-\theta) \ge \alpha - \mathrm{ord}_p\, \mathfrak{D}(f)\} ,$$

then $B_\alpha(\theta) \supset A_\alpha(\theta)$, by (18). Hence

(19) $$\text{card } A_\alpha(\theta) \le \text{card } B_\alpha(\theta) = p^{\text{ord}_p \mathfrak{D}(f)} .$$

As K is a splitting field for $f(X)$, $\theta = \xi_i$ for some i and so

$$V(f,p^\alpha) \subset \bigcup_{1\le i\le m} A_\alpha(\xi_i) \subset \bigcup_{1\le i\le m} B_\alpha(\xi_i) .$$

Hence, by (19)

$$N(f,p^\alpha) \le m\, p^{\text{ord}_p \mathfrak{D}(f)} .$$

4. Proof of the Theorem. Let

$$V_i(f,p^\alpha) = \{x \in V(f,p^\alpha) : \text{ord}_p(x-\xi_i) = \max_{1\le j\le m} \text{ord}_p(x-\xi_j)\}$$

and put $N_i(f,p^\alpha) = \text{card } V_i(f,p^\alpha)$. Then

(20) $$N(f,p^\alpha) \le \sum_{1\le i\le m} N_i(f,p^\alpha) .$$

By Lemma 1 (with 1 replaced by i)

$$x \in V_i(f,p^\alpha) \Longrightarrow \text{ord}_p(x-\xi_i) \ge (\alpha-\delta_i)/e_i .$$

Hence, if we define

$$W_i(\alpha) = \{x \in V(f,p^\alpha) : \text{ord}_p(x-\xi_1) \ge (\alpha-\delta_i)/e_i\}$$

then $V_i(f;p^\alpha) \subset W_i(\alpha)$ and so $N_i(f,p^\alpha) \le \text{card } W_i(\alpha) \le p^{\alpha-(\alpha-\delta_i)/e_i}$.

Then, by (20),

$$N(f,p^\alpha) \le \sum_{1\le i\le m} p^{\alpha-(\alpha-\delta_i)/e_i} \le m\, p^{\alpha-(\alpha-\delta)/e} .$$

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Dept. of Mathematics
University of Toronto
Toronto, Ontario Canada
M5S 1A1

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MAILING ADDRESSES

1.	S.J. Bhatt	Dept. of Mathematics, Sardur Patel University, Vallabh Vidyanagar 388120, Gujarat, India
2.	L.G. Brown	Dept. of Mathematics, Purdue University, West Lafayette, Indiana 47907, U.S.A.
3.	J.H.H. Chalk	Dept. of Mathematics, University of Toronto, Toronto, Ontario,Canada M5S 1A1
4.	G.F.D. Duff	Dept. of Mathematics, University of Toronto, Toronto, Ontario,Canada M5S 1A1
5.	G.A. Elliott	Mathematics Institute, Universitatsparken 5, 2100 Copenhagen, Denmark.
6.	W.H. Greub	Dept. of Mathematics, University of Toronto, Toronto, Ontario,Canada M5S 1A1
7.	J. Malzan	Dept. of Mathematics, University of Toronto, Toronto, Ontario,Canada M5S 1A1
8.	R. Padmanabhan	Dept. of Mathematics, University of Manitoba, Winnipeg, Manitoba,Canada R3T 2N2
9.	P. Penner	Dept. of Mathematics, University of Manitoba, Winnipeg, Manitoba,Canada R3T 2N2
10.	H.R. Petry	Institut fuer theoretische Kernphysik, Nussallee 14, Bonn, W. Germany
11.	W. Schempp	Lehrstuhl für Mathematik I, Universität Siegen, Holderlinstrasze 3, D-5900 Siegen 21, W. Germany
12.	E. Sinestrari	Instituto Matematico "Cantelnuovo", Universita di Roma, 00185, Roma, Italy
13.	R.A. Smith	Dept. of Mathematics, University of Toronto, Toronto, Ontario,Canada M5S 1A1