C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 6

C.R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 6

SCHUR INDICES, SUMS OF SQUARES AND SPLITTING FIELDS

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Presented by G. de B. Robinson, F.R.S.C.

The purpose of this paper is to provide a simple proof of:

Theorem: Let K be a field of characteristic zero such that -1 is a sum of two squares in K. Let χ be a complex irreducible character of a finite group of exponent n. If the Sylow 2-subgroup, $G(K(\varepsilon_n)/K)_2$ of the Galois group of $K(\varepsilon_n)$ over K is cyclic, where ε_n is a primitive n^{th} root of unity, then the Schur index, $m_K(\chi)$, of χ over K is odd.

In [5] Goldschmidt and Isaacs gave a relatively elementary proof of the above result under the restriction that ε_4 is in K rather than that -1 is a sum of two squares in K. The above theorem was conjectured in [5] and proved by Fein in [3]. However Fein's argument relies on some very deep results from algebraic number theory. In this paper we provide a proof using only properties of Schur algebras and we employ the Brauer-Witt theorem which is basic to any study of Schur indices.

Before proving the theorem we provide some comments and a lemma to set the stage. We let $A(\chi,K) = A$ denote the simple component of the group algebra KG corresponding to χ, (see [10, pp. 4-13]). We note that by Benard-Schacher [10, p. 92] if p and q are K-primes above the rational prime q then

$A \otimes_K K_p$ and $A \otimes_K K_{\mathfrak{q}}$ have the same index called the *q-local index* of A, denoted $\text{ind}_q A$, where $K_{\mathfrak{p}}$ and $K_{\mathfrak{q}}$ denote the completion of K at $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Henceforth tensor products shall be assumed to be taken over the center of the algebra in the left factor. For all relevant information pertaining to crossed product algebras the reader is referred to the beautifully written book by Reiner [9]. Furthermore we shall write K_q for $K_{\mathfrak{q}}$ and refer to the decomposition of q in any abelian extension K/Q rather than that of $\mathfrak{q}$ since the decomposition essentially depends on q and not on $\mathfrak{q}$. s(K) will denote the *stufe* of an algebraic number field K, i.e. the minimum number of squares required to represent -1 in K. Finally if m is an integer with $m = p^a t$ where p and t are relatively prime then $|m|_p = p^a$; i.e. $|m|_p$ is the highest power of the prime p dividing m. Equivalence in the Brauer group shall be denoted by $\sim$.

Schur Indices

Lemma 1

Let K be a field of characteristic zero, which does not contain $\sqrt{-1}$. Then $s(K) = 2$ if and only if $\Lambda \otimes K \sim K$ in B(K) where Λ denotes the ordinary quaternion algebra over Q.

Proof. Since $\Lambda = (Q(\sqrt{-1})/Q, -1)$ then: $\Lambda \otimes K = (K(\sqrt{-1})/K, -1)$. Hence $\Lambda \otimes K \sim K$ if and only if $s(K) = 2$ by [9, Th. (30.4), p. 260].

Q.E.D.

Corollary 1

Let K be an algebraic number field with no real infinite primes, and which does not contain $\sqrt{-1}$. Then $s(K) = 2$ if and only if $|K_q : Q_2| > 1$ for all K-primes q dividing 2 .

Proof. By lemma 1 we have that $s(K) = 2$ if and only if $\Lambda \otimes K \sim K$. Since C has no real infinite primes then $inv_q(\Lambda \otimes K) > 0$ if and only if q is a K-prime above 2. However; $inv_q(\Lambda \otimes K) \equiv |K_q : Q_2| inv_2\Lambda \pmod 1$ (see [2]). Thus $\Lambda \otimes K \sim K$ if and only if $|K_q : Q_2|_2 > 1$ for all q above 2.

Q.E.D.

We note that corollary 1 is [4, Th. 1, p. 310] and [1, Theorem, p. 20]. Now we are in a position to prove the main result.

Proof of the theorem

We initiate the theorem as in [3]. Since $m_K(\chi) = m_{K(\chi)}(\chi)$ we may assume that $K(\chi) = K$, and since we have [5] then we may assume ε_4 is not in K . We assume $|m_K(\chi)|_2 > 1$ and obtain a contradiction since we wish to establish that $|m_K(\chi)|_2 = 1$.

Let $E = Q(\varepsilon_n) \cap K$ and let L be the subfield of $Q(\varepsilon_n)$ such that $L \supset E$, $|L : E|_2 = 1$ and $|Q(\varepsilon_n) : L| = |Q(\varepsilon_n) : L|_2$. Since $Q(\varepsilon_n)$ splits χ then $|m_L(\chi)| = |m_L(\chi)|_2$. If $|m_L(\chi)|_2 = 1$ then $|m_{LK}(\chi)|_2 = 1$ which implies $|m_K(\chi)|_2 = 1$, contradicting the hypothesis.

On the other hand, if $|m_L(\chi)|_2 \geq 4$ then ε_4 is in L by [10, Prop. 6.2, p. 89]. Thus ε_4 is in LK. However, $|LK : L|_2 = |K : E|_2 = 1$. Therefore ε_4 is in K, a contradiction. Hence $m_L(\chi) = 2$.

Let $A = A(\chi, L)$. By theBrauer-Witt theorem $A \sim (L(\psi)/L,-1)$ for a suitable character ψ, (see [3]). If $\mathrm{ind}_p A = 2$ for a finite prime p then $p|n$ by [10, Th. 9.1, p. 143]. Therefore we restrict our attention to primes p dividing n. If L is real then since $G(Q(\varepsilon_n)/L)$ is cyclic we have $\mathrm{ind}_p A = 1$ for all $p|n$ by [11]. Therefore we may assume that L is non-real, and so only infinite primes concern us. Suppose first that $p \equiv 3 \pmod 4$. Since 4 divides n by [10, Th. 9.1, p. 143] and $G(Q(\varepsilon_n)/L)$ is cyclic by hypothesis then $L(\varepsilon_4)$ is the unique quadratic extension of L in $L(\psi)$. Therefore $L(\varepsilon_p) = L(\varepsilon_4)$ wnenever ε_p is not in L. Hence the tame

ramification index of p in L/Q is p-1, and so by [10, Th. 4.4, p. 43] we have $\mathrm{ind}_p A = 1$. If $p \equiv 1 \pmod 4$ then ε_4 is in L_p. By [5] $m_{L_p}(\chi) = 1$ and so $\mathrm{ind}_p A = 1$. The

only remaining prime to consider is 2. Thus $\mathrm{ind}_2 A = 2$ since $m_L(\chi) = 2$. Now we show $A \sim A \otimes L$.

By [10, Th. 5.14, p. 88] $Q_2(\varepsilon)/L_2$ must have non-cyclic inertia group for some root of unity ε. Therefore, if $|n|_2 = 2^a$ for $a > 2$ then $L = L(\sqrt{2})$ and $L(\varepsilon_4) = L(\varepsilon_{2^a})$. Thus $Q_2(\varepsilon_n)/L_2(\varepsilon_4)$ is unramified.

Let M be the inertia subfield of $Q(\varepsilon_n)$ over L at 2. Since $Q(\varepsilon_n)/L$ is cyclic then intermediate fields are linearly ordered. Therefore $L(\varepsilon_4) \subseteq M$ or $M \subseteq L(\varepsilon_4)$. But by [10, Th. 5.11, p. 81] $L_2(\varepsilon_4)/L_2$ is ramified, so $M \subseteq L(\varepsilon_4)$ which means $L = M$; i.e. 2 is totally ramified in $Q(\varepsilon_n)/L$. Thus $Q(\varepsilon_n) = L(\varepsilon_4) = L(\varepsilon)$.

Now $A(X, LK) \sim A \otimes LK$ by [6, Th. 3.4, p. 473]. Therefore $A(X, LK) \sim \Lambda \otimes LK$. Yet by lemma 1, $\Lambda \otimes LK \sim LK$, a contradiction.

Q.E.D.

We note that the above proof yields a proof of [7, Th. 5, p.113] and [8, Th. 3], which does not use the deep number theoretic results involved in [8].

The author thanks the referee for comments which helped to improve the readability of the paper.

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Received April 7, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

ON THE OPERATOR EQUATION AX-XB = Q WITH POSSIBLY UNBOUNDED A,B,Q

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ABSTRACT. The existence of a solution X of the equation, $AX\phi - XB\phi = Q\phi \ \forall \ \phi \in \mathcal{V}^B$, is considered where A,B,Q are possibly unbounded operators in a complex Banach space H, and A,B,Q are continuous on their domains $\mathcal{V}^A, \mathcal{V}^B, \mathcal{V}^B$ respectively. $\mathcal{V}^A$ is a Hilbert space algebraically and topologically contained in H. $\mathcal{V}^B$ is a separable pre-Hilbert space spanned by the eigenvectors of B. A,B are linear, Q is anti-linear. A unique continuous anti-linear solution $X_Q: \mathcal{V}^B \to \mathcal{V}^A$ exists if Q behaves like a Hilbert-Schmidt operator, or if Q has a one-dimensional range, under some suitable conditions on A, provided the relaxed coercivity condition is satisfied, which is: for all nonzero X, $\exists \ \phi_X \in \mathcal{V}^B$ such that $|AX\phi_X - XB\phi_X|_H > \beta \, \|X\| \, |\phi_X|_{\mathcal{V}^B}$ for some constant $\beta > 0$. Moreover, the map $Q \mapsto X_Q$ is continuous. Examples are given.

1. INTRODUCTION. We will obtain a set of sufficient conditions for existence and uniqueness of a solution X of the (Lyapunov) equation AX - XB = Q acting on a suitable space, where A,B,Q are possibly unbounded operators in a complex Banach space H. In literature ([1,3,4,5] and references therein) Q has usually been a bounded operator. In as much as we are presenting a different method of approach to the problem, our conditions and hypotheses are different from those found in the references mentioned. We will first formulate our results in a finite-dimensional setting, and then we will find that our sufficient conditions allow for a transition to the infinite-dimensional situation.

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With complexes $\mathbf{C}$ as the field of scalars, let V^A be a Hilbert space, V^B and K normed linear spaces with continuous inclusion injection $V^B \hookrightarrow K$. Let $V^A \subset H$. Let $A \in L(V^A,H)$, $B \in L(V^B,K)$, $B\phi \in V^B \ \forall \ \phi \in V^B$, $Q \in \mathcal{I}(V^B,H)$. Here, for example, $L(V^A,H)$ is the space of all continuous linear maps : $V^A \to H$, and $\mathcal{I}(V^B,H)$ is the space of all continuous anti-linear maps : $V^B \to H$. The norms on spaces H,K,V^A,V^B are denoted by $|\cdot|$ with suffixes H,K,A and B respectively. Let $W = \mathcal{I}(V^B,V^A)$. Our objective in this paper is to present conditions for existence and uniqueness of an $X = X_Q \in W$ such that

$$AX\phi - XB\phi = Q\phi \ \forall \ \phi \in V^B. \tag{1}$$

We define $\forall$ $X \in W$, the anti-linear map U_X: $V^B \to H$ and the linear map U on W by setting $U_X(\phi) = AX\phi - XB\phi \ \forall \ \phi \in V^B$, and $U(X) = U_X$.

2. A SUFFICIENT CONDITION. By the relaxed (one-sided) coercivity condition we will mean

$$\left.\begin{array}{l}\exists \text{ a constant } \beta > 0 \text{ such that } \forall \text{ nonzero } X \in W, \\ \exists \ \phi_X \in V^B \text{ satisfying } |AX\phi_X - XB\phi_X|_H > \beta \, \|X\| \, |\phi_X|_B .\end{array}\right\} \tag{2}$$

Theorem 2.1. Assume (2) and that H,W,V^B are finite-dimensional with $\dim W = (\dim V^B)(\dim H)$. Then Eq (1) has a unique solution $X_Q \in W$. Moreover, $\|X_Q\| \le \frac{1}{\beta}\|Q\|$.

Proof. Uniqueness is a direct consequence of (2). By finite-dimensionality, $U(X) \in \mathcal{I}(V^B,H)$. By (2), and definition of norm of $U(X)$, we have $\|U(X)\| > \beta \, \|X\| \ \forall \ X \neq 0$. By finite-dimensionality again, U is a bijection, and U^{-1} is continuous with $\|U^{-1}\| \le \frac{1}{\beta}$. $X_Q = U^{-1}(Q)$ satisfies the conclusions of this theorem. Q.E.D.

To generalize this theorem to infinite-dimensional cases, we assume that

$$\mathcal{V}^A \text{ is topologically included in } \mathcal{H} \tag{3}$$

so as to make $AX\phi - XB\phi$ a continuous function of X from $\mathcal{W}$ into $\mathcal{H}$; that

$$\left.\begin{array}{l} \mathcal{V}^B \text{ is a separable inner product space, and } \exists \text{ an} \\ \text{orthonormal basis } \mathcal{B} = \{b_i | i \in N\}^{\ddagger} \text{ of } \mathcal{V}^B \text{ such that each} \\ b_i \text{ is an eignvector of } B \text{ belonging to an eigenvalue } \lambda_i. \end{array}\right\} \tag{4}$$

Such an assumption is valid, for example, for certain compact operators B. Let us denote by $\mathcal{V}_n^B$ the subspace of $\mathcal{V}^B$ generated by $b_o, b_1, \ldots, b_n$. Thus, $\dim \mathcal{V}_n^B = n+1$. The subspace of $\mathcal{H}$ generated by $\{Q(b_i) | i = 0, \ldots, n\}$ is denoted by $[Q(b_i)]_{i=0}^n$, and is of dimension $\leq n+1$. We assume that $\forall\ n \in N$

$$\left.\begin{array}{l} \exists\ \mathcal{V}^A(n) \subset \mathcal{V}^A \text{ such that } \dim (\mathcal{V}^A(n)) < \infty, \\ A[\mathcal{V}^A(n)] = [Q(b_i)]_{i=0}^n, \text{ and } \mathcal{V}^A(n) \supset [Q(b_i)]_{i=0}^n.^{\ddagger\ddagger} \end{array}\right\} \tag{5}$$

Let $\mathcal{W}(n) = \{X$ restricted to $\mathcal{V}_n^B \mid X \in \mathcal{W}, X(b_i) = 0\ \forall\ i > n, X(b_i) \in \mathcal{V}^A(n)\ \forall\ i \leq n\} = \overline{\mathcal{L}}(\mathcal{V}_n^B, \mathcal{V}^A(n))$. To every $X_n \in \mathcal{W}(n)$ there corresponds exactly one $X \in \mathcal{W}$ coinciding with X_n on $\mathcal{V}_n^B$ but vanishing on $\{b_i | i > n\}$.

Next, let $\mathcal{H}(n)$ be a subspace of $\mathcal{H}$ such that $Q(b_i) \in \mathcal{H}(n)\ \forall\ i \leq n$, $\forall\ n \in N$, and $\dim \mathcal{H}(n) = \dim \mathcal{V}^A(n)$. Define, $\forall\ n \in N$, $A_n \in \mathcal{L}(\mathcal{V}^A(n), \mathcal{H}(n))$, $B_n \in \mathcal{L}(\mathcal{V}_n^B, \mathcal{K}), Q_n \in \overline{\mathcal{L}}(\mathcal{V}_n^B, \mathcal{H}(n))$ by setting $A_n v = Av$, $B_n u = Bu$ and $Q_n(b_i) = Q(b_i)\ \forall\ i \leq n$. Note that $B_n[\mathcal{V}_n^B] \subset \mathcal{V}_n^B$ (by (4)), and $\|Q_n\| \leq \|Q\|\ \forall\ n$.

The following results now follow from the assumptions (2) - (5).

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‡ For convenience we take N to be the set {0,1,2,3,...}.

‡‡ This condition may trivially hold for certain surjections A.

<u>Lemma 2.2</u>. For all nonzero $Y \in \mathcal{W}(n)$, $\exists$ $\phi_Y \in \mathcal{V}_n^B$ such that $|A_n Y \phi_Y - Y B_n \phi_Y|_{\mathcal{H}} > \frac{\beta}{M} \|Y\| \, |\phi_Y|_B$ for some constant $M \geq 1$ independent of n.

<u>Proof</u>. Let $X \in \mathcal{W}$ be such that $X(u) = Y(u)$ $\forall$ $u \in \mathcal{V}_n^B$, and $X(b_i) = 0$ $\forall$ $i > n$. Let $\phi_X = \sum_{j=0}^{\infty} \alpha_j b_j$, $\alpha_j \in C$, satisfy (2). By Nikol'skiĭ's theorem ([2]), $\exists$ a constant $M \geq 1$ such that $\left|\sum_{j=0}^{n} \alpha_j b_j\right|_B \leq M|\phi_X|_B$. Using (4), we obtain the lemma with $\phi_Y = \sum_{j=0}^{n} \alpha_j b_j$. Q.E.D.

<u>Corollary 2.3</u>. There exists a unique $Y_n \in \mathcal{W}(n)$ such that $A_n Y_n \phi - Y_n B_n \phi = Q_n(\phi)$ $\forall$ $\phi \in \mathcal{V}_n^B$. Moreover, $\|Y_n\| \leq \frac{M}{\beta}\|Q_n\| \leq \frac{M}{\beta}\|Q\|$.

<u>Theorem 2.4</u>. Assume that Q satisfies the regularity hypothesis, $\sum_{i \in N} |Q(b_i)|_{\mathcal{H}}^2 < \infty$[‡]. Then $\exists$ a unique solution $X_Q \in \mathcal{W}$ of Eq (1). Moreover, the map $Q \mapsto X_Q$ is continuous, the space of the Q's having the topology of $\mathcal{L}(\mathcal{V}^B, \mathcal{H})$.

<u>Proof</u>. Uniqueness follows readily from (2). So we go on to prove existence.

$\forall$ $n \in N$, with Y_n given by Corollary 2.3, define $X_n \in \mathcal{W}$ by $X_n(b_i) = Y_n(b_i)$ $\forall$ $i \leq n$, $X_n(b_i) = 0$ $\forall$ $i > n$. If $\phi = \sum_{i=0}^{\infty} \alpha_i b_i \in \mathcal{V}^B$, $\alpha_i \in C$, we have $|X_n \phi|_A = \left|\sum_{i=0}^{n} \bar{\alpha}_i Y_n(b_i)\right|_A \leq \|Y_n\| \left|\sum_{i=0}^{n} \alpha_i b_i\right|_B \leq (M/\beta)\,\|Q_n\| \cdot M\,|\phi|_B$ for some constant $M \geq 1$. Hence,

$$\|X_n\| \leq \frac{M^2}{\beta} \|Q\| \;\forall\; n \in N. \tag{6}$$

Also, $\forall$ $\phi \in \mathcal{V}_n^B$, we have $AX_n\phi - X_n B\phi = Q_n\phi$. In particular, by uniqueness of solution in the finite-dimensional case we have $X_n\phi = X_{n-1}\phi$ $\forall$ $\phi \in \mathcal{V}_{n-1}^B$. So,

$$(X_n - X_{n-1})b_i = 0 \;\forall\; i \neq n > 0, \text{ and } (X_n - X_{n-1})b_n = X_n b_n. \tag{7}$$

Either $X_n b_n = 0$ in which case $|X_n b_n|_A \leq (1/\beta)|Q(b_n)|_{\mathcal{H}}$, or else, $X_n b_n \neq 0$.

‡ Such conditions are satisfied by Hilbert-Schmidt operators.

Then $X_n - X_{n-1} \neq 0$ (by (7)), and so, by (2), $\exists\ \phi_o = \sum_{i=0}^{\infty} \alpha_i b_i \in V^B$ such that $|A(X_n-X_{n-1})\phi_o-(X_n-X_{n-1})B\phi_o|_H > \beta|(X_n-X_{n-1})\phi_o|_A$. Therefore, by (7), $\alpha_n \neq 0$, and $\beta|X_nb_n|_A < |A(X_n-X_{n-1})b_n-(X_n-X_{n-1})Bb_n|_H = |AX_nb_n-X_nBb_n|_H = |Q(b_n)|_H$. Clearly, $\beta|X_ob_o|_A < |AX_ob_o-X_oBb_o| = |Q(b_o)|_H$. Hence $\forall\ n \in \mathbb{N}$ we have,

$$|X_nb_i|_A < \frac{1}{\beta}\ |Q(b_i)|_H \quad \forall \quad i \leq n. \tag{8}$$

Let $\mathcal{X}$ be the set of all finite linear combinations of the X_n's. $\mathcal{X}$ is a sub-space of $\mathcal{W}$, and can be turned into a pre-Hilbert space by defining the inner product $(((.,.)))$ by $(((Y,Z))) = \sum_{i=0}^{\infty} (Y(b_i), Z(b_i))_A$. If $Y = \sum_{j=0}^{n} \beta_jX_j \in \mathcal{X}, \beta_j \in \mathbb{C}$, then the norm in $\mathcal{X}$ is given by $|||Y||| = \left[\sum_{i=0}^{n} \left(|\sum_{j=i}^{n} \beta_j|^2\ |X_nb_i|_A^2\right)\right]^{\frac{1}{2}}$. Moreover, $||Y|| \leq |||Y|||\ \ \forall\ \ Y \in \mathcal{X}$ where $||Y||$ is the norm of Y in $\mathcal{W}$; indeed, $\forall\ \phi = \sum_{j=0}^{\infty}\gamma_jb_j \in V^B$, $\gamma_j \in \mathbb{C}$, straightforward calculation shows that $|Y\phi|_A^2 \leq |||Y|||^2|\phi|_B^2$. Therefore, the closure $\overline{\mathcal{X}}$ of $\mathcal{X}$ in $|||\cdot|||$-topology is a Hilbert space, and is a subset of $\mathcal{W}$.

From (8), $\forall\ n \in \mathbb{N}$, $|||X_n|||^2 < \frac{1}{\beta^2}\sum_{i=0}^{\infty} |Q(b_i)|_H^2$. Hence, a subsequence of $\{X_n\}_{n\in\mathbb{N}}$, denoted again by $\{X_n\}_{n\in\mathbb{N}}$, converges weakly in $\overline{\mathcal{X}}$, and therefore, weakly in $\mathcal{W}$, to some $X_Q \in \mathcal{W}$ (actually, $X_Q \in \overline{\mathcal{X}}$). By (6), $||X_Q|| \leq \frac{M^2}{\beta}\ ||Q||$ so that the map $Q \mapsto X_Q$ is continuous. Also, we have $AX_n\phi - X_nB\phi$ converging weakly in H to $AX_Q\phi - X_QB\phi\ \ \forall\ \phi \in V^B$. On the other hand, if $\phi = \sum_{j=0}^{\infty}\gamma_jb_j \in V^B$, $\gamma_j \in \mathbb{C}$, then $AX_n\phi - X_nB\phi =$

$A_nY_n\left(\sum_{j=0}^{n}\gamma_jb_j\right) - Y_nB_n\left(\sum_{j=0}^{n}\gamma_jb_j\right) =$ (by Corollary 2.3) $Q_n\left(\sum_{j=0}^{n} \gamma_jb_j\right) =$ $Q\left(\sum_{j=0}^{n}\gamma_jb_j\right) \to Q\phi$ in H as $n \to \infty$. Q.E.D.

<u>Remark 2.5</u>. The hypothesis, $\sum_{i\in\mathbb{N}} |Q(b_i)|_H^2 < \infty$, in the statement of Theorem 2.4 may be replaced by the hypothesis, "Q has one-dimensional

range", and we still get the existence of a unique solution of Eq (1).

3. AN EXAMPLE. Let $\mathcal{H} = L^2([0,2\pi]; \mathbb{C})$ with orthonormal basis $\{a_i\}_{i=0}^{\infty}$ where $a_o(t) = (2\pi)^{-\frac{1}{2}}$, $a_{2n-1}(t) = \pi^{-\frac{1}{2}} \sin nt$, $a_{2n}(t) = \pi^{-\frac{1}{2}} \cos nt$. Let $\mathcal{V}^A = \mathcal{H}$, I = the identity operator : $\mathcal{V}^A \to \mathcal{H}$, $A = cI$ where c is a constant so that $c + n^2 \neq 0$ $\forall$ $n \in \mathbb{N}$. Define

$$b_p = (1 + i^2 + i^4 + i^6)^{-\frac{1}{2}} a_p \text{ for } p = 2i \text{ or } 2i-1 \quad \forall \ i \in \mathbb{N}.$$

Let $\mathcal{V}^B$ be the space of all finite linear combinations of the b_i's over $\mathbb{C}$, with the inner product structure of the Sobolev space $\mathcal{H}^3$. The sequence $\{b_i\}_{i=0}^{\infty}$ forms an orthonormal basis for $\mathcal{V}^B$. Let $B : \mathcal{V}^B \to \mathcal{K}$ be defined by $B\phi = \phi''$ (= the second derivative of ϕ), $\mathcal{K}$ being the Sobolev space $\mathcal{H}^1$. Theorem 2.4 allows us to deduce that the equation, $cX\phi - X\phi'' = Q\phi \ \forall \ \phi \in \mathcal{V}^B$, has a unique solution $X \in \mathcal{L}(\mathcal{V}^B, \mathcal{V}^A)$, wherein for $Q : \mathcal{V}^B \to \mathcal{H}$ we may take the map defined by $(Qg)(\omega_2) = \int_0^{2\pi} K(\omega_1,\omega_2) g'''(\omega_1) d\omega_1$, $K \in L^2([0,2\pi] \times [0,2\pi]; \mathbb{C})$ being a suitable kernel. All the hypotheses in the statement of Theorem 2.4 can be shown to be valid. Note that B and Q are unbounded operators in $\mathcal{H}$.

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Received April 14, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

FLOR TYPE REPRESENTATIONS FOR MATRICES SATISFYING $A^s = A^t$

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Presented by P. Ribenboim, F.R.S.C.

In [3] (also [2]), Flor generalized Doob's description of stochastic idempotent matrices to the nonnegative idempotent matrices. Flor's characterization has become well known and is a useful too when working with nonnegative matrices. Banerjee and Nagase [1] and Styan [5] have independently established that A is idempotent if and only if $A^2 = A^3$ and rank A = tr A. Khatri [4] extended that result to show that A is idempotent whenever $A^s = A^t$, for different positive integers s and t, and rank A = tr A. In this study we have derived Flor type representations for nonnegative matrices that satisfy $A^s = A^t$ for different positive integers s and t.

In this study matrices that satisfy $A^s = A^t$, where $s < t$, are called (s,t)-potent. Notice that a (1,2)-potent matrix is simply an idempotent matrix. The index, k, of A is the smallest positive integer for which rank A^{k+1} = rank A^k. Clearly, $s \geq k$ whenever A is (s,t)-potent. In fact whenever t is minimal,i.e., t is the smallest positive integer for which $A^s = A^t$, it can be shown, using the Jordan canonical representation for A, that A is also (u,v)-potent if and only if $t-s|v-u$ and $v > u \geq$ index A. Moreover, $v = u + j(t - s)$ where, j is a positive integer. Also working with the Jordan canonical representation for A it can be established that A is (s,s + n)-potent if and only if tr A^n = rank A^k (The condition tr A^n = rank A^k forces all the positive eigenvalues of A to satisfy $|\lambda^n| = 1$). Moreover, the smallest pair (s,t) of positive integers for which $A^s = A^t$ is (k, k + n). Actually, when tr A^{m-1} = rank A, where $m \geq 2$, then A is m-potent (i.e.,(1,m)-potent).

Let [m,n] denote the least common multiple of m and n. If A = diag

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AMS 1970 subject classifications. Primary 15A21, 15A48

$[A_1, \ldots, A_d]$, where for each i, A_i is an (s,t_i)-potent matrix with t_i minimal, then a straightforward argument shows that A is an (s,t)-potent matrix with t minimal if and only if $t - s = [t_1 - s, \ldots, t_d - s]$.

The nonnegative idempotent matrices, which Flor characterized, necessarily had an index of 1. Hence, Flor's representation was for the matrix A^k as well as the matrix A, where k = index A = 1. As noted earlier, an (s,t)-potent matrix does not necessarily have an index of 1. Moreover, an (s,t)-potent matrix A can be regarded as a (k, k+n)-potent matrix, where k = index A and n is the smallest positive integer for which rank A^k = tr A^n. Our ultimate goal in this paper is to provide related Flor type representations for both A and A^k.

Group the row indices of A into four sets according to whether the ith row and column are both nonzero, the ith row is zero and the ith column is nonzero, the ith row is nonzero and the ith column is zero, and both the ith row and column are zero. Then A is cogredient to

$$\begin{bmatrix} B & C & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D & E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

<u>Theorem</u> Let A be a nonnegative (k,k + n)-potent matrix such that B does not have any zero rows or columns. Then there exist permutation matrices Q and P such that

$$Q^T A Q = \begin{bmatrix} J & Y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X & Z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

and

$$P^TA^kP = \begin{bmatrix} J^k & J^{k+n-1}Y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ XJ^{k+n-1} & XJ^{k+n-2}Y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $J = \text{diag}\,[J_1, \ldots, J_d]$, each J_i is a nonnegative irreducible $(k,k+n_i)$-potent matrix with n_i minimal, $n = [n_1, \ldots, n_d]$, and X, Y, and Z are arbitrary nonnegative matrices of the appropriate size. Moreover, each J_i^k is cogredient to a matrix having one of the following forms:

(1) $\alpha\beta^T$, where α and β are positive column vectors and $\beta^T\alpha = 1$,

(2)

$$\begin{bmatrix} 0 & \alpha_1\beta_1^T & 0 & \ldots & 0 \\ 0 & 0 & \alpha_2\beta_2^T & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & \alpha_{m-1}\beta_{m-1}^T \\ \alpha_m\beta_m^T & 0 & 0 & \ldots & 0 \end{bmatrix},$$

where the α_i and β_i are positive column vectors for which $(\beta_1^T\alpha_2)(\beta_2^T\alpha_3)\ldots(\beta_m^T\alpha_1) = 1$,

(3) $\text{diag}\,[M_1, \ldots, M_d]$, where each matrix M_i has the form found in (1) or (2).

Flor's representation for nonnegative idempotents is obtained as a corollary when $n = k = 1$. Also, when n and k are relatively prime a simpler representation is possible.

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Received June 12, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

PRODUCTS EP_r PARTITIONED MATRICES

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Presented by H. Schwerdtfeger, F.R.S.C.

Abstract: This paper gives necessary and sufficient conditions for product of EP_r partitioned matrices to be EP_r. All matrices considered here are complex matrices. For A, let A^* denote the conjugate tranpose of A. Let A^+ denote the Moore-Penrose inverse of A [4]. A is called EP if $N(A) = N(A^*)$ or equivalently $R(A) = R(A^*)$ where $N(A)$ and $R(A)$ are the null space and range space of A respectively, [6]. A is EP_r if A is EP and $rk(A) = r$, where $rk(A)$ denotes the rank of A.

1. Introduction:

[1] contains answers to the following questions: (1) If A and B are EP_r matrices when is AB an EP_r matrix? (2) When is a matrix of rank r a product of EP_r matrices? Here we have extended these results to partitioned matrices. We give necessary and sufficient conditions for the product of two EP_r partitioned matrices to be EP_r and for the reverse order law to hold for the generalized inverse of the product of matrices. We have also given general conditions under which a matrix can be expressed as product of EP_r matrices, which includes as a special case the results found in [1].

Throughout this paper, we are concerned with $n \times n$ matrices M partitioned in the form

(1) $$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ with } rk(M) = rk(A) = r,$$

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where A is $k \times k$ and D is $(n-k)\times(n-k)$. By corollary in [2] it follows that M of the form (1) satisfies the following:

(2) $$N(A) \subseteq N(C), \cdot N(A^*) \subseteq N(B^*) \text{ and } D = CA^+B.$$

In the sequel, we make use of the following result obtained in [3] regarding a partitioned matrix to be EP.

Result 1: Let M be a partitioned matrix of the form (1). Then M is an EP_r matrix if and only if A is EP_r and $CA^+ = (A^+B)^*$

Lemma 1: If M is an EP_r matrix of the form (1), then there exists a $k \times (n-k)$ matrix X such that

(3) $$M = \begin{bmatrix} A & AX \\ X^*A & X^*AX \end{bmatrix} \text{ and } A \text{ is } EP_r.$$

Proof: Since M is of the form (1) it satisfies (2). Hence there is an $(n-k) \times k$ matrix Y and a $k \times (n-k)$ matrix X such that $C = YA$ and $B = AX$ (p.21, [5]). Since M is EP_r, by Result 1, A is EP_r (equivalently $AA^+ = A^+A$) and $CA^+ = (A^+B)^*$. Therefore $YAA^+ = (A^+AX)^* = X^*A^+A = X^*AA^+$; from which it follows that $YA = X^*A$. Thus $D = CA^+B = (YA)A^+(AX) = X^*AX$. Hence M is of the form (3).

2. Product of EP_r Matrices:

We extend the result found in [1] in the following Theorem.

Theorem 1: Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $L = \begin{bmatrix} F & G \\ H & K \end{bmatrix}$ be EP_r matrices both of the form (1) and ML be of rank r. Then the following are equivalent:

(i) ML is EP_r

(ii) AF is EP_r and $CA^+ = HF^+$

(iii) AF is EP_r and $A^+B = F^+G$

Proof: Since M and L are of the form (1), by Lemma 1 there exist $k \times (n-k)$ matrices X and Y such that

$$M = \begin{bmatrix} A & AX \\ X^*A & X^*AX \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} F & FY \\ Y^*F & Y^*FY \end{bmatrix} .$$

Now,

$$ML = \begin{bmatrix} AZF & AZFY \\ X^*AZF & X^*AZFY \end{bmatrix}, \quad \text{where} \quad Z = I + XY^* .$$

Clearly, $N(AZF) \subseteq N(X^*AZF)$; $N(AZF)^* \subseteq N(AZFY)^*$ and the Schur complement of AZF in ML is zero, hence by Theorem 1 in [2], $rk(AZF) = rk(ML) = r$. Thus ML is also of the form (1). By Result 1, A and F are both EP_r and

(4) $$CA^+ = (A^+B)^* \quad \text{and} \quad HF^+ = (F^+G)^* .$$

$R(AZF) \subseteq R(A)$; $R(AZF)^* \subseteq R(F^*) = R(F)$ and $rk(AZF) = rk(A) = rk(F) = r$.

Hence,

(5) $$R(AZF) = R(A); \quad R(AZF)^* = R(F) .$$

This implies,

(6) $$(AZF)(AZF)^+ = AA^+; \quad (AZF)^+(AZF) = FF^+ .$$

Now the proof runs as follows:

ML is $EP_r \Leftrightarrow AZF$ is EP_r and $X^*(AZF)(AZF)^+ = [(AZF)^+(AZF)\ Y]^*$

(By Result 1).

$\Leftrightarrow R(AZF) = R(AZF)^*$ and $X^*AA^+ = Y^*FF^+$ (By 6).

$R(A) = R(F)$ and $CA^+ = HF^+$ (By (5) and from the forms of M and L).

$\Leftrightarrow AF$ is EP_r and $CA^+ = HF^+$ (By Theorem 3 in [1]).

$\Leftrightarrow AF$ is EP_r and $A^+B = F^+G$ (By (4)).

Hence the Theorem

Theorem 2. Let M and L be EP_r matrices as stated in Theorem 1, and $rk(ML) = r$. Then $(ML)^+ = L^+M^+ \Leftrightarrow (AF)^+ = F^+A^+$ and $CA^+ = HF^+ \Leftrightarrow (AF)^+ = F^+A^+$ and $A^+B = F^+G$.

Proof. Since $rk(ML) = r$, using Theorems 4 and 5 in [1] and Theorem 1, we have,

$$(ML)^+ = L^+M^+ \Leftrightarrow ML \text{ is } EP_r$$
$$\Leftrightarrow AF \text{ is } EP_r \text{ and } CA^+ = HF^+$$
$$\Leftrightarrow (AF)^+ = F^+A^+ \text{ and } CA^+ = HF^+$$
$$\Leftrightarrow (AF)^+ = F^+A^+ \text{ and } A^+B = F^+G.$$

This proves the Theorem.

Remark 1: IF A and F are nonsingular, then Theorem 2 reduces to $(ML)^+ = L^+M^+ \Leftrightarrow CA^{-1} = HF^{-1} \Leftrightarrow A^{-1}B = F^{-1}G$. Theorem 2 fails if we relax the condition on the rank of ML. (refer to the example under Theorem 5 in [1].)

3. Factorization:

We give conditions under which a matrix can be expressed as product of EP_r matrices.

Lemma 2: Let M be a matrix of the form (1) and A be EP_r. Then M is a product of EP_r matrices.

Proof: Since M is of the form (1), it satisfies (2), hence there exists X and Y such that $C = YA$, $B = AX$ and $D = CA^+B = YAX$. Consider the matrices.

$$S = \begin{bmatrix} AA^+ & AA^+Y^* \\ YAA^+ & YAA^+Y^* \end{bmatrix}, \quad L = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} A^+A & A^+AX \\ X^*A^+A & X^*A^+AX \end{bmatrix}.$$

By Result 1, S, L and T are EP_r. Also $M = SLT$. Thus M is expressed as a product of EP_r matrices.

Theorem 3. If M is of rank r and has a principal submatrix that is EP_r, then M is a product of EP_r matrices.

Proof: Let A be a $k \times k$ principal submatrix of M and A be EP_r. Then there exists a permutation matrix P such that $\hat{M} = PMP^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A is EP_r. By Lemma 2, $\hat{M}$ is a product of EP_r matrices; and by Lemma 3, in [1], M is also a product of EP_r matrices. Hence the Theorem.

Corollary 1: If M is a P_r matrix, then M is a product of EP_r matrices. (Theorem 7 in [1].)

Proof: This is immediate from Theorem 3 and the definition of a P_r matrix.

Acknowledgement: The author wishes to thank Pl. Kannappan for his help in preparation of the manuscript and suggestions.

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Received October 8, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

NON-ISOMORPHIC C*-ALGEBRAS WITH ISOMORPHIC n by n MATRIX RINGS

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Abstract:

We construct a family $B(n,k)$ of non-isomorphic C*-algebras indexed on pairs of integers k,n with $0 \le k < n$. Their matrix algebras, up to isomorphism, are given by the formula $M_m(B(n,k)) = B(n,km)$, the product km being reduced mod n. In particular, the algebra $A(n) = B(n,0)$ is isomorphic to all matrix rings over it, and, at the same time, can be expressed as the n by n matrices over n non-isomorphic C*-algebras.

$B(n,k)$ has a large supply of projections, but it is not clear whether or not they form a lattice. There does, however, exist a subset $P(n,k)$ uniformly dense in the projectionsof $B(n,k)$ and such that $P(n,k)$ is an irreducible atomic lattice generating $B(n,k)$. $B(n,k)$ is not separable: in fact $B(n,k)/K$ is the Calkin algebra. We are able to explicitly calculate the equivalence classes of projections under unitary equivalence, labelling them by the dimension and codimension functions (derived from the atoms) and, in addition, a function r onto Z_n. In the language of K-theory, $K_o(B(n,k))$ can be identified with Z_n, and if the identification is specified by $[e] = 1$ for atomic projections e, then $[1] = k$.

General remarks: The construction closely resembles that of [1] or [2, p. 65]. Extra difficulties are presented by the projections of $B(n,k)$ probably not forming a lattice, by the $P(n,k)$ not being canonically determined by the $B(n,k)$, and finally by the $P(n,k)$ failing to be modular. We are, nevertheless, able to treat the set of projections geometrically by introducing the notion of "special perspectivity", written below $\overset{s.p.}{\sim}$. The projections of the B(n,k) are actually non-isomorphic as partially ordered sets, but we omit the proof of this.

The construction:

Let H be a Hilbert space of countably infinite dimension. Fix a subspace H' of H of infinite dimension and codimension and a vector ψ' independent of H' . Let H_n be a Hilbert space of dimension n with basis $\phi_1, \phi_2, \ldots, \phi_n$. Denote by K the compact operators on $H \otimes H_n$, and define, on $H \otimes H_n$:

$$A(n) = K + \mathcal{L}(H) \otimes 1$$

$$e(n,k) = pr[\psi'] \otimes pr[\phi_1, \phi_2, \ldots, \phi_k] + pr(H') \otimes 1$$

$$B(n,k) = e(n,k)A(n)e(n,k) .$$

Outline of the results:

THEOREM 1. A(n) is uniformly closed, and is therefore a C*-algebra.

DEFINITION 1. Let P(n) be the set of projections e in A(n) which can be written in the form $e = p \otimes 1 + e_1$, where p is a projection in $\mathcal{L}(H)$ and e_1 is a finite dimensional projection in $\mathcal{L}(H \otimes H_n)$. For e in P(n) define r(e) in Z_n to be the dimension of e_1 mod n .

THEOREM 2. Every projection in A(n) can be uniformly approximated by projections in P(n) . If e and f are in P(n) and if $\|ef\| < 1$, then

$$r(e \cup f) = r(e) + r(f) .$$

. . .

DEFINITION 2. Suppose that e and f are projections in $A(n)$:

1. $e \sim f$ means there exists a unitary u such that $e = ufu^*$

2. A projection g in $A(n)$ will be called a <u>special complement</u> for e if $\|eg\| < 1$ and $\|(1-e)(1-g)\| < 1$.

3. $e \overset{s.p.}{\sim} f$ means that there exists a projection g in $A(n)$ which is a special complement for both e and f .

LEMMA. If e and f are projections in $A(n)$, $\|e-f\| < 1$ implies both $e \sim f$ [4, p.206] and $e \overset{s.p.}{\sim} f$ with common special complement $1-e$.

COROLLARY. If e and f are in $P(n)$ then $\|e-f\| < 1$ implies $r(e) = r(f)$.

DEFINITION 3. For a projection e in $A(n)$ define $r(e) = r(f)$ for f in $P(n)$ with $\|e-f\| < \frac{1}{2}$.

THEOREM 3. Suppose that e and f are projections in $A(n)$:

1. If e is a special complement for f then $r(e) + r(f) = 0$
2. If $e \overset{s.p}{\sim} f$ then $r(e) = r(f)$.
3. If $ef = 0$ then $r(e+f) = r(e) + r(f)$.

THEOREM 4. If e is a projection in $A(n)$ of infinite dimension and codimension and with $k = r(e)$, then $e \sim e(n,k)$.

THEOREM 5. Let e be a projection in $A(n)$ of infinite dimension and codimension. Then $r(e) = 0$ if and only if there exist projections $e_1, e_2, \ldots, e_n$ in $A(n)$ such that $e_i \overset{s.p.}{\sim} e_j$ for all i,j , and $e = \sum_{i=1}^{n} e_i$.

THEOREM 6. If ϕ is an automorphism of $A(n)$ then $r(\phi(e)) = r(e)$ for all projections e of $A(n)$.

THEOREM 7. Suppose that e is a projection of $A(n)$ of infinite dimension and codimension. Then the following are equivalent:

1. $r(e) = k$
2. $e \sim e(n,k)$
3. $eA(n)e$ is isomorphic to $B(n,k)$.

Notes on proofs:

Theorem 1 can be demonstrated by taking π to be a contraction (see [5], for example) of $L(H \otimes H_n)$ onto $L(H) \otimes 1$: it suffices to take $\pi(x) = \int uxu^{-1}du$ with integration over the unitary group of $1 \otimes L(H_n)$. Then x is in $A(n)$ exactly when $\pi(x) - x$ is compact.

To prove the first statement of theorem 2 we use several facts:

1. If $x^* = x$ and $x^2 - x$ is compact, then $x = e + k$ for a projection e and a compact k (see [3], for example).

2. If $e - f$ is compact and $\varepsilon > 0$, then there exist finite projections $e_1 \le e$ and $f_1 \le f$ such that $\|(e-e_1)-(f-f_1)\| < \varepsilon$.

3. f finite-dimensional in $L(H \otimes H_n)$ implies $f \le \bar{f} \otimes 1$ with $\bar{f}$ finite-dimensional in $L(H)$.

The second statement of theorem 2 is an easy consequence of the fact that, for projections e and f onto subspaces M and N , $\|ef\| < 1$ implies that $M + N$ is closed and $M \cap N = \{0\}$.

To prove theorem 3 we note that, if e is a special complement for f , then choosing e_1 and f_1 close enough to e and f and in $P(n)$, we have e_1 and f_1 are special complements. Hence theorem 2 shows $r(e_1) + r(e_2) = 0$. 2. follows immediately. For 3. direct calculation shows that if $e_n \to e$ and $f_n \to f$ then $e_n \cup f_n \to e+f$.

In proving theorem 4, because of the lemma, we may assume e is in $P(n)$. Then we write $e = p \otimes 1 + e_1$ with $\dim e_1 = \ell n + k$ and $e_1 \le \overline{e_1} \otimes 1$ with $\overline{e_1}$ finite-dimensional in $L(H)$. Then on $(\overline{e_1} \otimes 1)H$ there is a unitary u_1 such that $u_1 e_1 u_1^*$ is of the form $p_1 \otimes 1 + pr[\psi] \otimes pr[\phi_1, \phi_2, \ldots, \phi_k]$. Then evidently $u_1 + (1-e_1)\otimes 1$ $= u_2$ is a unitary in $A(n)$, and letting $u = (v \otimes 1)u_2$ for a suitable unitary in $L(H)$ we have $ueu^* = e(n,k)$.

To prove theorem 5, suppose first that $r(e) = 0$. Then by theorem 4 there exists a unitary u in $A(n)$ such that $ueu^* =$ $e(n,0)$. Directly from the definition we see that $e(n,0) =$ $\sum_{i=1}^{n} p_i' \otimes 1$ where such p_i' has infinite dimension and codimension. It is easy to confirm that $p_i' \overset{s.p.}{\sim} p_j'$ in $L(H)$. Setting $p_i = u^*(p_i' \otimes 1)u$, we easily confirm that $p_i \overset{s.p.}{\sim} p_j$ and $e = \sum_{i=1}^{n} p_i$. This converse part of theorem 5 follows from theorem 3.

In proving theorem 6, we note first that $r(\phi(e)) = 0$ if and only if $r(e) = 0$. This follows from theorem 5 for e of infinite dimension and codimension. If e is finite dimensional $r(e) =$ $\dim e \pmod n$, which is preserved by ϕ because ϕ takes atoms to atoms. If e has finite codimension $r(e) = -r(1-e)$, by theorem 3. Thus in all cases, $r(\phi(e)) = 0$ if and only if $r(e) = 0$. Now every e can be written as $e_1 + a_1 + \ldots + a_k$ where the a_i are atoms, $r(e_1) = 0$ and $r(e) = k$. Then $r(\theta(e_1)) = 0$ so $r(\phi(e)) = k$.

In theorem 7, $1 \Rightarrow 2$ is theorem 4, and $2 \Rightarrow 3$ is evident. Suppose than that $eA(n)e$ is isomorphic to $B(n,k)$. By theorem 4, $eA(n)e$ is isomorphic to $B(n,k')$ where $k' = r(e)$. We can see directly that both $e(n,k)$ and $e(n,k')$ can be extended to systems of n by n matrix units for $A(n)$. Hence the isomorphism of $B(n,k)$ onto $B(n,k')$ extends to an automorphism ϕ of $A(n)$ such that $\phi(e(n,k)) = \phi(e(n,k'))$. Then theorem 6 shows that $k = k' = r(e)$.

The matrix algebras of the $B(n,k)$ can now be computed directly within $A(n)$. n is determined by the algebra $B(n,k)$ as the number of equivalence classes of projections with infinite dimension and codimension.

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Received September 10, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

A SYSTEMATIC NOTATION FOR THE COXETER GRAPH

H.S.M. Coxeter, F.R.S.C.

To Heinrich Heesch for his 75th birthday

Consider a trivalent graph whose 28 vertices are denoted by the $\binom{8}{2}$ unordered pairs of 8 symbols consisting of ∞ and the residues

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6 \qquad\qquad (\text{mod } 7),$$

while its 42 edges are determined by the following two rules:

For every arithmetic progression α, β, γ (mod 7), $\alpha\gamma$ *is joined to* $\beta\infty$, *and for every arithmetic progression* $\alpha, \beta, \gamma, \delta$ (mod 7), $\alpha\beta$ *is joined to* $\gamma\delta$.

These incidences are evidently preserved when we add 1 (or any other residue) to all the residues involved (including $6 + 1 = 0$ and $\infty + 1 = \infty$) and when we multiply them by any non-zero residue (including -1, which is 6). By an apparent miracle, the incidences are preserved also by reciprocation (replacing $\infty, 0, 1, 2, 3, 4, 5, 6$ by $0, \infty, 1, 4, 5, 2, 3, 6$). For instance, the vertex 01 is joined to 23, 56, 4∞, and the 'reciprocal' vertex 1∞ is joined to 45, 36, 02. We see in this manner that the automorphism group of the graph is (or at least includes) PGL(2, 7), the group of linear fractional transformations of determinant ± 1 in the field GF[7], of order 336.

By drawing the graph as in Figure 1 or 2, with three heptagons joined to seven 'extra' vertices $\beta\infty$, we recognize it to be the one which Tutte [4] ascribed to me. Tutte proved that it is, like the Peterson graph [2, p. 14], both non-Hamiltonian and 3-regular. Non-Hamiltonian means that there is no closed 28-route visiting all the vertices; 3-regular means that the automorphism group is transitive on the open 3-routes (such as AB, BC, CD) but not on the 4-routes (such as AB, BC, CD, DE). Since the order of the automorphism group of an s-regular trivalent graph is 2^s times the number of edges [3, p. 111], the group in the present case must have order

$$2^3\, 42 = 336,$$

which makes it precisely PGL(2, 7).

This 'Coxeter graph' was discovered independently by J.H. Conway and R.M. Foster. It was studied in detail by Norman Biggs [1], who proved that it shares with the Petersen graph, but with no other trivalent graph, the following three properties:

Its automorphism group acts primitively on the vertices and transitively on the pairs of vertices at each particular distance apart, and it is non-Hamiltonian.

Like the Petersen graph, the Coxeter graph 'only just' fails to be Hamiltonian: although it admits no closed route visiting all the vertices, there is an open route doing so, for instance,

03 5∞ 12 06 45 1∞ 36 04 2∞ 56 34 0∞ 16 35

02 46 13 05 24 3∞ 15 26 4∞ 01 23 6∞ 14 25.

For many other properties, see 'My graph' in a forthcoming issue of the Proceedings of the London Mathematical Society.

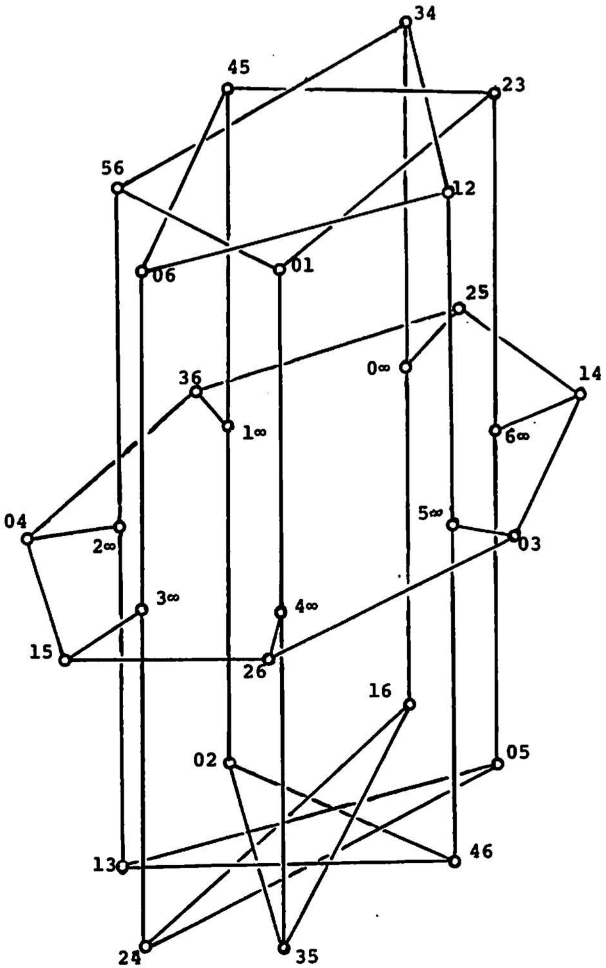

Figure 1: The Coxeter graph as drawn by Heinrich Heesch

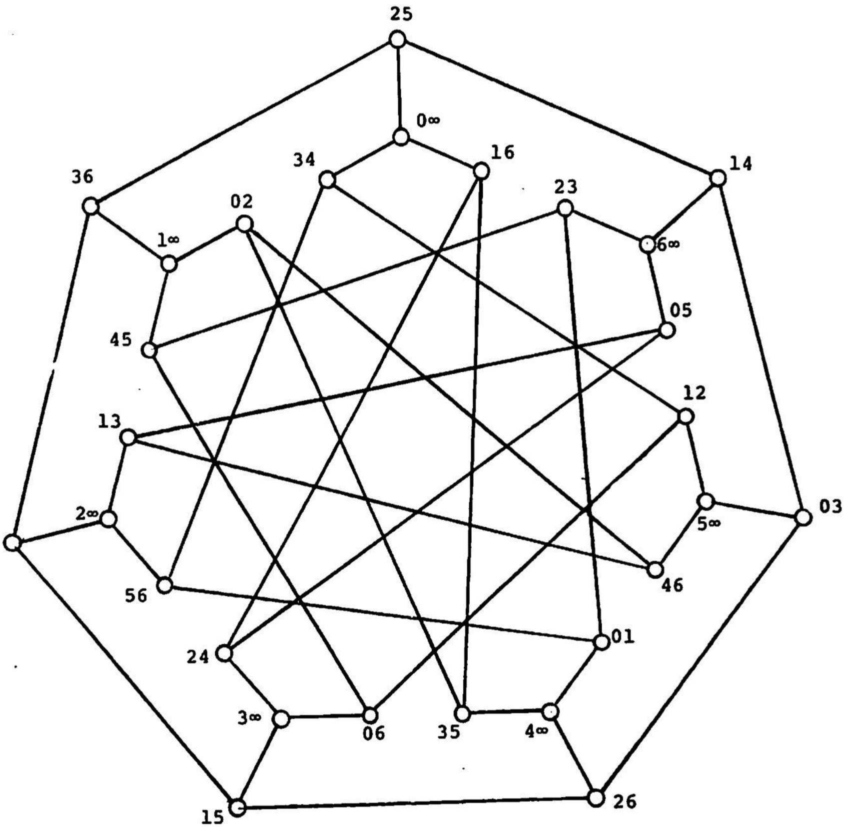

Figure 2: The Coxeter graph as drawn by Milan Randić

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Received September 15, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

ON PLANES OF CLASS I6

Peter Scherk, F.R.S.C.

Let $\mathfrak{A}$ denote an affine plane of the Lenz-Barlotti class I6. It possesses a parallel pencil π_0 with a bijection $x \longmapsto \pi(x)$ of π_0 onto the set of the other parallel pencils. This associates with every point $Y I x$ the line $h_Y \in \pi(x)$ through Y. For every $x \in \pi_0$, the group of the affinities with the axis x and the pencil of traces $\pi(x)$ is linearly transitive. There are no other linearly transitive groups of axial affinities, dilations, or translations; cf. [2;p.32].

It is essentially known that $\mathfrak{A}$ possesses a translation τ parallel to π_0 and characterized by the relation

(1) $$A I h_B \iff B I h_{A\tau} = h_A \tau \qquad \text{if } A \neq B.$$

It corresponds to the unit element 1 of the ternary coordinate ring of $\mathfrak{A}$. By [3; Theorem E], we have either 1+1=0 or 1+1+1=0. The relation (1) permits a simple geometric construction of τ. The following results can also be proved geometrically: (i) τ lies in the centre of the collineation group of $\mathfrak{A}$. (ii) ordτ = 2 or 3. (iii) Suppose τ is involutory. Then the order of every axial affinity is 1 or ∞ or an odd prime. If ordτ = 3, every axial affinity has the order 1,2, or ∞; cf. [3; Theorem 2.13]. The proof of (iii) is based on (iv): Suppose the subplane $\mathcal{L}$ of $\mathfrak{A}$ contains the points A,Aτ and the line h_A. Then either $\mathcal{L}$ belongs to the class I6 or it is desarguesian. In the latter case, we have either ord $\mathcal{L}$ = ord τ =2 or ord$\mathcal{L}$ =4 and ord τ =2 or ord$\mathcal{L}$ = ord τ =3; cf. [3; Theorem D].

Let α be an axial affinity; $\alpha^2 \neq 1$. Denote the axis of α by $[\infty]$. Furthermore let

$$p = \begin{cases} \text{ord}\alpha & \text{if} \quad \text{ord}\alpha < \infty \\ 0 & \phantom{\text{if}} \quad \text{ord}\alpha = \infty. \end{cases}$$

and let F_p denote the prime field of characteristic p. Then there is a group $G(\alpha) = \{\alpha^n \mid n \varepsilon F_p\}$ of affinities with the axis $[\infty]$ which is mapped isomorphically onto the additive group of F_p by $\alpha^n \mapsto n$; $\alpha^0=1$, $\alpha^1=\alpha$. Next let $[\infty] \neq [0] \varepsilon \tau_0$ and put $[n] = [0]\alpha^n$. For every $n\varepsilon F_p^*$ let δ^n denote the affinity with the axis [0] which maps $[\infty]$ onto $[n^{-1}]$; $\delta^0=1$. Then $n \mapsto \delta^n$ maps F_p isomorphically onto $G(\delta) = \{\delta^n \mid n\varepsilon F_p\}$. The two groups $G(\alpha)$ and $G(\delta)$ are connected by the identities

$$(2) \qquad \alpha^{-r}\,\delta^{\frac{1}{r}}\,\alpha^{n} = \delta^{-\frac{1}{r}}\,\alpha^{-r}\,\delta^{\frac{1}{r}} \qquad (r\varepsilon F_p^*).$$

Let $G = G(\alpha,\delta)$ denote the group generated by α and δ. Formula (2) readily implies that G is perfect.

If ordα = p is finite, - thus ord G = p, - then we can map G isomorphically onto SL(2,p) through

$$\alpha \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \delta \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The kernel of the induced homomorphism of G onto PSL(2,p), i.e. the centre of G, consists of the identity and an involutory translation

(G acts on the set

$$\{[\infty]\} \cup \{[n] \mid n \in F_p\} \subset \pi_0$$

precisely in the way PSL(2,p) acts on the projective line $F_p \cup \{\infty\}$ over F_p). It is possible to construct a finite configuration which is transformed by G into itself.

If ordα is infinite, there still exists a natural homomorphism of G onto PSL(2,Q). Its kernel is the centre of G and non-trivial. On account of (2), G is a homomorphic image of the Steinberg group St(2,Q). But nothing more is known at present about the location of G between St(2,Q) and PSL(2,Q); cf.[1; p.82].

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Received October 8, 1981

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C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

CUBATURE, QUADRATURE, AND GROUP ACTIONS

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Presented by G. de B. Robinson, F.R.S.C.

The notion of t-design is well established in combinatorics. Replacing the "discrete" sphere by the compact Euclidean unit sphere S_{n-1} of $\mathbb{R}^n (n \geq 2)$ and the action of the symmetric group by the action of the special orthogonal group $SO(n,\mathbb{R})$, the notion of spherical t-design in $\mathbb{R}^n$ emerges. Since spherical t-designs allow to measure certain regularity properties of finite subsets X of S_{n-1}, this notion has computational besides theoretical significance. In particular, spherical t-designs are useful for the explicit construction of cubature formulae for surface integrals over S_{n-1} by averaging over X (Section 1). The purpose of the present note is to deal mainly with the one-dimensional case (n=1). It will be shown that in this case the action of the real Heisenberg group $\tilde{A}(\mathbb{R})$ gives rise to a trapezoidal rule for improper integrals [8]. The error of this quadrature formula will be represented by a complex contour integral with noncompact integration path (Section 2).

1. Spherical t-Designs in $\mathbb{R}^n$ and Cubature

Let σ denote the surface measure of the compact Euclidean unit sphere $S_{n-1} = \{x \in \mathbb{R}^n \| \|x\| = 1\}$, $n \geq 2$. A non-empty finite subset X of S_{n-1} is called a spherical t-design in $\mathbb{R}^n$ if the approximation of the (standardized) surface integral

$$\frac{1}{\sigma(S_{n-1})} \int_{S_{n-1}} f(x)\,d\sigma(x)$$

by the average cubature formula

$$\frac{1}{\# X} \sum_{x \in X} f(x)$$

is exact for all homogeneous polynomials f in n variables of degree $k \leq t$; see Delsarte-Goethals-Seidel [2], Goethals-Seidel [4], [5], and the survey article [7]. Identify the unit sphere S_{n-1} with the compact homogeneous manifold $SO(n-1,\mathbb{R})\backslash SO(n,\mathbb{R})$. The central point of the harmonic analysis of the compact Gelfand pair $(SO(n,\mathbb{R}),SO(n-1,\mathbb{R}))$ (cf. [11]) is the Hilbert space decomposition

$$L^2(S_{n-1};\sigma) = \hat{\bigoplus_{k \in \mathbb{N}}} \mathcal{H}_k$$

such that any class 1 representation of $SO(n,\mathbb{R})$ is realizable on a vector space $\mathcal{H}_k$ of harmonic homogeneous polynomials of degree k on $\mathbb{R}^n$ (solid spherical harmonics) and that the degree k of homogeneity of these polynomials is determined by the highest weight vector of the representation. The finite dimensional Hilbert spaces $\mathcal{H}_k$ admit as their reproducing kernels the zonal spherical harmonics $P_k \in L^2(SO(n-1,\mathbb{R})\backslash SO(n,\mathbb{R})/SO(n-1,\mathbb{R}))$ which may be identified, up to a real constant factor, with the Gegenbauer polynomials (in the case n=2 with the Čebyšev polynomials of the first kind) of degree k. These facts of the large compact subgroups theory combined with some matrix techniques allow an explicit construction of spherical t-designs in $\mathbb{R}^n (n \geq 2)$.

<u>Example.</u> In the case n=3, the 12 vertices of the icosahedron, and the 20 vertices of the dodecahedron, provide a spherical 5-design in $\mathbb{R}^3$.

In the case n=1, we shall consider the action of the real Heisenberg group and the compact Heisenberg manifold instead of the action of the special orthogonal group $SO(n,\mathbb{R})$ and the compact Euclidean sphere S_{n-1}.

2. The Heisenberg Group $\tilde{A}(\mathbb{R})$ and Quadrature

As is well known, there is an intimate relationship between the Heisenberg uncertainty principle and the real Heisenberg group $\tilde{A}(\mathbb{R})$. The group $\tilde{A}(\mathbb{R})$ is formed by the triangular matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \qquad = (x,y,z) \in \mathbb{R}^3$$

and admits the Schrödinger representation

$$U: (x,y,z) \rightsquigarrow U(x,y,z)f(t) = e^{2\pi i(z+ty)}f(t+x)$$

on the complex Hilbert space $L^2(\mathbb{R})$ as an irreducible unitary representation. The central part of the harmonic analysis on $\tilde{A}(\mathbb{R})$ is the Stone-von Neumann-Mackey theorem (or uniqueness of the representation of the canonical Heisenberg commutation relations) [9]. According to this fundamental result, the Schrödinger representation U of $\tilde{A}(\mathbb{R})$ in $L^2(\mathbb{R})$ is, up to unitary isomorphy, the only irreduzible unitary linear representation of $\tilde{A}(\mathbb{R})$ such that

$$U((0,0,z)) = z.\mathrm{id}_{L^2(\mathbb{R})} \qquad (z \in \mathbb{R}).$$

As a consequence, the automorphism τ: $(x,y,z) \rightsquigarrow (y,-x,z-xy)$ gives rise to a second realization $U^\tau = U \circ \tau$ of the Schrödinger representation U of $\tilde{A}(\mathbb{R})$. The Fourier cotransform $\bar{\mathcal{F}}_{\mathbb{R}}$ is a unitary isomorphism of U onto U^τ. On the other hand, if we introduce the discrete cocompact τ-stable subgroup

$$P = \{(x,y,z) \mid x,y,z \in \mathbb{Z}\} = \mathbb{Z}^3,$$

the compact Heisenberg manifold $P\backslash\tilde{A}(\mathbb{R})$ can be considered as an analog of the compact unit sphere S_{n-1} in Section 1. The Hilbert space decomposition into a sequence $(H_N)_{N \in \mathbb{Z}}$ of primary subspaces

$$L^2(P\backslash\tilde{A}(\mathbb{R})) = \hat{\bigoplus}_{N \in \mathbb{Z}} H_N$$

gives rise to a third realization δ_1 of the Schrödinger representation U by restricting the right regular representation δ of $\tilde{A}(\mathbb{R})$ to the irreducible subspace

$$H_1 = \{ f \in L^2(P\backslash\tilde{A}(\mathbb{R})) \mid f(x,y,z+z') = e^{2\pi i z'} f(x,y,z),\ z' \in \mathbb{R} \}.$$

In this case, the Weil-Brezin isomorphism W: $f \rightsquigarrow ((x,y,z) \rightsquigarrow \sum_{n \in \mathbb{Z}} f(x+n)e^{2\pi i(z+ny)})$ defines a unitary isomorphism of U onto δ_1 and we have the following factorization of the intertwining operators involved:

$$W^{-1} \circ \tau \circ W = \bar{\mathcal{F}}_{\mathbb{R}}.$$

An application of this identity to the central basis spline M_1 of degree 0 furnishes the Whittaker-Shannon cardinal interpolation series. It follows by the classical Paley-Wiener theorem and termwise integration that the trapezoidal rule with respect to the step width $h > 0$ for improper integrals over the real line $\mathbb{R}$, i.e.,

$$\int_{\mathbb{R}} g(x)dx = h \sum_{n \in \mathbb{Z}} g(nh)$$

(cf. Engels [3], Chap. 7) is valid for all entire holomorphic functions g such that $g|\mathbb{R} \in L^2(\mathbb{R})$ and the estimate

$$|g(z)| \leqq Ce^{\pi|\mathrm{Im}\ z|/h} \qquad (z \in \mathbb{C})$$

holds for some constant $C > 0$.

In the case when g satisfies less restrictive conditions, the trapezoidal rule admits a quadrature error which can be represented by a complex contour integral with noncompact integration path. Suppose that g denotes a holomorphic function in the open horizontal strip $\{z \in \mathbb{C} \mid |\mathrm{Im}\ z| < b\}$ of width $2b > 0$ such that

$$\lim_{|x| \to \infty} \int_{-b}^{b} |g(x+iy)|\, dy = 0,$$

and that both the boundary integrals $\lim_{y \to b\pm} \int_{\mathbb{R}} |g(x+iy)|\, dx$ exist.

Denote the sum of these limits by $N_b(g)$. An application of the Cauchy residue theorem furnishes the complex contour integral representation of the quadrature error

$$\int_{\mathbb{R}} g(x)dx - h\sum_{n\in\mathbb{Z}} g(nh) = \frac{e^{-\pi b/h}}{2i}\int_{\mathbb{R}}\left(\frac{g(t+ib)e^{i\pi t/h}}{\sin(\pi(t+ib)/h)} - \frac{g(t-ib)e^{-i\pi t/h}}{\sin(\pi(t-ib)/h)}\right)dt$$

and the error bound (see, for instance, Stenger [12]):

$$\left|\int_{\mathbb{R}} g(x)dx - h\sum_{n\in\mathbb{Z}} g(nh)\right| \leq \frac{e^{-\pi b/h}}{2\sinh(\pi b/h)} N_b(g).$$

Thus, if $g|\mathbb{R}$ converges rapidly to zero as $|x|\to\infty$ (for instance, if g decreases exponentially), the trapezoidal rule admits a high degree of accuracy in approximating the improper integral $\int_{\mathbb{R}} g(x)dx$.

For interesting applications of the finite Heisenberg group, the reader is referred to the stimulating paper by Auslander-Tolimieri [1]. Also see [9], [10]. A systematic use of complex contour integral representations with noncompact integration path provides the monograph [6].

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Received October 19, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

ON A MODIFICATION OF THE MELLIN-STIELTJES TRANSFORM.

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Presented by J. Aczél, F.R.S.C.

Let F be a distribution function, then $M(s) = \int_0^\infty x^s dF(x)$ is called the Mellin-Stieltjes transform of F. This transform exists for all values of the complex variable s on some strip

$$\mathcal{D} = \{s: \sigma_1 \le \text{Re } s \le \sigma_2\}$$

where $\sigma_1 > 0.\sigma_2 > 0$. This strip contains the imaginary axis and may reduce to this axis. Let F_1 and F_2 be two distribution functions and let

$$F_0(z) = \int_{-\infty}^{\infty} F_1(\tfrac{z}{x})\, dF_2(x).$$

The distribution function F_0 has then the Mellin-Stieltjes transform

$$M_0(s) = M_1(s) M_2(s)$$

with the strip of convergence $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ where $\mathcal{D}_j$ is the strip of F_j $(j=1,2)$.

The following two theorems are proven.

Theorem 1. Let K be a function of the complex variable s and the real variable x which satisfies the following conditions:

(I) K is bounded and measurable for all $x > 0$ and all s which belong to a vertical strip $\mathcal{D}$ which contains the imaginary axis and which may reduce to this axis.

(II) $K(s;x) \neq 0$ for all x.

(III) F_1 and F_2 are two distribution functions such that $F_1(x) = F_2(x) = 0$ for $x < 0$ and that

$$\int_0^\infty K(s;x)\,dF(x) = \int_0^\infty K(s;x)\,dF_1(x) \int_0^\infty K(s;x)\,dF_2(x).$$

Then $K(s;x) = x^{iA(s)}$, where $x > 0$ and where $A(s)$ is real for all $s \in \mathcal{D}$.

For the proof of Theorem 1 we introduce the functions f_j by

$$f_j(s) = \int_0^\infty K(s;x)\,dF_j(x), \quad (j=0,1,2)$$

and conclude easily that

$$f_0(s) = f_1(s) f_2(s).$$

We can then derive the functional equation

$$K(s;\xi_1\xi_2) = K(s;\xi_1)K(s;\xi_2)$$

for the kernel $K(s;x)$. Its solution yields the statement of Theorem 1.

The next result deals with the set $\mathcal{D}$ of distribution functions F such that $F(x) = 0$ for $x < 0$. Let s be a complex variable and let $F \in \mathcal{D}$ and consider a mapping $\alpha(s;F)$. The functions produced by this mapping form a set $\mathcal{F}$.

The mapping α is said to be bounded if all functions $f \in \mathcal{F}$ are uniformly bounded. The mapping α is said to be linear if $\alpha(s;pF_1+qF_2) = p\alpha(s;F_1)+q\alpha(s;F_2)$ for all $p>0, q>0, p+q=1$ and all $F_1, F_2 \in \mathcal{D}$.

We say that a sequence $\{F_n\}$ of distribution functions converges weakly to a distribution function F if the sequence converges to F in all continuity points of F. We write then

$$\operatorname*{Lim}_{n\to\infty} F_n = F.$$

Theorem 2. Let $\alpha(s;F)$ be a bounded linear mapping of the set $\mathcal{D}$ onto the set $\mathcal{F}$. Suppose that

(I) the relations $F_1 \in \mathcal{D}$, $F_2 \in \mathcal{D}$, $F_0(x) = \int_0^\infty F_1(\frac{u}{x})\,dF_2(u)$ imply that $\alpha(s;F_0) = \alpha(s;F_1)\alpha(s;F_2)$,

(II) $\alpha(s;F) \neq 0$ for all $F \in \mathcal{D}$,

(III) $\operatorname*{Lim}_{n\to\infty} F_n = F$ if, and only if,

$$\lim_{n\to\infty} \alpha(s;F_n) = \alpha(s;F).$$

Then $\alpha(s;F) = \int_0^\infty x^{iA(s)}\,dF(x)$ for all $F \in \mathcal{D}$. Here A is a real valued function of s.

In proving Theorem 2, one considers first the case where the distribution functions F_1 and F_2 are suitably chosen degenerate distributions so that F_0 is also degenerate.

In the next step simple discrete distributions with a finite number of salti are studied and the corresponding

$\alpha(s;F_0)$ is determined. Then a finite, but not necessarily discrete, distribution which is defined on a finite interval is considered and the corresponding $\alpha(s;F)$ is derived. Transition to a limit leads then to the proof of the general statement of Theorem 2.

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Received October 23, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

A NEW FOCUS AND DIRECTRIX THEOREM FOR CONICS

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Presented by H.S.M. Coxeter, F.R.S.C.

The projections of the sections of a cone into a plane through the vertex of the cone and perpendicular to the axis of the cone yield a pleasingly simple, and apparently previously unnoticed, description of their foci, directrices, and eccentricities.

Let K be a right circular cone with vertex V and axis z. Let π be the plane through V perpendicular to z. Let σ (the sectioning plane) be a plane which misses V and is neither parallel nor perpendicular to z. Then $S = K \cap \sigma$ is a classical conic section (ellipse, parabola, or hyperbola) and the projection S' of S parallel to z into the plane π is a conic of the same kind as S.

<u>Theorem</u>. The focus of S' is V, the vertex of the cone; the directrix corresponding to V is the line $d = \pi \cap \sigma$; and the eccentricity of S' is the ratio between the slope of σ and the slope of a ruling of the cone K — both slopes taken relative to π.

The author's original proof of this theorem was analytic (computational). However, Howard Eves of Lubec, Maine, U.S.A., has produced the brief synthetic proof which is given below. It captures the essence of the result in a single line.

<u>Proof</u>. Referring to Figure 1, let P' be any point of S' and let P in S

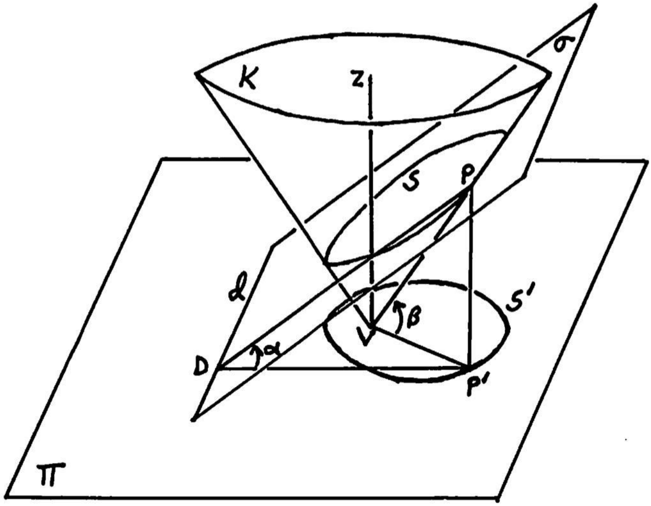

Figure 1

be the inverse projection of P'. Then

$$\frac{VP'}{DP'} = \frac{VP'}{PP'} \cdot \frac{PP'}{DP'} = \cot\beta \cdot \tan\alpha = \frac{\tan\alpha}{\tan\beta} = \varepsilon',\ \text{a constant.}$$

Although Figure 1 is for the case $\alpha < \beta$ (an ellipse), the proof is unchanged if $\alpha = \beta$ (a parabola) or $\alpha > \beta$ (an hyperbola).

There is, of course, an elegant nineteenth century theory of the focus and directrix of the (unprojected) conic section S — due to G. P. Dandelin [1, 27-29; 2, 331] — which involves inscribing spheres in K. The relative simplicity of the theory presented here for S' suggests that the context of the projections, or "shadows," of conic sections in the plane π is also an appropriate one for considering the ellipse, parabola, and hyperbola.

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Received October 27, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

ON THE PI CHARACTER OF ULTRAFILTERS

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Presented by G.F.D. Duff, F.R.S.C.

Abstract

If the continuum, $\underset{\sim}{c}$, is regular, then there is a point in $\beta N\setminus N$ whose π-character equals $\underset{\sim}{c}$. However, it is consistent that $\underset{\sim}{c}$ is singular and every point in $\beta N/N$ has π-character ω_1.

§0. Introduction. If X is a topological space, $p \in X$, and $\mathcal{B}$ is a family of non-empty open subsets of X, then $\mathcal{B}$ is called a local π-base at p iff for all open U containing p, there is a $B \in \mathcal{B}$ with $B \subset U$. $\mathcal{B}$ is a local base at p iff in addition each $B \in \mathcal{B}$ contains p. The π-character of p in X, $\pi\chi(p,X)$ is the least cardinality of a local π-base at p, whereas the character $\chi(p,X)$, is the least cardinality of a local base at p. Clearly $\pi\chi(p,X) \leq \chi(p,X)$.

We now restrict our attention to the space $N^* = \beta N\setminus N$, where N is the space of non-negative integers, ω, with the discrete topology, and βN is its Čech compactification. For $p \in N^*$, we set $\chi(p) = \chi(p,N^*)$ and $\pi\chi(p) = \pi\chi(p,N^*)$. (Note that $\chi(p,\beta N) = \chi(p,N^*)$, whereas $\pi\chi(p,\beta N) = \omega$.) It is easily seen that

$$\omega_1 \leq \pi\chi(p) \leq \chi(p) \leq \underset{\sim}{c} ,$$

where $\underset{\sim}{c} = 2^{\omega}$. Thus, problems regarding these cardinal functions only arise in the absence of the Continuum Hypothesis.

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*Research supported by NSF Grant MCS-7900824.

By a well-known theorem of Pospíšil (see [P], or Exercise A11, p. 289, of [K1]), there must be a $p \in N^*$ whose character equals $\underset{\sim}{c}$. In §1, we show that <u>if $\underset{\sim}{c}$ is regular</u>, the same result holds also for π-character. However, in §2, we show by a forcing argument that it is consistent with $\underset{\sim}{c} = \omega_{\omega_1}$ that every point in N^* has π-character ω_1. We also remark in §2 that related questions about χ and $\pi\chi$ are already answered by known forcing arguments.

Throughout, we identify N^* with the set of non-principal ultrafilters on ω. If $A,B \subset \omega$, $A \subset^* B$ means that $A \setminus B$ is finite. If $\mathcal{U}$ is a non-principal ultrafilter on ω and $\mathcal{B}$ is a family of infinite subsets of ω, we call $\mathcal{B}$ a <u>local π-base</u> for $\mathcal{U}$ if $\forall A \in \mathcal{U}\, \exists B \in \mathcal{B} (B \subset^* A)$; $\mathcal{B}$ is a <u>local base</u> iff in addition $\mathcal{B} \subset \mathcal{U}$. If we define $\pi\chi(\mathcal{U})$ to be the least cardinality of a local π-base and $\chi(\mathcal{U})$ to be the least cardinality of a local base, it is easily seen that these functions agree with our previous topological definitions when we regard $\mathcal{U}$ as a point in N^*.

§1. <u>Points of large π-character</u>. We construct our ultrafilters with the aid of a matrix of sets.

1.1. <u>Lemma</u>. There are sets $A_{\alpha\beta} \subset \omega$, for $\alpha,\beta < \underset{\sim}{c}$, such that

a) For each α, if $\beta_1 \neq \beta_2$, then $A_{\alpha\beta_1} \cap A_{\alpha\beta_2}$ is finite, and

b) For each $n < \omega$ and $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n < \underset{\sim}{c}$ with the $\alpha_1,\ldots,\alpha_n$ distinct, $A_{\alpha_1\beta_1} \cap \ldots \cap A_{\alpha_n\beta_n}$ is infinite.

<u>Proof</u>. Such a matrix is embedded in the more complicated matrix of [K2]. For a direct proof, let

$$I = \{\langle n,f\rangle : n \in \omega \text{ and } f \in \mathcal{P}(n)^{\mathcal{P}(n)}\},$$

and, for $X,Y \subset \omega$, let

$$A_{XY} = \{\langle n,f\rangle \in I: f(X\cap n) = Y\cap n\} .$$

Then $|I| = \omega$, and the A_{XY}, for $X,Y \in P(\omega)$, have the desired properties as a matrix of subsets of I. □

For any infinite cardinal κ, $cf(\kappa)$ denotes its cofinality; then κ is regular iff $\kappa = cf(\kappa)$. The first result stated in the abstract is now immediate from:

1.2. Theorem. There is a non-principal ultrafilter $\mathcal{U}$ on ω with $\pi\chi(\mathcal{U}) \geq cf(\mathfrak{c})$.

Proof. Let the $A_{\alpha\beta}$ be as in Lemma 1.1, and let D_ξ, for $\xi < \mathfrak{c}$, enumerate the infinite subsets of ω. For each α and ξ, 1.1(a) implies that there is at most one β such that $D_\xi \subset^* A_{\alpha\beta}$, so we may choose an $f(\alpha) < \mathfrak{c}$ such that for all $\xi \leq \alpha$, $D_\xi \not\subset^* A_{\alpha,f(\alpha)}$. 1.1(b) implies that there is a non-principal ultrafilter $\mathcal{U}$ on ω with each $A_{\alpha,f(\alpha)} \in \mathcal{U}$. If $\mathcal{B}$ is a π-base for $\mathcal{U}$, write $\mathcal{B} = \{D_\xi : \xi \in S\}$, where $|\mathcal{B}| = |S|$. For each α, there is a $\xi \in S$ such that $D_\xi \subset^* A_{\alpha,f(\alpha)}$, and, by our choice of $f(\alpha)$, this ξ must be greater than α. Thus, S is unbounded in $\mathfrak{c}$, so $|S| \geq cf(\mathfrak{c})$. □

§2. Points of small π-character. We assume familiarity with finite support iterated forcing constructions, such as occur in the proof of consistency of MA + ¬CH; see, e.g., [J] or [K1]. We use letters such as $\mathbb{P}$ or $\mathbb{Q}$ to denote partial orders, with the $\leq$ always understood. Thus, $\mathbb{P} \subset \mathbb{Q}$ is always taken to imply that the unmentioned partial orders of $\mathbb{P}$ and $\mathbb{Q}$ agree on $\mathbb{P}$. The notion $\mathbb{P} \subset_c \mathbb{Q}$ ($\mathbb{P}$ is completely contained in $\mathbb{Q}$) is a strengthening of $\mathbb{P} \subset \mathbb{Q}$ (see p.218 of [K1]) which has the following interpretation in terms of models. Let M be a countable transitive model (c.t.m.) for ZFC, and suppose $\mathbb{P}, \mathbb{Q} \in M$ with $\mathbb{P} \subset_c \mathbb{Q}$; let G be $\mathbb{Q}$-generic over M; then $G \cap \mathbb{P}$ is $\mathbb{P}$-generic over M and $M[G \cap \mathbb{P}] \subset M[G]$. In terms of complete Boolean algebras (which we do not use here), $\mathbb{P} \subset_c \mathbb{Q}$ holds iff the inclusion of $\mathbb{P}$ into $\mathbb{Q}$ defines a complete injective homomorphism from the completion of $\mathbb{P}$ into the completion of $\mathbb{Q}$.

2.1. Lemma. Let M be a c.t.m. for ZFC + GCH. Then in M, there is a sequence of partial orders, $\langle \mathbb{P}_\alpha : \alpha \leq \omega_1^M \rangle$ such that within M:

a) Each $\mathbb{P}_\alpha$ has the c.c.c.,

b) Each $|\mathbb{P}_\alpha| = \omega_\alpha$,

c) If $\alpha \leq \beta$ then $\mathbb{P}_\alpha \subset_c \mathbb{P}_\beta$,

d) If γ is a limit, then $\mathbb{P}_\gamma = \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$,

and, whenever G is $\mathbb{P}_{\omega_1}$-generic over M,

e) For each $\alpha < \omega_1$, MA and $c = \omega_{\alpha+1}$ hold in $M[G \cap \mathbb{P}_{\alpha+1}]$.

Proof. The $\mathbb{P}_\alpha$ are constructed in M via a finite support iteration, which we identify with an ascending chain under $\subset_c$ (see Chapter VIII, §5 of [K1]). At limits we take unions, which preserves (a)-(d), and $\mathbb{P}_0$ is an arbitrary countable partial order. Given $\mathbb{P}_\alpha$, $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{P}_\alpha * \pi$, where π is a $\mathbb{P}_\alpha$-name for a partial order. To describe π, set $\mathbb{P} = \mathbb{P}_\alpha$ and $\kappa = \omega_{\alpha+1}$. By (b), whenever G is $\mathbb{P}$-generic over M, $2^{<\kappa} = \kappa$ holds in $M[G]$, so there is, in $M[G]$, a c.c.c. $\mathbb{Q}$ such that

$$|\mathbb{Q}|=\kappa \text{ and } \mathbb{1} \Vdash_{\mathbb{Q}} (\text{MA and } \underset{\sim}{c}=\check{\kappa}),$$

where $\Vdash_{\mathbb{Q}}$ refers to forcing over $M[G]$ with $\mathbb{Q}$. G was arbitrary, so there is a $\mathbb{P}$-name, π, such that

$$\Vdash_{\mathbb{P}} [\,|\pi|=\check{\kappa} \text{ and } \mathbb{1} \Vdash_{\pi} (\text{MA and } \underset{\sim}{c}=\check{\kappa})] \;,$$

where $\Vdash_{\mathbb{P}}$ refers to forcing over M with $\mathbb{P}$. We may take $|\mathrm{dom}(\pi)| = \kappa$, which will ensure that (a)-(e) hold for $\mathbb{P}_{\alpha+1}$. □

2.2. Theorem. If ZFC is consistent, so is ZFC plus $\underset{\sim}{c} = \omega_{\omega_1}$ plus

$$\forall p \in N^*(\pi\chi(p)=\omega_1) .$$

Proof. Let M and the $\mathbb{P}_\alpha$ be as in Lemma 2.1, set $\mathbb{P} = \mathbb{P}_{\omega_1}$, and let G be $\mathbb{P}$-generic over M. It is easily seen that $\underset{\sim}{c} = \omega_{\omega_1}$ holds in $M[G]$. Now, in $M[G]$, fix a non-principal ultrafilter $\mathcal{U}$ on ω; we shall produce, in $M[G]$, a local π-base of size ω_1.

Let $\tau \in m$ be a $\mathbb{P}$-name such that $\mathcal{U} = \tau_G$. We may assume that

$$\tau = \cup \{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dom}(\tau)\} ,$$

where, for each σ, A_σ is an antichain in $\mathbb{P}$ and $\mathbb{1} \Vdash (\sigma \subset \check{\omega})$ (see p. 208 of [K1]). Since $\mathbb{P}$ has the c.c.c. in M, each A_σ is countable in M, and hence in $M[G]$. Likewise, we may assume that for each $\sigma \in \mathrm{dom}(\tau)$,

$$\sigma = \cup\{\{\check{n}\} \times B_n^\sigma : n \in \omega\} ,$$

where B_n^σ is a countable antichain in $\mathbb{P}$.

Now, for each $\alpha < \omega_1^M$, let

$$\mathcal{U}_\alpha = \{\sigma_G : \forall n (B_n^\sigma \subset \mathbb{P}_\alpha) \text{ and } \exists p \in G \cap \mathbb{P}_\alpha (\langle \sigma,p\rangle \in \tau)\} .$$

Then $\mathcal{U}_\alpha \in M[G \cap \mathbb{P}_\alpha]$,

$$\mathcal{U} = \cup \{\mathcal{U}_\alpha : \alpha < \omega_1^M\} ,$$

and

$$\langle \mathcal{U}_\alpha : \alpha < \omega_1^M\rangle \in M[G] .$$

Also, for each α, $|\mathcal{U}_\alpha| \le \omega_{\alpha+1}$ holds in $M[G \cap \mathbb{P}_\alpha]$ (since $\underset{\sim}{c} = \omega_{\alpha+1}$ in $M[G \cap \mathbb{P}_{\alpha+1}]$), so in $M[G \cap \mathbb{P}_{\alpha+2}]$ (and hence also in $M[G]$),

$$\exists X \subset \omega(|X| = \omega \text{ and } \forall Y \in \mathcal{U}_\alpha (X \subset^* Y))$$

(since $M[G \cap \mathbb{P}_{\alpha+2}]$ satisfies MA and $c = \omega_{\alpha+2}$). Thus, in $M[G]$, there is a sequence $\langle X_\alpha : \alpha < \omega \rangle$ of infinite subsets of ω such that

$$\forall \alpha < \omega_1 \; \forall Y \in \mathcal{U}_\alpha (X_\alpha \subset^* Y) ,$$

whence $\{X_\alpha : \alpha < \omega_1\}$ is a local π-base for $\mathcal{U}$. □

Some related questions on χ and $\pi\chi$ are easily answered by well-known forcing arguments. For example, the statement

$$\forall p \in N^* (\pi\chi(p) = \omega_1)$$

does not simply follow from $\underset{\sim}{c} = \omega_{\omega_1}$, since if one adds ω_{ω_1} Cohen generic sets to a model of GCH, the extension will satisfy $\underset{\sim}{c} = \omega_{\omega_1}$, plus

$$\forall p \in N^* (\pi\chi(p) = \underset{\sim}{c}) .$$

Also, if one wishes only to have some points with π-character (and even character) ω_1, $\underset{\sim}{c}$ can be anything; any M has a c.c.c. extension with the same $\underset{\sim}{c}$ satisfying

$$\exists p \in N^* (\chi(p) = \omega_1)$$

(see Exercise A10, p. 289, of [K1]).

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Received October 29, 1981

C.R. Math. Rep. Acad. Sci. Canada - Vol. III(1981) No. 6

ON SOME PROPERTIES OF PM-RINGS AND MP-RINGS

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All rings are commutative with unit and the proofs will appear elsewhere. Goal of this Note is a further investigation on the rings in the title.

§1

Definition 1.1 A ring A is said to be a pm-*ring* (or a ring with the (PM)-property) if one of the following equivalent conditions holds:

1. Every prime ideal is contained in a unique maximal ideal.

2. (PM)-property: $\forall m \in A$, $\exists a, b \in A$ such that $(1-am)(1-bm') = 0$, $m' = 1-m$ (see [7], [8]).

Propostion 1.2 *An ultraproduct of pm-rings is a pm-ring.*

Special cases of pm-rings

1. Soft rings.

Definition 1.3 A *soft* (or more)-*ring* is one with Jacobson radical zero and maximal spectrum T_2 (see [5], [13]).

Proposition 1.4 *An ultraproduct of soft rings is a soft ring.*

2. (TB)-rings.

For any ring A, let B(A) (or simply B) denote the boolean algebra of idempotent elements of A. The map ϕ : Max(A) $\to$ Spec(B(A)), $M \mapsto M \cap B(A)$, is always continuous, surjective and closed (see [1]).

Definition 1.5 A ring A is said to be *topologically boolean* (briefly, a (TB)-ring) if either the above map ϕ is injective (i.e., a homeomorphism) or the following property is satisfied:

(TB)-property: $\forall\, m, m' \in A$, $m+m'=1$, $\exists\, a, b, c, d, e \in A$,

$$e^2=e, \text{ such that } (1-am)(1-bm')=0,$$

$$1-am = ce \text{ and } 1-bm' = de' \text{ where } e'=1-e.$$

(see [7]).

<u>Proposition 1.6</u> *The (TB)-property transfers to ultraproduct.*

3. <u>Complete (TB)-rings.</u>

<u>Definition 1.7</u> A (TB)-ring whose boolean algebra of idempotent elements is complete is said to be a *complete (TB)-ring*. (see [12]).

<u>Definition 1.8</u> A topological space is said to be *extremally disconnected* if the closure of every open subset is an open subset (see [9], 1H).

<u>Lemma 1.9</u> *A boolean algebra B is complete if and only if the topological space Spec (B) is extremally disconnected* (see [10]).

<u>Lemma 1.10</u> *A closed subspace of an extremally disconnected space need not be extremally disconnected itself* (see [9], 1H:5).

<u>Lemma 1.11</u> *Let* $B = \prod_{\lambda\in\Lambda} B_\lambda$, *where every* B_λ *is a complete boolean algebra. Let U be an ultrafilter over* Λ *and let* a_U *denote the ideal of B associated to it. If the closed subspace* $V(a_U)$ *of* Spec(B) *is not extremally disconnected,*

then the ultraproduct $\overline{B} = r/a_U$ *is not complete.*

<u>Theorem 1.12</u> *Let* $R = \prod_{\lambda\in\Lambda} R_\lambda$ *, where each* R_λ *is a complete (TB)-ring. Set* $B_\lambda = B(R_\lambda)$ $\lambda\in\Lambda$ *, and let* $B = \prod_{\lambda\in\Lambda} B_\lambda$ *. Let* U *be an ultrafilter over* Λ *and let* a_U *denote the ideal of* B *associated to it. Set* $\overline{R} = R/\sim U$, $\overline{B} = B/\sim U$ and $\mathcal{B} = B(\overline{R})$

If the ultrafilter U *is such that the closed subspace* $V(a_U)$ *of* Spec(B) *is not extremally disconnected, then* $\overline{R}$ *is a* (TB)-*ring which is not complete.*

§2.

In this section we introduce a new kind of rings which appear "dual" to the ones of §1 by means of the reverse topology (see [11]).

<u>Definition 2.1</u> A ring is said to be an mp-*ring* if every prime ideal contains a unique minimal prime ideal.

<u>Remark 2.2</u> The "duality" between the (PM)-property and the (MP)-property established by means of the reverse topology is only with respect to the order of prime ideals. In fact, if a ring is pm, then its maximal spectrum is compact (see [8]), but the (MP)-property does not imply that the minimal spectrum of the ring is compact. Take as a counter-example C(X,R) the ring of all continuous real-valued functions on a topological space X which is an F-space but not basically disconnected (see [2], [9]).

<u>Propostion 2.3</u> *A direct product* (resp. *an ultraproduct*) *of*

reduced mp-rings is a reduced mp-ring.

Remark 2.4 If we drop the hypothesis that the rings are reduced, the above result does not hold any more as the following example shows.

Example 2.5 Let $A = \prod_{n=1}^{\infty} A_n$, where $A_n = K[X_n,Y_n]/(X_nY_n,X_n^n,Y_n^n)$ and K is a field.

Remark 2.6 Notice that for pm-rings the nilpotent elements do not have any influence on the result (see [7]), and therefore A in the above example is a pm-ring.

Special case of mp-rings

1. Weak Baer rings

Definition 2.7 A ring is said to be a *weak Baer ring* if the annihilator ideal of every element is principal and generated by an idempotent element (see [4], [15]).

Proposition 2.8 *A direct product* (resp. *an ultraproduct*) *of weak Baer ring is a weak Baer ring.*

2. Baer rings

Definition 2.9. A is said to be a *Baer ring* if the annihilator ideal of every ideal is principal and generated by an idempotent element (see [12]).

A useful characterization of the above rings is the following:

Propostion 2.10 A *is a Baer ring iff it is a weak Baer ring whose boolean algebra of idempotent elements is complete.*

Proposition 2.11 *A direct product of Baer rings is also a Baer ring.*

We end the section and the Note with a result "dual" to that of theorem 1.12.

Theorem 2.12 *With all notation as in theorem 1.12, but assume that every* R_λ *is a Baer ring. Then* $\bar{R}$ *may note not Baer ring.*

We wish to thank Professor P. Ribenboim and Professor C. Weibel for useful discussions.

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Received October 30, 1981

BRAUER GROUPS FOR KRULL DOMAINS

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We construct Brauer groups and class groups for suitable categories of divisorial lattices over a Krull domain or a Krull scheme, and derive exact sequences linking them. These sequences give a unified way of recovering diverse results about Brauer groups of noetherian integrally closed domains, while also extending these results to a more general setting.

The exposition here is an overview without proofs of work to appear in [6, 7, 9, 10]. Let R be a Krull domain, K its field of fractions. Let Z denote the set of height one primes of R, so that for each p in Z, R_p is a D.V.R. For any R-module M let $\tilde{M}$ denote the intersection of the images of M_p in $K\otimes_R M$, p ranging through Z. Call M divisorial if $M = \tilde{M}$. Write $M\tilde{\otimes}_R N$ for $\tilde{P}$, where $P = M\otimes_R N$. As noted in [13], $M\tilde{\otimes}_R N$ has a universal mapping property vis-a-vis divisorial R-modules L : Let h from $M\otimes_R N$ to $M\tilde{\otimes}_R N$ be the map satisfying $h(m\otimes n) = 1\otimes m\otimes n$ in $K\otimes_R M\otimes_R N$. Given any R-homomorphism f from $M\otimes_R N$ to L, there is a unique R-homomorphism g from $M\tilde{\otimes}_R N$ to L such that $gh = f$.

An R-lattice is a torsion-free R-module M such that $M \subseteq F \subseteq K\otimes_R M$ with F a finitely generated R-module. If M and N are divisorial R-lattices so are $M\tilde{\otimes}_R N$ and $\mathrm{Hom}_R(M,N)$. By the mapping property of $M\tilde{\otimes}_R N$ there exist R-module maps

$$f : M\tilde{\otimes}_R M^* \to \mathrm{End}_R(M), \quad g : \mathrm{End}_R(M)\tilde{\otimes}_R\mathrm{End}_R(N) \to \mathrm{End}_R(M\tilde{\otimes}_R N).$$

[1] Research supported by N.S.E.R.C. grant A7202

In fact, f and g are isomorphisms. This can be shown as follows: By divisoriality it is sufficient to show f_p and g_p are isomorphisms for each p in Z. The operation $\tilde{\otimes}$ commutes with localization. Both M_p and N_p are free R_p-modules of finite rank for p in Z, and for such modules $\tilde{\otimes} = \otimes$.

Let $\mathcal{D}(R)$ be the category of divisorial R-lattices. Suppose we have a subcategory $\mathcal{C}$ of $\mathcal{D}(R)$ which satisfies:

(A1) $R \in \mathcal{C}$.

(A2) M and N in $\mathcal{C}$ implies $M\tilde{\otimes}_R N$ and $\mathrm{Hom}_R(M,N)$ are in $\mathcal{C}$.

The class group $\mathrm{Cl}(\mathcal{C})$ is defined as the set of isomorphism classes of modules in $\mathcal{C}$ of rank one. The operations are given by $\{I\}\{J\} = \{I\tilde{\otimes}_R J\}$ and $\{I\}^{-1} = \{I^*\}$. The Brauer group $\mathrm{Br}(\mathcal{C})$ is defined as $\mathrm{Az}(\mathcal{C})/\sim$: $\mathrm{Az}(\mathcal{C})$ consists of R-algebras A which have center R, are in $\mathcal{C}$ as R-modules, and for which $A\tilde{\otimes}_R A^o \to \mathrm{End}_R(A)$ is an isomorphism. The relation $\sim$ is given by $A \sim B$ if $A\tilde{\otimes}_R \mathrm{End}_R(P) \simeq B\tilde{\otimes}_R \mathrm{End}_R(Q)$ with P and Q in $\mathcal{C}$. $\mathrm{Br}(\mathcal{C})$ is a group with operations $[A][B] = [A\tilde{\otimes}_R B]$, $[A]^{-1} = [A^o]$.

If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{D}(R)$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy axioms (A1) and (A2) we define a group $\mathrm{BCl}(\mathcal{C}_1,\mathcal{C}_2)$. Its elements are equivalence classes M where M in $\mathcal{C}_2$ is such that $\mathrm{End}_R(M)$ is in $\mathcal{C}_1$, and the equivalence relation is given by $M \sim N$ when $M\tilde{\otimes}_R P \simeq N\tilde{\otimes}_R Q$ for P and Q in $\mathcal{C}_1$. The operations on $\mathrm{BCl}(\mathcal{C}_1,\mathcal{C}_2)$ are $\langle M\rangle\langle N\rangle = \langle M\tilde{\otimes}_R N\rangle$ and $\langle M\rangle^{-1} = \langle M^*\rangle$. We shall say $\mathcal{C}$ satisfies (A3) if M and $M\tilde{\otimes}_R N$ in $\mathcal{C}$ and N in $\mathcal{D}(R)$ implies N is in $\mathcal{C}$.

Theorem 1. Let $\mathcal{C}_1 \subseteq \mathcal{C}_2$ be subcategories of $\mathcal{D}(R)$ satisfying axioms (A1), (A2) and (A3). Then we have an exact sequence

$$1 \to Cl(C_1) \to Cl(C_2) \to BCl(C_1,C_2) \to Br(C_1) \to Br(C_2).$$

Let $i : R \hookrightarrow S$ be an inclusion of Krull domains. If S is divisorial as an R-module, the correspondence $M \to S\tilde{\otimes}_R M$ gives a functor $\mathcal{D}(i) : \mathcal{D}(R) \to \mathcal{D}(S)$ (see [13]). It is shown in [10] that S is divisorial over R if and only if i is a Krull morphism, i.e., it satisfies the condition called (PDE) or (NBU), which says that primes of height one in S contract to primes of R having height at most one. With this fact one can show that $\mathcal{D}(\ )$ is a fibered category over the category K of Krull domains and Krull morphisms. In particular $\mathcal{D}(ji) = \mathcal{D}(j)\mathcal{D}(i)$ for $i : R \hookrightarrow S$ and $j : S \hookrightarrow T$ in K. Suppose $C(\ )$ is a fibered subcategory of $\mathcal{D}(\ )$ over K, with axioms (A1) and (A2) valid in all $C(R)$, R in K. We get that $Cl(C(\ ))$, $Br(C(\ ))$ are then functors from K to Ab. If $C_1(\ ) \subseteq C_2(\ )$ are suitable categories then $BCl(C_1(\ ),C_2(\ ))$ gives a functor as well.

The results above remain valid for a Krull scheme $(X,\mathcal{O}_X)$ and we shall now consider, in that setting, several candidates for C : 1. $\mathcal{D}$ = the divisorial $\mathcal{O}_X$-lattices. 2. Let $Z \subseteq Y$ X. P_Y = the $\mathcal{O}_X$-lattices locally free on Y. We write P for P_X. 3. Y as in 2. I_Y = the $\mathcal{O}_X$-lattices M such that for each y in Y, M_y is a direct sum $I(y)^n$ for some $\mathcal{O}_{X,y}$-module $I(y)$ of rank one. The corresponding notation for $Cl(C)$ and $Br(C)$ is: 1. $Cl(X)$ and $\beta(X)$. 2. $Pic_Y(X)$ and $Br_Y(X)$ (for $Y = X$ we write $Pic(X)$ and $Br(X)$). 3. $Cl(X)$ and $Br(I_Y(X))$. We write $BCl(X)$ for $BCl(P,\mathcal{D})$.

<u>Theorem 2</u>. Let $(X,\mathcal{O}_X)$ be a Krull scheme with function field K, Y a subset of X containing Z.

(a) $\beta(X) \to Br(K)$ is one-one; in fact $\beta(X) = \bigcap_{x\in Z} Br(\mathcal{O}_{X,x})$.

(b) $Br(I_Y(X)) \to \prod_{p \in Y} Br(O_{X,y})$ is one-one.

(c) Assume that for each x in X the strict henselization of $O_{X,x}$ is factorial. Then $Br(X) \to Br(K)$ is one-one.

(d) There are exact sequences

$$1 \to Pic(X) \to Pic_Y(X) \to BCl(P, P_Y) \to Br(X) \to Br_Y(X),$$

$$1 \to Pic(X) \to Cl(X) \to BCl(X) \to Br(X) \to Br(K),$$

$$1 \to Pic(X) \to Cl(X) \to BCl(P, I_Y) \to Br(X) \to \prod_{y \in Y} Br(O_{X,y}).$$

These results extend and unify work of Auslander [1,2]. Some generalizations are from noetherian normal domains to Krull schemes, others from the case $Y = X$ to that of any Y as described. The result in (c) was obtained cohomologically by Grothendieck [5].

Let $C(\ )$ be a fibered subcategory of $D(\ )$ over K. Let G be a finite (or profinite) group of automorphisms of S, $R = S^G$, with R and S Krull domains. We'll call S a C-Galois extension of R if three conditions hold: 1. The map j from $\Delta(S,G)$ (the trivial crossed-product) to $End_R(S)$ given by $j(su_\sigma)(t) = s\sigma(t)$ is an isomorphism. 2. $S \in C(R)$ (in particular $R \to S$ is a Krull morphism). 3. If M is in $C(S)$ then M is in $C(R)$. To extend this to Krull schemes X and Y with $X = Y^G$ we require that $Y \to X$ be an affine morphism. Write T^* for the units of T, $Br(C(Y/X))$ for $Kernel(Br(C(X)) \to Br(C(Y)))$.

<u>Theorem 3</u>. Let $Y \to X$ be a C-Galois extension of Krull schemes. There is an exact sequence

$$1 \to H^1(G, O_Y(Y)^*) \to Cl(C(X)) \to Cl(C(Y)^G) \to H^2(G, O_Y(Y)^*) \to$$
$$\to Br(C(Y/X)) \to H^1(G, Cl(Y)) \to H^3(G, O_Y(Y)^*).$$

This theorem specializes to familiar results: to a theorem of Auslander and Brumer or Chase and Rosenberg (see [3, Ch. IV]) by taking $X = \mathrm{Spec}(R)$, $C = P$; to a theorem of Rim [12] by taking $X = \mathrm{Spec}(R)$, $C = D$ and using (a) of Theorem 2 to substitute $\cap \mathrm{Br}(S_p/R_p)$ (p in Z) for $\beta(S/R)$; to a theorem of Lichtenbaum [8] by taking X a complete, smooth, geometrically connected curve over k and $G = \mathrm{Gal}(k^s/k)$. The exact sequence of Theorem 3 can be used to compute $\mathrm{Br}(X)$ in special cases. For example, let k be a field of characteristic zero. If F is a non-degenerate n-ary quadratic form with n at least 5 and X is the projective variety in $P^{n-1}(k)$ or $P^n(k)$ defined by F, then $\mathrm{Br}(X) = \mathrm{Br}(k)$.

The Brauer groups we have constructed are examples of Brauer groups of monoidal categories as constructed by Fröhlich and Wall [4] and by Pareigis [11]. Our considerations about Krull schemes and morphisms are what is necessary to show that we are indeed in the context developed by these authors.

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Received November 2, 1981

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