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# ANALYTIC ITERATION AND POWER ROOTS

Ludwig Reich and Arnold R. Kräuter

*Presented by J. Aczél, F.R.S.C.*

In our papers [5] and [6] we considered the connection between the existence of a sequence of roots of a formally biholomorphic mapping $F$ and the analytic iterability of $F$ under different points of view. In [2] the first author moreover investigated the possibility of replacing the fractional iterates (i.e. roots) by rational iterates in the case of shrinking biholomorphic mappings. This suggests to look for analogous results for formally biholomorphic mappings. This is the object of the present paper. Instead of "rational iterates" we shall speak of "power roots". (For the notation we refer to [5].)

Theorem 1. Suppositions:

(a) Let $F \in \Gamma$ and $\lambda = (\lambda_1,\ldots,\lambda_n)$ be an arbitrary but fixed choice of the logarithms of the eigenvalues of $\operatorname{lin}(F)$.

(b) Let $(F^{p_j/q_j})_{j\in N}$ be a sequence of power roots of $F$ with $[\frac{q_j}{p_j}] < [\frac{q_{j+1}}{p_{j+1}}]$, such that $\operatorname{lin}(F^{p_j/q_j})$ has the eigenvalues $\exp(\frac{p_j}{q_j}\lambda_1),\ldots,\exp(\frac{p_j}{q_j}\lambda_n)$ for all $j \in \mathbb{N}$.

(c) For all $j \in \mathbb{N}$ and for all $m \in \mathbb{N}\setminus\{1\}$, let $R_j^{(m)}$ denote the set

$$R_j^{(m)} := \{n_k^{(j,m)}(\beta) \in \mathbb{Z} : \frac{p_j}{q_j}\lambda_k = \sum_{\nu=1}^{n} \beta_\nu \frac{p_j}{q_j}\lambda_\nu + 2\pi i n_k^{(j,m)}(\beta),\ \beta \in \mathbb{N}_0^n,\ 2 \le |\beta| \le m,\ k=1,\dots,n\}\ .$$

Then we assume that $p_j$ divides $q_j n_k^{(j,m)}(\beta)$ for all $n_k^{(j,m)}(\beta) \in R_j^{(m)}$.

(d) For all $m \in \mathbb{N}\setminus\{1\}$ and for all $j \in T(m)$ (where $T(m)$ has the same meaning as in the proof of Theorem 1 in [5]) the following assumption holds: If the m-jet of $W(j,m)^{-1}F^{p_j}W(j,m)$ has at most smooth additional monomials with respect to $\Lambda$ then the same holds for $W(j,m)^{-1}FW(j,m)$.

(e) Let $F^{p_j/q_j}$ and $F^{p_{j+1}/q_{j+1}}$ commute for all $j \in \mathbb{N}$.

Assertion: Then there exists an analytic iteration of $F$ with respect to $\Lambda$.

The proof is quite similar to that of Theorem 1 in [5]. However, to apply the methods used there it was necessary to assume the additional conditions (c) and (d), respectively. An analogous theorem can be formulated for Theorem 3 in [5].

Remark 1. Theorem 1 in [5] is contained in Theorem 1 above.

Considering the circumstances in the case of a single power root $F^{p/q}$ of $F$, we can only deduce the iterability of $F^p$:

Theorem 2. Suppositions:

(a) as in Theorem 1.

(b) There exists a power root $F^{p/q}$ of $F$ with respect to $(\exp(\frac{p}{q}\lambda_1),\dots,\exp(\frac{p}{q}\lambda_n))$ for $\frac{p}{q} > M(\Lambda)$ where $M(\Lambda) \in \mathbb{N}$

denotes a constant depending only on $\Lambda$ such that the set $R_0$ (defined as in Theorem 1 in [6]) is bounded by $M(\Lambda)$, i.e. $|n_{k\ell}| < M(\Lambda)$ for all $n_{k\ell} \in R_0$.

(c) Let $R_1$ denote the set

$$R_1 := \{m_k(\beta) \in \mathbb{Z} : \frac{p}{q}\lambda_k = \sum_{\nu=1}^{n} \beta_\nu \frac{p}{q}\lambda_\nu + 2\pi i m_k(\beta) ,$$

$$\beta \in \mathbb{N}_0^n, \quad |\beta| \geq 2;\ k=1,\dots,n\} .$$

Then assume that $p$ divides $qm_k(\beta)$ for all $m_k(\beta) \in R_1$.

Assertion: Then there exists an analytic iteration of $F^p$ with respect to $p\Lambda$.

Proof. The condition $|n_{k\ell}| < M(\Lambda)$ for all $n_{k\ell} \in R_0$ yields (together with (c)) that all monomials additional to $\exp(\frac{p}{q}\lambda_k)$ are at the same time smooth monomials additional to $\exp(\lambda_k)$ with respect to $\Lambda$. But these are also smooth monomials additional to $\exp(p\lambda_k)$ with respect to $p\Lambda$ and vice versa. By assumption there exists a q-th root $F^{p/q}$ of $F^p$. Now, by [4], $F^p$ is conjugate to a normal form $N : x \to N(x) = Jx + \mathfrak{N}(x)$ such that in $\mathfrak{N}_k(x)$ at most monomials additional to $\exp(\frac{p}{q}\lambda_k)$ occur. According to what we have mentioned above, $N$ has therefore at most smooth additional monomials with respect to $p\Lambda$. This implies (see [3], p. 219) the existence of an analytic iteration of $F^p$ with respect to $p\Lambda$. $\square$

Remark 2. Theorem 1 in [6] is contained in Theorem 2 above.

Remark 3. In connection with Theorem 2 a result by D.C. Lewis Jr. in [1] is worth noting:

Let $F \in \Gamma$. Then there always exists an exponent $p \in \mathbb{N}$

such that $F^p$ is analytically iterable with respect to a suitable choice of logarithms of the eigenvalues of $\mathrm{lin}(F^p)$, $\tilde{\Lambda}$.

An elegant proof of this result, using methods from the theory of normal forms, has been obtained recently by J. Schwaiger.

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# CONGRUENCE PROPERTIES OF THE SOLUTIONS OF PELL'S EQUATION

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Every known solution of the congruence subgroup problem (cf. [1] & [2]) depends on some deep results from class field theory. We give here an elementary proof in the case of the torus T defined by Pell's equation

$$x^2 - my^2 = 1, \tag{1}$$

where m is a square-free integer. In fact, we prove a stronger result than any known so far. The problem is interesting only when $m > 0$.

The multiplication in T is given by

$$(x,y)(x',y') = (xx' + myy', xy' + x'y). \tag{2}$$

Let $T(\mathbb{Z}) = T \cap \mathbb{Z} \times \mathbb{Z}$ and for an integer $N \neq 0$, let

$$T(N) = \{g \in T(\mathbb{Z}) \mid g \equiv e (\text{mod } N)\},$$

where $e = (1,0)$ is the identity of the group $T(\mathbb{Z})$. We prove the following

<u>THEOREM</u>. <u>Let</u> $\Gamma$ <u>be a subgroup of</u> $T(\mathbb{Z})$ <u>with finite index</u> $[T(\mathbb{Z}) : \Gamma]$. <u>Then</u> $\Gamma$ <u>contains a subgroup</u> $T(N)$ <u>for some</u> $N > 0$.
<u>Moreover, if we put</u>

$$i(\Gamma) = \inf\{[\Gamma : T(N)] \mid T(N) \subseteq \Gamma\}.$$

<u>then</u>

$$i(T) \overset{\text{def}}{=} \sup\{i(\Gamma) \mid [T(\mathbb{Z}) : \Gamma] < \infty\} = 1 \text{ or } 2.$$

<u>Proof</u>. It is well known that $T(\mathbb{Z})$ is a free group generated by a single element, namely the smallest positive solution $(x_1, y_1)$ of (1). Since $x_1, y_1$ and m are all positive, it is obvious from (2) that

$$o < y_1 < y_2 < y_3 < \cdots . \tag{3}$$

For $n \in \mathbb{Z}$, let $(x_n, y_n) = (x_1, y_1)^n$. Then for $i = 0, 1$ and $0 < j < n$, we see that

$$(x_{ni+j}, y_{ni+j}) \not\equiv (1,0) \pmod{y_n}. \tag{4}$$

If $i = 0$, this is obvious by (3). Let $i = 1$. Then

$$\begin{aligned}(x_{n+j}, y_{n+j}) &= (x_n y_j + m x_j y_n,\ x_n y_j + x_j y_n) \\ &\equiv (1,0) \pmod{y_n}\end{aligned}$$

implies that $y_n \mid x_n y_j$. But $x_n$ and $y_n$ are coprime. Therefore $y_n \mid y_j$, contradicting (3) again. Now we put $N = y_n$, where $n = [T(\mathbb{Z}):\Gamma]$. If $x_n \equiv 1 \pmod N$, then by (4), $\Gamma = T(N)$. Otherwise

$$\begin{aligned}(x_{2n}, y_{2n}) &\equiv (x_n^2, 0) \\ &\equiv (1,0) \pmod N.\end{aligned}$$

By (4) again, $\Gamma$ contains $T(N)$ as a subgroup of index 2.

Remark. For an algebraic group $G$ defined over a number field $k$, one can define $i(G)$ as above. It would be intersting to know, when is $i(G)$ finite.

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# ON A STABILITY THEOREM

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ABSTRACT: We give a simple proof for a generalization of a stability theorem of Baker concerning the cosine equation.

John A. Baker in [1] proves the following stability property of the cosine equation: if $G$ is an Abelian group and $f: G \to C$ is a complex valued function with the property, that the function $(x,y) \to f(x+y)+f(x-y) -2f(x)f(y)$ is bounded then either $f$ is bounded or $f(x+y)+f(x-y)=2f(x)f(y)$ holds for all $x,y$ in $G$. In this note we extend and greatly simplify this result.

If $G$ is a semigroup, $F$ is a field and $V$ is a linear space of functions defined on $G$ and having values in $F$, then we call $V$ right invariant, if $f$ belongs to $V$ implies that the function $x \to f(xy)$ belongs to $V$ for every $y$ in $G$. Similarly, we define left invariant spaces, and we call $V$ invariant, if it is right and left invariant.

Our main result is the following:

THEOREM. Let $G$ be a group, $F$ a field and $V$ a linear space of functions on $G$ and having values in $F$.

*Let* $f,g\colon G\to F$ *be functions such that* $f(xyz)=f(xzy)$ *holds for all* $x,y,z$ *in* $G$. *If for every* $y$ *in* $G$ *the function* $x\to f(xy)+f(xy^{-1})-2f(x)g(y)$ *belongs to* $V$ *and* $V$ *is right invariant, or for every* $x$ *in* $G$ *the function* $y\to f(xy)-f(xy^{-1})-2f(y)g(x)$ *belongs to* $V$ *and* $V$ *is left invariant, then either* $f$ *belongs to* $V$ *or* $g(xy)+g(xy^{-1})=2g(x)g(y)$ *holds for every* $x,y$ *in* $G$.

*PROOF*. Let, for $x,y$ in $G$

$$A(x,y)=f(xy)+f(xy^{-1})-2f(x)g(y)$$

and

$$B(x,y)=f(xy)-f(xy^{-1})-2f(y)g(x)\ .$$

Then one can easily check the equations

$$A(xy,z)+A(xy^{-1},z)-A(x,yz)-A(x,yz^{-1})+2g(z)A(x,y)$$

$$=2f(x)(g(yz)+g(yz^{-1})-2g(y)g(z))$$

and

$$B(xy,z)+B(xy^{-1},z)-B(x,yz)+B(x,yz^{-1})-2g(x)B(y,z)$$

$$=-2f(z)(g(xy)+g(xy^{-1})-2g(x)g(y))$$

for all $x,y$ in $G$, and hence our statement follows.

REMARK. Let $G$ be a group and $f:G\to C$ a function for which $f(xyz)=f(xzy)$ holds for all $x,y,z$ in $G$. Suppose that for every $y$ in $G$ the function $x\to f(xy)+f(xy^{-1})-2f(x)f(y)$ is bounded, and here the bound can depend on $y$. Then, choosing $V$ to be the linear space of all bounded complex functions on $G$ in the above theorem, we get that either $f$ is bounded, or it is a solution of the equation $f(xy)+f(xy^{-1})=2f(x)f(y)$. Using the results of [2], we obtain Baker's result: either $f$ is bounded, or it is the even part of an exponential. We remark that by choosing other "function-properties" instead of "boundedness" (for instance "continuity", "measurability", "integrability" whenever $G$ is a locally compact topological group), we can obtain further interesting results analoguous to that of Baker.

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# THE DETERMINATION OF $\pi$-WEIGHT BY SUBSPACES OF SINGULAR CARDINALITY

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*Presented by G. de B. Robinson, F.R.S.C.*

ABSTRACT

If $\kappa$ is a singular cardinal of countable cofinality and X is a Hausdorff topological space of $\pi$-weight $\geq \kappa$ then X has a subspace Y of cardinality $< \kappa$ with $\pi$-weight $\geq \kappa$. The situation for singular cardinals of uncountable cofinality is discussed. The analagous theorem for regular cardinals was earlier proven by Hajnal and Juhász.

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The weight of a topological space is determined by its small subspaces [1]. Precisely, $w(X) < \kappa$ if $w(Y) < \kappa$ for all $Y \subseteq X$ with $|Y| \leq \kappa$. The proof appears in the book [2] which we use as a reference for terminology and results used below. There it is also noted that a similar result holds for $\pi$-weight instead of weight provided that $\kappa$ is a regular cardinal. The situation for singular cardinals is, however, an open problem.

We show below that the $\pi$-weight situation for a particular topological space X and singular cardinal $\kappa$ may, in contrast to the case of weight, depend upon the topological properties of X and the cofinality of $\kappa$. The following problem, however, remains open.

Problem Is there a singular cardinal $\kappa$ and a regular topological space X such that $\pi(X) \geq \kappa$ but for each subspace $Y \subseteq X$ of cardinality $\leq \kappa$ we have $\pi(Y) < \kappa$?

A (not necessarily regular) topological space having the above property will for the purposes of this article be called a $\kappa$-example.

E. van Douwen has shown [2] that for every singular cardinal $\kappa$ there is a $T_1$ $\kappa$-example and also gave consistent $T_2$ $\kappa$-examples for $\kappa$ with $\omega < cf(\kappa) < \kappa$.

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Recalling the result of Hajnal and Juhász, [1], about weight and noting that for a $T_2$ space X, $w(X) \leq \exp(\exp(\pi(X)))$, we easily observe that if $\kappa$ is a strong limit cardinal, then there is no $T_2$ $\kappa$-example.

Our key observation concerning $\kappa$-examples is the following.

Lemma If X is a $\kappa$-example, then X has an open subspace which is also a $\kappa$-example and furthermore doesn't contain $cf(\kappa)$ many pairwise disjoint open subsets.

Proof Let $\mathcal{U}$ be a maximal pairwise disjoint family of open subsets of a $\kappa$-example X such that for each $U \in \mathcal{U}$, $\pi(U) < \kappa$. We must have $|\mathcal{U}| < \kappa$ since if $|\mathcal{U}| \geq \kappa$ we could form $Y \subseteq X$ such that $|Y \cap U| = 1$ for each of $\kappa$ many $U \in \mathcal{U}$ and contradicting that X is a $\kappa$-example.

Now note that $\pi(\cup\mathcal{U}) < \kappa \cdot \kappa = \kappa$. Hence $\pi(X - \overline{\cup\mathcal{U}}) > \kappa$ and we claim that this open set satisfies the lemma.

Let $\{\kappa_\alpha : \alpha < cf(\kappa)\}$ be a sequence of regular cardinals cofinal in $\kappa$ and suppose $\{V_\alpha : \alpha < cf(\kappa)\}$ is a pairwise disjoint collection of nonempty open subsets of $X - \overline{\cup\mathcal{U}}$. By the maximality of $\mathcal{U}$, for each $\alpha < cf(\kappa)$ we have $\pi(V_\alpha) \geq \kappa$. By the result of Hajnal and Juhász [1], $V_\alpha$ is not a $\kappa_\alpha$-example so there is $Z_\alpha \subseteq V_\alpha$ of cardinality $\leq \kappa_\alpha$ such that $\pi(Z_\alpha) \geq \kappa_\alpha$. $Z = \cup\{Z_\alpha : \alpha < cf(\kappa)\}$ contradicts that X is a $\kappa$-example, since $|Z| \leq \kappa$ and $\pi(Z) \geq \kappa$.

As an immediate consequence we obtain:

Theorem If $\omega = cf(\kappa) < \kappa$ then there is no $T_2$ $\kappa$-example.

We now combine the lemma with some non-trivial results of other authors to obtain the following.

Theorem If $\kappa = \kappa^{cf\kappa}$, then there is no locally compact $\kappa$-example.

Proof In defiance of the theorem, let Y be such a $\kappa$-example. By the lemma we have an open $X \subseteq Y$ such that $\hat{c}(X) \leq cf(\kappa)$. Now

$$\pi(X) \leq \pi\chi(X)^{\hat{c}(X)}$$

by a result of B. Šapirovski [2]; hence $\pi\chi(X) > \kappa$. By another result of B. Šapirovski [2],

$$\pi\chi(X) \leq t(X) \ ;$$

hence $t(X) > \kappa$. Now by a result of A.V. Archangel'skii [2],

$$t(X) \leq s(X)$$

and hence $s(X) > \kappa$ and there is a discrete subspace Z of X of cardinality $\kappa$, contradicting that Y is a $\kappa$-example.

Theorem Assume Martin's Axiom. If $cf(\kappa) = \omega_1$ and $\kappa^+ < 2^\omega$ then there is no locally compact $T_2$ $\kappa$-example.

Proof Again suppose that Y is such a $\kappa$-example. By the lemma there is an open $X \subseteq Y$ with $c(X) = \omega$. Since X is a $\kappa$-example $s(X) \leq \kappa$ so by Archangel'skii's result $\tau(X) \leq \kappa$. Therefore $\tau(X)^+ < 2^\omega$. We now have by Martin's Axiom that X is separable (see [3] page 507). This gives the required contradiction.

By R. Jensen's Covering Lemma (see [4] page 356), if $cf(\kappa) = \omega_1$ and $2^\omega < \kappa$, then $\kappa^{cf\kappa} = \kappa$. The Covering Lemma is now known to follow from the non-existence of measurable cardinals. This leads to the following bizarre result.

Corollary Assume Martin's Axiom and $cf(\kappa) = \omega_1$.

Then either (i) there is a locally compact $\kappa$-example,

(ii) $2^\omega = \kappa^+$, or

(iii) there is a measurable cardinal.

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# ON THE SLOW MOTION OF A FLUID COATED SPHERE IN AN IMMISCIBLE VISCOUS FLUID

M.E. O'Neill and K.B. Ranger, F.R.S.C.

Abstract. The flow of a viscous fluid past a sphere coated with an immiscible viscous fluid is considered and an application to the translation of a coated sphere in a plane interface between immiscible fluids is given.

## 1. Introduction.

Two phase flow problems involving two immiscible viscous fluids are of considerable importance in chemical engineering science and one of the most fundamental of these problems concerns the motion of a rigid body towards and through the interface between two immiscible fluids of arbitrary viscosities. The mathematical difficulties encountered in attempting to provide an analytical solution are formidable and there has recently been renewed interest in presenting theoretical models for those flow situations when the displacement of one fluid into the region formerly occupied by the second fluid is small and the interface can be regarded as effectively flat. This can be achieved by modifying the usual non-slip boundary condition on the rigid body so that a degree of slip is permitted. This enables a finite force acting on the body to be predicted and thus the body can pass across the interface. Such a study has recently been carried out by O'Neill, Ranger and Brenner (1981) for the case of a sphere moving half submerged through a free surface, and O'Neill and Ranger (1981) have presented a generalised theory describing the motion of an axisymmetric body which straddles a flat interface between immiscible fluids.

Another way of circumscribing the difficulties associated with the non-slip boundary condition when a sphere moves through an interface is to take into account the displaced fluid by regarding it as a coating, and in this communication we consider the slow flow past such a coated sphere of an immiscible viscous fluid. The non-slip condition is applied at the surface of the rigid sphere while continuity of velocity and stress components apply at the surface of the coating. An application to the translation of a coated sphere in an interface is given.

## 2. Flow past a coated rigid sphere.

A rigid sphere of radius $\lambda a$ $(0 < \lambda < 1)$ is surrounded by a concentric coating of fluid of thickness $(1-\lambda)a$ and viscosity $\mu_1$. Exterior to the coating an immiscible fluid of viscosity $\mu_0$ is streaming with constant speed $U$ in the direction $\theta = \pi$ of a system of spherical polar coordinates $(ar, \theta, \phi)$ with pole at the centre of the sphere.

At the surface $r = 1$ of the coating, the normal component of velocity vanishes and the tangential components of velocity and stress are continuous. In the first instance we shall suppose that the rigid sphere $r = \lambda$ is at rest in which case to satisfy the non-slip condition, the velocity of the fluid vanishes when $r = \lambda$. The motions of the two immiscible fluids are axially symmetric about the axis $\theta = 0$ and suitable forms for the stream functions $\psi_1$ and $\psi_0$ within the fluids of viscosity

$\mu_1$ and $\mu_0$ respectively are

$$\psi_1 = U\left(A r^4 + Br^2 + Cr + D/r\right)\sin^2\theta\,, \quad (\lambda < r < 1)\,, \tag{1}$$

$$\psi_0 = U\left(\tfrac{1}{2} r^2 + Er + F/r\right)\sin^2\theta\,, \quad (r \geq 1)\,. \tag{2}$$

Application of the boundary conditions leads to the following six equations involving the constants A,B,C,D,E,F:

$$\left.\begin{aligned} &A + B + C + D = 0 = \tfrac{1}{2} + E + F\,,\\ &4A + 2B + C - D = 1 + E + F\,,\\ &4A - 2B - 2C + 4D = \mu[-1-2E+4F]\,,\\ &A\lambda^5 + B\lambda^3 + C\lambda^2 + D = 0\,,\\ &4A\lambda^5 + 2B\lambda^3 + C\lambda^2 - D = 0\,, \end{aligned}\right\} \tag{3}$$

where $\mu = \mu_0/\mu_1$. The solution of these equations is

$$[\lambda^5-5\lambda^3+5\lambda^2-1]A = -\tfrac{1}{2}(\lambda^3-\lambda^2) + F[2(\lambda^3-\lambda^2) - \mu(2\lambda^3-3\lambda^2+1)]\,, \tag{4}$$

$$[\lambda^5-5\lambda^3+5\lambda^2-1]B = \tfrac{1}{2}(\lambda^5-1) - F[2(\lambda^5-1) - \mu(2\mu^5-5\mu^2+3)]\,, \tag{5}$$

and

$$C = -\mu F - B\,, \quad D = \mu F - A\,, \quad E = \tfrac{1}{2} - F\,, \tag{6}$$

where $F = \frac{1}{2}(2+\alpha\mu)^{-1}$ with

$$\alpha = (4+3\lambda-3\lambda^2-4\lambda^3)/(2+3\lambda+3\lambda^2+2\lambda^3)\,. \tag{7}$$

The force acting on the coated sphere is $F_z \hat{\underset{\sim}{k}}$, where

$$F_z = \pi\mu_0 a \int_0^\pi \varpi^3 \frac{\partial}{\partial r}\left[\frac{L_{-1}\psi_0}{\varpi^2}\right] d\theta\,, \quad L_{-1} \equiv \frac{\partial^2}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi} + \frac{\partial^2}{\partial z^2} \tag{8}$$

and the integrand is evaluated on the surface of the coating $r=1$ and $(a\varpi, \phi, az)$ are cylindrical polar coordinates related to the spherical polar coordinates in the usual way. Substitution of $\psi_0$ leads to

$$-\frac{F_z}{6\pi\mu_0 Ua} = -\frac{4}{3}E = \frac{1+\alpha\mu/3}{1+\alpha\mu/2} \tag{9}$$

The function $\alpha=2$ when $\lambda=0$ and $\alpha=0$ when $\lambda=1$. Other values are displayed in the table

| $\lambda$ | $\alpha$ |
|---|---|
| 0.0 | 2.0000 |
| 0.2 | 1.6257 |
| 0.4 | 1.1723 |
| 0.6 | 0.9444 |
| 0.8 | 0.3312 |
| 1.0 | 0.0000 |

When $\lambda=0$, equation (9) gives

$$-\frac{F_z}{6\pi\mu_0 Ua} = \frac{1+2\mu/3}{1+\mu} \tag{10}$$

which is the well known formula for the force acting on a spherical droplet in a uniform stream. When $\lambda=1$, the right hand side of (9) becomes unity, corresponding to Stokes law for the force acting on a rigid sphere in a uniform stream. Such results can be found in Happel & Brenner (1965).

Solutions can be similarly constructed if the rigid sphere has a non-zero velocity and the surface of the coating has a different translational velocity from that of the rigid sphere. Of particular interest among such solutions are those for which the tangential stress vanishes on $r=1$. For instance, if the rigid sphere has an instantaneous velocity $KU$ along the z-axis, where

$$K = (2\lambda^6 - 3\lambda^5 + 3\lambda - 2)/\lambda(2\lambda^5 - 5\lambda^2 + 3) ,$$

the non-slip condition can be satisfied on $r = \lambda$ together with vanishing tangential component of stress and normal component of velocity on $r = 1$. The solution in the outer fluid phase is then just that for flow past a spherical bubble, and the stream function is accordingly

$$\psi_0 = \frac{1}{2} r(r-1)\sin^2\theta . \qquad (11)$$

The flows interior and exterior to the coating are now uncoupled and it follows from O'Neill and Ranger (1981) that equation (11) is also the stream function in the exterior flow of two immiscible fluids with viscosities $\mu_1$ and $\mu_2$ when the coated sphere moves parallel to the interface which coincides with an azimuthal plane. The force on the coated sphere is then $F_z \hat{\underset{\sim}{k}}$ where

$$F_z = 2\pi Ua(\mu_1 + \mu_2) .$$

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# ITERATED TILTED ALGEBRAS OF TYPES $\mathbb{B}_n$ and $\mathbb{C}_n$

Ibrahim Assem

*Presented by V. Dlab, F.R.S.C.*

Abstract: We classify the iterated tilted algebras of types $\mathbb{B}_n$ and $\mathbb{C}_n$ in terms of their bounden species.

Let A be a finite-dimensional algebra (associative, with an identity) over a commutative field k. Throughout, by a module we mean a finite-dimensional right module. A module $T_A$ is called a tilting module (cf. [2], [5] or [6]) if:

(T1) pd $T_A < 1$ ,

(T2) $\mathrm{Ext}_A^1 (T,T) = 0$,

(T3) there is a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T', T'' direct sums of summands of $T_A$.

A tilting module $T_A$ is called splitting if every indecomposable B-module $N_B$, where $B = \mathrm{End}\, T_A$, is such that either $N \otimes_B T = 0$, or $\mathrm{Tor}_1^B (N,T) = 0$. A finite-dimensional k-algebra B is iterated tilted (called "generalized tilted" in [1]) if:

1) there exists a sequence of algebras $A_o, A_1, \ldots A_m = B$ with $A_o$ hereditary,

2) there exists a sequence of splitting tilting modules $T_{A_i}^{(i)}$ $(0 \le i \le m-1)$ such that End $T_{A_i}^{(i)} = A_{i+1}$.

B is said to be <u>of type Δ</u> for a (non-oriented) valued graph Δ, if $A_o$ is the tensor algebra of an oriented valued graph, with a non-oriented underlying graph Δ (cf. [3]).

Let $\Sigma = (F_i, {}_iM_j)_{i,j\in I}$ be a k-species, and $T(\Sigma)$ its tensor algebra (cf. [3], [4]). An ideal R of $T(\Sigma)$ contained in $(\text{rad } T(\Sigma))^2$ will be called a relation ideal. For a perfect field k, any finite-dimensional basic k-algebra can be written as $T(\Sigma)/R$, where Σ is a k-species, and R a relation ideal. Writing ${}_i\hat{M}_j = F_iT(\Sigma)F_j$ and ${}_iR_j = F_i\, RF_j$, R can be defined by the bimodule epimorphims ${}_i\rho_j : {}_i\hat{M}_j \to {}_i\hat{M}_j/{}_iR_j$.

The following theorems hold:

<u>Theorem (1)</u>: Let k be a perfect field. A connected basic finite-dimensional k-algebra is an iterated tilted algebra of type $\mathbb{B}_n$ if an only if it is given by a k-species $\Sigma = (F_i, {}_iM_j)_{i,j\in I}$ with a relation ideal $R = \bigoplus_{i,j} {}_iR_j$ satisfying the following conditions:

(1) The graph G of Σ is a tree.

(2) There is a vertex $i_o$ such that $F_{i_o} = F$, and for all $i_o \neq i$, $F_i = E$, where $E$ and $F$ are two skew fields, finite-dimensional over $k$ and such that $\dim {}_FE = 2$. Moreover, if ${}_iM_j \neq 0$, ${}_iM_j = {}_FE_E$ for $i = i_o$, ${}_iM_j = {}_EE_F$ for $j = i_o$, and ${}_iM_j = {}_EE_E$ otherwise:

(3) The vertex $i_o$ has at most two neighbours $i$ and $j$, and, if it is so, then $i \to i_o \to j$ and there is a relation on the subspecies

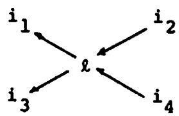
$$E \xrightarrow{{}_EE_F} F \xrightarrow{{}_FE_E} E$$

given by an epimorphism ${}_EE_F \otimes {}_FE_E \to {}_EE_E$.

(4) All relations are of length two, and the only relations besides the one in (3) are zero-relations.

(5) Each vertex of $G$ has at most four neighbours.

(6) If a vertex $\ell$ has four neighbours, then $G$ contains a full connected subgraph of the form




with the zero-relations ${}_{i_2}M_\ell \otimes {}_\ell M_{i_1}$ and ${}_{i_4}M_\ell \otimes_\ell M_{i_3}$.

(7) If a vertex $\ell$ has three neighbours, then $G$ contains a full connected subgraph of one of the forms


with the zero-relation ${}_{i_3}M_{\ell} \otimes {}_{\ell}M_{i_1}$.

<u>Theorem (2)</u>: Let $k$ be a perfect field. A connected basic finite-dimensional k-algebra is an iterated tilted algebra of type $C_n$ if and only if it is given by a k-species $\Sigma = (F_i, {}_iM_j)_{i,j\in I}$ with a relation ideal $R = \bigoplus_{i,j} {}_iR_j$ satisfying the conditions (1), (4), (5), (6), (7) of theorem (1) and:

(2) there is a vertex $i_o$ such that $F_{i_o} = E$, and for all $i \neq i_o$, $F_i = F$, where $E$ and $F$ are two skew fields, finite-dimensional over $k$, and such that $\dim_F E = 2$. Moreover, if ${}_iM_j \neq 0$, ${}_iM_j = {}_FE_E$ for $j = i_o$, ${}_iM_j = {}_EE_F$ for $i = i_o$ and ${}_iM_j = {}_EE_E$ otherwise.

(3) the vertex $i_o$ has at most two neighbours $i$ and $j$ and, if it is so, then $i \rightarrow i_o \rightarrow j$ and there is a relation on the subspecies

$$F \xrightarrow{{}_FE_E} E \xrightarrow{{}_EE_F} F$$

defined by an epimorphism ${}_FE_E \otimes {}_EE_F \rightarrow {}_FF_F$.

In the course of the proof, yet another characterization of these algebras is given in terms of their Auslander-

Reiten graphs. These results constitute a part of the author's Ph.D. thesis and will appear in full in Communications in Algebra.

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# THE HILBERT FUNCTION AND COHEN MACAULAY TYPE OF ORDINARY SINGULARITIES

S.K. Gupta and L.G. Roberts

*Presented by P. Ribenboim, F.R.S.C.*

Abstract. We use Cartesian squares to construct ordinary singular points on algebraic curves, and to discuss their Hilbert function and Cohen-Macaulay type.

We construct the co-ordinate ring $A$ of a curve with one singular point $P$ by means of a cartesian square

$$(1)\qquad \begin{array}{ccc} A & \longrightarrow & \bar{A} \\ \downarrow & & \downarrow \pi \\ D & \xrightarrow{\ f\ } & \Pi_{i=1}^{s} k[t_i]/t_i^{n} \end{array}$$

where the horizontal arrows are inclusions, $\pi$ is an onto map, and $\bar{A}$ is non-singular. (In general the singular locus of $A$ equals Spec $D$. To ensure that there is only one singular point we take Spec $D$ connected.) Full details appear in [3], and will be published elsewhere. This paper continues the investigations of [2], [5].

One of our aims is to describe properties of the singular local ring $A_P$ in terms of the bottom horizonal arrow $f$ of (1). To this end let $\tau : \Pi_{i=1}^{s} k[t_i] \to \Pi_{i=1}^{s} k[t_i]/t_i^{n}$ be the canonical projection, and set $\mathfrak{D} = \tau^{-1}(D)$, $\mathfrak{P} = \tau^{-1}(M)$, $M$ the maximal ideal of $D$. Clearly $(\mathfrak{D}, \mathfrak{P})$ depends only on the inclusion $f$ and we have

Theorem 1. The completion of $A$ at $P$ is isomorphic to the completion of $\tilde{D}$ at $\tilde{P}$.

Corollary. The Hilbert function $H(i) = \dim_k(P^i/P^{i+1})$ depends only on $f$.

Theorem 2. If $\bar{A} = k[t]$ then $K_i(A) \cong K_i(k) \oplus (s-1)K_{i+1}(k) \oplus K_{i+1}(P)$ $(i \leq 1)$ where $K_{i+1}(P) = \operatorname{coker}(\mathcal{K}_{i+1}(D) \to \oplus_{j=1}^{s} \mathcal{K}_{i+1}k[t_j]/t_j^n)$.

If $D$ is homogeneous, and char $k = 0$ or is sufficiently large, then $K_1(P) = Nk$ and $K_2(P) = N\Omega_k/V$ where $N = \operatorname{codim}_k(\Pi_{j=1}^{s} k[t_j]/t_j^n, D) - s + 1$ and $V$ is a finite dimensional vector space over $k$.

Theorem 3. Let $r$ denote Cohen-Macaulay type. Then $r(\tilde{D}) = r(A)$.

The leading term of $f \in D$ (or $\tilde{D}$) is the non-zero graded piece of lowest degree. The possible leading terms of a given degree form a vector space, which we can identify with a subspace of $k^s$ by leaving out the powers of $t_i$. Choose a basis for the leading terms of degree one, and write these $r$ basis vectors as the rows of an $r \times s$ matrix $C$. Then we have

Theorem 4. The non-zero columns of $C$ are the tangent directions at the non-singular branches of $P$. $P$ is an ordinary singularity if and only if the columns of $C$ represent distinct points in $\mathbb{P}^{r-1}$. The reduced tangent cone $B$ of $A$ at $P$ is the subalgebra of $\Pi_{i=1}^{s} k[t_i]$ generated by the rows of $C$.

We compare properties of $A_P$ with the corresponding property of $B$. We say that $A$ (or $P$) is minimal ordinary if $A$ is minimal among all those ordinary $A \subset \bar{A}$ with the same reduced tangent cone, and that $A$ (or $P$) is homogeneous if $D$ is homogeneous. Then we have

Theorem 5. If $A$ is minimal homogeneous ordinary then $\tilde{D}$ is the reduced tangent cone of $A$ (at $P$), so in this case $A$ and its reduced tangent cone have the same Hilbert function, and the same Cohen-Macaulay type.

Theorem 6. If $A$ is minimal ordinary (but not necessarily homogeneous) and the tangent directions are in generic s-position, then $A$ and its reduced tangent cone have the same Hilbert function and the same Cohen-Macaulay type. If the tangent directions are not in generic s-position, then $r(A)$ need not equal $r(B)$, and the Hilbert functions can differ as well.

The Cohen-Macaulay part of Theorems 5, 6 answers some questions posed in [1].

Motivated by [7] p. 40, we give an algorithm for computing the Hilbert function $H(i)$ of an ordinary singularity $P$, and discuss when $H(i)$ can decrease. By Theorem 1, $H(i) = \dim_k(\tilde{\mathfrak{P}}^i/\tilde{\mathfrak{P}}^{i+1})$. Furthermore $\tilde{\mathfrak{P}}^i$ is homogeneous with $j^{th}$ graded piece $V_{ij}$ equal to the subspace of $k^s t^j \subset \Pi_{\ell=1}^s k[t_\ell]$ $(t=(t_1,\ldots,t_s))$ spanned by monomials of degree $\geq i$ (as a monomial) and degree $j$ (in $t$) in some set of homogeneous generators for $\tilde{D}$ . Let $b_{ij} = \dim_k(V_{ij}/V_{i+1,j})$ . Then $\tilde{P}^i/\tilde{P}^{i+1} \cong \oplus_{j\geq i} V_{ij}/V_{i+1,j}$ so $H(i) = \Sigma_{j\geq i} b_{ij}$ . We arrange the numbers $b_{ij}$ in an array

$$
\begin{array}{cccccc}
b_{11} & b_{22} & b_{33} & b_{44} & b_{55} & \cdots \\
 & b_{12} & b_{23} & b_{34} & b_{45} & \cdots \\
 & & b_{13} & b_{24} & b_{35} & \cdots \\
 & & & b_{14} & b_{25} & \cdots \\
 & & & & b_{15} & \cdots
\end{array}
$$

arranged so that columns correspond to dimensions of subspaces of $k^s t^j$ . ($b_{ii} = b_i$ is the Hilbert function of the reduced tangent cone of P .) H(i) is now the diagonal sum beginning with $b_{ii}$ . The numbers $b_{ij}$ are most easily computed in the generic case $b_{ij} = \inf(c_{ij}, s - \Sigma_{\ell>i} b_{\ell j})$ (if $\ell>j$ set $b_{\ell j} = 0$ ), ($c_{ij}$ = number of monomials of degree i as monomials, and degree j in t in a set of homogeneous generators for $\tilde{D}$ ). This is an inductive definition going down each column. We prove that the generic case exists, and that a generic singularity is ordinary so long as there are at least two generators of degree one. We have

Example 7. Let generic P be constructed with $s = 21$ from D which has 3 homogeneous generators of degree 1 (in t), and one generator of degree 2 (in t) and $n \geq 5$ (in the original cartesian square). Then $\tilde{D}$ is generated by the same homogeneous generators and our array is

| | | | | |
|---|---|---|---|---|
| 3 | 6 | 10 | 15 | 21 |
| | 1 | 3 | 6 | |

so that $H(1) = 4$ , $H(2) = 9$ , $H(3) = 16$ , $H(4) = 15$ , $H(i) = 21$ for $i \geq 5$ . Thus it is possible to have an ordinary singularity with $H(1) = 4$ , and a Hilbert function with a temporary decrease. This improves on examples 1f, 1g in [4], which give a decrease when $H(1) = 5$ or $6$ . In fact, if we make $s$ large enough we can get as many intervals of decrease as we wish. Thus

Example 8. Let $s = 274$ . Suppose there are 3 generators of degree 1 and one generator of degree 2 . Then the Hilbert function is 4, 10, 20, 35, 56, 84, 116, 145, 164, 202, 199, 232, 229, 226, 256, 256, 255, 254, 253, 252, 253, 274 (then remaining at 274). This has three separate intervals of decrease, generated by the first three rows of the array.

Similar examples can be given with $H(1) \geq 5$ . We have been unable to characterize the Hilbert functions that arise in this way (although the conditions described in [8] Theorem 2.2 are clearly necessary). Nor have we been able to find an example with $H(1) = 3$ and a Hilbert function that has a decrease. However we prove

Theorem 9. Let $P$ be homogeneous ordinary (but not necessarily generic) such that $H(1) = 3$ . Then the Hilbert function of $P$

can never decrease.

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# IDENTIFYING A RATIONAL FUNCTION

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*Presented by P. Ribenboim, F.R.S.C.*

Abstract. We discuss a remarkable result concerning rational functions which implies that, up to obvious ambiguities, one may recognise a rational function from any infinite subset of its Taylor coefficients.

1. Let $\sum_{h\geq 0} f_h X^h$ be the Taylor expansion of a rational function $r(X)/s(X)$ defined over a field $\mathbb{K}$ of characteristic zero; thus $r, s$ in $\mathbb{K}[X]$. We shall suppose that the degree of $r$ is less than that of $s$. Without loss of generality we may write $s(X) = 1 - \sum_{j=1}^{n} s_j X^j$, $s_n \neq 0$, whence the sequence $(f_h)$ satisfies a linear homogeneous recurrence relation of order $n$ with constant coefficients in $\mathbb{K}$:

$$f_{h+n} = s_1 f_{h+n-1} + s_2 f_{h+n-2} + \dots + s_n f_h, \qquad h \geq 0 .$$

Then the *initial values* $f_0, f_1, \dots, f_{n-1}$ of the *recurrence* $(f_h)$ are given by $r(X)$ by way of

$$r(X) = \sum_{j=0}^{n-1} \left( f_j - \sum_{i=1}^{j} s_i f_{j-i} \right) X^j .$$

We may factor $s(X)$ over $\overline{\mathbb{K}}$, an algebraic closure of $\mathbb{K}$, to obtain

$$s(X) = \prod_{i=1}^{n'} \left(1 - \alpha_i X\right)^{n(i)}$$

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with distinct $\alpha_1,\ldots,\alpha_{n'}$ of respective multiplicities $n(1),\ldots,n(n')$, so $\sum n(i) = n$. Hence

$$r(X)/s(X) = \sum_i \sum_{k=1}^{n(i)} a_{ik}(1-\alpha_i X)^{-k} = \sum_{h\geq 0}\sum_i \sum_k a_{ik}\binom{h+k-1}{k-1}\alpha_i^h X^h$$

$$= \sum_{h\geq 0}\left(\sum_i a_i(h)\alpha_i^h\right) X^h ,$$

where the $a_i(X)$ are polynomials of degree at most $n(i)-1$ respectively. Thus we have

$$f_h = \sum_i a_i(h)\,\alpha_i^h, \quad h = 0,1,2,\ldots.$$

a generalised power sum. Fixing branches of the logarithms $\log\alpha_i$ we see that the recurrence $(f_h)$ is given by the *exponential polynomial*

$$f(z) = \sum_i a_i(z)\exp(z\log\alpha_i)$$

at the non-negative integers. Should we wish to drop the condition on the degrees of $r$ and $s$ we need only add a polynomial term $a_0(z)$. All this is well known. We mention it to set notation.

2. We wish to discuss the matter of identifying the rational function $r(X)/s(X)$ given only an infinite subset of its Taylor coefficients $f_h$ presented in any order. Plainly this is hopeless if any quotient $\alpha_i/\alpha_j$, $i\neq j$ is a root of unity, or indeed if an $\alpha_i$ is a root of unity. In the former case the recurrence $(f_h)$ will collapse to a recurrence of lower order on arithmetic subprogressions $(dh+r)$; in the latter case we risk always 'recognising' a constant multiple of $(1-X)^{-1} = \sum X^h$ or some such similar irritation. In either case the recurrence $(f_h)$ is said to be *degenerate*, and we may say

the same of the associated rational function. Incidentally, the concept of 'degeneracy' is not contrived for the present purpose; see for example [2]. Next, let $f_{dh+r} = g_h$, $h = 0,1,2,\dots$ for integers $d > 0$, $r \geq 0$. Then the power series $\sum_{h\geq 0} g_h X^h$ again represents a rational function. We note that, in the notation used above, we have:

$$g(z) = \sum_i a_i\,(dz+r)\,\alpha_i{}^r \exp(z \log \alpha_i{}^d)\,.$$

We will say that two rational functions $\sum k_h X^h$ and $\sum g_h X^h$ are *similar* if both are related to a nondegenerate rational function $\sum f_h X^h$ in the manner that the exponential polynomial $g(z)$ is related to $f(z)$ as described. Again it is plain that we cannot hope to distinguish amongst similar rational functions. On the other hand we note that such functions really are 'very similar'. This said, it is remarkable that a recent result [1] of ours implies that *any infinite subset of the Taylor coefficients of a nondegenerate rational function identifies the similarity class of that function.* It may be more straightforward to rephrase this claim: *let* $\sum f_h X^h$ *and* $\sum g_h X^h$ *represent dissimilar nondegenerate rational functions; write* $F = \{f_h : h = 0,1,2,\dots\}$ *and* $G = \{g_h : h = 0,1,2,\dots\}$. *Then the set-theoretic intersection* $F \cap G$ *has finite cardinality.* We remind the reader that the underlying field is to have characteristic zero.

3. Let $(f_h)$ be a nondegenerate recurrence and let $m$ be a map of some infinite subset $S$ of the non-negative integers into the non-negative integers. There need be no *a priori* condition on $m$ other than that it be injective. Set $f_{m(h)} = g(h)$ for $h$ in $S$. We show in [1] that if $(g_h)$ is again a recurrence then for $h > H$,

say, thus from a certain point on, $m(h)$ is linear in $h$. The reader will readily see that this result implies the assertions made above. By supposing, as we may, that $m$ is defined on all of $\mathbb{N} \cup \{0\}$ and has only finitely many fixed points the title of [1] is justified: one sees that a nondegenerate recurrence has finite total multiplicity. At most finitely many values appear more than once in the sequence $(f_h)$; and any given value appears at most finitely many times. This last fact is the celebrated theorem of Skolem-Mahler-Lech-Mahler; see the survey [2].

4. The following is a loose sketch of our proof. Suppose first that $f_{m(h)} = g_h$ for all $h = 0,1,2,\ldots$ . In brief, we show that for each of infinitely many primes $p$ we lose little generality in supposing that we have $f \circ M(t) = g(t)$ for p-adic exponential polynomials $f$ and $g$ defined on the disk $D_p \subset \mathbb{Q}_p$ of those $t$ so that $\mathrm{ord}_p t > -1 + 1/p-1$. For each integer $r$, $0 \le r < p-1$, we have such $M$ which, in effect, coincide with $m$ on infinitely many integers of the shape $h(p-1) + r$. Now, supposing that $(f_h)$ is non-degenerate, we can conclude that $M$ is a p-adic power series converging in $D_p$. The upshot is that we may obtain finite sets of primes $P$ so that the product

$$\prod_{p \in P} p^{1/p-1}$$

is as large as we wish and so that the maps $M : h \longmapsto m(h(p-1) + r)$, for $h > H(P)$, can be continued to p-adic power series converging in $D_p$. With the more limited original data we can only prove these claims for at least one $r$ corresponding to each $p$. The product becomes

$$\prod_{p \in P} p^{1/(p-1)^2},$$

and of course is greater than 1. By an extra assumption, which we describe below, and a straightforward argument involving only well known and frequently used ideas, it follows in either case that $M$ is an exponential polynomial defined over $\mathbb{C}$; thus an entire function! But now consideration of $f \circ M(z) = g(z)$ near infinity implies that $M$ is linear, as claimed.

5. The additional requirement is that $m$ must not grow too fast. Specifically, for all large integers $h$, $m(h)$ should be less than the h-th power of the relevant product displayed above. In the event these conditions are implied by weak growth hypotheses for nondegenerate recurrences defined over an algebraic number field. In the present situation appropriate specialisation loses no generality and we may presume that the parameters defining $f$ belong to an algebraic number field $\mathbb{L}$. Set $A = \max |\alpha_i|$, and note that we may suppose that $\alpha_1, \ldots, \alpha_n$, have no common (ideal) factor since otherwise $m$ already satisfies the required growth conditions.

Much more than we need, namely the growth hypotheses for non-degenerate recurrences defined over an algebraic number field (see [2]) can be shown [3] to follow from the results of Schmidt-Schlickewei: for all large integers $h$ (and all prime ideals $p$ of $\mathbb{L}$),

$$|f(h)| > A^{(1-\epsilon)h} \qquad \text{and} \qquad \mathrm{ord}_p f(h) < \delta h .$$

By virtue of the $p$-adic inequalities we may suppose the $\alpha_i$ are integers of $\mathbb{L}$ since otherwise, once again, $m$ already satisfies our requirements. Then, if necessary after taking, as we may, a new embedding of $\mathbb{L}$ in $\mathbb{C}$, we can suppose $A \geq 1$. Moreover $A = 1$ is excluded because $f$ is nondegenerate so the $\alpha_i$ are not roots of unity. Thus $A > 1$ and then the lower bound for $|f(h)|$ amply implies the required upper bounds for $m(h)$.

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# VON NEUMANN'S COORDINATIZATION THEOREM

Israel Halperin, F.R.S.C.

## 1. Notation and equivalence of lattice inequalities

$L$ denotes a complemented, modular lattice with homogeneous basis $a_1, \dots, a_N$, $N \geq 4$ [2, Part II, Def. 3.1]; $A^j \equiv a_1 \vee \dots \vee a_j$; $ab$ means $a \wedge b$; $a \overset{\circ}{\vee} b$ means $a \vee b$ if $ab = 0$; $L_{ji} \equiv (b \in L : b \overset{\circ}{\vee} a_j = a_i \vee a_j)$. If $R$ is a ring and $m \leq N$, then $R^N(m)$ denotes the right $R$-module $((\alpha_1, \dots, \alpha_N) : \text{all } \alpha_i \in R \text{ and } \alpha_i = 0 \text{ for } m < i \leq N)$; $(\alpha_1, \dots, \alpha_m)$ is an abbreviation for $(\alpha_1, \dots, \alpha_m, 0, \dots, 0) \in R^N(m)$; $R^N \equiv R^N(N)$; $L(R^N(m))$ denotes the set of finitely generated submodules of $R^N(m)$, ordered by inclusion.

A lattice inequality: $Ea \leq Fb$ implies $Ea \vee c \leq Fb \vee c$; and by modularity, this implies $E(a \vee c) \leq F(b \vee c)$ hence $E \leq F$, if $c \leq E \leq a \vee c$, $c \leq F \leq b \vee c$; so, if such $c$ exists, then $Ea \leq Fb$ is equivalent to $E \leq F$.

2. Von Neumann's theorem In each $L_{ji}(j \neq i)$, addition and multiplication can be defined so that:

(2.1) The $L_{ji}$ become regular rings with unit, isomorphic to a common regular ring $R$ [2, Part II, Theorem 9.2].

(2.2) For each $j$ the sublattice $(b \in L : b \leq a_j)$ is isomorphic to $L(R)$, the lattice of principal right ideals of $R$ [2, Part II, Theorem 9.2].

(2.3) $L$ is isomorphic to $L(R^N)$ [2, Part II, Theorem 14.1].

## 3. Outline of von Neumann's proof

(3.1) Choose $c_{j1}$, $2 \leq j \leq N$ so that $c_{j1} \overset{\circ}{\vee} a_j = c_{j1} \overset{\circ}{\vee} a_1 = a_j \overset{\circ}{\vee} a_1$ set $c_{ji} = (c_{j1} \vee c_{i1})(a_j \vee a_i)$ for all $1 \neq i \neq j$.

(3.2) Call a family $\alpha \equiv (\alpha_{ji} \in L_{ji} : i \neq j)$ an $L$-number if $(\alpha_{ji} \vee c_{jk})(a_k \vee a_i) = \alpha_{ki}$ and $(\alpha_{ji} \vee c_{ik})(a_j \vee a_k) = \alpha_{jk}$.
Note: For every $b \in L_{ji}$ there is a unique $\alpha$ with $\alpha_{ji} = b$ [2, Part II, Lemma 6.1].

(3.3) Let $R$ denote the set of $L$-numbers with operations:
$(\alpha + \beta)_{ji} = (\alpha_{jk} \vee (\beta_{ji} \vee a_k)(a_j \vee c_{ik}))(a_j \vee a_i)$
$(\alpha\beta)_{ji} = (\alpha_{jk} \vee \beta_{ki})(a_j \vee a_i)$.

(3.4) For each $\alpha \in R$ and $1 \leq j \leq N$, define the <u>reach</u> of $\alpha$ into $a_j$:
$\alpha_j^{(r)} \equiv (\alpha_{ji} \vee a_i)a_j$ (does not depend on $i$, $i \neq j$).

(3.5) Prove: $\alpha\gamma = \beta$ has a solution $\gamma$ if and only if $\beta_j^{(r)} \leq \alpha_j^{(r)}$ (holds for all $j$ if for some $j$) [2, Part II, Lemma 9.4].

(3.6) Prove: For each $b \leq a_j$: $b = e_j^{(r)}$ for some idempotent $e \in R$ [2, Part II, Theorem 9.3].

(3.7) Deduce: Parts (2.1), (2.2) of the theorem hold [2, Part II, Theorem 9.2].

$(3.8)_m$ Prove: For $1 \leq m \leq N$ there exists an isomorphism $\phi_m : (b \in L : b \leq A^m) \to L(R^N(m))$ with $\phi_1 \subset \phi_2 \subset \ldots \subset \phi_N$.
Note: $\phi_N$ establishes Part (2.3) of the theorem.
The outstanding difficulty in von Neumann's proof is to establish the $\phi_m$.

## 4. <u>Von Neumann's strategy to prove</u> $(3.8)_m$

(4.1) Call $b$ an m-element if (i) $m = 1$ and $b \leq a_1$, or (ii) $2 \leq m \leq N$ and $b \vee A^{m-1} \leq A^m$.

(4.2) For each m-element $b$ define $\phi(b)$, an element of $L(R^N(m))$, as follows:
(i) If $b \leq a_1$ define $\phi(b) = (e, 0, \ldots, 0)R$ with $e$ idempotent and $e_1^{(r)} = b$.

(ii) If $2 \le m \le N$ define $\phi(b) = (-\alpha_1, \ldots, -\alpha_{m-1}, e)R$ with $e$ idempotent and $e_m^{(r)} = (A^{m-1} \vee b)a_m$, with $\alpha_i \in R$ and $(\alpha_i)_{im} = (b \vee \bar{b} \vee A^{i-1} \vee a_{i+1} \vee \ldots \vee a_{m-1})(a_i \vee a_m)$ and $\bar{b} \vee e_m^{(r)} = a_m$.

Note: $\phi(b)$ is determined uniquely by $b$ though $e$, $\bar{b}$, and the $\alpha_i$ may not be; also, $\alpha_i = \alpha_i e$.

(4.3) For each $x \in L$ and decomposition $x = \bigvee_{i=1}^{N} x_i$ with $x_i$ an i-element, (such decompositions exist for all $x$), assign to $x$ the submodule $\phi(x_1) + \ldots + \phi(x_N)$.

$(4.4)_m$ Prove: the set $(\phi(x_1) + \ldots + \phi(x_m) : x \le A^m) = L(R^N(m))$.

$(4.5)_m$ Prove: For decompositions $x = \bigvee_{i=1}^{m} x_i$, $y = \bigvee_{i=1}^{m} y_i$:

$x \le y$ if and only if $\sum_{i=1}^{m} \phi(x_i) \le \sum_{i=1}^{m} \phi(y_i)$.

Note: $(4.5)_m$ implies that $\phi_m(x) \equiv \sum_{i=1}^{m} \phi(x_i)$ has the same value for all decompositions of $x$; then $(4.4)_m$, $(4.5)_m$ establish $(3.8)_m$.

Von Neumann established $(4.4)_m$ without difficulty [2, Part II, Theorem 11.2]; $(4.5)_1$ follows immediately from (3.5), (3.6). But von Neumann's proof of $(4.5)_m$, $2 \le m \le N$ [2, Part II, pages 168-208], is a virtuoso demonstration of mathematical technique.

5. <u>A new proof of</u> $(4.5)_m$, $2 \le m \le N$. We shall give a direct lattice calculation for the case $m = 2$, and, by a simple technical device, reduce the case $m$ to the case $m-1$ when $3 \le m \le N$.

We require the following properties of $L$-numbers.

(5.1) $(\alpha-\beta)_{jk} = (\alpha_{ji} \vee (a_k \vee \beta_{ji})(a_j \vee c_{ik}))(a_j \vee a_k)$ [1, (2.3)]

hence

(5.2) $(\alpha-\beta)_j^{(r)} = (\alpha_{ji} \vee \beta_{ji})a_j$.

(5.3) $(\alpha+\beta\gamma)_{ji} = (\gamma_{jk} \vee (\alpha_{ji} \vee a_k)(\beta_{ki} \vee a_j))(a_j \vee a_i)$ [1, (5.2)].

6. <u>Proof of</u> $(4.5)_2$. We assume $x_1 \le a_1$, $y_1 \le a_1$, $x_2 \diamond a_1 \le a_2 \vee a_1$, $y_2 \diamond a_1 \le a_2 \vee a_1$ and we need to prove:

(6.1) $x_1 \vee x_2 \le y_1 \vee y_2$ if and only if $\phi(x_1) + \phi(x_2) \le \phi(y_1) + \phi(y_2)$.

Because of modularity we need consider only the case $x_1 = 0$, $\phi(x_1) = 0$ (use $(4.5)_1$).

Now the inequality $\phi(x_2) \le \phi(y_1) + \phi(y_2)$ is equivalent, in turn, to each of:

(6.2) $(-\alpha_1(x_2), e(x_2)) = (e(y_1), 0)\beta_1 + (-\alpha_1(y_2), e(y_2))\beta_2$ for some $\beta_1, \beta_2 \in R$

(6.3) $e(y_2)e(x_2) = e(x_2)$ and $(\alpha_1(y_2)e(x_2) - \alpha_1(x_2)e(x_2))_1^{(r)} \le \le (e(y_1))_1^{(r)}$

(6.4) $(a_1 \vee x_2)a_2 \le (a_1 \vee y_2)a_2$ and, (use (5.2)), $((\alpha_1(x_2)e(x_2))_{13} \vee (\alpha_1(y_2)e(x_2))_{13})a_1 \le y_1$

(6.5)(i) $a_1 \vee x_2 \le a_1 \vee y_2$ and

(ii) $(\alpha_1(x_2)e(x_2))_{13} \le y_1 \vee ((\alpha_1(y_2)e(x_2))_{13}$.

The inequality (6.5)(ii) is equivalent to each of:

(6.6) $((\alpha_1(x_2))_{12} \vee (e(x_2))_{23})(a_1 \vee a_3) \le y_1 \vee ((\alpha_1(y_2))_{12} \vee (e(x_2))_{23}$

(6.7) $(\alpha_1(x_2))_{12} \vee (e(x_2))_{23})(a_1 \vee a_3 \vee (e(x_2))_{23}) \le y_1 \vee (\alpha_1(y_2))_{12} \vee (e(x_2))_{23}$

(6.8) $(\alpha_1(x_2))_{12}(a_1 \vee (a_3 \vee (e(x_2))_{23})a_2) \le y_1 \vee (\alpha_1(y_2))_{12}$

(6.9) $(\alpha_1(x_2))_{12}(a_1 \vee (a_1 \vee (e(x_2))_{21})a_2) \le y_1 \vee (\alpha_1(y_2))_{12}$

(6.10) $(\alpha_1(x_2))_{12}(a_1 \vee x_2) \le y_1 \vee (\alpha_1(y_2))_{12}$.

Now (6.5)(i) and (6.10) together are equivalent to:

(6.11) $x_2 \le y_1 \vee y_2$

which establishes (6.1), i.e. $(4.5)_2$.

7. <u>Proof of</u> $(4.5)_m$ <u>assuming</u> $(4.5)_{m-1}$, $3 \le m \le N$.

We assume $x_1 \le A^{m-1}$, $y_1 \le A^{m-1}$, $x_2 \diamond A^{m-1} \le A^m$, $y_2 \diamond A^{m-1} \le A^m$

and we need to prove: $x_1 \vee x_2 \le y_1 \vee y_2$ if and only if $\phi_{m-1}(x_1) + \phi(x_2) \le \phi_{m-1}(y_1) + \phi(y_2)$ where $\phi_{m-1}$ is the isomorphism on $A^{m-1}$ determined by $\phi$ (existing since $(4.5)_{m-1}$ is assumed to hold). We may assume that $x_1 = y_1 (= z$, say), so we need to prove:

(7.1) $z \vee x_2 \le z \vee y_2$ if and only if $\phi_{m-1}(z) + \phi(x_2) \le \phi_{m-1}(z) + \phi(y_2)$.

We recall that [2, Part II, Lemma 13.2] states: <u>if</u> $a \le b$ <u>then every</u> $x$ <u>can be expressed as</u> $(x \vee a)(x \vee c)$ <u>for some</u> $c$ <u>with</u> $a \dot{\vee} c = b$. Repeated application of this Lemma shows that our $z$ can be expressed as $z^{(1)} \wedge z^{(2)} \wedge \ldots \wedge z^{(m)}$ where, for each $j$: $z^{(j)} \vee a_i = A^{m-1}$ for some $i$. It is clearly sufficient to establish (7.1) with $z$ replaced by $z^{(j)}$; $j = 1, \ldots, m$; <u>thus, in</u> (7.1) <u>we may assume that</u> $z \vee a_i = A^{m-1}$ <u>for some</u> $i < m$.

Consider first the case that $i = 1$, i.e. $z \vee a_1 = A^{m-1}$. Set $\bar{z} = zA^{m-2}$ and chose $z_{m-1}$ so that $z_{m-1} \dot{\vee} A^{m-2} = z \vee A^{m-2}$; in fact, $z_{m-1}$ may be chosen so that $\phi(z_{m-1}) = (-\beta, 0, \ldots, 0, 1, 0, \ldots, 0)R$ with $\beta \in R$ and 1 in the (m-1)st place.

Set $\bar{x} = (x_2 \vee z_{m-1})(A^{m-2} \vee a_m)$, $\bar{y}_2 = (y_2 \vee z_{m-1})(A^{m-2} \vee a_m)$. Then $\bar{x}_2 A^{m-1} = 0 = \bar{y}_2 A^{m-1}$; $z \vee \bar{x}_2 = z \vee x_2$; $z \vee \bar{y}_2 = z \vee y_2$ and so the inequality: $z \vee x_2 \le z \vee y_2$ can be expressed as: $z \vee \bar{x}_2 \le z \vee \bar{y}_2$.

If $\phi(x_2) = (-\beta_1, \ldots, -\beta_{m-2}, -\beta_{m-1}, f)R$ and $\phi(y_2) = (-\alpha_1, \ldots, -\alpha_{m-2}, -\alpha_{m-1}, e)R$ then (use (5.3)):
$\phi(\bar{x}_2) = (-\beta_1 - \beta\beta_{m-1}, -\beta_2, \ldots, -\beta_{m-2}, 0, f)R$ and
$\phi(\bar{y}_2) = (-\alpha_1 - \beta\alpha_{m-1}, -\alpha_2, \ldots, -\alpha_{m-2}, 0, e)R$, so the inequality:
$\phi_{m-1}(z) + \phi(x_2) \le \phi_{m-1}(z) + \phi(y_2)$ can be expressed as:
$\phi_{m-1}(z) + \phi(\bar{x}_2) \le \phi_{m-1}(z) + \phi(\bar{y}_2)$ (use: $(-\beta, 0, \ldots, 0, 1)(\alpha_{m-1} - \beta_{m-1})$

with 1 in the (m-1)st place, as in $\phi_{m-1}(z)$). Thus we need only prove (7.1) with $z, x_2, y_2$ replaced by $z, \bar{x}_2, \bar{y}_2$ respectively; then we may even replace $z$ by $\bar{z}$.

Now apply $(4.5)_{m-1}$ with $a_1, \ldots, a_{m-2}, a_m$ in place of $a_1, \ldots, a_{m-2}, a_{m-1}$, replacing $(\alpha_1, \ldots, \alpha_{m-1}) \in L(R^N(m-1))$ by $(\alpha_1, \ldots, \alpha_{m-2}, 0, \alpha_{m-1}) \in L(R^N(m))$. The equivalence (7.1) follows for the case $z \vee a_i = A^{m-1}$, $i = 1$. A similar argument applies if $i \neq 1$ (exact statement of induction is essential).

This proves $(4.5)_m$, hence von Neumann's theorem.

## 8. Supplementary remark

Call $a_1, \ldots, a_N$, $N \geq 3$ a Desarguesian basis for $L$ if
(i) $a_i$ is perspective to some $b_i \leq a_1$ for $i \geq 2$ with $b_2 = b_3 = a_1$, and
(ii) $a_2 a_1 = a_3(a_2 \vee a_1) = 0$ and $a_1 \vee \ldots \vee a_N = 1$, and
(iii) the formulae (3.3) make $R$ a regular ring if, in the definition of $L$-number, $i, j$ are restricted to $\{1,2,3\}$.
Then $a_i$, $i > 3$ can be altered so that $\{a_1, \ldots, a_N\}$ becomes an independent basis for $L$ and, with minor changes, the proof of von Neumann's theorem holds; the condition (iii) can be replaced by certain Desarguesian-type lattice conditions [K.D. Fryer and Israel Halperin, Acta Math. Szeged, XVII(1956), pages 203-249; Bjarni Jónsson, Trans. Amer. Math. Soc. 97(1960), 64-94].

The proof is simplified when, in the definition of $L$-number the $i, j$ are further restricted to $j < i$, but then the use of $e_N^{(r)}$ in (4.2) above and $(e(x))_{21}$ in (6.9) above, must be (and can be) adjusted.

References 1. K.D. Fryer & I. Halperin, Can. J. Math. 7(1955), 432-444
2. John von Neumann, Continuous Geometry, Princeton U. Press, 19

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# HOMOGENEOUS BASES IN COMPLEMENTED MODULAR LATTICES

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We give a simple class of examples of irreducible complemented modular lattices with homogeneous bases $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ such that $[0, a_1]$ is not isomorphic to $[0, b_1]$. This phenomenon is, of course, impossible in a continuous geometry [3], or indeed in any lattice with a reasonable dimension function. We obtain also, for a fixed integer $n$, examples of lattices which possess homogeneous bases of degree $m$ if and only if $m$ and $n$ are relatively prime.

In more detail, we construct a family $M(n,k)$ of non-isomorphic lattices indexed on integers $n$ and $k$ with $0 \le k < n$. Each is an atomic irreducible complemented modular lattice. We can label the projectivity classes by the ordinary dimension and codimension functions plus a $Z_n$-valued function $r$. Hence the elements of infinite dimension and codimension form $n$ distinct projectivity classes: at the same time if $x$ and $y$ are such elements then $x$ is projective to $y_1 \le y$, so the Schroeder-Bernstein-like lemma for perspectivity fails. In each case we are able to compute explicitly the number of inequivalent homogeneous bases of any given degree $m$.

Each $M(n,k)$ has homogeneous bases of arbitrarily large degree, in particular $\ge 4$, so it can be co-ordinatized by a regular ring $R(n,k)$. This family of non-isomorphic rings resembles the examples given by Goodearl [2: p.65]. We obtain the apparently stronger result:

$$M_m(R(n,k)) \cong R(n,km) \quad \text{for all integers } m \ge 1,$$

where the product $km$ is reduced mod $n$. In particular this means that for a fixed integer $n > 1$, there exists a ring $R$ which can be written as $n \times n$ matrices over $n$ non-isomorphic regular rings.

The contents of this note are arranged as follows: §1 is a statement of the results as theorems 1-5; §2 describes the construction of the $M(n,k)$ ; §3 is a sequence of lemmas which can be proved by the reader using the simplest methods (except for lemma 13, which is a simple consequence of the co-ordinatization theorem), and which yield the theorems of §1 .

It is a pleasure for me to acknowledge my indebtedness to Israel Halperin for his many contributions to the writing of this paper.

## §1 Statement of results.

$k$ and $n$ are integers with $0 \le k < n$ . The $L(n)$ and $M(n,k)$ are complemented modular lattices which are irreducible and atomic. $b(k)$ is an element of $L(n)$ and $M(n,k) = [0,b(k)]$, the lattice of all elements $x$ of $L(n)$ with $x \le b(k)$ . Because $M(n,k)$ is atomic, dim and codim can be defined in the usual way. $r$ is a function from $M(n,k)$ to $Z_n$ which is additive on disjoint elements.

Note that we say $x$ is an m-part of $M$ to mean that $x$ belongs to a homogeneous basis for $M$ of degree $m$ , that is that there exists an independent family $a_1, a_2, \ldots, a_m$ of mutually perspective elements of $M$ with $x = a_1$ and $1 = a_1 \cup a_2 \cup \ldots \cup a_m$. We follow the notations of [1] or [3].

THEOREM 1. $L = L(n)$ has the following properties:

1. $\dim b(k) = \operatorname{codim} b(k) = +\infty$  $r(b(k)) = k$

2. $[0,b(0)]$ is isomorphic to $L$ and $b(0)$ is an m-part of

$L$ for all integers $m > 1$ .

3. For $k \neq 0$ , $[0,b(k)]$ is not isomorphic to $L$ ; $b(k)$ is an m-part of $L$ if and only if $n$ divides $mk$ .

4. $L$ has inequivalent homogeneous bases of degree $m$ (ie. there exist $x$ and $y$ both m-parts of $L$ with $[0,x]$ not isomorphic to $[0,y]$) if and only if $(m,n) \neq 1$ .

5. Suppose that $x$ and $y$ are in $L$ . Then the following conditions are equivalent:

(i) $x \approx y$ (i.e. $x=x_1 \sim x_2 \sim \ldots \sim x_n = y$)

(ii) $\dim x = \dim y$ , $\operatorname{codim} x = \operatorname{codim} y$ and $r(x) = r(y)$ .

(iii) There exists an isomorphism $\phi$ of $L$ onto itself such that $\phi x = y$ .

If, in addition, $\dim x = \dim y = \operatorname{codim} x = \operatorname{codim} y = \infty$ , the above conditions are equivalent to:

(iv) $[0,x]$ is isomorphic to $[0,y]$ .

THEOREM 2. $M = M(n,k)$ has the following properties:

1. There exists an m-part of $M$ if and only if $(m,n)$ divides $k$ .

2. There exists an m-part $x$ of $M$ such that $[0,x]$ is isomorphic to $M$ if and only if $n$ divides $(m-1)k$ .

3. There exists an m-part $x$ of $M$ such that $[0,x]$ is isomorphic to $M(n,s)$ if and only if $ms \equiv k \pmod{n}$ .

4. Property 5 of theorem 1 holds as stated.

THEOREM 3. Let $p$ be a prime.

1. If $m$ is a multiple of $p$ , $L(p)$ has $p$ inequivalent homogeneous bases of degree $m$ . If $p$ doesn't divide $m$ then all homogeneous bases of degree $m$ are equivalent.

2. Let $0 < k < p$ . Then $M(p,k)$ has a homogeneous basis

of degree m if and only if p doesn't divide m . All homogeneous bases of degree m are equivalent: x is an m-part if and only if $x \approx b(m^{-1}k)$ where $m^{-1}k$ is calculated in $Z_p$ .

THEOREM 4. $M(n,k)$ isomorphic to $M(n',k')$ implies $n = n'$ and $k = k'$ .

THEOREM 5. Suppose that K is a fixed field. For all integers n and k with $0 \le k < n$ there exists a regular ring $R(n,k)$ such that:

1. $R(n,k)$ isomorphic to $R(n',k')$ implies $n = n'$ and $k = k'$ .

2. Fix $n > 1$ and write $S(k)$ for $R(n,k)$ and R for $S(0)$ . Then the $S(k)$ are a family of n non-isomorphic regular rings with the properties:

$$M_n(S(k)) \cong R$$

$$M_m(R) \cong R \quad M_m(S(k)) \cong S(km) \qquad \text{for all integers } m \geq 1$$

Here the multiplication km is carried out mod n , $M_m(S)$ denotes the m×m matrix ring of S , and ≅ denotes ring isomorphism.

3. The centre of $R(n,k)$ is K .

## §2 Construction of $L$, $M$, a, b

Suppose that n is an integer >1 and that K is a field. Let V be a vector space of countably infinite dimension over K , and let $K^n$ denote the vector space of dimension n over K with basis $\phi_1,\phi_2,\ldots\phi_n$ . Let $L = L(n)$ be the set of subspaces E of $V \otimes K^n$ of the form

$$E = M \otimes K^n + F$$

where $M$ is a subspace of V and F is a finite-dimensional subspace of $V \otimes K^n$ . Evidently F can be taken with $F \cap (M \otimes K^n) = 0$ ;

under that condition $E$ uniquely determines the dimension of $F$ modulo $n$. Define $r : L \to Z_n$ by:

$$r(E) = \dim F \pmod n$$

where $F \cap (M\otimes K^n) = \{0\}$.

Let $V_1$ be a fixed subspace of $V$ with infinite dimension and codimension, and let $\psi$ be a fixed vector in $V$ independent of $V_1$. Then define $a = b(0)$ and $b(k)$ for $0 < k < n$ by:

$$a = b(0) = V_1\otimes k^n$$

$$b(k) = V_1\otimes K^n + [\psi]\otimes[\phi_1,\phi_2,\ldots\phi_k] .$$

§3 Outline of Proof

LEMMA 1. Suppose $F$, $M$, $N$ are subspaces of a vector space with $M\cap N = \{0\}$ and $F$ finite-dimensional. Then $M\cap(N+F)$ is finite-dimensional.

LEMMA 2. If $E_1$ and $E_2$ are in $L$ then $E_1\cap E_2$ is in $L$.

LEMMA 3. Every $E$ in $L$ can be written as

$$E = M\otimes K^n + F$$

where $F \subset N\otimes K^n$, $M \cap N = \{0\}$ and $\dim N < \infty$

LEMMA 4. $L$ is a complemented modular lattice, the lattice operations being intersection and addition of subspaces.

LEMMA 5. Suppose that $E_1$ and $E_2$ are in $L$ :

(i) If $E_1\cap E_2 = \{0\}$ then $r(E_1+E_2) = r(E_1) + r(E_2)$

(ii) If $E_1 \sim E_2$ then $r(E_1) = r(E_2)$ .

LEMMA 6. $a$ is an m-part of $L$ for all integers $m > 1$ and $[0,a]$ is isomorphic to $L$ .

LEMMA 7. Let $b = b(n,k)$ . Then $r(b) = k$ . $b$ is an m-part of $L$ if and only if $n$ divides $mk$ .

LEMMA 8. The lattice $M = M(n,k) = [0,b]$ has an m-part if and only if $(m,n)$ divides $k$ .

LEMMA 9. For $k \neq 0$, $M(n,k)$ does not have an n-part.

LEMMA 10. For $k \neq 0$, $M(n,k)$ is not isomorphic to $L$.

LEMMA 11. Let $x$ in $L(n)$ satisfy $\dim x = \operatorname{codim} x = \infty$ and let $k = r(x)$. Then:

(i) $x \approx b(n,k)$

(ii) There exists an isomorphism $\phi$ of $L$ onto itself such that $\phi(x) = b(n,k)$.

LEMMA 12. If $\phi$ is an isomorphism of $L(n)$ onto itself then $r(\phi(x)) = r(x)$ for all $x$ in $L(n)$.

LEMMA 13. Suppose that $x$ and $y$ are m-parts of $L(n)$ and that $[0,x]$ is isomorphic to $[0,y]$. Then there exists an isomorphism $\phi$ of $L$ onto itself such that $\phi(x) = y$.

LEMMA 14. $M(n,k)$ isomorphic to $M(n,k')$ implies $k = k'$.

LEMMA 15. Suppose that $x$ and $y$ are in $L(n)$, that $\dim x = \dim y = \operatorname{codim} x = \operatorname{codim} y = \infty$, and that $[0,x]$ is isomorphic to $[0,y]$. Then $r(x) = r(y)$.

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