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A SEQUENCE OF IRRATIONAL ROTATION ALGEBRAS

by Alexander Kumjian

Presented by G. de B. Robinson F.R.S.C.

Abstract: A certain AF algebra is shown to be the norm closure of an increasing sequence of unital subalgebras each isomorphic to the irrational rotation algebra.

Since the communication of Rieffel's explicit formula for a non-trivial projection in an irrational rotation algebra [4], the structure of these algebras has come under intense study. If $\theta \in [0,1]$ is irrational, then the rotation of the unit circle by $\exp(2\pi i\theta)$ yields a free and minimal action of the integers. The cross-product C*-algebra of this action is denoted A_θ and is called the irrational rotation algebra. This algebra is simple and has a unique normalized trace which induces a map $K_0(A_\theta) \rightarrow \mathbb{R}$. The range of this map is $\mathbb{Z} + \mathbb{Z}\theta$ [4,5] and the kernel is trivial [6]. Pimsner and Voiculescu [5] have shown that A_θ embeds unitaly in an AF algebra (that is a C*-inductive limit of finite dimensional semi-simple algebras):

$$i: A_\theta \rightarrow B_\theta$$

where $K_0(B_\theta) = \mathbb{Z} + \mathbb{Z}\theta$ (this is a dimension group [1] with order inherited from \mathbb{R}) and B_θ has dimension range $[0,1] \cap (\mathbb{Z} + \mathbb{Z}\theta)$ [1]. So, in particular, B_θ is also simple and has a unique normalized trace.

In this note we exhibit B_θ as the inductive limit of a sequence of copies of A_θ ; this will follow from the fact that a converse embedding exists. More explicitly we prove the following:

Theorem: There is a sequence of increasing unital sub-
algebras $A_i \subseteq B_\theta$ such that:

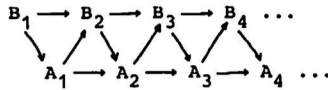
- i) $A_i \simeq A_\theta$
- ii) $B_\theta = \overline{\bigcup_i A_i}$.

Proof: There is a unital embedding $j: B_\theta \rightarrow A_\theta$ [2]. Composing the two embeddings gives $ij: B_\theta \rightarrow B_\theta$ which is a unital embedding of B_θ into itself which leaves the dimension of projections fixed (since there is a unique normalized trace). This means that the induced map of the K_0 groups, $K_0(ij): K_0(B_\theta) \rightarrow K_0(B_\theta)$, is the identity. Consider a sequence of such embeddings:

$$(*) \quad B_1 \xrightarrow{s_1} B_2 \xrightarrow{s_2} B_3 \rightarrow \dots$$

where each $B_i \simeq B_\theta$ and $s_i: B_i \rightarrow B_{i+1}$ is given by the embedding ij . The inductive limit $B = \varinjlim B_i$ is an AF-algebra and so is completely specified by its dimension group $K_0(B)$ and its dimension range. But $K_0(B) = \varinjlim K_0(B_i) = K_0(B_\theta)$ since each map is simply the identity, while the dimension range also remains the same (namely $[0,1] \cap (\mathbb{Z} + \mathbb{Z}\theta)$) whence $B \simeq B_\theta$.

By construction each embedding factors through a copy of A_θ , say A_1 ; so we have the following commutative diagram wherein each arrow denotes a unital embedding:



Both sequences have the same inductive limit, whence $\varinjlim A_i \cong B_\theta$.

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ALGEBRAIC HOMOTOPY THEORY AND SOME HOMOTOPY GROUPS OF ALGEBRAIC GROUPS

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In an article [8] which appeared in *Topology* in 1977, Kan and Miller showed that, if k is a unique factorization domain, then the homotopy type of a finite simplicial set K can be recovered from its k -algebra $\Lambda^0 K$ of Sullivan-de Rham 0-forms. More generally [4], one may associate to each simplicial set X a pro- k -algebra $\hat{A}X$, which coincides with $\Lambda^0 X$ if X is finite, and from which the entire integral homotopy type of X may be recovered. In fact, for each unique factorization domain k , the category $\text{pro-}\underline{M}_k$ of pro-algebras over k is a model for all of integral homotopy theory in that $\text{pro-}\underline{M}_k$ has a closed model structure in the sense of Quillen [10, 11] in such a way that its homotopy category $\text{Ho}(\text{pro-}\underline{M}_k)$ is contravariantly equivalent to the homotopy category $\text{Ho}(\underline{S})$ which is associated to the category \underline{S} of simplicial sets. This equivalence is induced by the contravariant functors $\hat{A}: \underline{S} \rightarrow \text{pro-}\underline{M}_k$ and $\hat{F}: \text{pro-}\underline{M}_k \rightarrow \underline{S}$; these functors are adjoint on the right.

Let ind-Aff_k be the category of pro-representable functors from the category \underline{M}_k of k -algebras to the set category \underline{E} . Using the contravariant equivalence of $\text{pro-}\underline{M}_k$ with ind-Aff_k , one may construct covariant functors $R_k: \underline{S} \rightarrow \text{ind-Aff}_k$ and $S_k: \text{ind-Aff}_k \rightarrow \underline{S}$, and push the closed model structure of $\text{pro-}\underline{M}_k$ over to ind-Aff_k to show

Theorem 1: For any unique factorization domain k , ind-Aff_k is a closed model category in such a way that $\text{Ho}(\text{ind-Aff}_k)$ is equivalent to $\text{Ho}(\underline{S})$. This equivalence is induced by S_k and R_k .

For T in ind-Aff_k , define the i^{th} homotopy group $\pi_i(T)$ of T by

$\pi_1(T) = \pi_1(S_k T)$, where $\pi_1(S_k T)$ is the i^{th} homotopy group of the geometric realization of the simplicial set $S_k T$. These are the groups which determine the weak equivalences of ind-Aff_k . An attempt has been made to understand what is measured by these groups in the case where T is an algebraic group over an algebraically closed field k .

A first result in this direction is

Theorem 2: Suppose that G is a connected algebraic group which is defined over an algebraically closed field k of arbitrary characteristic. Then G is path-connected in the sense that $\pi_0(G)$ is trivial if and only if its group $G(k)$ of k -rational points is generated by unipotent elements.

One may show that the group $\pi_0(G)$ of path-components of a connected algebraic group G is the group of rational points of a torus which has the same rank as a maximal torus of the solvable radical $R(G)$.

A next step is to find enough homomorphisms of algebraic groups $\pi: G \rightarrow H$ which are fibrations of ind-Aff_k in the sense that they induce fibrations $S_k \pi: S_k G \rightarrow S_k H$ of simplicial groups. Say that a homomorphism π is surjective if it is surjective on rational points. Then it may be seen that, if π is surjective, then $S_k \pi$ is a fibration if and only if every k -scheme homomorphism $A_k^n \rightarrow H$ lifts to G , and this for all $n \geq 1$, where A_k^n is affine n -space over k . This lifting property is equivalent to the vanishing of the induced fpqc K -torsor over A_k^n , where K is the group-scheme kernel of π . This observation is the starting point for a number of calculations. The most useful general result is

Theorem 3: Suppose that k is algebraically closed of arbitrary characteristic

and that $\pi: G \rightarrow H$ is a surjective algebraic group homomorphism over k with multiplicative group-scheme kernel K . Then π is a fibration.

There is a well-known result which says that, if X is an affine k -scheme, and U is a connected unipotent algebraic group, then the set $\mathcal{H}^1(X; U)$ of U -torsors over X is trivial. This, together with Theorem 3 and the usual exact sequence techniques, may be used to show

Proposition 4: Suppose that G is a connected algebraic group over k . Then the canonical map $G \rightarrow G/R(G)$ is a fibration.

Let T be a maximal torus of $R(G)$. Then $\pi_0(R(G)) = T(k)$ and $\pi_1(R(G)) = 0$ for $i > 0$, so that it is easy to calculate $\pi_1(G)$ in terms of the homotopy groups of the semi-simple group $G/R(G)$ by using a long exact sequence.

Suppose now that G is a semi-simple algebraic group over k , and that G_1, \dots, G_r are its simple algebraic subgroups. The "Smoothness of centralizers Theorem" [2] may be used together with Theorem 3 to show

Theorem 5: Multiplication in G defines a homomorphism $m: G_1 \times \dots \times G_r \rightarrow G$ of algebraic groups which is a fibration.

Every simple algebraic group is isomorphic to some Chevalley group G_ρ which comes from a representation ρ of a simple Lie algebra L that is defined over the complex numbers. Suppose that ϕ is the root system of L with respect to a fixed maximal toral subalgebra H , that Γ_ρ is the weight lattice of ρ , and that Γ_1 is the abstract weight lattice. Write $\Pi(G_\rho) = \Gamma_1/\Gamma_\rho$. Then there is a covering map $\lambda: G_\phi \rightarrow G_\rho$, where G_ϕ is the group of universal type for ϕ . One uses the Bruhat decomposition for the

respective groups to prove

Theorem 6: The covering homomorphism $\lambda: G_\phi \rightarrow G_\rho$ is a fibration.

If K is the group-scheme kernel of λ , then $\pi_0(K) = \Pi(G_\rho)$, mod its p -primary component if $\text{char}(k) = p$, and $\pi_1(K) = 0$ for $i > 0$.

It should also be pointed out that one may show

Theorem 7: Suppose that $\text{char}(k) = 0$. Then every surjective homomorphism of algebraic groups over k is a fibration.

It follows from these results that the homotopy groups of a path-connected group G over k of arbitrary characteristic coincide with a direct sum of the homotopy groups of the universal covers of the simple algebraic subgroups of $G/R(G)$, up to a finite twist in π_1 . The homotopy groups of Chevalley groups of universal type are, in turn, strongly related to various algebraic K -theories of the underlying field k .

It has been known for some time [3], for example, that there are isomorphisms

$$(1) \quad \pi_i(S_k \text{Gl}) \cong K_{i+1}(k) \quad \text{for } i \geq 0,$$

where the K -groups are those of Quillen and Gl is the infinite general linear group. One may prove this result by identifying the classifying space $BS_k \text{Gl}$ of the simplicial group $S_k \text{Gl}$ with the space $B\text{Gl}(k)^+$, up to homotopy equivalence. One uses a spectral sequence which converges to the integral homology $H_*(BG; \mathbb{Z})$ of the classifying space BG of a simplicial group G , and which has $E_{p,q}^1 = H_q(G_p; \mathbb{Z})$. Similar techniques show that, provided k contains $1/2$, there are isomorphisms

$$(2) \quad \pi_1(S_k Sp) \cong {}_{-1}L_{1+1}(k) \text{ for } 1 \geq 0,$$

where Sp is the infinite symplectic group, and the ${}_{-1}L$ -theory is that of Karoubi [7]. $S_k SI$ is the path-component of the identity e of $Gl(k)$ in $S_k Gl$. Recall also that $SI_{n+1}(k)$ is universal of type A_n and that $Sp_{2m}(k)$ is universal of type C_m . The isomorphisms (1) and (2) hold for much more general rings k .

If ϕ is a fixed irreducible root system, then "almost all fields" will be taken to mean the list of fields k for which the Steinberg group $St_\phi(k)$ is the universal central extension of $G_\phi(k)$ (see [12]). In this context, one may use the spectral sequence referred to above to prove:

Theorem 8: There are isomorphisms $\pi_1(S_k SI_n) \cong H_2(SI_n(k); Z) \cong K_2(k)$ for all $n \geq 3$ and almost all fields k .

Theorem 9: There are isomorphisms $\pi_1(S_k Sp_{2m}) \cong H_2(Sp_{2m}(k); Z) \cong {}_{-1}L(k)$ for all $m \geq 1$ and almost all fields k such that $\text{char}(k) \neq 2$.

A slightly different method proves

Theorem 10: There are isomorphisms $\pi_1(S_k Sp_{2m}) \cong H_2(Sp_{2m}(k); Z) \cong K_2(k)$ for all $m \geq 1$ if k is algebraically closed of arbitrary characteristic.

Theorem 8 was previously obtained by Krusemeyer in the context of unstable Karoubi-Villamayor K -theory [9].

$K_2(k)$ of an algebraically closed field k is usually non-trivial. One may use results of Bass and Tate [1] to show

Theorem 11: Suppose that k is an algebraically closed field. Then $K_2(k) = 0$

if the Kronecker dimension $\delta(k)$ of k satisfies $\delta(k) \leq 1$. $K_2(k)$ is a non-trivial uniquely divisible abelian group if $\delta(k) \geq 2$:

These results are proven in [5]. A detailed account of the fibration theory for algebraic groups is to appear in [6].

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CARDINAL SPLINES AND NILPOTENT HARMONIC ANALYSIS

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As is well known, the Heisenberg group represents the group-theoretic embodiment of the canonical commutation relations of quantum mechanics (cf. Gross [1]). However, apart from quantum mechanics, harmonic analysis on this two-step nilpotent Lie group admits a variety of different applications. The purpose of the present note is to examine two specific applications: (i) the classical Whittaker-Shannon cardinal interpolation series (i.e., the sampling theorem for band-limited signal functions) and (ii) the Subbotin-Schoenberg theorem concerning the existence and uniqueness of cardinal spline interpolants. For some applications of nilpotent harmonic analysis to the theory of periodic spline functions the reader is referred to the preceding note [3], to the papers [4], [6], and to the forthcoming monograph [7].

1. The Heisenberg Groups $A(\mathbb{R})$ and $\tilde{A}(\mathbb{R})$

Let $T = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional compact torus group. The (reduced) Heisenberg group $A(\mathbb{R})$ is formed by the product space $\mathbb{R} \times \mathbb{R} \times T$ and the law of composition $(x_1, y_1, \zeta_1) \cdot (x_2, y_2, \zeta_2) = (x_1 + x_2, y_1 + y_2, \zeta_1 \zeta_2 e^{2\pi i x_1 y_2})$. Its universal covering group $\tilde{A}(\mathbb{R})$ is called the real Heisenberg group (Weil [14], Igusa [2]). Obviously $\tilde{A}(\mathbb{R})$ is isomorphic to the group of triangular matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad (x, y, z \in \mathbb{R})$$

and represents therefore a connected, simply connected, nilpotent Lie group. The Lie algebras of $A(\mathbb{R})$ and $\tilde{A}(\mathbb{R})$ are the same (de-

noted by \mathcal{H}) and give rise to the canonical commutation relations of quantum mechanics. Moreover, the exponential map defines a real analytic diffeomorphism of \mathcal{H} onto $\tilde{A}(\mathbb{R})$.

2. The Schrödinger Representation

The center of $\tilde{A}(\mathbb{R})$ is formed by the subgroup $\{(0,0,z) \mid z \in \mathbb{R}\}$ which is isomorphic to \mathbb{R} . Observe that $\tilde{A}(\mathbb{R})$ is the semidirect product of the subgroups

$$N_1 = \{(0,y,z) \mid y \in \mathbb{R}, z \in \mathbb{R}\}, \quad T_1 = \{(x,0,0) \mid x \in \mathbb{R}\}.$$

If the one-dimensional irreducible unitary representation $z \mapsto e^{2\pi iz} \text{id}_{\mathbb{C}}$ of the center is induced from N_1 to $\tilde{A}(\mathbb{R})$ we obtain according to the Mackey theory an irreducible unitary representation U of $\tilde{A}(\mathbb{R})$. The action of U on the complex Hilbert space $L^2_{\mathbb{C}}(\mathbb{R})$ is given by the rule

$$U(x,y,z)f(t) = e^{2\pi i(z+ty)} f(t+x) \quad (t \in \mathbb{R})$$

and quantum mechanics suggests to call U the Schrödinger representation of $\tilde{A}(\mathbb{R})$ (cf. Igusa [2]). In view of the Kirillov correspondence, U is the prototype of an "essential" continuous irreducible unitary representation of $\tilde{A}(\mathbb{R})$ (having 2π as Planck's constant). The "inessential" continuous irreducible unitary representations of $\tilde{A}(\mathbb{R})$ are one-dimensional and act trivially on the center of $\tilde{A}(\mathbb{R})$.

It should be observed that the Heisenberg group $\tilde{A}(\mathbb{R})$ is also the semidirect product of the subgroups

$$T_2 = \{(0,y,0) \mid y \in \mathbb{R}\}, \quad N_2 = \{(x,0,z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}.$$

If the one-dimensional representation $z \mapsto e^{2\pi iz} \text{id}_{\mathbb{C}}$ of the center is induced from N_2 to $\tilde{A}(\mathbb{R})$ we obtain by an application of the Mackey theory an irreducible unitary representation U^σ of $\tilde{A}(\mathbb{R})$. According to the uniqueness theorem of Stone-von Neumann-Mackey U^σ is unitarily isomorphic to U and we have $U^\sigma = U \circ \sigma$ where the automorphism $\sigma: (x,y,z) \mapsto (y,-x,z-xy)$ of $\tilde{A}(\mathbb{R})$ maps the abelian normal subgroup N_1 onto the subgroup N_2 .

Another useful realization of the Schrödinger representation U of $\tilde{A}(\mathbb{R})$ can be obtained as follows: The discrete subgroup (lattice) $P = \{(x, y, z) \mid x, y, z \in \mathbb{Z}\}$ satisfies $\sigma(P) = P$. The quotient manifold $P \backslash \tilde{A}(\mathbb{R})$ of right cosets mod P is a compact nilmanifold (called the Heisenberg manifold) which admits the half-open cube $[-\frac{1}{2}, +\frac{1}{2}]^3$ as a fundamental domain. Consider the spectral decomposition

$$L^2(P \backslash \tilde{A}(\mathbb{R})) = \hat{\bigoplus}_{N \in \mathbb{Z}} H_N$$

of the complex Hilbert space $L^2(P \backslash \tilde{A}(\mathbb{R}))$ into the bi-infinite sequence $(H_N)_{N \in \mathbb{Z}}$ of closed subspaces which are stable with respect to the right regular representation δ of $\tilde{A}(\mathbb{R})$. Then the restriction $\delta_1 = \delta|_{H_1}$ forms an irreducible unitary representation of $\tilde{A}(\mathbb{R})$ which is unitarily isomorphic to U as well as to U^σ . Consequently there exist intertwining operators. In fact, the Fourier transform $\mathcal{F}_{\mathbb{R}}$ (cf. [9]) defines a unitary isomorphism of U onto U^σ and the Weil-Brezin isomorphism

$$f \mapsto W(f)(x, y, z) = \sum_{n \in \mathbb{Z}} f(x+n) e^{2\pi i(z+ny)}$$

defines a unitary isomorphism of U onto δ_1 . Moreover, the factorization

$$W^{-1} \circ \sigma \circ W = \mathcal{F}_{\mathbb{R}}$$

holds.

3. A First Consequence: The Whittaker-Shannon Cardinal Series

The central basis spline M_1 of degree 0 on \mathbb{R} (cf. Schoenberg [10]) represents the indicator function of the fundamental domain of the Heisenberg manifold $P \backslash \tilde{A}(\mathbb{R})$ with respect to one direction of its three coordinate axes. An application of the factorization $W^{-1} \circ \sigma \circ W = \mathcal{F}_{\mathbb{R}}$ yields for any function $f \in L^2_{\mathbb{C}}([-\frac{1}{2}, +\frac{1}{2}])$ the identities (cf. [8])

$$\mathcal{F}_{\mathbb{R}} f(x) = \sum_{n \in \mathbb{Z}} \mathcal{F}_{\mathbb{R}} f(n) \int_{\mathbb{R}} e^{2\pi i(x-n)y'} M_1(y') dy'$$

$$\begin{aligned}
 &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)} \quad (x \in \mathbb{R}) \\
 &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} f(n) \operatorname{sinc}(x-n).
 \end{aligned}$$

On the right hand side of this formula the classical Whittaker-Shannon cardinal interpolation series occurs (see Young [15]) which plays an important rôle in medical tomography (cf. Schwierz-Härer-Wiesent [12]) and in signal theory (sampling theorem for band-limited signal functions; see, for instance, Spletstößer [13]). Since the function $g = \int_{\mathbb{R}} f$ on the left hand side can be characterized by an application of the Paley-Wiener theorem, we have established by group-theoretic methods the following result of complex analysis: Any entire holomorphic function g of exponential type $\leq \pi$ which vanishes at all the integers is necessarily the trivial function $g = 0$ (theorem of Carlson).

4. Schwartz Kernels

Another proof of the cardinal interpolation series which is based on harmonic analysis on the Heisenberg group $\tilde{\mathbb{A}}(\mathbb{R})$ proceeds as follows: The Gårding space of \mathcal{C}^∞ -vectors of the Schrödinger representation U is the Fréchet space $\mathcal{V}(\mathbb{R})$ of infinitely differentiable complex-valued functions on \mathbb{R} which are rapidly decreasing at infinity. The complex Hilbert space $L^2_{\mathbb{C}}(\mathbb{R})$ on which U acts is embedded continuously into the locally convex topological vector space $\mathcal{V}'(\mathbb{R})$ of tempered distributions on \mathbb{R} . Thus U admits a N_1 -invariant cyclic distribution vector (cf. Schwartz [11]) and the sampling theorem is a consequence of the theory of Schwartz kernels. Some details of this aspect will be pursued in a forthcoming paper.

5. A Second Consequence: The Subbotin-Schoenberg Theorem

To be more general, let M_m ($m \geq 1$) denote the m -th convolution power of M_1 . Then M_m is the so-called central basis spline of degree $m-1$ on \mathbb{R} (cf. Schoenberg [10]) and $x \mapsto M_m(x - \frac{1}{2}m)$ represents the ordinary (forward) basis spline of degree $m-1$ on \mathbb{R} with support

$[0, m]$ (cf. [5]). Let $c \in \ell_c^\infty$ be a given (bounded) bi-infinite sequence of interpolation data. An application of the factorization $W^{-1} \circ \sigma \circ W = \int_{\mathbb{R}} \cdot$ to M_{m+1} combined with the Calderón-Spitzer-Widom characterization of invertible Toeplitz matrices shows that when m is odd the cardinal interpolation problem for c admits a unique solution of degree m . In the case when m is even and the knots are shifted by $1/2$, the midpoint cardinal interpolation problem for the bounded data c also admits a unique solution of degree m . (existence and uniqueness theorem of Subbotin-Schoenberg; see Schoenberg [10]). For details, the reader is referred to the monograph [7].

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AN ERGODIC THEOREM IN HARMONIC ANALYSIS¹

by

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Presented by M. Shinbrot F.R.S.C.

Abstract. Given a locally compact abelian group G with dual group G^* , we obtain a space C of functions which includes the continuous almost periodic and the (not necessarily continuous) positive definite functions on G . It is shown that, given a directed set D and a net $\{L_\delta; \delta \in D\}$ of positive uniformly bounded linear transformations on C (with supremum norm) such that for all $z \in G$, $L_\delta(\hat{z})(z)$ converges to 1 if \hat{z} is the identity in G^* , and to 0 for all other \hat{z} in G^* , then $L_\delta(g)(z)$ converges to a limit (which we identify by an integral) which is independent of $z \in G$. This ergodic theorem yields a generalization of a result due to J. R. Blum et al. [2] for unitary representations of G on a Hilbert space.

1. Given a locally compact abelian group G with dual group G^* , let $AP(G)$ be the space of continuous almost periodic functions on G and let $M(g)$ denote the mean value of g for all $g \in AP(G)$. Then the linear span C of all functions $f: G \rightarrow \mathbb{R}$ for which

$$\inf \{M(h-g); g \in C, h \in AP(G)\} = 0$$

contains not only $AP(G)$ but also the (not necessarily continuous) positive definite functions on G . A linear transformation $L: C \rightarrow C$ is said to be

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positive if $L(f)$ is nonnegative for all nonnegative $f \in C$. Denote by m the normalized Haar measure on the Borel subsets $B(\bar{G})$ of the Bohr compactification \bar{G} of G , and let $R = \{A \in B(\bar{G}) : m(\bar{A}) = m(\bar{A})\}$ (\bar{A} and \bar{A} being the interior and closure of A in \bar{G} , respectively). It is easy to show that R is an algebra of sets in \bar{G} . Consequently, its trace over G : $R_G = \{A \cap G : A \in R\}$ is an algebra of sets contained in the Borel subsets of G . As shown in [1, proposition 3.10], the function m_G on R_G given by $m_G(A \cap G) = m(A)$ ($A \in R$) is a well defined finitely additive real-valued set function. Furthermore, for all $f \in C$, the integral $\int_G f dm_G$ exists in the sense of Moore-Smith convergence (see [4, pp. 183-191] or [5, pp. 401-404]) or equivalently, by the Dunford-Schwartz method [3, pp 101-125]. From these facts we get the following ergodic theorem.

Theorem 1. Given a locally compact abelian group G with dual group G^\wedge , and given a directed set D , let $\{L_\delta, \delta \in D\}$ be a net of positive uniformly bounded (in supremum norm) linear transformations on C for which

$$\lim_{\delta} L_\delta(\xi)(z) = \begin{cases} 0 & \text{if } \xi \neq \text{identity in } G^\wedge \\ 1 & \text{if } \xi = \text{identity in } G^\wedge \end{cases}$$

for all $z \in G$. Then $\lim_{\delta} L_\delta(f)(z)$ exists and is given by

$$\lim_{\delta} L_\delta(f)(z) = \int_G f dm_G$$

for all $f \in C$ and all $z \in G$; the integral being of the Moore-Smith type.

Example 1. Let $G = \mathbb{Z}$, let $D = \mathbb{N}$ (with usual order), and let $L_n: C \rightarrow C$ ($n \in \mathbb{N}$) be the positive uniformly bounded linear transformation given by

$$L_n(g)(k) = \frac{1}{n} \sum_{j=0}^{n-1} g(k+j) \quad (g \in C)$$

for all $k \in G$. Then $\lim_{n \rightarrow \infty} L_n(\xi)(k)$ exists and is given by

$$\lim_{n \rightarrow \infty} L_n(\xi)(k) = \begin{cases} 0 & \text{if } \xi \neq \text{identity in } G \\ 1 & \text{if } \xi = \text{identity in } G \end{cases}$$

and so the following limit exists and is given by

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(k+j) = \int_G g \, dm_G$$

for all $g \in C$ and all $z \in G$. These transformations are a particular example of a regular matrix $\{a_{nj}, i, j=0, \pm 1, \pm 2, \dots; n=1, 2, 3, \dots\}$ with $a_{nj} \geq 0$ and $\sum_{j=-\infty}^{\infty} a_{nj} = 1$ ($n=1, 2, 3, \dots$), which is strongly regular; i.e. for which

$$\lim_{n \rightarrow \infty} \sum_{j=-\infty}^{\infty} |a_{nj} - a_{n(j-1)}| = 0$$

These matrices have the property

$$\lim_{n \rightarrow \infty} \sum_{j=-\infty}^{\infty} a_{nj} = \begin{cases} 0 & \text{if } t \in (0, 2\pi) \\ 1 & \text{if } t = 0. \end{cases}$$

To see this note that, by definition, $\sum_{j=-\infty}^{\infty} a_{nj} = 1$, while

$$(1 - e^{it}) \sum_{j=-\infty}^{\infty} a_{nj} e^{ij t} = \sum_{j=-\infty}^{\infty} (a_{nj} - a_{n(j-1)}) e^{ij t}$$

for all $t \in (0, 2\pi)$, and so

$$|(1 - e^{it}) \sum_{j=-\infty}^{\infty} a_{nj} e^{ij t}| \leq \sum_{j=-\infty}^{\infty} |a_{nj} - a_{n(j-1)}|$$

which converges to 0 as $n \rightarrow \infty$. Since any $\xi \in G^*$ is of the form $\xi(j) = e^{ij t}$ ($j \in \mathbb{Z}$) for some $t \in [0, 2\pi)$, it follows that

$$\lim_{n \rightarrow \infty} \sum_{j=-\infty}^{\infty} a_{nj} \xi(j) = \begin{cases} 0 & \text{if } \xi \neq \text{identity in } G \\ 1 & \text{if } \xi = \text{identity in } G \end{cases}$$

and so, by theorem 1, the following limit exists and is given by

$$\lim_{n \rightarrow \infty} \sum_{j=-\infty}^{\infty} a_{nj} g(k+j) = \int_G g \, dm_G$$

for all $g \in C$ and all $k \in G$.

Example 2. Let $G = (-\infty, \infty)$, let $D = (0, \infty)$ (with usual order) and let $L_T: C \rightarrow C$ ($T \in D$) be the uniformly bounded linear transformations given by

$$L_T(g)(t) = \frac{1}{T} \int_0^T g(t+s) ds \quad (g \in C)$$

for all $t \in G$. Then

$$\lim_{T \rightarrow \infty} L_T(\xi)(t) = \begin{cases} 0 & \text{if } \xi \neq \text{identity in } G \\ 1 & \text{if } \xi = \text{identity in } G \end{cases}$$

and so $\lim_{T \rightarrow \infty} L_T(g)(t)$ exists and is given by

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(t+s) ds = \int_G g d\mu_G$$

for all $g \in C$ and all $t \in G$.

2. Given a locally compact abelian group G , let $\{U_z: z \in G\}$ be a unitary representation of G on a Hilbert space H with inner product (\cdot, \cdot) . For any given $x \in H$, the complex-valued function $z \rightarrow (U_z x, x)$ on G is easily shown to be positive definite. So, by polarization (see [5, p. 322]), given $x, y \in H$, the complex-valued function $z \rightarrow (U_z x, y)$ on G , lies in C . Furthermore, we have the following:

Theorem 2. Given a locally compact abelian group G and given a unitary representation $\{U_z: z \in G\}$ of G on a Hilbert space H , the integral $\int_G (U_z x, y) d\mu_G(z)$ exists in the sense of Moore-Smith convergence and is equal to (Px, y) for all $x, y \in H$; where P is the orthogonal projection of H onto the space $\{x \in H: U_z x = x \text{ for all } z \in G\}$.

We can now apply theorem 1 to theorem 2 to obtain:

Theorem 3. Given a unitary operator U on a Hilbert space H , and given a sequence $\{n_k, k=1, 2, 3, \dots\}$ of integers for which

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n z^{n_k} = \begin{cases} 0 & \text{if } z \neq 1, z \neq -1 \\ 1 & \text{if } z = 1 \end{cases}$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (U^{n_k} x, y) = (Px, y)$$

for all $x, y \in H$; where P is the orthogonal projection of H onto $\{x \in H : Ux = x\}$.

These two theorems generalize results obtained by Blum et al. [2] in that i) the unitary representation need not be strongly continuous or have pure point spectrum (which amounts to assuming that the complex-valued function $z \rightarrow (U_z x, y)$ on G lies in $AP(G)$), and ii) to obtain (Px, y) in the form of an integral, the complex-valued function $z \rightarrow (U_z x, y)$ on G need not be extended to a complex-valued function on \bar{G} .

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ON ELLIPTIC DIFFERENTIAL EQUATIONS WITH SMALL PARAMETER

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Let D be a bounded domain in an r -dimensional space \mathbb{R}^r with a smooth boundary ∂D . We consider the Dirichlet problem

$$\begin{aligned}
 (1) \quad L^\epsilon u^\epsilon(x) &\equiv \epsilon \left(\frac{1}{2} \sum_{i,j=1}^r a^{ij}(x) \frac{\partial^2 u^\epsilon(x)}{\partial x^i \partial x^j} + \sum_{i=1}^r b^i(x) \frac{\partial u^\epsilon(x)}{\partial x^i} \right. \\
 &\quad \left. + \frac{1}{2} \sum_{i,j=1}^r A^{ij}(x) \frac{\partial^2 u^\epsilon(x)}{\partial x^i \partial x^j} + \sum_{i=1}^r B^i(x) \frac{\partial u^\epsilon(x)}{\partial x^i} \right) \\
 &\equiv (\epsilon L_1 + L_0) u^\epsilon(x) = 0 \quad \text{for } x \in D, \\
 \lim_{x \rightarrow x_0} u^\epsilon(x) &= \psi(x_0) \quad \text{for } x_0 \in \partial D.
 \end{aligned}$$

Here ϵ is a small positive parameter. The coefficients of the operators L_0 and L_1 are assumed to be smooth enough, e.g. twice continuously differentiable, defined for $x \in \mathbb{R}^r$. The operator L_1 is assumed to be elliptic: $\sum_{i,j=1}^r a^{ij}(x) \lambda_i \lambda_j \geq a \sum_{i=1}^r \lambda_i^2$ for some $a > 0$. The form $\sum_{i,j=1}^r A^{ij}(x) \lambda_i \lambda_j$ is assumed non-negative (all A^{ij} may vanish identically; in that case, problem (1) reduces to a known problem for elliptic equations with small parameter in high derivatives [1], [2]).

If $\sum A^{ij}(x) \lambda_i \lambda_j \not\equiv 0$, new effects occur in the behavior of u^ϵ , as $\epsilon \rightarrow 0$, and these are treated in the present paper.

In the simplest case: $\lim_{\epsilon \rightarrow 0} u^\epsilon = u_0$ exists, u_0 does not depend on the perturbation term L_1 and u_0 is a solution of the problem $L_0 u_0(x) = 0$, $x \in D$, with appropriate boundary con-

ditions. If the A^{ij} vanish identically, then these assertions hold whenever Levinson's conditions [1] are fulfilled: the characteristics of the first order operator L_0 , starting from any $x \in D$, arrive at ∂D in a sufficiently correct way, e.g. by crossing the boundary at a non-zero angle. The limit function u_0 is a solution of the degenerate problem

$$L_0 u_0(x) = 0, \quad x \in D, \quad \lim_{x \rightarrow x_0} u_0(x) = \psi(x_0), \quad x_0 \in \widetilde{\partial D}$$

where $\widetilde{\partial D}$ is that part of the boundary through which the characteristics leave the domain D .

To study problem (1) in the general case, we introduce a family of the Markov processes $(X_t^\epsilon, P_x^\epsilon)$ depending on a parameter $\epsilon \geq 0$ [3]. Trajectories of these processes are defined by the stochastic differential equations

$$(2) \quad dx_t^\epsilon = \epsilon \sigma_1(x_t^\epsilon) dW_t^1 + \sigma_0(x_t^\epsilon) dW_t^2 + (\epsilon b(x_t^\epsilon) + B(x_t^\epsilon)) dt,$$

where W_t^1, W_t^2 are independent Wiener processes in \mathbb{R}^r ; $\sigma_1(x), \sigma_0(x)$ are $r \times r$ -matrices, $\sigma_1^2(x) = (a^{ij}(x))$, $\sigma_0^2(x) = (A^{ij}(x))$, $b(x) = (b^1(x), \dots, b^r(x))$, $B(x) = (B^1(x), \dots, B^r(x))$.

In the general case the trajectories of the process $(X_t^\epsilon, P_x^\epsilon)$ play the role of characteristics. If the A^{ij} vanish identically then for $\epsilon = 0$ equations (2) become the characteristics equations of the operator $L_0 = \sum_{i=1}^r B^i(x) \frac{\partial}{\partial x^i}$. We denote $\tau^\epsilon = \inf\{t: X_t^\epsilon \notin D, \epsilon \geq 0, x \in D\}$. Then we shall say that generalized Levinson's conditions are satisfied, if:

1. $\lim_{t \rightarrow \infty} P_x^0\{\tau^0 > t\} = 0$ uniformly in $x \in D$,
2. $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and

$$\prod_{i,j=1}^r A^{ij}(x) \eta_i(x) \eta_j(x) = 0, \quad \sum_{i=1}^r B^i(x) \eta_i(x) < -\beta < 0$$
 for $x \in \Gamma_1$

$$\sum_{i,j=1}^{\Gamma} A^{ij}(x) \eta_i(x) \eta_j(x) = 0, \quad \sum_{i=1}^{\Gamma} B^i(x) \eta_i(x) > \beta > 0 \quad \text{for } x \in \Gamma_2,$$

$$\sum_{i,j=1}^{\Gamma} a^{ij}(x) \eta_i(x) \eta_j(x) > \beta > 0 \quad \text{for } x \in \Gamma_3,$$

where $(\eta_i(x))$ are direction cosines of the outward normal. Each of the sets Γ_1 and Γ_2 is assumed to coincide with the closure of the set of its points interior with respect to ∂D .

If L_0 is the first order operator, then these conditions are converted into classic Levinson's conditions. In order for condition 1 to be fulfilled it is sufficient, e.g. for at least one coefficient of the operator L_0 to be nonzero in the set $D \cup \partial D$.

Let A designate an infinitesimal operator of a Markov process $(\bar{X}_t^0, \bar{P}_x^0)$ in $D \cup \partial D$ derived from (X_t^0, P_x^0) by stopping at the first exit time τ^0 from the domain D . If the generalized Levinson conditions hold, the following problem

$$(3) \quad Au_0(x) = 0, \quad \lim_{x \rightarrow x_0} u_0(x) = \psi(x_0), \quad x_0 \in \Gamma_2 \cup \Gamma_3$$

has a unique solution [4]. The solution of this problem is naturally referred to as a generalized solution of the Dirichlet problem for the equation $L_0 u(x) = 0$ in D (see [4]).

THEOREM 1. Suppose that the generalized Levinson conditions hold and that the boundary function is Hölder continuous. Then for any closed domain $G \subset D \cup \partial D$ which is a positive distance from Γ_1 , one can find $C, k > 0$ such that

$$(4) \quad \sup_{x \in G} |u^\epsilon(x) - u_0(x)| < C\epsilon^k,$$

where u_0 is a solution of problem (3).

The proof of this theorem uses a representation of a solution of problem (1) in the form of the functional integral $u^\epsilon(x) = E_x \psi(X_{\tau^\epsilon}^\epsilon)$ and direct examination of trajectories of equation

(2) as $\varepsilon \rightarrow 0$.

If L_0 is the first order operator, then for sufficiently smooth data of the problem, the difference $u^\varepsilon(x) - u_0(x)$ is of the order ε and one can write for it the next terms of the asymptotic expansion. In general, this is not the case. The bound presented by Theorem 1 cannot be improved in the class of all equations of the foregoing type.

EXAMPLE. Let $\tilde{D} = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}$ and let D be a domain in the plane \mathbb{R}^2 with infinitely differentiable boundary, symmetric with respect to the X -axis, and such that $\tilde{D} \subset D \subset \{(x, y) \in \mathbb{R}^2 : |x| < 1\}$. Let us consider the Dirichlet problem in D :

$$L^\varepsilon u^\varepsilon(x, y) = \frac{\varepsilon}{2} \Delta u^\varepsilon + \frac{\alpha^2}{2} \frac{\partial^2 u^\varepsilon}{\partial x^2} + \beta y \frac{\partial u^\varepsilon}{\partial y} + \gamma \frac{\partial u^\varepsilon}{\partial x} = 0, \quad (x, y) \in D, \quad (5)$$

$$u^\varepsilon(x, y) = y^2 \quad \text{for } (x, y) \in \partial D.$$

Here α, β, γ are positive constants. It is easily seen that the generalized Levinson conditions are fulfilled. Therefore, by Theorem 1, a solution of problem (5) converges to a generalized solution of the problem

$$\frac{\alpha^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \beta y \frac{\partial u_0}{\partial y} + \gamma \frac{\partial u_0}{\partial x} = 0, \quad (x, y) \in D, \quad (6)$$

$$u_0(x, y) \Big|_{(x, y) \in \partial D} = y^2 \quad \text{as } \varepsilon \rightarrow 0.$$

Let λ_0 be the first eigenvalue of the problem

$$\frac{\alpha^2}{2} \frac{d^2 v(x)}{dx^2} + \gamma \frac{dv(x)}{dx} = \lambda_0 v(x), \quad x \in (-1, 1), \quad v(-1) = v(1) = 0.$$

One can show that for any $\mu > \frac{\lambda_0}{2\beta}$ there is a $C > 0$ such that $|u^\varepsilon(0,0) - u_0(0,0)| > C\varepsilon^\mu$ for sufficiently small ε . This implies the impossibility of expanding $u^\varepsilon(x,y)$ in preassigned powers of the small parameter. One can prove that, given $k < 1 \wedge \frac{\lambda_0}{2\beta}$, then $|u^\varepsilon(x,y) - u_0(x,y)| < C\varepsilon^k$, i.e., the lower bound cited in this example cannot be improved.

Now we state conditions which ensure the existence of the first k terms of asymptotic expansion into integer powers of small parameter. We put

$$\alpha(L_0, D) = -\limsup_{t \rightarrow \infty} \sup_{x \in D} \frac{1}{t} \ln P_x^0\{\tau^0 > t\},$$

$$K = \sup_{x \in \mathbb{R}^r; i, j, k, \ell = 1, \dots, r} \left(2r^{3/2} \left| \frac{\partial^2 A^{ij}(x)}{\partial x^k \partial x^\ell} \right|^{1/2}, \left| \frac{\partial B^i(x)}{\partial x^k} \right| \right).$$

We will introduce the recursive sequence $\beta_{k,m}$:

$$\beta_{1,m} = (2m-1)mr^3K^2 + 2mrK, \quad m = 1, 2, \dots,$$

$$\beta_{k,m} = \beta_{1,m} + \beta_{k-1,km} + 4m^2, \quad k > 1.$$

THEOREM 2. Assume that the coefficients of the operators L_1 and L_0 are infinitely differentiable, $\partial D = \Gamma_1 \cup \Gamma_3$ (see the generalized Levinson conditions), where Γ_1 and Γ_3 are closed $(r-1)$ -dimensional manifolds of class C^∞ , and let ψ be an infinitely differentiable function on ∂D . If $\alpha(L_0, D) > \beta_{4k,4k}$ then

$$u^\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \dots + \varepsilon^k u_k(x) + O(\varepsilon^k), \quad \varepsilon \neq 0,$$

where u_0 is a solution of problem (3), and u_i , $1 \leq i \leq k$, are defined successively by the equations:

$$(7) \quad L_0 u_i(x) = L_1 u_{i-1}(x), \quad x \in D; \quad u_i(x) \Big|_{x \in \Gamma_3} = 0.$$

Each of problems (7) for $1 \leq i \leq k$ has a unique (classic) solution.

The proof of Theorem 2 relies on bounds presented in [4] and [5]. Presumably, a sequence growing not so fast can be taken in place of $\beta_{k,m}$, if one evaluates the bounds more thoroughly.

REMARK. If the operator L_0 is replaced by $\tilde{L}_0 = L_0 - c(x)$, $c(x) \geq c_0 > 0$, where $c(x)$ is an infinitely differentiable function, $\left| \frac{\partial c(x)}{\partial x^i} \right| \leq K$, then, in the hypotheses of Theorem 2, $\alpha(L_0, D)$ may be replaced by $\alpha(L_0, D) + c_0$. For the operator \tilde{L}_0 , the requirement in the generalized Levinson condition, to leave the domain uniformly fast, may be dropped.

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$\phi(n)$, $\mu(n)$ being respectively the Euler totient function and the Möbius function.

A short proof, following the lines of Hooley's method, is supplied in Lemmas 2, 3 and 4. In preparation for these, we define

$$(6) \quad k = (h^2 - a)q^{-1},$$

so that, by hypothesis, $k > 0$ for all sufficiently large h , and note that

$$(7) \quad 0 < \mu = \rho\sigma = q^{-1}[4p(x_1^2 + x_2^2) - a] \leq q^{-1}(h^2 - a) = k$$

holds for the summation condition in (2). Thus, we may split the sum S into three such sums (as in the elementary Dirichlet divisor problem), where $S = S_1 + S_2 - S_3$ and

$$(8) \quad S_1 = \sum_{\rho \leq k^{\frac{1}{2}}} \quad , \quad S_2 = \sum_{\sigma \leq k^{\frac{1}{2}}} \quad , \quad S_3 = \sum_{\rho \leq k^{\frac{1}{2}}, \sigma \leq k^{\frac{1}{2}}} .$$

Alternative expressions for these sums are given in (11), (12) and (15), respectively. As explained in [1], the main tool is a revised version of Smith's theorem on the circle problem in arithmetic progressions:

"If $(b, m) = (e, m) = (c, b) = 1$, $m \ll X^{\frac{2}{3}}$ then, for any $\delta > 0$,

$$(9) \quad \sum_{\substack{n \leq X, \\ (n, b) = 1 \\ cn \equiv b \pmod{m}}} r(n) - \frac{\pi}{4} \frac{X}{m} M(b, m) \ll_{\delta} (X^{\frac{2}{3}} m^{-\frac{1}{2}})^{1+\delta} |b|^{\delta},$$

$$(10) \quad \text{where } M(b, m) = \phi(b)b^{-1}H_m(1)H_b(1)."$$

2. Certain elementary estimates are required for Lemmas 3 and 4; these are given without proof in Lemma 1:

LEMMA 1. (i)
$$\sum_{\substack{\rho \leq N \\ (\rho, 2A)=1}} \chi(\rho) H_{q\rho}(1) \ll H_q(1) d(Aq) \log N, \text{ as } N \rightarrow \infty$$

(ii)
$$\sum_{\substack{\rho \leq N \\ (\rho, 2A)=1}} \chi(\rho) \rho^{-1} H_{q\rho}(1) - H_q(1) L_{Aq}(1) \zeta_{2Aq}(2)^{-1} \ll H_q(1) d(A) N^{-1} \log N \text{ as } N \rightarrow \infty.$$

LEMMA 2. $S_2 = S_1.$

Proof: Observe that S_1 and S_2 can be written as repeated sums involving congruential conditions. Thus

(11)
$$S_1 \stackrel{\text{def.}}{=} \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ 4p(x_1^2+x_2^2)-q\rho\sigma=a, (\rho\sigma, a)=1 \\ 4p(x_1^2+x_2^2) \leq h^2, \rho \leq k^{\frac{1}{2}}}} \chi(\rho) = \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ (\rho, 2pa)=1}} \chi(\rho) \sum_{\substack{1 \\ 4p(x_1^2+x_2^2) \equiv a \pmod{q\rho} \\ 4p(x_1^2+x_2^2) \leq h^2, (x_1^2+x_2^2, a)=1}}$$

(12)
$$S_2 \stackrel{\text{def.}}{=} \sum_{\substack{\sigma \leq k^{\frac{1}{2}} \\ 4p(x_1^2+x_2^2)-q\rho\sigma=a, (\rho\sigma, a)=1 \\ 4p(x_1^2+x_2^2) \leq h^2, \sigma \leq k^{\frac{1}{2}}}} \chi(\rho) = \sum_{\substack{\sigma \leq k^{\frac{1}{2}} \\ (\sigma, 2pa)=1}} \chi(\rho) \sum_{\substack{1 \\ 4p(x_1^2+x_2^2)-q\rho\sigma=a, (\rho, a)=1 \\ 4p(x_1^2+x_2^2) \leq h^2}} \\ = -\chi(aq) \sum_{\substack{\sigma \leq k^{\frac{1}{2}} \\ (\sigma, 2pa)=1}} \chi(\sigma) \sum_{\substack{1 \\ 4p(x_1^2+x_2^2) \equiv a \pmod{q\sigma}, (x_1^2+x_2^2, a)=1, \\ 4p(x_1^2+x_2^2) \leq h^2}}$$

since $(\sigma, a) = (q\rho\sigma, a) = (4p(x_1^2 + x_2^2), a) = (x_1^2 + x_2^2, a)$ by the hypotheses of the theorem. Since, also, $-q\rho\sigma \equiv a \pmod{4}$ and $(\rho, a) = 1$, we have $\chi(\rho) = -\chi(aq)\chi(\sigma)$. But, for a solution of (1), with p replaced by $4p$, to exist, we have

$$\chi(a) = \chi(-q(x_1^2 + x_2^2)) = \chi(-q).$$

or $\chi(aq) = -1$. Thus $S_1 = S_2$.

3. Lemma 3. For any $\delta > 0$, $a = o(h^2)$ and $h^2/p \gg (pq)^3$ as $pq \rightarrow \infty$,

$$S_1 - \frac{\pi}{4^2} \frac{h^2}{pq} W(p, q, a) \ll_{\delta} \left[h^{1/6} p^{-2/3} q^{-3/6} \right]^{1+\delta} |a|^{\delta}.$$

Proof: By (11),

$$S_1 = \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ (\rho, 2pa)=1}} \chi(\rho) \sum_{\substack{n \leq X, (n,b)=1 \\ cn \equiv b \pmod{m}}} r(n)$$

where $m = qp$, $X = h^2/4p$, $c = 4p$, $b = a$. Since $(x_1^2 + x_2^2, a) = 1$, $(a, 4p) = 1$ we have $(b, m) = 1$; also $(c, m) = (4p, qp) = 1$, $(c, b) = (4p, a) = 1$. Now, for the hypothesis $m \ll X^{2/3}$ concerning (9), we note that

$$m = qp \ll (h^2/4p)^{2/3} = X^{2/3}$$

holds when $qk^{\frac{1}{2}} \ll (h^2/p)^{2/3}$ or when $h^2/p \gg (pq)^3$ and $a = o(h^2)$.

Thus (9) holds in the form

$$(13) \quad \sum_{\substack{n \leq X, (n,b)=1 \\ cn \equiv b \pmod{m}}} r(n) - \frac{\pi}{4} \frac{h^2}{4pq} M(a, qp) \ll_{\delta} \left[(h^2/p)^{2/3} (qp)^{-\frac{1}{2}} \right]^{1+\delta} |a|^{\delta}.$$

Hence

$$S_1 - \frac{\pi}{4^2} \frac{h^2}{pq} \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ (\rho, 2pa)=1}} \chi(\rho) \rho^{-1} M(a, qp) \ll_{\delta} E_1,$$

where

$$(14) \quad \begin{aligned} E_1 &= \left[h^{1/3} p^{-2/3} q^{-1/2} \right]^{1+\delta} |a|^{\delta} \sum_{\rho \leq k^{\frac{1}{2}}} \rho^{-\frac{1}{2}(1+\delta)} \\ &\ll_{\delta} \left[h^{1/3} p^{-2/3} q^{-1/2} \right]^{1+\delta} |a|^{\delta} k^{\frac{1}{2}(1-\delta)} \\ &\ll_{\delta} \left[k^{\frac{1}{2}} h^{1/3} p^{-2/3} q^{-1/2} \right]^{1+\delta} |a|^{\delta} \\ &\ll_{\delta} \left[h^{1/6} p^{-2/3} q^{-3/4} \right]^{1+\delta} |a|^{\delta}. \end{aligned}$$

On using the formula (10) for $M(a, qp)$, where $(a, qp) = 1$, then Lemma 1(ii) with $N = k^{\frac{1}{2}}$, $A = pa$, and noting that

$$\frac{h}{k^{\frac{1}{2}}} \frac{1}{(pq)^{\frac{1}{2}}} \ll h^{1/6} p^{-2/3} q^{-3/4}, \text{ if } h^2/p \gg (pq)^3.$$

(cf. (19)), we obtain the required estimate.

Lemma 4. For any $\delta > 0$, $a = o(q^{\frac{1}{2}}h)$ as $h \rightarrow \infty$, $h^2/p \gg (pq)^3$ as $pq \rightarrow \infty$,

$$S_3 \ll_{\delta} \left[h^{1/6} p^{-2/3} q^{-3/4} \right]^{1+\delta} |a|^{\delta}.$$

Proof:

$$(15) \quad S_3 = \sum_{\substack{\chi(\rho) \\ 4p(x_1^2+x_2^2)-q\rho=a, (\rho, a)=1 \\ 4p(x_1^2+x_2^2) \leq h^2, \rho \leq k^{1/2}, \sigma \leq k^{1/2} \\ (\rho, 2pa)=1}} \chi(\rho) = \sum_{\substack{\chi(\rho) \\ \rho \leq k^{1/2} \\ (\rho, 2pa)=1}} \chi(\rho) \sum_{\substack{1 \\ 4p(x_1^2+x_2^2) \equiv a \pmod{q\rho}, (x_1^2+x_2^2, \rho)=1 \\ 4p(x_1^2+x_2^2) \leq \min(h^2, a+q\rho k^{1/2})}} 1$$

where $(\sigma, a) = (x_1^2 + x_2^2, a)$, as remarked in the proof of Lemma 2.

For $\rho \leq k^{1/2}$

$$(16) \quad a + q\rho k^{1/2} \leq a + qk = h^2$$

and so

$$(17) \quad S_3 = \sum_{\substack{\chi(\rho) \\ \rho \leq k^{1/2} \\ (\rho, 2pa)=1}} \chi(\rho) \sum_{\substack{r(n) \\ n \leq X, (n, b)=1 \\ cn \equiv b \pmod{m}}} r(n)$$

with

$$(18) \quad m = q\rho, \quad X = (a + q\rho k^{1/2})/4p, \quad c = 4p, \quad b = a.$$

Since $(x_1^2 + x_2^2, a) = 1$, $(a, 4p) = 1$, we see that (b, m)

$= (a, q\rho) = 1$; also $(c, m) = (4p, q\rho) = 1$, $(c, b) = (4p, a) = 1$.

For the remaining hypothesis $m \ll X^{2/3}$ concerning (9), it

suffices to prove that, with $a = o(q^{\frac{1}{2}}h)$,

$$m^{3/2} = (q\rho)^{3/2} \ll X - \frac{a}{4p} = \frac{q\rho}{4p} k^{1/2} \ll X$$

and this is certainly satisfied for all $\rho \leq k^{1/2}$ and

$h^2/p \gg (pq)^3$. Hence, by (9) and (18),

$$\sum_{\substack{r(n) \\ n \leq X, (n, b)=1 \\ cn \equiv b \pmod{m}}} r(n) - \frac{\pi}{4} \left[\frac{a + q\rho k^{1/2}}{4p q\rho} \right] M(a, q\rho) \ll_{\delta} \left[(h^2/p)^{2/3} (q\rho)^{-1/2} \right]^{1+\delta} |a|^{\delta},$$

on using (16). Thus, from (17) and (14),

$$\begin{aligned}
S_3 &<< \frac{|a|}{pq} \left| \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ (\rho, 2pa)=1}} \chi(\rho) \rho^{-1} M(a, q\rho) \right| + \frac{k^{\frac{1}{2}}}{p} \left| \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ (\rho, 2pa)=1}} \chi(\rho) M(a, q\rho) \right| + E_1 \\
&<< \frac{|a|}{pq} H_a(1) \left| \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ (\rho, 2pa)=1}} \chi(\rho) \rho^{-1} H_{qp}(1) \right| + \frac{k^{\frac{1}{2}}}{p} H_a(1) \left| \sum_{\substack{\rho \leq k^{\frac{1}{2}} \\ (\rho, 2pa)=1}} \chi(\rho) H_{qp}(1) \right| + E_1 \\
&<< \frac{|a|}{pq} H_a(1) H_q(1) L_{pqa}(1) \zeta_{2pqa}(2)^{-1} + \frac{k^{\frac{1}{2}}}{p} H_a(1) H_q(1) d(2A) \log k + E_1
\end{aligned}$$

on using Lemma 1(i) and (ii). By (14), $S_3 \ll E_1$ as required, since

$$(19) \quad \frac{|a|}{pq} \ll \frac{qk^{\frac{1}{2}}}{pq} = \frac{k^{\frac{1}{2}}}{p} \ll \frac{h}{p^{\frac{1}{2}}} \frac{1}{(pq)^{\frac{1}{2}}} < \left(\frac{h}{p^{\frac{1}{2}}}\right)^{\frac{1}{3}} p \ll h^{\frac{1}{6}} p^{-\frac{2}{3}} q^{-\frac{1}{6}}$$

when $h^2/p \gg (pq)^3$ and

$$\begin{aligned}
H_a(1) H_q(1) L_{pqa}(1) \zeta_{2pqa}(2)^{-1} &<< |a|^{\frac{1}{2}\delta} q^{\frac{1}{2}\delta} (pq|a|)^{\frac{1}{2}\delta} \cdot 1 \ll (pq|a|)^\delta, \\
H_a(1) H_q(1) d(2A) \log k &<< |a|^{\frac{1}{2}\delta} q^{\frac{1}{2}\delta} (p|a|)^{\frac{1}{2}\delta} \left(\frac{h}{q^{\frac{1}{2}}}\right)^\delta = \left(\frac{h}{p^{\frac{1}{2}}}\right)^\delta p^\delta |a|^\delta \\
&< \left[\left(\frac{h}{p^{\frac{1}{2}}}\right)^{\frac{1}{3}} p\right]^\delta |a|^\delta.
\end{aligned}$$

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ROOTS AND ANALYTIC ITERATION OF
FORMALLY BIHOLOMORPHIC MAPPINGS

Ludwig Reich and Arnold R. Kräuter

Presented by J. Acaél F.R.S.C.

The first author has shown in [2] and [3] that there is a close relation between the existence of roots of a shrinking biholomorphic mapping and the analytic iterability of this mapping (for these and further definitions cf. [1]-[4], and the bibliography mentioned there). More exactly he investigated, under what conditions the existence of a sequence of roots implies the existence of an analytic iteration (the converse is trivially true). That nontrivial conditions can be expected follows from an example given by J. Tritthart in [6] where he constructed a non-iterable mapping with roots of arbitrarily high orders. The present note deals with generalizations of the results in [2] and [3] for arbitrary formally biholomorphic mappings. First we make some remarks on the notation. Let Γ denote the group of formally biholomorphic mappings $F : x \mapsto F(x) = Ax + \mathfrak{P}(x)$, $x = {}^t(x_1, \dots, x_n)$, $A =: \text{lin}(F) \in \text{GL}_n(\mathbb{C})$, $\mathfrak{P}(x) \in (\mathbb{C}[x])^n$, $\text{ord } \mathfrak{P}(x) \geq 2$, with the composition \circ of formal power series as group operation (we write FG instead of $F \circ G$ for F, G). By definition, $\Gamma^{(m)} := \{\mathfrak{P}(x) \in (\mathbb{C}[x])^n : \text{ord } \mathfrak{P}(x) > m\}$; for $F(x), G(x) \in (\mathbb{C}[x])^n$ we define $F(x) \equiv G(x) \pmod{\Gamma^{(m)}}$ iff $F(x) - G(x) \in \Gamma^{(m)}$. If an r -th root of F , $F^{1/r}$, is congruent $\pmod{\Gamma^{(m)}}$ to a normal form, we call $F^{1/r}$ a "normal form $\pmod{\Gamma^{(m)}}$ ". (The term "normal form" used here is slightly more

general than that in [1].)

Theorem 1. Suppositions:

(a) Let $F \in \Gamma$ and $\Lambda = (\lambda_1, \dots, \lambda_n)$ be an arbitrary but fixed choice of the logarithms of the eigenvalues of $\text{lin}(F)$.

(b) Let $(F^{1/r_j})_{j \in \mathbb{N}}$ be a sequence of roots of F with $r_j < r_{j+1}$, such that $\text{lin}(F^{1/r_j})$ has the eigenvalues $\exp(\frac{1}{r_j} \lambda_1), \dots, \exp(\frac{1}{r_j} \lambda_n)$ for all $j \in \mathbb{N}$.

(c) Let F^{1/r_j} and $F^{1/r_{j+1}}$ commute for all $j \in \mathbb{N}$.

Assertion: Then there exists an analytic iteration of F with respect to Λ .

Proof. Assumption (b) implies, for arbitrary $m \in \mathbb{N} \setminus \{1\}$, the existence of a lower bound $T(m) \in \mathbb{N}_0$, such that for all $j_1 > T(m)$ each normal form $N_{j_1} = S^{-1} F^{1/r_{j_1}} S$ of $F^{1/r_{j_1}}$ is already smooth mod $\Gamma^{(m)}$ with respect to Λ . Now, repeated application of (c) (cf. [5], p. 76) yields that, for arbitrary $j_2 > j_1$,

$M_{j_2} = S^{-1} F^{1/r_{j_2}} S$ is also smooth mod $\Gamma^{(m)}$ with respect to Λ . Furthermore we choose j_2 so large that each normal form of

$F^{1/r_{j_2}}$ is smooth mod $\Gamma^{(m+1)}$ with respect to Λ . According to a lemma in the theory of normal forms there exists a transformation $T_{j_1} \in \Gamma$ with $T_{j_1}(x) = Ex + (1) \tilde{\zeta}(x)$, $\text{ord}_{(1)} \tilde{\zeta}(x) \geq m+1$, bringing M_{j_2} to a normal form $N_{j_2} = T_{j_1}^{-1} S^{-1} F^{1/r_{j_2}} S T_{j_1}$. By the definition of j_2 , N_{j_2} is smooth mod $\Gamma^{(m+1)}$ with respect to Λ ,

and this holds for $N_{j_2}^{r_{j_2}} = T_{j_1}^{-1} S^{-1} F S T_{j_1}$ too. Continuing in this way we get a sequence of transformations $(S T_{j_1} \dots T_{j_k})_{k \in \mathbb{N}}$ converging to a $U \in \Gamma$, due to the particular form of the T_{j_k} , such that $N = U^{-1} F U$ is a smooth normal form with respect to A . Then the assertion follows with [4], p. 219. \square

As a very important special case of Theorem 1 we mention the following

Corollary. Theorem 1 holds in particular for consecutive roots (i.e. $r_j = s_1 \dots s_j$, $s_j \in \mathbb{N} \setminus \{1\}$ for all $j \in \mathbb{N}$) without the supposition (c).

In the following theorems the condition (c) in Theorem 1 will be varied and weakened, respectively.

Theorem 2. Suppositions:

(a), (b) as in Theorem 1.

(c) Let $T^{-1}(\text{lin}(F^{1/r_j}))T$ be of "canonical structure" with respect to A for all $j \in \mathbb{N}$, where $T \in GL_n(\mathbb{C})$ denotes a suitable transformation of $\text{lin}(F)$ to the Jordan normal form.

(d) For all $k \in \mathbb{N}$ the following holds: If $G_{(k)} \in \Gamma$, with $\text{lin}(G_{(k)}) = \text{lin}(F^{1/r_k})$, satisfies the congruence $F(x) \equiv (G_{(k)}(x))^{r_k} \pmod{\Gamma^{(m_k)}}$ for sufficiently large $m_k \in \mathbb{N} \setminus \{1\}$, then the m_k -jet of $G_{(k)}$ can be continued to an r_k -th root of F .

Assertion: Then there exists an analytic iteration of F with respect to A .

In (c) the term "canonical structure" refers to a certain

block structure of the matrix depending only on Λ .

Theorem 3. Suppositions:

(a), (b) as in Theorem 1.

(c) For all $m \in \mathbb{N} \setminus \{1\}$ and for all $j > T(m)$ (where $T(m)$ has the same meaning as in the proof of Theorem 1), there exists a $V(j,m) \in \Gamma$, such that $V(j,m)^{-1} F^{1/r^j} V(j,m)$ and $V(j,m)^{-1} F^{1/r^{j+1}} V(j,m)$ are mod $\Gamma^{(m)}$ in normal form.

Assertion: Then there exists an analytic iteration of F with respect to Λ .

Remark. Theorem 3 implies Theorem 1.

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FURTHER CONNECTIONS BETWEEN
FRACTIONAL AND ANALYTIC ITERATION

Ludwig Reich and Arnold R. Kräuter

Presented by J. Aczél F.R.S.C.

In [4] we have considered the connection between an infinite sequence of roots of a formally biholomorphic mapping F and the analytic iterability of F . Here we shall treat this problem assuming only the existence of a single root. Moreover we will study the question when is a given sequence of roots contained in an analytic iteration. (For notation refer to [4].)

Theorem 1. Suppositions:

- (a) Let $F \in \Gamma$ and $\Lambda = (\lambda_1, \dots, \lambda_n)$ be an arbitrary but fixed choice of the logarithms of the eigenvalues of $\text{lin}(F)$.
- (b) Suppose that there exists an s -th root $F^{1/s}$ of F with respect to $(\exp(\frac{1}{s} \lambda_1), \dots, \exp(\frac{1}{s} \lambda_n))$ for $s \in M(\Lambda)$ where $M(\Lambda) \in \mathbb{N}$ denotes a constant depending only on Λ such that the set

$$R_0 := \{n_{k\ell} \in \mathbb{Z} : \lambda_k = \sum_{j=1}^n \alpha_j^{(k,\ell)} \lambda_j + 2\pi i n_{k\ell}, \alpha_j^{(k,\ell)} \in \mathbb{N}_0^n, \\ |\alpha_j^{(k,\ell)}| \geq 2; k=1, \dots, n; \ell = \ell(k) \in \mathbb{N}\}$$

is bounded by $M(\Lambda)$, i.e. $|n_{k\ell}| < M(\Lambda)$ for all $n_{k\ell} \in R_0$.

Assertion: Then there exists an analytic iteration of F with respect to Λ .

Proof. According to [2] the existence of $F^{1/s}$ is equivalent to the following: F is conjugate to a normal form

$N : x \rightarrow N(x) = Jx + \mathfrak{N}(x)$, such that in $\mathfrak{N}_k(x)$ at most monomials additional to $\exp(\frac{1}{s}\lambda_k)$ occur. The condition $|n_{k\ell}| < M(\Lambda)$ for all $n_{k\ell} \in \mathbb{R}_0$ just implies that each monomial additional to $\exp(\frac{1}{s}\lambda_k)$ is at the same time a smooth monomial additional to $\exp(\lambda_k)$ with respect to Λ . Therefore N is a smooth normal form with respect to Λ . Applying [1], p. 219, we get the assertion. \square

Remark. Theorem 1 is, in particular, true for shrinking biholomorphic mappings (i.e. the absolute value of all eigenvalues of $\text{lin}(F)$ is smaller than 1).

Corollary. Theorem 1 holds in the special case which we get by replacing (b) by the following assumption:
Suppose that there exists an s -th root $F^{1/s}$ of F with respect to $(\exp(\frac{1}{s}\lambda_1), \dots, \exp(\frac{1}{s}\lambda_n))$ for $s > M(\Lambda)$ where $M(\Lambda) \in \mathbb{N}$ is a suitable constant depending only on Λ . Furthermore assume that almost all monomials additional to $\exp(\lambda_k)$ are smooth with respect to Λ for $k = 1, \dots, n$.

Theorem 2. Suppositions:

- (a) as in Theorem 1.
- (b) There exists an s -th root $F^{1/s}$ of F with respect to $(\exp(\frac{1}{s}\lambda_1), \dots, \exp(\frac{1}{s}\lambda_n))$ where s does not divide any $n_{k\ell} \in \mathbb{R}_0 \setminus \{0\}$ (\mathbb{R}_0 defined as in Theorem 1).

Assertion: Then there exists an analytic iteration of F with respect to Λ .

Theorem 3. Suppositions:

- (a) as in Theorem 1.

(b) Let $(F^{1/r_1 \cdots r_j})_{j \in \mathbb{N}}$ be a sequence of consecutive roots of F with $r_j \in \mathbb{N} \setminus \{1\}$ for all $j \in \mathbb{N}$.

(c) Suppose that $\text{lin}(F^{1/r_1 \cdots r_j})$ belongs to an analytic iteration of $\text{lin}(F)$ with respect to Λ for all $j \in \mathbb{N}$.

Assertions: (1) There exists an analytic iteration \mathcal{G}' of F with respect to Λ .

(2) $F^{1/r_1 \cdots r_j} \in \mathcal{F}$ for all $j \in \mathbb{N}$.

Assumption (c) implies that $\text{lin}(F^{1/r_1 \cdots r_j})$ has the eigenvalues $\exp(\frac{1}{r_1 \cdots r_j} \lambda_1), \dots, \exp(\frac{1}{r_1 \cdots r_j} \lambda_n)$. Therefore all conditions of the Corollary to Theorem 1 in [4] are satisfied and (1) follows immediately. For the proof of (2) one needs the following important

Lemma. Suppose that $\mathcal{G}^{(j)}$ is a (smooth) analytic iteration of $F^{1/r_1 \cdots r_j}$ with respect to $\frac{1}{r_1 \cdots r_j} \Lambda$ modulo a parameter transformation $t \mapsto \tau = \frac{t}{r_1 \cdots r_j}$ for an arbitrary but fixed $j \in \mathbb{N}$ (for the expression "smooth iteration" cf. [3]). Then $\mathcal{G}^{(j)}$ is a (smooth) analytic iteration of F with respect to Λ and $F^{1/r_1 \cdots r_j} \in \mathcal{G}^{(j)}$.

Theorem 4. Suppositions:

(a) as in Theorem 1.

(b) Suppose that there exists an s -th root $F^{1/s}$ of F for $s > M(\Lambda)$ where $M(\Lambda) \in \mathbb{N}$ denotes a constant depending only on Λ , such that the set R_0 defined as in Theorem 1 is bounded by $M(\Lambda)$, i.e. $|n_{k\ell}| < M(\Lambda)$ for all $n_{k\ell} \in R_0$.

(c) $\text{lin}(F^{1/s})$ belongs to an analytic iteration of $\text{lin}(F)$ with respect to Λ .

Assertions: (1) There exists an analytic iteration \mathcal{U} of F with respect to A .

(2) $F^{1/s} \in \mathcal{F}$.

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ON A THEOREM OF S. BERBERIAN AND I. HALPERIN

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Presented by P. G. Rooney F.R.S.C.

In a privately circulated paper, S. Berberian has proved that for x, y bounded operators on a Hilbert space H , with $xy = 1$ and $yx \neq 1$, $x - 1$ (respectively, $y - 1$) cannot be compact. I. Halperin has extended this, by a geometric argument, to the case in which H is allowed to be any Banach space [2]. [In the course of his argument, Halperin makes the following useful and easily verified observation: from the same hypotheses, it follows more generally that $x - a$ (respectively, $y - a$) cannot be compact, if a is any bounded invertible operator on H . However, our concern is with the Banach space theorem and $a = 1$].

In this note, we present two successive improvements on this theorem of Berberian and Halperin, replacing non-compactness of the operator x by stronger, spectral properties of x , thereby transporting the context of the problem from operator algebras to general Banach algebras.

§1. First, B. Aupetit, in Québec, obtains an extension of the theorem as a corollary to the following interesting lemma, which is a simplification of corollary 2, page 21, in [1].

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Lemma. Let A be a Banach algebra. Suppose that a, b in A satisfy $a(ab - ba) = 0$ or $(ab - ba)a = 0$ and suppose that the spectrum of a has Lebesgue planar measure zero. Then the spectral radius of $ab - ba$ is zero.

Proof. For instance we suppose that $(ab - ba)a = 0$, the other case being proved by a similar argument. If a is invertible we have $ab - ba = 0$ so the argument is finished. So we suppose that 0 is in the spectrum of a . We have:

$$(\lambda - a)b = \left\{ b - \frac{1}{\lambda}(ab - ba) \right\} (\lambda - a) \quad \text{for } \lambda \neq 0,$$

so $(\lambda - a)b(\lambda - a)^{-1} = b - \frac{1}{\lambda}(ab - ba)$ for $\lambda \notin \text{Sp } a$. Consequently, if ρ denotes the spectral radius:

$$|\lambda| \rho(b) = |\lambda| \rho \left\{ (\lambda - a)b(\lambda - a)^{-1} \right\} = \rho(\lambda b - (ab - ba)) \quad \text{for } \lambda \notin \text{Sp } a.$$

The function $\lambda \mapsto |\lambda| \rho(b)$ is subharmonic on \mathbb{C} , while Vesentini's theorem (see [1], p.9) says in part that if on a domain D of \mathbb{C} , $f: D \rightarrow A$ is analytic, then $\rho \circ f: D \rightarrow \mathbb{R}$ is subharmonic; so $\lambda \rightarrow \rho(\lambda b - (ab - ba))$ is also subharmonic on \mathbb{C} . By hypothesis, these two subharmonic functions agree almost everywhere on \mathbb{C} ; but this implies they are identical (see for instance the lemma of §10, page 64, in [3]). Taking $\lambda = 0$, we obtain that $\rho[ab - ba] = 0$.

Corollary. *Let A be a Banach algebra with identity. Suppose that x, y in A satisfy $xy=1$ and $yx \neq 1$. Then x and y have spectra with non zero planar measure. In particular these spectra are not countable.*

Proof. Let $p = yx \neq 1$. Of course $p^2 = y(xy)x = p$. We have

$$x(xy - yx) = x(1 - p) = x - xp = 0$$

$$(xy - yx)y = (1 - p)y = y - py = 0$$

If $\text{Sp } x$ or $\text{Sp } y$ has measure zero, by the lemma we conclude that the idempotent $1 - p = xy - yx$ is quasi-nilpotent. But then

$$\| (1 - p)^n \|^{1/n} = \| 1 - p \|^{1/n}$$

goes to zero when n goes to infinity. So $p=1$, which is a contradiction.

§2. Then, by the most elementary Banach algebra methods, L. T. Gardner in Toronto proves a stronger result.

Theorem. *Let A be a Banach algebra with identity, and let x, y in A satisfy $xy=1$, $yx \neq 1$. Then the spectrum of x (respectively, of y) contains a neighborhood of 0.*

Proof. We need only prove the assertion for x , since the roles of x and y are interchanged in the opposed algebra

A^0 , while the spectra do not change; $\text{Sp}_{A^0} a = \text{Sp}_A a$ ($a \in A$).

It is clear that $0 \in \text{Sp } x$. As before, let the idempotent yx be denoted by p . For $\lambda \in \mathbb{C}$, we have

$$(1) \quad (x - \lambda 1)y = xy - \lambda y = 1 - \lambda y, \text{ and}$$

$$(2) \quad y(x - \lambda 1) = yx - \lambda y = p - \lambda y \neq 1 - \lambda y.$$

If $1 - \lambda y$ is invertible, we then have

$$(1') \quad (x - \lambda 1)(y(1 - \lambda y)^{-1}) = 1, \text{ and}$$

$$(2') \quad (y(1 - \lambda y)^{-1})(x - \lambda 1) = (1 - \lambda y)^{-1}(y(x - \lambda 1)) = (1 - \lambda y)^{-1}(p - \lambda y) \neq 1.$$

So, as above, $0 \in \text{Sp}(x - \lambda 1)$, or $\lambda \in \text{Sp } x$. In particular this holds if $|\lambda|^{-1} > \rho(y)$, the spectral radius of y , or $|\lambda| > \rho(y)^{-1}$.

Corollary. (to the proof of the Theorem)

Let A be a Banach algebra with identity, and let x, y, a in A satisfy

i) a is invertible in A ,

ii) $ay = ya$,

iii) $xy = a = yx$.

Then $\text{Sp } x$ contains a disk about 0 , and y is not quasi-nilpotent.

Remark. It is reasonable to ask whether, on the hypotheses of the theorem, $\text{Sp } x$ must be a disk about 0 , as is clearly the case if $\rho(x) \cdot \rho(y) = 1$. The answer is negative..

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SOME HOMOLOGICAL CHARACTERIZATIONS OF REGULAR RINGS

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*Presented by J. Lambek F.R.S.C.*1. Introduction.

A ring R is regular if for each a in R there exists an x in R such that $a = axa$. Regular rings were introduced by von Neumann in [26], where he showed that the set of principal right (left) ideals of such a ring forms a complemented, modular lattice. von Neumann also used the notion of regular rings in his study of continuous geometries. Various ring theoretical characterizations of regular rings can be found in [2]. In 1956 Kaplansky proved that a commutative ring R is regular if and only if each simple R -module is injective (see [23] for a published proof of this result). Several important homological characterizations of regular rings have since appeared in the literature. For instance, Auslander [3] proved that a ring R (commutative or not) is regular if and only if the weak global dimension of R is zero. Another well-known result which is again due to Kaplansky [18], states that R is a regular ring if and only if each finitely generated submodule of a projective right (left) R -module P is a direct summand of P . Some work in generalizing this result has recently been done in [1], where it is shown that a commutative ring R is regular if and only if each finitely generated submodule of a finitely generated quasi-projective R -module is again quasi-projective. Regular rings have also been characterized in terms of flat modules, by Auslander in his above cited paper. In particular, Auslander proved that a ring R is regular if and only if each right (left) R -module is flat. In fact, in order to prove that R is regular, it is sufficient to assume that each cyclic right (left) R -module is flat. Thus there is an important connection between regular rings and flat modules over such rings. In some recent publications the concept of flatness itself has been generalized and also dualized. We refer to a paper of Hill [16] for a generalization of flatness called 'quasi-flatness', and to a paper of Damiano [9] in which a concept termed 'coflatness', dual to that of flatness has been introduced. The present authors have, recently, obtained several characterizations of regular rings in terms of these new notions. The purpose of this report is to give a summary of their results. After making a preliminary discussion in Section 2, a brief resume' of the main results is stated successively in Sections 3, 4 and 5.

2. Preliminaries.

Throughout this report we shall assume that rings are associative and have identity element. Also, every module is right and unitary. For a fixed module M over a ring R , the notions of M -injective and M -projective modules, generalizing the definitions of injective and projective modules respectively, were defined and studied by Azumaya [4] (see also Anderson and Fuller [2] for basic facts about these modules). We have used properties of M -injective and M -projective modules in the proofs of some of our theorems. Also, we need the definition of M -flat modules which generalizes the notion of flat modules. For basic facts about flat and M -flat modules, we may again refer to [2]. We also state the following important characterization of flat modules due to Lambek [19]. A right R -module M is flat if and only if M^* is injective, as a left R -module where $M^* = \text{Hom}_Z(M, Q/Z)$. Motivated by this characterization, Hill [16] introduced the concept of quasi-flat modules. A right R -module M is quasi-flat in case for every divisible abelian group D , M is M_D^* -flat, where $M_D^* = \text{Hom}_Z(M, D)$ is a left R -module. In other words, M is quasi-flat if and only if for every divisible abelian group D , and every left R -submodule $K \subseteq K_D^*$, the natural map $: K \otimes_R M \rightarrow M_D^* \otimes_R M$ is a monomorphism. A right R -module M is ' χ_0 -injective' if given $f \in \text{Hom}_R(I, M)$, I a finitely generated right ideal of R , there exists $g \in \text{Hom}_R(R, M)$ such that $g|_I = f$ (see Colby [7], and also [10]). According to Damiano [9], a module M_R is coflat if and only if it is χ_0 -injective. Clearly, all injective modules are coflat but not every coflat module is injective. A finitely generated module M_R is finitely presented in case every exact sequence:

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

with F finitely generated and free, the kernel K is also finitely generated. A module M_R is ' FP -injective' in case for every exact sequence

$$0 \rightarrow K_R \rightarrow L_R \rightarrow N_R \rightarrow 0$$

such that N_R is finitely presented, the sequence

$$0 \rightarrow \text{Hom}_R(N_R, M_R) \rightarrow \text{Hom}_R(L_R, M_R) \rightarrow \text{Hom}_R(K_R, M_R)$$

is exact (see Stenström [25]). Every injective module is indeed FP -injective but the converse may not be true. It may however be noted that every FP -injective module is right coflat (see Prop. 1.14, p. 354, [9]). On the other hand,

it is not known whether right coflat modules are right FP-injective. Nevertheless, if R is a right coherent ring then a right R -module is coflat if and only if it is FP-injective. Recall that a ring R is right coherent if every finitely generated right ideal of R is finitely presented. We also use the notions of torsionless modules, semiperfect (and perfect rings) and semihereditary (and hereditary rings). For the definitions of these notions, and their various properties we refer to Goodearl [15] and Lambek [19].

3. Regular rings characterized by coflat modules.

We have obtained the following characterizations of regular rings in terms of coflat modules.

Theorem 3.1. Let R be any ring. TFAE:

1. R is regular.
2. Each cyclic right R -module is coflat.
3. R is semihereditary and R_R is coflat.

Theorem 3.2. Let R be any ring. Then each torsionless right R -module is coflat $\Leftrightarrow R$ is regular.

Theorem 3.3. Let R be a selfinjective ring. Then R is regular \Leftrightarrow for each essential right ideal E of R , R/E is coflat.

A ring R is a right QI-ring if every quasi-injective right R -module is injective (see A.K. Boyle [5]). We generalize this concept and call a ring R 'generalized QI' if each quasi-injective right R -module is coflat.

The following theorem connects generalized QI-rings with regular rings in the commutative case.

Theorem 3.4. Let R be a commutative ring. Then R is generalized QI $\Leftrightarrow R$ is regular.

Remark: The above characterization of regular rings is not true in the non-commutative case (see Cozzens [8] for an example).

4. Regular rings characterized by quasi-flat modules.

Our main result in this section is stated in the form of the following theorem.

Theorem 4.1. Let R be a commutative ring. Then R is regular \Leftrightarrow each submodule of a quasi-flat module is quasi-flat.

Remark: We do not know whether the above result is also valid in the non-commutative case.

5. Semihiereditary rings characterized by quasi-flat and coflat modules.

Recall that a ring R is semihiereditary if R is both right and left semihiereditary. We have obtained the following characterizations of semihiereditary rings, using the notions of quasi-flat and coflat modules.

Theorem 5.1. Let R be any ring. Then R is semihiereditary \Leftrightarrow each torsionless left and right R -module is quasi-flat.

Theorem 5.2. Let R be any ring. Then R is right semihiereditary \Leftrightarrow each homomorphic image of an injective right R -module is coflat.

Theorem 5.3. Let R be any ring. Then R is right semihiereditary \Leftrightarrow each sum of injective submodules of an R -module is coflat.

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