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R-COMPLETIONS ET ACTIONS DE GROUPE

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Presented by Paulo Ribenboim, F.R.S.C.

Résumé. Si le groupe Q agit sur le groupe N, nous définissons dans certains cas une action de $\hat{Q}_R$ sur $\hat{N}_R$, où $\hat{\ }_R$ désigne la R-complétion. Nous étudions quelques propriétés de cette action et en tirons des applications, notamment aux variétés de groupes.

1. R-complétion d'une action

1.1. R est soit la P-localisation Z_P de Z, pour un ensemble de premiers P, ou Z/n pour un naturel n non nul. Si m est un entier, nous écrivons $m \in P$ si tous ses facteurs premiers sont dans P et $m \in P'$ si aucun de ses facteurs premiers n'appartient à P.

Pour un groupe G, on définit $\Gamma_i^R G$ inductivement par $\Gamma_1^R G = G$ et

$$\Gamma_{i+1}^R G = \begin{cases} \{x \in G \mid x^m \in [G, \Gamma_i^R G] \text{ pour un } m \in P'\} & \text{si } R = Z_P \\ \mathrm{sgr}\{[G, \Gamma_i^R G], \{x^n \mid x \in \Gamma_i^R G\}\} & \text{si } R = Z/n . \end{cases}$$

La R-complétion de G, $\hat{G}_R$, est alors définie comme

$$\varprojlim (G/\Gamma_i G)_P \quad \text{si } R = Z_P \quad \text{et} \quad \varprojlim (G/\Gamma_i^R G) \quad \text{si } R = Z/n .$$

Nous notons l'application canonique de G dans $\hat{G}_R$ par η_G^R (ou simplement η).

Les détails peuvent être trouvés dans [2], §11 et §12.

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1.2. Soit ω une action d'un groupe Q sur un groupe N . ω induit une action $\bar{\omega}$ de Q sur $N_{ab} \otimes R$. Nous notons $cl_R\omega$ la classe de nilpotence de $\bar{\omega}$ (donc $cl_R\omega = j$ si dans la chaîne $\Gamma_1 = N_{ab} \otimes R, \dots, \Gamma_{i+1} = \mathrm{sgr}\{\bar{\omega}(q)[x] \cdot x^{-1} \mid x \in \Gamma_i,\ q \in Q\}$, on a $\Gamma_j \neq 1$, $\Gamma_{j+1} = 1$; $cl_R\omega = \infty$ si un tel j n'existe pas).
Si X et Y sont des groupes, $A(Y,X)$ désigne l'ensemble des actions de Y sur X et $A_R(Y,X)$ l'ensemble des actions de cl_R finie.

1.3. Par [2], 14.1 et 14.2, si $\omega \in A_R(Q,N)$, la R-complétion est exacte sur $N \rightarrowtail N]_\omega Q \twoheadrightarrow Q$ où $N]_\omega Q$ est le produit croisé suivant ω . La suite $\hat{N}_R \rightarrowtail (\widehat{N]Q})_R \twoheadrightarrow \hat{Q}_R$ est donc exacte et scindée et la section permet de définir par conjugaison une action de $\hat{Q}_R$ sur $\hat{N}_R$ que nous désignerons par $\lambda\omega$. Nous avons aussi une action $\varepsilon\omega$ de Q sur $\hat{N}_R$ définie par $Q \to \mathrm{Aut}\, N \to \mathrm{Aut}\, \hat{N}_R$ et pour $\hat{\omega} \in A(\hat{Q}_R,\hat{N}_R)$ une action de Q sur $\hat{N}_R$ définie par restriction et notée $\rho\hat{\omega}$. Nous nous trouvons dans une situation analogue à cette étudiée dans [4], [5], [6] mais plus générale et nous commençons par quelques résultats dans la ligne de ces travaux.

1.4. THEOREME. Le diagramme

$$(1) \qquad \begin{array}{ccc} A_R(Q,N) & \xrightarrow{\ \varepsilon\ } & A(Q,\hat{N}_R) \\ & {\scriptstyle\lambda}\searrow \qquad \nearrow{\scriptstyle\rho} & \\ & A(\hat{Q}_R,\hat{N}_R) & \end{array}$$

est commutatif.

1.5. THEOREME. Si $N_{ab} \otimes R$ et $Q_{ab} \otimes R$ sont de type fini comme R-modules, (1) peut se réduire au diagramme

$$\begin{array}{ccc} A_R(Q,N) & \xrightarrow{\ \varepsilon\ } & A_R(Q,\hat{N}_R) \\ & {\scriptstyle\lambda}\searrow \quad \nearrow{\scriptstyle\varrho} & \\ & A_R(\hat{Q}_R,\hat{N}_R) & \end{array}$$

et ϱ est une bijection.

1.6. THEOREME. Supposons $N_{ab}\otimes R$ et $Q_{ab}\otimes R$ de type fini comme R-modules. Si $\hat{\omega}\in A_R(\hat{Q}_R,\hat{N}_R)$ et $x\in \hat{N}_R$, alors x est fixé pour $\hat{\omega}$ si et seulement si il est fixé pour $\varrho\hat{\omega}$. Si $\omega\in A_R(Q,N)$ et y est un élément de N fixé pour ω, alors ηy est fixé pour $\lambda\omega$.

1.7. COROLLAIRE. Supposons $N_{ab}\otimes R$ et $Q_{ab}\otimes R$ de type fini comme R-modules. Si $\hat{\omega}\in A_R(\hat{Q}_R,N_R)$, alors $cl_R(\varrho\hat{\omega}) = cl_R\hat{\omega}$ et si $\omega\in A_R(Q,N)$ alors $cl_R\omega = cl_R\lambda\omega = cl_R\epsilon\omega$.

1.8. $I(R)$ désigne la classe des groupes N pour lesquels η_N^R est injective.

PROPOSITION. $I(Z_p)$ est la classe des groupes résiduellement nilpotents sans P'-torsion; $I(Z/p)$ contient tout groupe de rang fini et résiduellement p-fini.

1.9. Pour une action ω de Y sur X,

$\mathrm{Fix}\ \omega = \{y\in Y \mid \omega(y)[x] = x,\ \forall x\in X\}$.

THEOREME. Si $N\in I(R)$ et $\omega\in A_R(Q,N)$ alors $\mathrm{Fix}\ \omega = \eta_N^{R\,-1}(\mathrm{Fix}\ \lambda\omega)$. Donc

$$\mathrm{Ker}\ \eta_Q^R \subset \mathrm{Fix}\ \omega\ .$$

2. Applications

Dans cette section, N est un sous-groupe normal de G et ω l'action de conjugaison.

2.1. LEMME. $cl_R\omega<\infty$ si et seulement si $N/\Gamma_2^R N \subset Z^t(G/\Gamma_2^R N)$ pour un certain t .

Si N remplit les conditions du lemme, nous dirons qu'il est R-semi-central.

2.2. Soit CN le centralisateur de N dans G .

THEOREME. Si $N\in I(R)$ et R-semicentral, alors $G/CN\in I(R)$. En particulier, si N est résiduellement nilpotent sans P'-torsion et Z_p-semicentral, alors CN est P'-isolé dans G .

Exemple. Un sous-groupe normal sans torsion d'un groupe nilpotent a un centralisateur isolé.

2.3. Grâce à 1.9., nous avons des situations où la seule action de cl_R finie est l'action triviale. Pour simplifier nos énoncés, nous nous limitons aux deux cas où:

$R = Z_p$, X résiduellement nilpotent sans P'-torsion, Y_{ab} de P'-torsion.

et $R = Z/p$, X de rang fini et résiduellement p-fini, Y_{ab} p-divisible (= p-radicable).

Nous dirons alors que (X,Y) est R-opposé.

THEOREME. Si N est R-semicentral et (N,G) R-opposé, alors N est central. De même, si N est abélien R-semicentral et $(N,G/N)$ R-opposé, alors N est central.

2.4. Remarque. Une conséquence immédiate de 2.3. est un théorème de Baer (cf. [8], V. 10.2)

"Si $N\lhd G$, $N\leqslant Z^tG$ pour un certain t , N sans torsion et G_{ab} de torsion, alors N est central".

2.5. COROLLAIRE. Si G est hypercentral sans P'-torsion, N abélien Z_p-semicentral et G/N de P'-torsion, alors G est abélien. (G est hypercentral si $G = \bigcup_{\alpha} Z^{\alpha}(G)$, α ordinal)

2.6. COROLLAIRE. Si G est nilpotent de rang fini et résiduellement p-fini, N abélien et G/N p-divisible, alors G est abélien.

3. Variétés de groupes

3.1. Si X est un groupe, VX est le sous-groupe verbal correspondent à la variété $\mathfrak{D}$; var X et varo X désignent respectivement la variété engendrée par X et la variété d'exposant 0 engendrée par X.

3.2. Voici d'abord une conséquence simple de 2.3.

COROLLAIRE. Si N est abélien R-semicentral, $(N,G/N)$ R-opposé et $H_2(G/N)=0$ alors varo G = varo N.
(Ce résultat peut être généralisé en utilisant la notion de "absolutely $\mathfrak{D}$-group" introduite par Beyl [1])

3.3. Du point de vue d'une variété $\mathfrak{D}$, la plus importante propriété d'une action de Y sur X est de faire de X un $\mathfrak{D}Y$-groupe, c'est-à-dire que $V(X]Y) \subset Y$ (voir [7], p. 108).
Notons $A_{\mathfrak{D}}(Y,X)$ (respectivement: $A_{R\mathfrak{D}}(Y,X)$) le sous-ensemble de $A(Y,X)$ (respectivement: de $A_R(Y,X)$) formé des actions qui ont cette propriété.

3.4. THEOREME. $\lambda(A_{R\mathfrak{D}}(Q,N)) \subset A_{\mathfrak{D}}(\hat{Q}_R,\hat{N}_R)$ et $\varepsilon(A_{R\mathfrak{D}}(Q,N)) \subset A_{\mathfrak{D}}(Q,\hat{N}_R)$
si a) $R = Z/n$ ou Z,
ou b) $\mathfrak{D}$ est la variété de tous les groupes polynilpotents de classe $\leq (c_1,c_2,\dots,c_t)$,

ou c) $R = Z_p$, $P \neq \emptyset$, Q et N de type fini.

La démonstration dépend de

3.5. LEMME. Dans les cas a) et b) ci-dessus, ou si $R = Z_p$, $P \neq \emptyset$ et X de type fini, $X \in \mathfrak{D}$ entraîne $\hat{X}_R \in \mathfrak{D}$.

3.6. Terminons par un résultat hors du contexte mais dont la démonstration est analogue.

THEOREME. Si G est résiduellement nilpotent sans torsion et N un sous-groupe normal d'indice fini, de type fini ou divisible, alors var G = var N .

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LES APPLICATIONS ANALYTIQUES DE L'ESPACE QUI CONSERVENT LES ANGLES SOLIDES

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Presented by H.S.M. Coxeter F.R.S.C.

RESUME

Le potentiel d'une couche double de moment constant $\mu = 1$ est une intégrale de la forme

$$I = \iint_S \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\sigma .$$

Cette intégrale égale (au signe près) l'angle solide sous lequel apparaît le morceau de surface S (vu d'un point P). Dans cet article, nous déterminons les applications analytiques de l'espace qui conservent les angles solides.

SUMMARY

The potential of a double layer of constant moment $\mu = 1$ can be represented as an integral of the form

$$I = \iint_S \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\sigma .$$

Up to a sign this integral equals the solid angle determined by the surface piece as it is seen from a point P. In this paper we determine the analytical maps of the space that preserve solid angles.

1. L'ANGLE SOLIDE

Soient γ un arc de courbe (défini sur un intervalle fermé) et P un point arbitraire du plan $\mathbb{R}^2$, $P \notin \gamma$. Lorsque l'on projette de centre P l'arc γ sur la frontière du cercle unité autour de P, les deux points extrêmes de la projection, X et Y, déterminent l'angle radial $\measuredangle XPY$. Nous accordons la mesure radiale de cet angle au couple (γ,P) et le désignons par $\omega(\gamma,P)$. La mesure radiale de (γ,P) égale donc la longueur du plus petit des deux arcs de cercle qui sont déterminés par X et Y sur la frontière du cercle unité de centre P.

De façon analogue, nous considérons dans l'espace $\mathbb{R}^3$ un morceau de surface S (défini sur un domaine fermé) et un point arbitraire P, $P \notin S$. Nous désignons par $\Omega(S,P)$ l'aire de la projection de centre P du morceau S sur la superficie de la boule unité autour de P. $\Omega(S,P)$ est dit la mesure radiale de l'angle solide déterminé par le couple (S,P) (c.f. [1], pg. 418 ff, où l'on trouve aussi des remarques quant au contexte physique).

2. APPLICATIONS CONFORMES DE L'ESPACE

Soit D et D' des domaines dans l'espace $\mathbb{R}^3$. Une application analytique $f:D\to D'$ de D sur D', dont le Jacobien ne s'annule en aucun point de D, est appelée conforme, si elle conserve la mesure radiale de l'angle entre deux droites quelconques. D'après le théorème de Liouville, les applications conformes de l'espace sont exactement les transformations de Moebius.

3. APPLICATIONS CONFORMES PAR RAPPORT A L'ANGLE SOLIDE

Une application analytique $f:D\to D'$ $(D,D'\subset \mathbb{R}^3)$, dont le Jacobien ne s'annule en aucun point, est dite conforme par rapport à l'angle solide, si $\Omega(S',P') = \Omega(S,P)\forall(S,P)\subset D$, où $S' = f(S)$, $P' = f(P)$, c'est-à-dire si la mesure radiale de l'angle solide de chaque couple $(S,P)\subset D$ reste conservée. Le terme conforme est utilisé dans ce sens par K. Arbenz.

Nous introduisons la notion de la <u>tranche</u>. Soit P∈D. Nous posons par P deux demi-plans ε_1 et ε_2 de frontière commune. $\gamma_i(r)$ désigne le demi-cercle qui est l'intersection de ε_i et de la superficie de la boule de centre P et de rayon r : $\gamma_i(r) = S(P,r) \cap \varepsilon_i$, $i=1,2$. La courbe fermée $\gamma(r) = \gamma_1(r) \cup \gamma_2(r)$ détermine deux morceaux de surface sur S(P,r). Le plus petit de ces morceaux, plus petit par rapport à l'aire, est appelé <u>tranche</u> de centre P et de rayon r et désigné par $\mathcal{T}(P,r)$ (c.f. fig. 1).

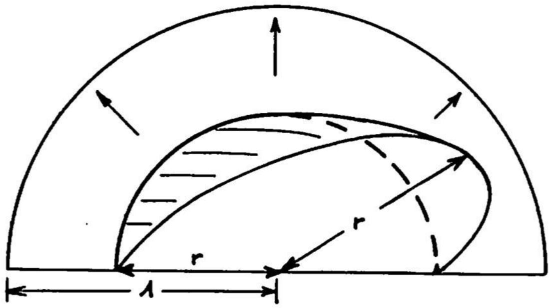

Fig. 1

On a le

<u>THEOREME</u> :

Une application analytique $f:D \to D'$ $(D,D' \subset \mathbb{R}^3)$, dont le Jacobien ne s'annule en aucun point, est conforme par rapport à l'angle solide si et seulement si elle se compose d'un nombre fini des transformations élémentaires suivantes :

i) translation : $f(x) = x+a$

ii) homothétie : $f(x) = rx$, $r>0$

iii) application orthogonale : f linéaire et $|f(x)| = |x|$.

DEMONSTRATION :

On vérifie facilement que chaque composition finie de transformations de la forme i), ii) et iii) est conforme par rapport à l'angle solide.

Inversément, nous supposons que f soit conforme par rapport à l'angle solide. On démontre que cela entraîne la conformité de f dans le sens habituel, c'est-à-dire que f conserve la mesure radiale de l'angle entre deux droites quelconques. La démonstration se fait par antithèse, en utilisant la notion de la tranche.

Chaque application conforme par rapport à l'angle solide est donc conforme dans le sens usuel, c'est-à-dire est une transformation de Moebius. Reste encore à examiner si chaque transformation de Moebius est une application conforme par rapport à l'angle solide. On construit facilement des exemples qui nous montrent qu'il faut exclure les inversions aux sphères. Chaque application conforme par rapport à l'angle solide est donc une composition finie des transformations élémentaires i), ii) et iii).

COROLLAIRE :

f est conforme par rapport à l'angle solide si et seulement si $\Omega(\mathcal{T}',P') = \Omega(\mathcal{T},P)\forall(\mathcal{T}(P,r),P)\subset D$, c'est-à-dire si la mesure radiale de l'angle solide de toute tranche reste conservée.

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SUBGROUPS OF SIMPLE ALGEBRAS AND THE ZETA FUNCTION

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Presented by J.G. Arthur F.R.S.C.

Let K be a field of characteristic zero and suppose that D is a K-<u>division ring</u>; i.e. a finite dimensional division algebra over K with center K . In [1] we proved: If K contains no non-trivial odd order roots of unity, then every finite odd order subgroup of D* (the multiplicative group of D) is cyclic. The first main result of this paper is to generalize the above result as follows: Suppose K contains no non-trivial odd order roots of unity. If G is a finite odd order multiplicative subgroup of a full matrix algebra $M_n(D)$ for a fixed n where D is a K-division ring and if $n < p$ for the minimum prime divisor p of $|G|$, then G is abelian with invariants of length less than or equal to n .

Now let K/k be a normal extension of the number fields. The next result provides a decomposition of the <u>Dedekind zeta function</u> $\zeta_K(s)$ of K in terms of a product of <u>Artin L-functions</u> over k , determined by the decomposition of unramified primes in K over k . We prove that one particular form of this decomposition is equivalent to the Galois group of K over k having cyclic Sylow subgroups. Moreover we provide the equivalence of this specific decomposition with three other results, one of which is a generalization of a result in [2].

Finally for an algebraic number field k we consider k-central simple algebras A which have an abelian maximal subfield K . Since G(K/k), the Galois group of K over k , is a direct product of cyclic groups $<\sigma_1> \times \ldots \times <\sigma_t>$ it is natural to ask whether A has a similar decomposition $A \cong A_1 \otimes \ldots \otimes A_t$ where the A_i's are k-central simple cyclic algebras. We give necessary and sufficient conditions for this to occur.

We need some definitions before we highlight the main results. If k is any field and K is a finite extension of k then K is (n,k) - adequate if and only if there is a simple algebra $M_n(D)$ central over k containing K as a maximal subfield. A finite group G is (n,k) - admissible if and only if there is a Galois extension K of k with G = G(K/k) and K is (n,k) - adequate. When n = 1 we say G is k-admissible and K is k-adequate. A finite group G is called totally n-admissible for a fixed positive integer n if and only if for every pair of number fields K and k with K Galois over k and G = G(K/k) it follows that K is (n,k) - adequate.

The first result generalizes [1, Th. 3.4, p. 242].

Theorem 1. Let E/K be finite Galois. If K is an (n,K) - adequate, E-division ring with $\sigma \in G(E/K)$ then σ extends to Aut(D).

The following is immediate from theorem 1 and generalizes [1, Corollary 3.5, p. 242]. A primitive mth root of unity is denoted by ϵ_m.

Corollary 1. Let E/K be finite Galois and let D be an (n,K) - adequate E-division ring. If $[D] \in S(E)$ with exponent m then ϵ_m is in K.

Theorem 2. Suppose G is (n,k)-adequate with $n < p$ for the minimum odd prime divisor p of $|G|$. Then all odd Sylow subgroups G_p of G are abelian with invariants of length $\leq n$. If G_2 is K-adequate then either:

(1) G_2 is cyclic with $V(G_2) \cong Q(\epsilon_{2^a})$ where $|G_2| = 2^a$ and $V(G) = \{\Sigma\, \alpha_i g_i : \alpha_i \in Q,\ g_i \in G\}$ or

(2) G_2 is generalized quaternion and $V(G_2) \cong (Q(\epsilon_4)/Q,-1) \otimes Q(\epsilon_{2^a} + \epsilon_{2^a}^{-1})$ where $(Q(\epsilon_4)/Q,-1)$ denotes the ordinary quaternion division algebra over Q.

The above results are used in proving the first main result which generalizes [1, Th. 3.6, p. 243].

Theorem 3. Assume K contains no non-trivial odd order roots of unity. If G is any odd order (n,K) - adequate group with $n < p$ for the minimum prime divisor of $|G|$ then G is abelian with invariants of length $\leq n$.

Now we need to set the stage for a crucial lemma required for the next main result.

Let K be a finite normal extension of an algebraic number field k, and set $G = G(K/k)$. Suppose $G = \prod_{i=1}^{t} p_i^{a_i} = n$ where the p_i's are distinct primes. Let σ_i be an element of G of maximal p_i - power order; i.e. $|\sigma_i| = p_i^{b_i}$ where $b_i \leq a_i$ and $|\tau|_{p_i} \leq p^{b_i}$ for $1 \leq i \leq t$, $\tau \in G$, where $|\tau|_{p_i}$ denotes the p_i - part of τ. Let $H_i = < \sigma_i >$ for $1 \leq i \leq t$ and let X_i be a simple character of H_i; i.e. the character of an irreducible representation. Corresponding to X_i there is an induced character X_i^* of G defined by:

$$X_i^*(\mu) = \sum_j X_i\,(\alpha_j \mu \alpha_j^{-1})$$

with $\mu \in G$ where the sum ranges over all j such that $\alpha_j \mu \alpha_j^{-1} \in H_i$. In what follows $\psi_i = X_i^*$.

<u>Lemma 1</u>. Let $\underline{\ell} = \ell.c.m.\{f(\underline{p})\}$ which is the least common multiple ranging over the inertial degrees of all unramified k-primes $\underline{p}$ in K/k, and let $n = |K : k|$. Then:

$$\zeta_K(s) = \psi_0(s) \prod_{i=1}^{\ell} L(s,\psi_i)^{n/\ell}$$

where $\psi_0(s)$ is a multiplicative function embodying only the ramified primes and $L(s,\psi_i)$ is the Artin L-function.

<u>Theorem 4</u>. The following are equivalent:

(1) For each pair of number fields K and k with K normal over k and $G = G(K/k)$, $\zeta_K(s) = \psi_0(s) \prod_{i=1}^{n} L(s,\psi_i)$ where $\psi_0(s)$ is a multiplicative function embodying only the ramified primes.

(2) G is totally m-admissible for all $m \mid |G|$.

(3) All Sylow p-subgroups G_p of G are cyclic.

(4) For each pair of number fields K and k with K being normal over k and $G = G(K/k)$ we have $\underline{\ell} = |K : k| = |G| = n$ where $\underline{\ell} = \ell.c.m.\{f(\underline{p})\}$ is the least common multiple ranging over the inertial degrees of all k-primes $\underline{p}$ in K/k.

(5) Let m,r be two relatively prime integers. Put $s = (r-1,m)$, $t = m/s$ and n_0 = minimal integer satisfying $r^{n_0} \equiv 1 \pmod{m}$.

Then $G = \langle \alpha, b : a^m = 1, b^{n_0} = a^t, b^{-1}ab = a^r \rangle$ where $|G| = mn_0$ and $(n_0, t) = 1$.

The final main result is as follows.

Theorem 5. Let G be a finite abelian group. Then G is (n,K) - admissible for some number field K and some n dividing $|G|$. Suppose that $A = (L/K, B)$ is a K-central simple crossed product algebra with $G = G(L/K)$, and having β as factor set. Then $A \cong A_1 \otimes \ldots \otimes A_t$ where each A_i is cyclic K-central simple with maximal subfield L_i if and only if $B(\sigma, \tau) = B(\tau, \sigma)$ for all $\sigma, \tau \in G$.

Proofs of the above results as well as examples and applications shall appear elsewhere.

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RATIONAL GROUP DECISION MAKING GENERALIZED: THE CASE OF SEVERAL UNKNOWN FUNCTIONS

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Abstract: A fixed amount of money or other identifiable resource σ is to be distributed among a fixed number ($m>2$) of competing projects (cf. [5,6]). Each member of the group of n decision makers makes a recommendation on the distribution of σ to the m projects. The amalgamated (or consensus) allocation to each project will be a function ϕ_i of the allocations recommended for that project. The ϕ_i may, a priori, be all different.

As indicated, both the recommended and the amalgamated allocations are nonnegative and exhaust σ (the latter can in some cases be replaced by 'add up to not more than σ' if we 'make up' one more project whose allocations are the hitherto unallocated balances). Else we will suppose only that 'consensus of rejections should be followed', that is if, for a project, all recommended allocations are 0, then the consensus allocation will be 0 too.

Generalizing results of [2,3,6], we prove that, under these conditions, each ϕ_i is the same weighted arithmetic mean.

We unite the allocations recommended for the ith project by the n decision makers into an n-dimensional vector x_i ($i=1,2,...,m$). Latin letters (other than subscripts and integers describing dimensions, etc.) will denote such vectors, while Greek letters denote scalars (real numbers). However, we will write $\underline{0} = (0,0,\ldots,0)$ and $\underline{\sigma} = (\sigma,\sigma,\ldots,\sigma)$ (n equal components) for convenience. Our suppositions are $\phi_i : [0,\sigma]^n \to [0,\sigma]$, in particular

(1) $\phi_i(x) \geq 0$ for all $x \in [\rho,\tau]^n$, $(0 \leq \rho < \tau \leq \sigma)$,

(2) $\phi_i(\underline{0}) = 0$ $(i=1,2,\ldots,m)$,

(3) if $\sum_{i=1}^{m} x_i = \underline{\sigma}$, then $\sum_{i=1}^{m} \phi_i(x_i) = \sigma$.

__Theorem.__ __The conditions (1), (2), (3) are satisfied iff there exist weights__ ω_j ($\omega_j \in \mathbb{R}_+$, i.e. $\omega_j \geq 0$, $j=1,2,\ldots,n$; $\sum_{j=1}^{n} \omega_j = 1$) __such that__

$$\phi_i(x) = \phi_i(\xi_1,\xi_2,\ldots,\xi_n) = \sum_{j=1}^{n} \omega_j \xi_j, \text{ \underline{in particular} } \phi_1=\phi_2=\ldots=\phi_m.$$

__Proof.__ The 'if' part is obvious. - As to the 'only if' part, we write (3) as

(4) $\sum_{i=2}^{m} \phi_i(x_i)+\phi_1(\underline{\sigma}-\sum_{i=2}^{m} x_i) = \sigma$ $\quad (x_2,\ldots,x_m, \sum_{i=2}^{m} x_i \in [0,\sigma]^n)$.

Putting here $x_2=x_3=\ldots=x_m=\underline{0}$, we get, in view of (2), $\phi_1(\underline{\sigma}) = \sigma$ and, similarly,

(5) $\phi_i(\underline{\sigma}) = \sigma$ $(i=1,2,\ldots,m)$.

Also, substituting $x_3=\ldots=x_m=\underline{0}$, $x_2=x$ into (4), we obtain

(6) $\phi_1(\underline{\sigma}-x) = \sigma - \phi_2(x)$

and, putting $x_4=x_5=\ldots=x_m=0$, $x_2=x$, $x_3=y$ into (4), we get, in view of (2) and (6),

(7) $\phi_2(x+y) = \phi_2(x) + \phi_3(y)$ whenever $x,y,x+y \in [0,\sigma]^n$.

Finally, the substitution $x=0$ in (7) gives $\phi_2(y) = \phi_3(y)$ for all $y \in [0,\sigma]^n$ and, similarly,

(8) $\phi_1 = \phi_2 = \ldots = \phi_m =: \phi$,

so that (7) goes over into

(9) $\phi(x+y) = \phi(x)+\phi(y)$ whenever $x,y,x+y\in[0,\sigma]^n$.

This, of course, is Cauchy's equation for vectors [1], but on a restricted domain. However, it is easy to extend (7) to $x,y\in\mathbb{R}_+^n$ (cf. [4,2]). Indeed, (9) implies $\phi(kx) = k\phi(x)$ for $kx\in[0,\sigma]^n$, that is,

(10) $\phi(\frac{y}{k}) = \frac{1}{k}\phi(y)$ for all $y\in[0,\sigma]^n$, $k\in\mathbb{N}$.

Let $t\in\mathbb{R}_+^n$ be arbitrary. There exists a $k\in\mathbb{N}$ such that $x = \frac{1}{k} t \in [0,\sigma]^n$. Define

(11) $\psi(t) := k\phi(x) = k\phi(\frac{t}{k})$ $(t\in\mathbb{R}_+^n)$.

Note that, in particular,

(12) $\psi(t) = \phi(t)$ for all $t\in[0,\sigma]^n$.

If $t = kx = \ell y$ $(x,y\in[0,\sigma]^n)$ then, by (10),

$$\ell\phi(y) = k\ell\phi(\frac{y}{k}) = k\ell\phi(\frac{x}{\ell}) = k\phi(x) ,$$

so that the definition (11) is unambiguous. Also, by (11) and (9), if $\frac{x}{k},\frac{y}{k},\frac{x+y}{k} \in [0,\sigma]^n$,

(13) $\psi(x+y)=k\phi(\frac{x+y}{k})=k\phi(\frac{x}{k})+k\phi(\frac{y}{k})=\psi(x)+\psi(y)$ for all $x,y\in\mathbb{R}_+^n$.

In view of (1), (8) and (11), ψ is nonnegative on $[\rho,\tau]^n$, so there exists a $w\in\mathbb{R}_+^n$ such that

$$\psi(x) = (w,x) \quad (x\in\mathbb{R}_+^n) \text{ and so } \phi(x) = (w,x) = \sum_{j=1}^{n}\omega_j\xi_j$$

$$(\xi_j\in[0,\sigma], \quad \omega_j\geq 0, \ j=1,2,\ldots,n),$$

(cf. [1], where (13) holds on $\mathbb{R}^n$, but the proof is the same). Because of (5) and (8) $\sum_{j=1}^{n} \omega_j = 1$, $\phi_i(x) = \sum_{j=1}^{n} \omega_j \xi_j$ $(i=1,2,\ldots,m)$. □

The case $m=1$ is trivial and $m=2$ is easy (cf. [2]): the solution contains a (slightly restricted) arbitrary function of n variables [of (n-1) variables, if σ is variable too in (3) or (4)].

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A MODEL FOR EQUIVARIANT K-THEORY

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Abstract: When G is a compact, abelian Lie group the equivariant K-functor, $K_G(X)$, is constructed in terms of equivariant stable homotopy of projective spaces of G-representations.

§1: In this note representations, projective spaces and K-theories will be unitary. In [1;2;9] the equivariant K-functor, $K_G(X)$, is introduced and developed. Since its introduction $K_G(X)$ has been important in many developments (notably [3;5]; see also [4] for example). From the "splitting principle" [1;2] in equivariant K-theory comes the intuition that line bundles, and hence projective spaces, are fundamental in K_G-theory. The object of this note is to make this intuition more explicit by describing a model for $K_G(X)$, when G is abelian, defined in terms of the equivariant stable homotopy of projective spaces. The construction is a generalisation of a model for K(X) (the case when G = {1}) which I discovered in [10, II §9]. See also [11; §2.12]. The case in which G is an arbitrary abelian, compact Lie group will be dealt with by reduction to the case G = {1}.

This paper was written while the author was enjoying the hospitality of The University of Chicago. I am indebted to Arunas Liulevicius, Peter May and Paul Sally for valuable discussions. Peter May's lectures on G-stable homotopy suggested to me the main result (Theorem 3.1) and it was he who pointed out to me that the result is false for non-abelian groups (see §3.4 (ii)).

§2: The Model: Let G be a compact Lie group. For simplicity, assume that all spaces X,Y,... lie in the category CW_G of finite G-CW complexes [6;8]. CW_G^0 will denote the corresponding category of spaces with (G-fixed) base-point. Let S^1, the unit circle, have trivial G-action and base-point, 1.

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For $X,Y \in CW_G^0$ let

$$(X,Y)_G = \varinjlim_n [S^n \wedge X, S^n \wedge Y]_G$$

where the direct limit is taken over the suspension maps ($f \longrightarrow 1_{S^1} \wedge f$) using $S^n \cong S^1 \wedge S^1 \wedge \ldots \wedge S^1$ (n factors).

A less naive definition of stable homotopy classes of G-maps might be given by means of the direct limit over suspensions, $S(V) \wedge (\_)$, where $S(V)$ is the one-point compactification of a non-trivial G-representation. This sort of stable G-category has been developed by G. Lewis and J. P. May.

<u>2.1: Definition</u>: Now let $V_1 \subset V_2 \subset \ldots$ be a sequence of finite dimensional G-representations having the property that for any finite dimensional representation, W, of any closed subgroup, $H \subset G$, there exists an integer, $N(W)$, such that $V_{N(W)}$ contains (as an H-representation) a copy of W. Such a sequence of representations $\{V_i;\ i \geq 1\}$ will be called <u>cofinal</u>. For example, if G is finite, the choice $V_m = m(\mathbb{C}[G])$ gives a cofinal sequence. Since any H-representation, W, can be realised as an H-subspace of a G-representation [7] we have the following result.

<u>2.2: Lemma</u>: A compact Lie group has a cofinal sequence of representations in the sense of §2.1.

<u>2.3</u>: Let X_+ denote the disjoint union of X with a base-point. Set $V^\infty = \mathbb{C}^\infty \otimes_{\mathbb{C}} V$ for a representation V and let $P(V^\infty)$ denote the associated projective space. Let $\underline{n}$ denote the trivial, n-dimensional representation. Tensor product of line bundles induces a map

$$P(\underline{2}) \times P(V^\infty) \longrightarrow P(\underline{2} \otimes_{\mathbb{C}} V^\infty) \cong P(V^\infty). \tag{2.4}$$

Since $X_+ \wedge Y_+ = (X \times Y)_+$ (2.4) yields a map

$$m_V : P(\underline{2})_+ \wedge P(V^\infty)_+ \longrightarrow P(V^\infty)_+. \tag{2.5}$$

Now in the stable homotopy category $P(\underline{2})_+ \simeq P(\underline{2}) \vee S^0 = \mathbb{C}P^1 \vee S^0$ so that the generator, $\beta : S^2 \longrightarrow \mathbb{C}P^1$, of $\pi_2(\mathbb{C}P^1)$ induces an element $\beta_V \in \{S^2 \wedge P(V^\infty)_+, P(V^\infty)_+\}_G$ represented by the S-map composition of

$$S^2 \wedge P(V^\infty)_+ \xrightarrow{\beta \wedge 1} (\mathbb{C}P^1 \vee S^0) \wedge P(V^\infty)_+ \simeq P(\underline{2})_+ \wedge P(V^\infty)_+$$

with m_V of (2.5).

Finally we may define a functor on (non-based) G-spaces

(2.6) $$h_G(\_) = \varinjlim_i \{S^{2i} \wedge (\_)_+, P(V_i^\infty)_+\}_G$$

a direct limit taken over composition with β_{V_i} $(i = 1, 2, \ldots)$ and the inclusions $P(V_i^\infty) \subset P(V_{i+1}^\infty)$ associated to some cofinal chain of G-representations, $V_1 \subset V_2 \subset \ldots$. Clearly $h_G(\_)$ is independent of the choice of the cofinal chain of G-representations.

2.7: In $\tilde{K}_G^0(P(V^\infty)_+) \cong K_G^0(P(V^\infty))$ the Hopf bundle, H_V, is a canonical element [1;2]. Let $B \in \tilde{K}^0(S^2)$ denote the Bott periodicity element. Suppose that $f : S^{2k} \wedge S^{2i} \wedge (X_+) \longrightarrow S^{2k} \wedge P(V_i^\infty)_+$ is a G-map representing an element of $h_G(X)$. Form $f^*(B^k \otimes H_{V_i}) \in \tilde{K}_G^0(S^{2k} \wedge S^{2i} \wedge (X_+))$ and define $\lambda_f \in \tilde{K}_G^0(X_+) \cong K_G^0(X)$ by $B^{i+k} \otimes \lambda_f = f^*(B^k \otimes H_{V_i})$. If $H \longrightarrow \mathbb{C}P^1 = P(\underline{2})$ is the Hopf bundle then the map of (2.4) induces $H \otimes H_V$ from H_V so that $m_V^*(H_V) = H \otimes H_V$ in (2.5). Let g be the S-map given by the following composition

$$S^2 \wedge S^{2k} \wedge S^{2i} \wedge (X_+) \xrightarrow{m_{V_i}(\beta \wedge f)} S^{2k} \wedge (P(V_i^\infty)_+) \longrightarrow S^{2k} \wedge (P(V_{i+1}^\infty)_+),$$ so

that g and f represent the same class in $h_G(X)$. Also

$$\begin{aligned} B^{i+k+1} \otimes \lambda_g &= g^*(B^k \otimes H_{V_{i+1}}) \\ &= (\beta \wedge f)^*(H \otimes B^k \otimes H_{V_i}) \\ &= B \otimes Bf^*(B^k \otimes H_{V_i}) \\ &= B^{i+k+1} \otimes \lambda_f \end{aligned}$$

so that $\lambda_g = \lambda_f$. We obtain therefore a well-defined homomorphism (of rings, in fact [10, II §9])

$$\lambda_G(X) : h_G(X) \longrightarrow K_G^0(X). \tag{2.8}$$

§3: In this section we prove the following result.

3.1: Theorem: If $X \in CW_G$ then $\lambda_G(X)$ is an isomorphism, when G is an abelian, compact Lie group.

Proof: Firstly recall that for an abelian, compact Lie group all irreducible representations are one-dimensional. h_G and K_G extend to equivariant cohomology theories and λ_G extends to a stable cohomology operation. Therefore it suffices by [6;8] to verify for all closed subgroups, H, of G that $\lambda_G(G/H)$ is an isomorphism. Suppose that V is a finite dimensional G-representation. Considered as an H-representation write $V = \bigoplus_\alpha n_\alpha V_\alpha$ $(0 \le n_\alpha < \infty)$ where $\{V_\alpha\}$ runs through representatives of all the distinct isomorphism classes of irreducible H-representations. Then, by restriction to the identity coset, we obtain an isomorphism

$$\{S^{2i} \wedge (G/H)_+, P(V^\infty)_+\}_G \cong \{S^{2i}, P(V^\infty)_+\}_H.$$

Since S^{2i} has trivial action and so do the suspension coordinates the above group is isomorphic to $\{S^{2i}, (P(V^\infty)^H)_+\}$ where $(\ )^H$ denotes the H-fixed point set. If $z \in V^\infty$ represents a point of $P(V^\infty)^H$ then z belongs to the summand $n_\alpha V_\alpha$ for some α. For suppose $z = \sum_\alpha z_\alpha (z_\alpha \in n_\alpha V_\alpha)$ then $h(z) = \mu_h z$ and $\mu_{h_1 h_2} = \mu_{h_1}\mu_{h_2}$ for $h, h_1, h_2 \in H$. The function, μ, gives a one-dimensional H-representation, L. For each α with $z_\alpha \neq 0$ we have an embedding of L into $n_\alpha V_\alpha$ by sending $\lambda \in L(\cong \mathbb{C})$ to λz_α but, by Schur's lemma this can happen for at most one suffix, α, so $z = z_\alpha$ as required. Hence $(P(V^\infty)_+)^H = \vee_\alpha (P((n_\alpha V_\alpha)^\infty)_+)$ and we have an isomorphism

$$\{S^{2i} \wedge (G/H)_+, P(V^\infty)_+\}_G \cong \bigoplus_\alpha \pi_{2i}^S(P(V_\alpha^\infty)_+). \tag{3.2}$$

All these isomorphisms are compatible with the direct limit of (2.6). Hence additively

$$h_G^s(G/H) \cong \bigoplus_{V_\alpha} h_{\{1\}}^s(\text{point}). \tag{3.3}$$

By [10, II §9; 11] $h^*_{\{1\}}(X) \cong KU^*(X)$, the isomorphism being $\lambda_{\{1\}}$. Finally $K^*_G(G/H) \cong K^*_H(\text{point})$ is R(H) or zero as * is even or odd respectively. Here R(H) is the representation ring of H, which is the free abelian group on the V_α of (3.3). Hence the range and domain of $\lambda_G(G/H)$ are isomorphic. The fact that $\lambda_G(G/H)$ is an isomorphism results from the fact that the Hopf bundle on $P(V^\infty)$ restricts to the Hopf bundle on each subspace $P((n_\alpha V_\alpha)^\infty)$ so that $\lambda_G(G/H)$ decomposes, compatibly with (3.3), into the direct sum of copies of the isomorphism $\lambda_{\{1\}}$ - one for each irreducible, V_α. This completes the proof of Theorem 3.1.

3.4: Remark

(i) When $G = \{1\}$ we have the following related result. If $Q(\_) = \varinjlim_n \Omega^n \Sigma^n(\_)$ then $\{X, \mathbb{C}P_+^\infty\} = [X, Q(\mathbb{C}P_+^\infty)]$ and the map of (2.8) is induced by a map

$$(3.5) \qquad \lambda : Q(\mathbb{C}P_+^\infty) \longrightarrow Z \times BU$$

where $KU^0(X) = [X, Z \times BU]$. J. C. Becker and G. B. Segal independently showed that λ is a split surjection of spaces [10, I §3]. Since (3.4) factors through (2.8) Theorem 3.1 is an equivariant analogue of a weakening of the Becker-Segal result.

(ii) If H is not abelian $\lambda_G(G/H)$ will not be an isomorphism since, in the identification of $h_G^*(G/H)$ by means of (3.2) only the one-dimensional V_α contribute.

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THE CHARACTER GENERATOR OF Sp(2k)

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Abstract A simple combinatorial method of writing down the character generator of Sp(2k) is described.

1. Introduction

Recently Stanley [4], motivated by the work of Patera and Sharp [3], has derived a fundamental theorem which enables the character generator of SU(n) to be written down. The stumbling block in attempting to extend this work to other simple Lie groups was stated to be the lack of a suitable combinatorial description of the group characters analogous to that provided by the Young tableaux appropriate to SU(n). However, at least in the case of Sp(2k), such a combinatorial description is available [1].

2. Irreducible characters of Sp(2k)

Each inequivalent irreducible representation $\langle\lambda\rangle$ of Sp(2k) is labelled by means of a partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_k)$ into not more than k non-vanishing parts. The character in the irreducible representation $\langle\lambda\rangle$ of an element of Sp(2k) in the class labelled by $\underline{\phi}$ takes the form:

$$\chi^{\langle\lambda\rangle}(\underline{\phi}) = \sum_{\underline{w}} M_{\underline{w}}^{\langle\lambda\rangle} \exp(i\, \underline{w}.\underline{\phi}) \;. \tag{1}$$

The weight multiplicities $M_{\underline{w}}^{\langle\lambda\rangle}$ may be evaluated in several ways and have been tabulated [2] for all partitions λ of n with $n \leq 6$ as functions of $\underline{k}$.

The Young diagram, F^{λ}, corresponding to λ consists of n boxes arranged in left-adjusted rows of lengths $\lambda_1,\lambda_2,\ldots,\lambda_k$. Numberings, or arrays, $N^{\langle\lambda\rangle}$ are obtained by inserting positive and negative integers, chosen from the set $\{\pm1,\pm2,\ldots,\pm k\}$, into the boxes of F^{λ} in such a way that: (i) the entries are non-decreasing across each row from left to right, (ii) the entries are strictly increasing down each column from top to bottom and (iii) no entry $+i$ or $-i$ (denoted, respectively, by i and $\bar{i}$ for convenience) may appear in any row lower than the i th. The ordering used in applying these rules is defined by $\bar{1}<1<\bar{2}<2\ldots<\bar{k}<k$. A typical array is given by:

$$\begin{matrix} \bar{1} & \bar{1} & 1 & \bar{2} & \bar{2} & \bar{3} & 3 & 3 & 3 \\ \bar{2} & 2 & 2 & 2 & \bar{3} \\ 3 & 3 & 3 \end{matrix}$$

The combinatorial description of the character (1) required here is simply that the multiplicity $M_{\underline{w}}^{<\lambda>}$ of the weight $\underline{w} = (w_1, w_2, \ldots, w_k)$ is the number of distinct arrays $N^{<\lambda>}$ such that:

$$w_i = n_i - n_{\bar{i}} \qquad \text{for } i = 1,2,\ldots,k \tag{2}$$

where n_i and $n_{\bar{i}}$ are the number of i's and $\bar{i}$'s, respectively, in the array [1].

3. The character generator of Sp(2k)

To make contact with the notation of Stanley [4] it is then only necessary to replace i and $\bar{i}$ by $2k+1-2i$ and $2k+2-2i$, respectively, for $i=1,2,\ldots,k$. In the case of the previous array this gives

$$\begin{matrix} 6 & 6 & 5 & 4 & 4 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 3 & 2 \\ 1 & 1 & 1 \end{matrix}$$

More generally the resulting array is such that: (i) the entries are non-increasing across each row, (ii) the entries are strictly decreasing down each column and (iii) no entry in row i exceeds $m_i = 2k + 2 - 2i$ for $i = 1,2,\ldots,k$. Thus each new array P^λ is a column-strict plane partition of type $(\underline{m},\underline{r})$ with $m = (2k,2k-2,\ldots,2)$, with $r_i = \lambda_i - \lambda_{i+1}$ for $i = 1,2,\ldots,k$ and $\lambda_{k+1} = 0$.

Each partition $\underline{m}$ of m into unequal parts serves to define a shifted Young diagram $Z^{\underline{m}}$ in which the rows of length m_i are left-adjusted to a diagonal line, in that $Z^{\underline{m}}$ is obtained from $F^{\underline{m}}$ by moving the i th row $(i-1)$ places to the right for $i=1,2,\ldots,k$. To each $Z^{\underline{m}}$ there corresponds a set of standard shifted Young tableaux (SSYT) [4,5]. In the case $\underline{m}=(4,2)$, for example, this set of SSYT is:

$$\begin{matrix}1234\\56\end{matrix}\qquad\begin{matrix}1235\\46\end{matrix}\qquad\begin{matrix}1245\\36\end{matrix}\qquad\begin{matrix}1236\\45\end{matrix}\qquad\begin{matrix}1246\\35\end{matrix}\tag{3}$$

The generating formula for the required plane partitions is then that of Stanley's fundamental theorm [4]:

$$F_{\underline{m}}(\underline{A},\underline{X}) = \sum_{\pi}\left\{\prod_{j\in K_\pi}\Gamma(\pi^{(j)})\Big/\prod_{i=1}^{m}\left[1-\Gamma(\pi^{(i)})\right]\right\}\tag{4}$$

where π ranges over all SSYT of shape $\underline{m}$, K_π is the set of those j for which $j+1$ appears in π in a row above j, $\pi^{(i)}$ is obtained from π by deleting all entries greater than i, and

$$\Gamma(\pi^{(i)}) = A_{k(i)}\,X_{m^{(i)}_1}\,X_{m^{(i)}_2}\cdots X_{m^{(i)}_{k(i)}}$$

where $\underline{m}^{(i)}$ is the shape of the SSYT $\pi^{(i)}$.

It follows from (1), (2) and (4) that the Sp(2k) character generator is given by:

$$F_{\underline{m}}(\underline{A},\underline{X}) = \sum_{\lambda} c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_k^{\lambda_k}\,\chi^{<\lambda>}(\underline{\phi})\tag{5}$$

with $\underline{m}=(2k,2k-2,\ldots,2)$, $\underline{A}=(c_1,c_1c_2,\ldots,c_1c_2\ldots c_k)$ and $\underline{X}=(e^{i\phi_k},e^{-i\phi_k},\ldots,e^{i\phi_2},e^{-i\phi_2},e^{i\phi_1},e^{-i\phi_1})$. In general the denominator is a product of $m=k(k+1)$ factors, and the number of terms in the sum is [5]

$$g^{(2k,2k-2,\dots,2)} = \frac{[k(k+1)]!\ (k-1)!\ (k-2)!\ \dots\ 2!\ 1!}{(2k)!\ (2k-1)!\ \dots\ (k+1)!} \ . \qquad (6)$$

4. Example and conclusions

In the case of the group Sp(4) $\underline{m}$ is (4,2), and (3) yields, from (4), the character generator:

$$\begin{aligned} F_{(4,2)}(\underline{A},\underline{X}) = & 1/(1-A_2X_4X_2)(1-A_2X_4X_1)(1-A_1X_4)(1-A_1X_3)(1-A_1X_2)(1-A_1X_1) \\ & + A_2X_3X_1/(1-A_2X_4X_2)(1-A_2X_4X_1)(1-A_2X_3X_1)(1-A_1X_3)(1-A_1X_2)(1-A_1X_1) \\ & + A_2X_2X_1/(1-A_2X_4X_2)(1-A_2X_4X_1)(1-A_2X_3X_1)(1-A_2X_2X_1)(1-A_1X_2)(1-A_1X_1) \qquad (7) \\ & + A_2X_3X_2/(1-A_2X_4X_2)(1-A_2X_3X_2)(1-A_2X_3X_1)(1-A_1X_3)(1-A_1X_2)(1-A_1X_1) \\ & + A_2X_2X_1A_2X_3X_2/(1-A_2X_4X_2)(1-A_2X_3X_2)(1-A_2X_3X_1)(1-A_2X_2X_1)(1-A_1X_2)(1-A_1X_1) \ . \end{aligned}$$

It is to be expected that character generators of SO(2k+1) and SO(2k) may be established in the same way from appropriate Young tableaux [1] . However it is to be noted that the local isomorphism of Sp(4) and SO(5) is such that $F_{(4,2)}(\underline{A},\underline{X})$ is not only the Sp(4) character generator (5) under the substitutions $\underline{A} = (c_1, c_1c_2)$ and $\underline{X} = (e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1})$, but also the SO(5) character generator

$$F_{(4,2)}(\underline{A},\underline{X}) = \sum_{\mu_1,\mu_2} b_1^{\mu_1} b_2^{\mu_2} \chi^{[\mu_1\mu_2]}(\underline{\phi}) \qquad (8)$$

with $\underline{X} = (e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1})$, as before, but $\underline{A} = (\sqrt{b_1b_2}, b_1)$ where now $\mu_1 \geq \mu_2 \geq 0$ with μ_1 and μ_2 either both integral or both half-odd-integral.

Setting $\underline{\phi} = \underline{0}$, so that $\underline{X} = (1,1,\dots,1)$ in (5), furnishes the generating function for the dimensions of the irreducible representations of Sp(2k). This may be done in the example (7) to give the dimensions of the irreducible representations of Sp(4) and SO(5). The result is in accordance with the generating function given previously by Patera and Sharp [3] for the dimensions of irreducible representations of O(5).

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TENSOR CONCOMITANTS OF ORDER ZERO

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§1. Introduction. The classical definition of a concomitant of a given tensor field, Φ, is a tensor field, Ψ, which is built up out of the given field by differentiation and the usual tensor operations. Thus, for instance, curvature tensors are concomitants of metric tensors. The requirement that Ψ be a tensor places severe restrictions on the components of Ψ as functions of the components of Φ. This fact was used by É. Cartan in 1922 to characterize Einstein's field equations of general relativity ([1]).

The purpose of this paper is to consider concomitants of arbitrary type but of order zero, in the sense that no derivatives occur.

§2. Algebraic Preliminaries. Let E be an n-dimensional real vector space and let $E^p_q = \otimes^p E \otimes \otimes^q E^*$, $p,q = 0,1,\ldots$, where E^* denotes the dual Of E. Each space E^p_q provides a representation of GL_E via the definition, $\phi \mapsto \lambda_\phi$, where, for example, λ_ϕ acts on E^1_1, by

$$\lambda_\phi(x \otimes x^*) = \phi x \otimes \tilde{\phi} x^* \qquad x \in E, \quad x^* \in E^*, \quad \phi \in GL_E, \quad \tilde{\phi} = (\phi^*)^{-1}.$$

If $\Phi \in E^p_q$, $\Psi \in E^r_s$ and $\Phi \otimes \Psi \in E^{p+r}_{q+s}$ is defined in the usual way, then it is easily checked that

$$\lambda_\phi(\Phi \otimes \Psi) = \lambda_\phi \Phi \otimes \lambda_\phi \Psi , \qquad \phi \in GL_E . \tag{1}$$

Next, suppose that $\Phi \in E^p_q$ and $\Psi \in E^{q+r}_{p+s}$, with $r , s \geq 0$. Then we define a contraction operator, i, by the conditions

(i): $i(\Phi)\Psi \in E^r_s$ is bilinear in Φ and Ψ and (ii):

$$\begin{aligned} (2) \quad & i(x_1 \otimes \cdots \otimes x_p \otimes x^{*1} \otimes \cdots x^{*q})(y_1 \otimes \cdots \otimes y_{q+r} \otimes y^{*1} \otimes \cdots \otimes y^{*p+s}) \\ & = \langle x^{*1} , y_1 \rangle \cdots \langle x^{*q} , y_q \rangle \langle y^{*1} , x_1 \rangle \\ & \quad \cdots \langle y^{*p} , x_p \rangle (y_{q+1} \otimes \cdots \otimes y_{q+r} \otimes y^{*p+1} \otimes \cdots \otimes y^{*p+s}) , \end{aligned}$$

for decomposible Φ and Ψ.

The interaction between i and λ_ϕ is given by

$$i(\lambda_\phi \Phi)(\lambda_\phi \Psi) = \lambda_\phi [i(\Phi)\Psi] , \qquad \phi \in GL_E , \tag{3}$$

as may be checked by assuming that Φ , Ψ are decomposible and noting that $\langle \tilde{\phi} x^* , \phi x \rangle = \langle x^* , x \rangle$.

§3. <u>O-th Order Concomitants</u>. Henceforth fix E^h_k and consider smooth maps $f : E^h_k \to E^p_q$. We call such a map a <u>O-th order concomitant of</u> E^h_k <u>of type</u> (p,q), if

$$\lambda_\phi \circ f = f \circ \lambda_\phi , \qquad \phi \in GL_E . \tag{4}$$

(If this condition holds only for $\phi \in GL^+_E = \{\phi \in GL_E \mid \det \phi > 0\}$ we will call f an <u>oriented</u> concomitant.) The real vector space of such maps will be denoted by T^p_q.

For example, if $(h,k) = (p,q) = (1,0)$, and we write $f(x) = f^i(x)e_i$, $\phi e_i = \phi^j_i e_j$, relative to a basis $\{e_i\}$ of E, then (4) reads

$$f^i(\phi x) = \phi^i_j f^j(x) .$$

Interpreting x as the components of a vector-field at 0, relative to one coordinate system, ϕx as its components, relative to another and ϕ as the Jacobian of the coordinate change at 0, we obtain the classical definition of a concomitant of this type ([2]).

§4. Operations with 0-th order Concomitants. As noted above, concomitants of the same type may be added and multiplied by scalars. Also, if $f \in T^p_q$, $g \in T^r_s$, then the map $f \otimes g : E^h_k \to E^{p+r}_{q+s}$, defined by

$$(5) \qquad (f \otimes g)(\Phi) = f(\Phi) \otimes g(\Phi) , \qquad \Phi \in E^h_k ,$$

yields an element of T^{p+r}_{q+s}. This follows from equation (1).

Similarly, from equation (3), we conclude that the map $i(f)g : E^h_k \to E^r_s$, given by

$$(6) \quad [i(f)g](\Phi) = i\big(f(\Phi)\big)g(\Phi) , \quad \Phi \in E^h_k , \ f \in T^p_q , \ g \in T^{q+r}_{p+s} ,$$

yields an element of T^r_s.

Thus the space of all 0-th order concomitants of E^h_k is a graded algebra, $\bigoplus_{p,q\geq 0} T^p_q$, which is closed under contraction.

§5. Invariance Identities. To obtain necessary conditions that $f : E^h_k \to E^p_q$ be a concomitant, we consider smooth paths in GL_E through the identity; namely, smooth maps $\mathbb{R} \to GL_E$, written $t \mapsto \phi_t$ such that $\phi_0 = \iota$ and $\dot{\phi}_0 = \eta \in L_E$, say.

From equation (1) it follows that the derivative of λ_{ϕ_t} at $t = 0$ is a derivation, θ_η, of the tensor algebra over E.

In fact, it is easily checked, for example, that

$$\theta_\eta(x \otimes x^*) = \eta(x) \otimes x^* - x \otimes \eta^*(x^*) , \quad x \in E , \; x^* \in E^* ,$$

where $\eta^* \in L_{E^*}$ is the dual of $\eta \in L_E$. Thus $\theta_\iota(\Phi) = 0$, if $\Phi \in E_1^1$.

In general, if we identify E_1^1 with L_E by means of the linear isomorphism which maps $x \otimes x^* \in E_1^1$ onto the endomorphism of E given by $y \mapsto \langle x^*, y\rangle x$, $y \in E$, then

$$\theta_\eta(\xi) = \eta \circ \xi - \xi \circ \eta , \quad \xi \in L_E .$$

Also, from the expression for the action of θ_η on a decomposible element of E_q^p, it follows readily that

$$(7) \qquad \theta_\iota(\Phi) = (p-q)\Phi , \quad \Phi \in E_q^p .$$

Now, suppose that $f \in T_q^p$ so that equation (4) holds with ϕ replaced by ϕ_t, $t \in \mathbb{R}$. Differentiation with respect to t at $t = 0$ then yields

$$(8) \qquad \theta_\eta[f(\Phi)] = f'(\Phi ; \theta_\eta\Phi) , \quad \eta \in L_E , \; \Phi \in E_k^h .$$

In this equation, $f'(\Phi ; \Psi)$ denotes the value of the derivative of $f : E_k^h \to E_q^p$ at Φ (a <u>linear</u> map of E_k^h into E_q^p) when evaluated at $\Psi \in E_k^h$.

The system of partial differential equations represented intrinsically by equation (8) constitutes necessary conditions on f in order that it belong to T_q^p. They are also necessary conditions for f to be an <u>oriented</u> concomitant (cf. §3).

As was remarked in the introduction the condition (8) places severe restrictions on the form of f. We content ourselves here with noting that it implies (when $\eta = \iota$)

$$(p-q)f(\Phi) = (h-k)f'(\Phi ; \Phi) . \tag{9}$$

Thus, if $h = k$, the only non-zero concomitants of E^h_h are of type (p,p). On the other hand, if $h \neq k$, then (9) shows that f must be homogeneous of degree $(p-q)/(h-k)$. Thus, if we make the additional assumption that f is C^∞ near $0 \in E^h_k$, then $m = (p-q)/(h-k)$ must be a non-negative integer and f must be a polynomial in Φ, homogeneous of degree m. In particular, if $p=h=0$, $k=2$, we must have q even and f a polynomial of degree $q/2$. This is consistent with Weyl's result about invariants of an inner product ([3], [4]).

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AN EXAMPLE OF LOCAL ERGODIC DIVERGENCE

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Presented by M. Ackoglu F.R.S.C.

1. Akcoglu and Krengel [1] gave an example of a continuous group $\{T_t : t \text{ real}\}$ of unitary complex L_2- operators for which

$$\lim_{t \to 0+} \frac{1}{t} \int_0^t T_s f \, ds \tag{1}$$

diverges a.e. In this example T_0 = the identity. This example answered a problem of Kubokawa [3] for p=2 (and complex scalars). Kubokawa showed that (1) exists a.e. when $\{T_t : t \geq 0\}$ is a continuous semigroup of <u>positive</u> operators in L_p (for $1 \leq p < \infty$). "Operator" here means "bounded and linear operator". In this note we construct a bounded continuous group $\{T_t : t \text{ real}\}$ of operators on L_p (for $1 < p \leq 2$) such that the limit (1) diverges a.e. By a simple modification we may assume that the scalor field is either real or complex.

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2. Let $\{e_n\}$ be an orthonormal basis of the complex ℓ_2 space and let $P_n : \ell_2 \to \ell_2$ be the orthogonal projection onto the linear span of $e_1, \ldots e_n$. Akcoglu and Krengel proved:

<u>Lemma</u>. There is a group $\{T_t : t \text{ real}\}$ of unitary operators on the complex ℓ_2 space and a sequence $\{t_k\}$, $t_k \downarrow 0$ such that for $S_t = \int_0^t T_s \, ds$

$$\text{(2)} \qquad \Sigma \| P_n - t_n^{-1} S_{t_n} \| < \infty$$

<u>Remark</u>. If we replace P_n by P_{2n} in (2) then we may assume that the scalar field is real.

<u>Proof</u>. Let A : complex $\ell_2 \to$ real ℓ_2 be the real-linear isometry $(x_1, x_2, \ldots) \mapsto (\operatorname{Re} x_1, \operatorname{Im} x_1, \operatorname{Re} x_2, \operatorname{Im} x_2, \ldots)$. Take $\hat{T}_t = AT_tA^{-1}$ instead of T_t.

In [2] a sequence $\{f_n\} \subset L_p$ (when $1 < p \leq 2$) was constructed so that

1. $B : \ell_2 \to \overline{\text{span}}\, \{f_n\}$ given by $B(x_1, x_2, \ldots) = \Sigma x_i f_i$ is an isomorphism.
2. There exists a projection P of L_p onto $\overline{\text{span}}\, \{f_i\}$.
3. $\sum_{i=1}^{2n} x_i f_i$ diverges a.e. for some $(x_1, x_2, \ldots) \in \ell_2$.

Define $R_t : L_p \to L_p$ by $R_t = BT_tB^{-1} + I - P$; the T_t are from the Akcoglu-Krengel construction (in the complex case, and from our remark, in the real case and I = the identity). B^{-1} exists on the image of P thus R_t is well defined.

Now, $\{R_t : t \text{ real}\}$ is a bounded continuous group on L_p, $R_0 = I$ and $\lim_{t\to 0+} t^{-1}\int_0^t R_s f\, ds$ diverges a.e. for some f in L_p.

Problem. Can a similar construction be made for $p > 2$?

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ON SYMMETRY CLASSES OF TENSORS

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The result which follows characterizes symmetric, skew-symmetric and "curvature" tensors as the only solutions of a simple eigenvalue problem. It is understood throughout that the underlying field has characteristic 0. Thus when we deal with the representation theory of S_m, the symmetric group, we need deal only with the ordinary representations.

Theorem: Let Φ be a non-zero (0,m)-type tensor acted on by S_m in the usual way (permutation of indices). Suppose that for some scalar β

$$\sum_{\substack{j=1\\ j\neq i}}^{m} (i,j)\Phi = \beta\Phi \quad \text{for all} \quad i.$$

Then either

(i) $\beta = m-1$ and Φ is symmetric,

(ii) $\beta = -(m-1)$ and Φ is skew-symmetric, or

(iii) $m = 4$, $\beta = 0$ and Φ is of symmetry class [2,2]. That is, Φ is the sum of two tensors, each of which has the symmetries of a curvature tensor (but on different indices).

Proof: Denote by γ_i the sum

$$\sum_{\substack{j=1\\ j\neq i}}^{m} (i,j)$$

used in the theorem, and let V_β denote the linear space of tensors Φ satisfying $\gamma_i\Phi = \beta\Phi$ for all i. For all $\sigma \in S_m$

we have, for $\Phi \in V_\beta$,

$$\gamma_i(\sigma\Phi) = \sigma(\sigma^{-1}\gamma_i\sigma)\ \Phi = \sigma\gamma_{\sigma^{-1}(i)}\Phi = \sigma\beta\Phi = \beta(\sigma\Phi)\ .$$

Hence V_β is invariant under S_m action, and provides a module for S_m. We assume V_β non-zero, and take an irreducible submodule W_β of V_β. It will suffice to show that $W_\beta = 0$ unless Φ is as described in the theorem.

The irreducible representations of S_m are labelled $[\lambda]$ where λ is a young diagram (see [1]). If we denote by γ_i^λ the matrix representation for γ_i in the group algebra of S_m in the representation $[\lambda]$ we have

$$\gamma_i^\lambda = \beta I_k \text{ for all } i,\ 1 \le i \le m$$

where k is the degree of $[\lambda]$. In fact, it is easy to see that it is sufficient to assume this for $i = 1$, the other cases following by conjugation in S_m.

Define an element, T^λ, of the group algebra, $A(S_m)$, of S_m as follows:

$$T^\lambda = \frac{1}{m!} \sum_{\sigma\in S_m} \chi^\lambda(\sigma)\sigma$$

where $\chi^\lambda(\sigma)$ is the character of σ in $[\lambda]$. That γ_1 is represented by a scalar matrix in the representation afforded by $[\lambda]$ is equivalent to the assertion that $T^\lambda\gamma_1$ is in the centre of $A(S_m)$, since T^λ projects γ_1 to the simple subalgebra of $A(S_m)$ associated with $[\lambda]$.

Now $T^\lambda\gamma_1 \in Z(A(S_m))$ implies that $T^\lambda\gamma_1$ is a (weighted) sum of full conjugacy classes. In the sequel we will consider only the conjugacy class of transpositions in the product $T^\lambda\gamma_1$.

First, an observation concerning $[\lambda]$ needs to be made. Let $[\mu] = [\lambda]S_m \downarrow S_{m-1}$, the restriction of $[\lambda]$ to S_{m-1}. If $[\mu]$ is reducible, we have, since

$$(1,m)^{\lambda} = \beta I_k - (\sum_{j=2}^{m-1}(1,j))^{\lambda}$$

that $(1,m)^{\lambda}$ holds invariant the same subspaces as those held invariant by $(1,j)^{\lambda}$, $j=2,\ldots, m-1$ in S_{m-1}. Since S_m is generated by $(1,m)$ together with S_{m-1}, this contradicts the irreducibility of $[\lambda]$. We conclude that $[\lambda]S_m \downarrow S_{m-1}$ is irreducible. It follows [1] that there is only one way to remove a node from $[\lambda]$ so that the result is a Young diagram. The necessary and sufficient condition that this be the case is, as is easily seen, that $[\lambda]$ be rectangular with, let us say, a columns and b rows. (Here $ab = m$.)

We return now to the condition that all of the transpositions in $T^{\lambda}\gamma_1$ should have the same coefficient. Now in order that $\sigma(1,j) = (r,s)$ (with, say, $r < s$) we must have $\sigma = (r,s)(1,j)$. We have the following cases:

<u>Case 1</u>. $r = 1$, $s = j$, $\sigma = 1$.

<u>Case 2</u>. $r = 1$, $s \neq j$, $\sigma = (1js)$.

<u>Case 3</u>. $r,s \neq 1,j$, $\sigma = (rs)(1j)$.

<u>Case 4</u>. $r \neq 1$, $s = j$, $\sigma = (1rj)$.

<u>Case 5</u>. $r = j$, $\sigma = (1sj)$.

Hence the coefficient for $(1s)$ in $m!T^{\lambda}\gamma_1$ is

$$\chi^{\lambda}(1) + (m-2)\chi^{\lambda}(123)$$

while the coefficient for (rs) when $r>1$ is

$$(m-3)\chi^\lambda(12)(34) + 2\chi^\lambda(123)$$

and so we must have

$$\chi^\lambda(1) + (m-2)\chi^\lambda(123) = (m-3)\chi^\lambda(12)(34) + 2\chi^\lambda(123) \text{ or}$$

$$\chi^\lambda(1) = (m-3)\chi^\lambda(12)(34) - (m-4)\chi^\lambda(123) \quad (1)$$

$\chi^\lambda(1)$ is the degree of the representation $[\lambda]$ and so is given by $m!/H(\lambda)$, where $H(\lambda)$ denotes the product of the hook lengths in the diagram λ [1]. For the rectangular λ in question we have

$$H(\lambda) = [(a+b-1) \ldots (b)][(a+b-2) \ldots (b-1)] \ldots [(a) \ldots (1)].$$

The calculation of the characters $\chi^\lambda(12)(34)$ and $\chi^\lambda(123)$ is a similar, although more complicated calculation. Doing this, and simplifying, equation (1) becomes

$$m^3 - 6m^2 - (m-4)(a-b)^2 + 9m - 4 = 0 \quad (2)$$

Now it is clear that the conditions of the theorem are satisfied with $a = m$, $b = 1$ (i.e., Φ symmetric, and $\beta = m-1$) or $a = 1$, $b = m$ (i.e., Φ skew-symmetric, and $\beta = -(m-1)$). As well, $[2,2]$ has kernel $\{1, (12)(34), (13)(24), (14)(23)\} = K$ in S_4 and $S_4/K \cong S_3$. The image of γ_1 under this homomorphism is $(12) + (13) + (23)$, a full conjugacy class of S_3. The exception arises in this way.

For the rest, we can take $m > 4$, and $|a-b| < m/2$, since $[\lambda]$ has at least two rows. Hence we have from (2)

$$m^3 = 6m^2 + (m-4)c^2 + 9m - 8$$
$$< 6m^2 + \frac{m^2}{4}(m-4) + 9m$$

and so

$$\frac{3}{4}m^3 < 5m^2 + 9m \quad \text{or}$$

$$\frac{3}{4}m^2 < 5m + 9 < 7m \text{ (using } m \geq 5)$$

and so

$$\frac{3}{4}m < 7 \quad \text{and} \quad m < 10$$

Thus we need only check $m = 6$, $|a-b| = 1$; $m = 8$, $|a-b| = 2$; $m = 9$, $|a-b| = 0$. Substituting these in (2) yields

$$6^3 - 6^3 - 2 + 9.6 - 4 \neq 0.$$

$$8^3 - 6.8^2 - 4.4 + 9.8 - 4 \neq 0.$$

$$9^3 - 6.9^2 + 9^2 - 4 \neq 0.$$

These inequalities complete the proof.

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A LIMIT-POINT CRITERION FOR A THREE-TERM RECURRENCE RELATION

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Presented by F.V. Atkinson, F.R.S.C.

1. We are concerned here with the limit-point classification of the second order difference equation

$$-\Delta(c_{n-1}\Delta y_{n-1}) + b_n y_n = 0 \tag{1.1}$$

where $n = 0, 1, 2, \ldots$; $c_n > 0$ for $n = -1, 0, 1, \ldots$ and b_n is real and defined for $n = 0, 1, 2, \ldots$ Here Δ represents the forward difference operator, $\Delta y_n = y_{n+1} - y_n$. A glance at (1.1) shows that it is equivalent to the three-term recurrence relation

$$c_n y_{n+1} + c_{n-1} y_{n-1} - b'_n y_n = 0 \tag{1.2}$$

where $b'_n = b_n + c_n + c_{n-1}$.

(1.1) is said to be in the *limit-point* case, (LP), at infinity if there exists a solution (y_n) such that

$$\sum_{n=-1}^{\infty} |y_n|^2 = \infty \tag{1.3}$$

(i.e. $y_n \notin \ell^2$).

Recently, at the Spectral Theory Conference of Differential Operators held in Birmingham, Alabama, Prof. F.V. Atkinson reported that no "Levinson-type" criterion for the LP classification of (1.1) was known.

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The purpose of this note is to formulate a discrete analog of Levinson's LP criterion.

Theorem 1. Let $M_n > 0$ be a sequence of real numbers and let $c_n > 0$, b_n be as above. Assume that for all sufficiently large n, there exists constants K_1, K_2 such that

a) $b_n \geq - K_1 M_n$

b) $\left| \dfrac{c_n^{1/2} \Delta M_{n-1}}{M_n^{1/2} M_{n-1}} \right| \leq K_2$

and

c) $\sum_{n=N}^{\infty} (c_{n-1} M_n)^{-1/2} = \infty$.

Then (1.1) is LP at infinity.

The continuous analog of theorem 1 is originally due to Levinson [2]. Before proceeding with the proof we will state some basic results required in the sequel.

(i) Partial Summation Formula.

$$\sum_{K=M}^{N} u_K \Delta v_K = [u_{K-1} v_K]_M^{N+1} - \sum_{K=M}^{N} v_K \Delta u_{K-1}$$

(ii) Wronskian-type identity [1].

$$c_{n-1}(\phi_{n-1}\psi_n - \phi_n\psi_{n-1}) = \text{const} \qquad (1.4)$$

whenever ϕ_n, ψ_n are any two linearly independent solutions of (1.1).

2. PROOF OF THEOREM 1.

This proceeds, as usual, by contradiction. Let $x_n \in \ell^2$ be a real solution of (1.1). Then $\Delta(c_{n-1}\Delta x_{n-1}) = b_n x_n$ implies

$$\sum M_n^{-1} x_n \Delta(c_{n-1}\Delta x_{n-1}) = \sum b_n M_n^{-1} x_n^2 \tag{2.1}$$

where in general the summations will be over the range $n = m_1$ to $n = m$ where $m \geq m_1$ and m_1 is sufficiently large. Applying (i) to the left side of (2.1) we obtain, for some constant K_3,

$$x_{m-1} M_{m-1}^{-1} c_{m-1}\Delta x_{m-1} - K_3 - \sum c_n \Delta x_{n-1} \Delta\left(\frac{x_{n-1}}{M_{n-1}}\right) . \tag{2.2}$$

We note that the last term in (2.2) equals

$$\sum c_n (\Delta x_{n-1})^2 M_n^{-1} + \sum c_n x_{n-1} \Delta x_{n-1} \Delta (M_{n-1}^{-1}) . \tag{2.3}$$

Inserting (2.3) into the appropriate place in (2.2) and substituting the resulting expression for the left side of (2.1) we find

$$M_{m-1}^{-1} x_{m-1} c_{m-1}\Delta x_{m-1} - \sum c_n M_n^{-1} (\Delta x_{n-1})^2 - K_3 -$$

$$\sum c_n x_{n-1} (\Delta x_{n-1}) \Delta (M_{n-1}^{-1}) = \sum b_n M_n^{-1} x_n^2 . \tag{2.4}$$

Since $x_n \in \ell^2$ we see that, on account of (a), the right side of (2.4) is bounded below by

$$-K_1 \sum |x_n|^2 \geq -K_4 \tag{2.5}$$

where we may choose $K_4 = -K_1 \|x_n\|_2^2$.

Combining (2.4) with the bound (2.5) we obtain

$$-x_{m-1} M_{m-1}^{-1} c_{m-1} \Delta x_{m-1} + \sum c_n M_n^{-1} (\Delta x_{n-1})^2 -$$

$$\sum \frac{c_n x_{n-1} (\Delta x_{n-1}) (\Delta M_{n-1})}{M_n M_{n-1}} < K_5 \tag{2.6}$$

where $K_5 = -K_3 + K_4$.

We now set $H_m = \sum_{n=m_1}^{m} c_n M_n^{-1} (\Delta x_{n-1})^2$. Then

$$\left| \sum \frac{c_n x_{n-1} (\Delta x_{n-1}) (\Delta M_{n-1})}{M_n M_{n-1}} \right|^2 =$$

$$\left| \sum \left(\frac{c_n^{1/2} \Delta M_{n-1}}{M_n^{1/2} M_{n-1}} \right) \cdot \left(\frac{c_n^{1/2} x_{n-1} \Delta x_{n-1}}{M_n^{1/2}} \right) \right|^2 \tag{2.7}$$

Applying (b) to the right-side of (2.7) we find the upper bound

$$K_2^2 \left| \sum \frac{c_n^{1/2} x_{n-1} \Delta x_{n-1}}{M_n^{1/2}} \right|^2 ,$$

and an application of the Schwarz inequality now yields the bound

$$K_2^2 H_m \sum_{n=m_1}^{m} |x_{n-1}|^2 . \tag{2.8}$$

Consolidating these results we arrive at

$$\left| \sum \frac{c_n x_{n-1} (\Delta x_{n-1}) \Delta M_{n-1}}{M_n M_{n-1}} \right| \le K_2 H_m^{1/2} (\sum |x_{n-1}|^2)^{1/2} \tag{2.9}$$

It now follows from (2.6) that

$$-x_{m-1} c_{m-1} M_{m-1}^{-1} \Delta x_{m-1} + H_m \le K_5 + K_2 H_m^{1/2} \|x_n\|_2$$

i.e.

$$-x_{m-1} c_{m-1} M_{m-1}^{-1} \Delta x_{m-1} + H_m - K_6 H_m^{1/2} \le K_5 \tag{2.10}$$

If we assume that $H_m \to +\infty$ as $m \to \infty$ then for sufficiently large m we must have

$$M_{m-1}^{-1} x_{m-1} c_{m-1} \Delta x_{m-1} > \frac{H_m}{2}$$

and so $x_{m-1}\,\Delta x_{m-1} > 0$ for large m. The latter now implies that $x_n \notin \ell^2$ which is a contradiction. Hence

$$\sum_{n=m_1}^{\infty} c_{n-1} M_n^{-1} (\Delta x_{n-1})^2 < \infty. \tag{2.11}$$

Now let ϕ_n, ψ_n be two solutions of (1.1) satisfying (1.4) with const $\equiv 1$, and both in ℓ^2. Then

$$1 = c_{n-1}(\phi_{n-1}\Delta\psi_{n-1} - \psi_{n-1}\Delta\phi_{n-1})$$

and so

$$(M_n\, c_{n-1})^{-1/2} = \phi_{n-1}\,\frac{c_{n-1}^{1/2}\,\Delta\psi_{n-1}}{M_n^{1/2}} - \psi_{n-1}\,\frac{c_{n-1}^{1/2}\,\Delta\phi_{n-1}}{M_n^{1/2}}\,.$$

Now

$$\sum (M_n c_{n-1})^{-1/2} \le \left(\sum |\phi_{n-1}|^2\right)^{1/2} \left(\sum \frac{c_{n-1}(\Delta\psi_{n-1})^2}{M_n}\right)^{1/2} + \left(\sum |\psi_{n-1}|^2\right)^{1/2} \left(\sum \frac{c_{n-1}(\Delta\phi_{n-1})^2}{M_n}\right)^{1/2},$$

and so since ϕ_n, $\psi_n \in \ell^2$ and (2.11) holds for any ℓ^2-solution, we obtain a contradiction to (c) which proves the theorem.

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PONTRJAGIN CLASSES AND WEYL TENSORS

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Abstract: The Pontrjagin classes of a smooth manifold are certain cohomology classes which can be represented by differential forms which arise from the curvature tensor of a Riemannian (or pseudo-Riemannian) metric (cf. [1]). In this paper it is shown that these differential forms can be expressed in terms of the Weyl tensor (cf. [2]) and hence are conformal invariants.

1. A decomposition theorem. Let E, E^* be a pair of n-dimensional vector spaces and let $\wedge(E^*, E)$ denote the mixed exterior algebra over E (cf. [3]). It is bigraded by the subspaces $\wedge^p E^* \otimes \wedge^q E$ and contains the diagonal subalgebra

$$\Delta(E^*, E) = \sum_p \wedge^p E^* \otimes \wedge^p E$$

as a commutative subalgebra. Left multiplication in $\wedge(E^*, E)$ will be denoted by μ. Thus for every $a \in \wedge(E^*, E)$ we have a linear map

$$\mu(a): \wedge(E^*, E) \longrightarrow \wedge(E^*, E).$$

Let

$$i(a): \wedge(E^*, E) \longleftarrow \wedge(E^*, E)$$

denote the dual operator.

The unit tensor $t \in E^* \otimes E$ is defined by

$$t = \sum_\nu e^{*\nu} \otimes e_\nu$$

where e_ν, $e^{*\nu}$ $(\nu = 1, \ldots n)$ is a pair of dual bases. It determines a pair of dual maps

$$\mu(t): \wedge(E^*, E) \longrightarrow \wedge(E^*, E)$$

and

$$i(t): \wedge(E^*, E) \longleftarrow \wedge(E^*, E),$$

which are homogeneous of bidegree (1,1) and (-1 , -1) respectively.

Now assume that $n \geqq 3$ and fix an element $\underline{\Psi} \in \Lambda^2 E^* \otimes \Lambda^2 E$. Then there is a unique decomposition of the form

(1) $$\underline{\Psi} = \underline{\Psi}_0 + t \cdot \underline{\Psi}_1 + t^2 \cdot \underline{\Psi}_2,$$

where $\underline{\Psi}_0 \in \Lambda^2 E^* \otimes \Lambda^2 E$, $\underline{\Psi}_1 \in E^* \otimes E$ satisfy

$$i(t)\,\underline{\Psi}_0 = 0 \quad , \quad i(t)\,\underline{\Psi}_1 = 0$$

and $\underline{\Psi}_2$ is a scalar. $\underline{\Psi}_1$ and $\underline{\Psi}_2$ are explicitly given by the formulae

and $$\underline{\Psi}_1 = \frac{1}{n-2}\left(i(t)\,\underline{\Psi} - \frac{1}{n}\, i(t)^2 (t\,\underline{\Psi})\right)$$

$$\underline{\Psi}_2 = \frac{1}{n(n-1)}\, i(t)^2\, \underline{\Psi}.$$

This is a special case of the decomposition theorem given in [4].

2. <u>The operator θ</u>. Fix vectors x^* and y^* in E^* and set

$$\theta(x^* \otimes y^*)(u^* \otimes v) = \mu(x^*)u^* \otimes i(y^*)v$$

Then $\theta(x^* \otimes y^*)$ is a linear map from $\Lambda(E^*, E)$ to itself homogeneous of bidegree (1 , -1). By linearity we extend the definition of θ to elements $z \in E^* \otimes E$.

A straightforward calculation shows that $\theta(z)$ is an antiderivation in the algebra $\Lambda(E^*, E)$,

$$\theta(z)(\underline{\Phi} \cdot \underline{\Psi}) = \theta(z)\,\underline{\Phi} \cdot \underline{\Psi} + (-1)^{p+q}\,\underline{\Phi} \cdot \theta(z)\,\underline{\Psi}$$

$$\underline{\Phi} \in \Lambda^p E^* \otimes \Lambda^q E, \quad \underline{\Psi} \in \Lambda(E^*, E).$$

Suppose now that a non-degenerate inner product is given in E and set

$$(a^* \otimes b,\, u^* \otimes v)_F = (b, v)_E\; a^* \wedge u^*$$

$$a^* \in \Lambda^p E^*, \quad u^* \in \Lambda^q E^*, \quad b, v \in \Lambda^k E,$$

where $(\, , \,)_E$ denotes the induced inner product in $\Lambda^k E$. This defines a bilinear map

$$(\, , \,)_F : (\Lambda^p E^* \otimes \Lambda^k E) \times (\Lambda^q E^* \otimes \Lambda^k E) \longrightarrow \Lambda^{p+q} E^* .$$

On the other hand, if $g \in E^* \otimes E^*$ denotes the inner product in E, we have the operator

$$\theta(g) : \Lambda^p E^* \otimes \Lambda^q E \longrightarrow \Lambda^{p+1} E^* \otimes \Lambda^{q-1} E$$

Proposition: Let $\Phi \in \Lambda^p E^* \otimes \Lambda^q E$ and $\Psi \in \Lambda^{q-1} E^* \otimes \Lambda^{k+1} E$. Then

$$(\Phi, t \cdot \Psi)_F = (-1)^p (\theta(g) \Phi, \Psi)_F$$

3. Bianchi tensors. Define an operator $Z : E^* \otimes E^* \otimes E^* \longrightarrow E^* \otimes E^* \otimes E^*$ by

$$Z(x^* \otimes y^* \otimes z^*) = x^* \otimes y^* \otimes z^* + y^* \otimes z^* \otimes x^* + z^* \otimes x^* \otimes y^* .$$

A tensor $\Phi \in E^* \otimes E^* \otimes E^* \otimes E$ will be called a Bianchi tensor, if $(Z \otimes \iota)\Phi = 0$.

Let $T : E^* \otimes E^* \otimes E^* \otimes E \xrightarrow{\cong} E^* \otimes E^* \otimes E \otimes E$ be the operator defined by $T = \iota \otimes \iota \otimes \tau \otimes \iota$ where $\tau : E^* \xrightarrow{\cong} E$ is the isomorphism induced by the inner product. Denote the canonical projections $\otimes E^* \longrightarrow \Lambda E^*$ and $\otimes E \longrightarrow \Lambda E$ by π_Λ.

Lemma: With the notation above the following diagram commutes:

$$\begin{array}{ccccc}
\Lambda^2 E^* \otimes \Lambda^2 E & \xleftarrow{\pi_\Lambda \otimes \pi_\Lambda} & E^* \otimes E^* \otimes E \otimes E & \underset{\cong}{\xleftarrow{T}} & E^* \otimes E^* \otimes E^* \otimes E \\
\theta(g) \downarrow & & & & \downarrow Z \otimes \iota \\
\Lambda^3 E^* \otimes E & & \xleftarrow{\quad \pi_\Lambda \otimes \iota \quad} & & E^* \otimes E^* \otimes E^* \otimes E
\end{array}$$

In particular, if Φ is a Bianchi tensor, then the tensor

$\underline{\psi} = (\pi_A \otimes \pi_A)\, T(\underline{\Phi})$ satisfies the relation

$$\theta(g)\underline{\psi} = 0$$

4. Pontrjagin classes. Let M be a Riemannian (or pseudo-Riemannian) manifold of dimension $n \geq 3$ and let $A^p_q(M)$ denote the space of mixed skew symmetric tensor fields of type (p,q) on M . Regard the curvature tensor as an element of $A^2_2(M)$ (in components $\mathcal{R}^{\alpha\beta}_{\gamma\delta}$) and denote it by $\dot{\mathcal{R}}$. Then formula (1) yields the direct decomposition

$$\dot{\mathcal{R}} = W + t \cdot S + t^2 \cdot K \tag{2}$$

where $W \in A^2_2(M)$, $S \in A^1_1(M)$ and K is a scalar function. Moreover, W , S and K are called Weyl tensor, trace-free Ricci tensor and scalar curvature respectively.

Let $\dot{\mathcal{R}}^k$ denote the k-th power of $\dot{\mathcal{R}}$ in the algebra $A(M)$ (the direct sum of the spaces $A^p_q(M)$). Then $\dot{\mathcal{R}}^k \in A^{2k}_{2k}(M)$ and so

$$\mathcal{P}_k = \frac{(-1)^k}{(2\pi)^{2k}(k!)^2}\, (\dot{\mathcal{R}}^k, \dot{\mathcal{R}}^k)_F$$

is a differential form of degree $4k$. This differential form is closed (cf. [1]) and so it represents a cohomology class p_k of degree $4k$. p_k is called the k-th Pontrjagin class of M .

Theorem: The differential forms $\mathcal{P}_k$ depend only on the Weyl tensor.

Proof: Let $\mathcal{R}$ denote the curvature tensor of M regarded as a tensor of type $(3,1)$ (in components $\mathcal{R}^{\alpha}_{\beta\gamma\delta}$). Then the tensors $\dot{\mathcal{R}}$ and $\mathcal{R}$ are connected by the relation

$$\dot{\mathcal{R}} = (\pi_A \otimes \pi_A)\, T(\mathcal{R})_g$$

as is easily checked. Now $\mathcal{R}$ is a Bianchi tensor and so the lemma in sec. 3 implies that $\theta(g)\dot{\mathcal{R}} = 0$. .

Next observe that $\theta(g)\Phi = 0$ for every element $\Phi \in E^* \otimes E$ which is invariant under the map $x^* \otimes y \overset{\cong}{\longrightarrow} \tau^{-1}y \otimes \tau x^*$. In particular, $\theta(g)S = 0$ and $\theta(g)t = 0$. Now the antiderivation property of $\theta(g)$ shows that

(4) $$\theta(g)W = 0.$$

Now write (2) in the form

(5) $$\dot{\mathcal{R}} = W + t \cdot \Phi \quad \text{where} \quad \Phi = S + t \cdot K.$$

Since the diagonal subalgebra of $\Lambda(E^*, E)$ is commutative we may apply the binomial formula to (5) to obtain

(5) $$\dot{\mathcal{R}}^k = \sum_{j=0}^{k} W^{k-j} \cdot t^j \cdot \Phi^j = W^k + t \cdot \Omega,$$

where

$$\Omega = \sum_{j=1}^{k} t^{j-1} \cdot \Phi^j \cdot W^{k-j}$$

Thus,

(6) $$(\dot{\mathcal{R}}^k, \dot{\mathcal{R}}^k)_F = (W^k, W^k)_F + 2(W^k, t\Omega)_F + (t\Omega, t\Omega)_F$$

Now, by the proposition in sec. 2,

$$(W^k, t\Omega)_F = (\theta(g)W^k, \Omega)_F$$

But, by (4),

$$\theta(g)W^k = kW^{k-1}\theta(g)W = 0$$

and thus

(7) $$(W^k, t\Omega)_F = 0$$

On the other hand,

$$(t\Omega, t\Omega)_F = (\theta(g)(t\Omega), \Omega)_F$$

and

$$\theta(g)(t\Omega)=t\cdot\theta(g)\Omega=0,$$

(since $\theta(g)t=0$, $\theta(g)I=0$ and $\theta(g)W=0$).

Hence,

$$(t\Omega,t\Omega)_F=0 \tag{8}$$

Relations (6), (7) and (8) imply that

$$(\dot{R}^k,\dot{R}^k)_F=(W^k,W^k)_F \tag{9}$$

Finally, combining (3) and (9) we obtain the formula

$$P_k=\frac{(-1)^k}{(2\pi)^{2k}(k!)^2}(W^k,W^k)_F$$

which expresses the differential forms P_k in terms of the Weyl tensor.

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