The purpose of **Mathematical Reports** is to provide quick publication of short papers summarizing completed research. Typescripts will be photo-reproduced and papers are limited at most, to **six** typed pages. Detailed instructions for typing will be found inside the back cover.

Subscriptions at the rate of $8 per annum should be sent to the Editorial Office (1).

An issue should appear every two months. Authors should check their typescripts carefully since no proofs will be provided. On acceptance of a paper, a page charge at the rate of $10 per page will be requested. Of each paper 50 free reprints will be furnished by the journal.

A paper, written in English or French, should be sent either to one of the Fellows listed on the back cover or to Professor P. Ribenboim (2), accompanied by a detailed version (not for publication), since papers will be refereed.

Comptes rendus mathématiques est le titre d'un journal dans lequel seront publiés des articles courts résumant les dernières recherches effectuées. Les textes dactylographiés seront photocopiés. Ils ne doivent pas compter plus de **six** pages, et leur présentation doit être conforme aux instructions détaillées qui se trouvent à l'intérieur de la couverture arriére.

L'abonnement coûte $8 par année et doit être acquitté au bureau de la rédaction (1).

Un numéro paraîtra tous les deux mois. On conseille aux auteurs de relire leurs textes attentivement, puisqu'aucune épreuve ne leur sera envoyée. Les frais de publication de tout article accepté seront de $10 la page. Les auteurs recevront gratuitement cinquante exemplaires de leur tirage part.

Accompagnés d'une version détailée (non destinée à la publication), les articles rédigés en francais ou en anglais devront être addressés à l'un des membres de l'Académie des Sciences, dont la liste figure en derniére page, ou au professeur P. Ribenboim (2); ils seront soumis à une évaluation avant d'être acceptés.

(1) C.R. Math, Rep., Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1.

(2) Prof. P. Ribenboim, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L 3N6.

Sponsored by the departments of mathematics of Queen's University, the Universities of Toronto and Waterloo.
Subventionné par les sections de mathématiques des Universités Queen's, de Toronto et de Waterloo.

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No.2

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

THE BECKMAN-QUARLES THEOREM IN MINKOWSKI SPACE FOR A SPACELIKE SQUARE-DISTANCE

J.A. Lester

Presented by P. Scherk F.R.S.C.

Abstract: Transformations of Minkowski space which preserve a single spacelike square-distance are shown to be Lorentz transformations.

In 1953, Beckman and Quarles [1] proved that a (possibly multivalued) transformation of Euclidean n-space ($2 \leq n < \infty$) which preserves a single non-zero length must be single-valued and a Euclidean motion. We give an extension of this theorem to Minkowski space. Let $(\, , \,)$ denote the Minkowskian metric on $\mathbb{R}^n$, i.e. with respect to some basis of $\mathbb{R}^n$,

$$(x,y) := \sum_{i=1}^{n-1} x^i y^i - x^n y^n$$

for all $x,y \in \mathbb{R}^n$.

Theorem: For a fixed $\rho > 0$, let T be a (possibly multivalued) transformation of $V := (\mathbb{R}^n, (\, , \,))$ ($3 \leq n < \infty$) which preserves the square-distance ρ, i.e. to each $x \in V$, T assigns a non-empty subset $Tx \subseteq V$ such that for all $x,y \in V$ with $(x-y,x-y) = \rho$, all $\bar{x} \in Tx$, $\bar{y} \in Ty$ satisfy $(\bar{x}-\bar{y},\bar{x}-\bar{y}) = \rho$. Then T is single-valued (i.e. for all $x \in V$, Tx contains exactly one point) and the function $f : V \to V$, given by $\{f(x)\} = Tx$, is a Lorentz transformation (including a translation).

Note: The same theorem but with $\rho < 0$ has been proved by Benz [4], as has the two-dimensional case [3].

The multivalued nature of T may be easily removed (see [4]) thus we consider below only the single-valued function f.

Lemma 1: f is 1-1

Outline of proof: The maximum number of points $p_1,\dots,p_k$ in V with $(p_i-p_j,p_i-p_j) = \rho$ is n. If f were not 1-1, it would map a certain configuration of points into $n+1$ points satisfying this condition.

Definition 1: A prism is a set of $n-1$ parallel null lines $L_1,\dots,L_{n-1}$ such that for $i \neq j$, all $x \in L_i$, $y \in L_j$ satisfy $(x-y,x-y) = \rho$. (A null line is one whose direction vector d satisfies $(d,d) = 0$).

Any null line in V can be made part of some prism.

Definition 2: The subsets $A_1,\dots,A_{n-1}$ of V are equidistant if each contains at least 3 points and if for all $x \in A_i$, $y \in A_j$, $i \neq j$, $(x-y,x-y) = \rho$.

The lines of a prism are equidistant, as are their images (since f is 1-1). By using well-known geometric properties of metric vector spaces equidistant sets may be characterized as follows:

Lemma 2: All but one of the equidistant sets $A_1,\dots,A_{n-1}$ must be contained in the corresponding lines of some prism $M_1,\dots,M_{n-1}$. The remaining set, say A_{n-1}, is contained in $M_{n-1} \cup \bar{M}_{n-1}$, where $\bar{M}_{n-1}$ is line parallel to M_n such that all

$a \in \bar{M}_n$ satisfy $(a-b,a-b) = 2(n-1)\rho/(n-2)$, for all $b \in M_n$.

Except for the possibility that one of its lines is split into two, then, any prism is mapped into a prism. This problem can be removed by considering more than one prism; thus we have the following corollary.

Corollary 1: f maps null lines into null lines.

By first showing that f preserves the midpoints of line segments of 'square-length' 4ρ, it is easily shown that f maps any plane determined by two intersecting null lines into a like plane. Sine any line in V is the intersection of two such planes, f preserves lines. Known results now show that f is linear [6], and is thus a Lorentz transformation ([5], Lemma 3.6).

Bibliography

[1] F.S. Beckman and D.A. Quarles Jr., On Isometries of Euclidean Spaces. Proc. Amer. Math. Soc. 4 (1953), 810-815.

[2] W. Benz, Zur Charakterisierung der Lorentztransformationen, J. Geometry 9 (1977), 29-37.

[3] W. Benz, A Beckman-Quarles-Type-Theorem for Plane Lorentz Transformations, Math. Z. (to appear).

[4] W. Benz, Eine Beckman-Quarles-Charakterisierung der Lorentztransformationen des $\mathbb{R}^n$, Arch. Math. (to appear).

[5] J.A. Lester, Cone-Preserving Mappings for Quadratic Cones over Arbitrary Fields, Canad. J. Math. 29 (1977), 1247-1253.

[6] G. Martin, The Fundamental Theorem of Affine Geometry (1976), (unpublished notes)

Math. Seminar der Univ. Hamburg
Bundesstr. 55
2 Hamburg 13, W. Germany

Received December 20, 1980

This work was completed with the financial assistance of the Alexander von Humboldt Foundation.

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

THE STABILITY OF LINEAR FUNCTIONAL EQUATIONS

László Székelyhidi

Presented by J. Aczél F.R.S.C.

ABSTRACT: We prove a general stability theorem for linear functional equations which is used to give all solutions of (1) with bounded right hand side.

In this paper we consider the functional equation

$$f(x) + \sum_{i=1}^{n+1} c_i f(\phi_i(x)+\psi_i(y)) = F(x,y) \tag{1}$$

and we call

$$f(x) + \sum_{i=1}^{n+1} c_i f(\phi_i(x)+\psi_i(y)) = 0 \tag{2}$$

the homogeneous equation corresponding to (1). Here x,y are elements of an Abelian group G, $\phi_i,\psi_i : G \to G$ are homomorphisms for which the range of ϕ_i is contained in the range of ψ_i, c_i is a complex number $(i=1,\dots,n+1)$, and $F : G \times G \to C$ is a bounded complex valued function. Our main result is that every solution $f : G \to C$ of (1) is the sum of a bounded solution of (1) and a solution of (2). For the proof we use a stability theorem ([1], [3], [7]) and also our result can be considered as a stability theorem.

In this paper C denotes the set of complex numbers. If G is an Abelian group and $P : G \to C$ is a function for which $\Delta^{n+1}_{t_1,\dots,t_{n+1}} P(x) = 0$ holds for every $x,t_1,\dots,t_{n+1}$ in G, then we call P a polynomial of degree at most n. It is well known (see e.g. [2], [4], [5], [6]) that a polynomial on an Abelian group can be expressed as the sum of the diagonalizations of multi-additive functions.

Concerning polynomials on groups see e.g. [2], [4], [6].

First we prove a more general stability theorem:

THEOREM 1. *Let* G, ϕ_i, ψ_i *be as above and let* $f, f_i : G \to C$ *be functions, for which the function*

$$(x,y) \to f(x) + \sum_{i=1}^{n+1} f_i(\phi_i(x)+\psi_i(y))$$

is bounded. Then $f = f_0 + P$ *where* $f_0 : G \to C$ *is bounded and* $P : G \to C$ *is a polynomial of degree at most* n.

PROOF. Let, for x, y in G,

$$F(x,y) = f(x) + \sum_{i=1}^{n+1} f_i(\phi_i(x)+\psi_i(y)) \tag{3}$$

and let t be a fixed element in G. First we choose an element s in G such that $\phi_{n+1}(t) + \psi_{n+1}(s) = 0$. This is possible, because the range of ϕ_{n+1} is contained in the range of ψ_{n+1}. Then substituting $x+t$ for x and $y+s$ for y in (3) we have

$$\begin{aligned} F(x+t,y+s) &= f(x+t)+\sum_{i=1}^{n+1} f_i(\phi_i(x+t)+\psi_i(y+s)) = \\ &= f(x+t)+\{\sum_{i=1}^{n} f_i(\phi_i(x)+\psi_i(y)+\phi_i(t)+\psi_i(s))\}+f_{n+1}(\phi_{n+1}(x)+\psi_{n+1}(y)). \end{aligned} \tag{4}$$

Subtracting (3) from (4) we have

$$F(x+t,y+s)-F(x,y) = \Delta_t f(x)+\sum_{i=1}^{n} \Delta_{\phi_i(t)+\psi_i(s)} f_i(\phi_i(x)+\psi_i(y)). \tag{5}$$

Here the left hand side is bounded. Repeating this argument we have that the function $(x,t_1,\dots,t_{n+1}) \to \Delta^{n+1}_{t_1,\dots,t_{n+1}} f(x)$

is bounded and this implies (see [1], [7]) that $f = f_0 + P$, where $f_0 = G \to C$ is bounded and $\Delta^{n+1}_{t_1,\dots,t_{n+1}} P(x) = 0$ for all $x, t_1, \dots, t_{n+1}$ in G; that is, P is a polynomial of degree at most n.

THEOREM 2. *Let* G, ϕ_i, ψ_i *be as above*, c_i *be a complex number* $(i=1,\dots,n+1)$ *and* $F : G \times G \to C$ *be a bounded function. If* $F : G \to C$ *is a solution of (1), then* $f = f_0 + P$ *where* $f_0 : G \to C$ *is a bounded solution of (1) and* $P : G \to C$ *is a polynomial solution of (2) of degree at most* n.

PROOF. By the preceding theorem we have $f = f_0 + P$, where $f_0 : G \to C$ is bounded and $P : G \to C$ is a polynomial of degree at most n. Substituting $f_0 + P$ into (1) we get

$$\begin{aligned} P(x) + \sum_{i=1}^{n+1} c_i P(\phi_i(x)+\psi_i(y)) &= \\ = F(x,y) - [f_0(x) + \sum_{i=1}^{n+1} c_i f_0(\phi_i(x)+\psi_i(y))]. \end{aligned} \tag{6}$$

Here the left hand side is a polynomial in x for all fixed y, and the right hand side is bounded. Since all bounded polynomials are constant, we have

$$P(x) + \sum_{i=1}^{n+1} c_i P(\phi_i(x)+\psi_i(y)) = K \tag{7}$$

where K is a complex number. Now we have two possibilities. If $\sum_{i=1}^{n+1} c_i = -1$, then $K = 0$. Indeed, we can write $P(x) = Q(x) + P(0)$ where Q is a polynomial of degree at most n and $Q(0) = 0$. Substituting into (7) we have

$$Q(x)+P(0)+\sum_{i=1}^{n+1} c_i Q(\phi_i(x)+\psi_i(y)) + (\sum_{i=1}^{n+1} c_i)\cdot P(0) = K\ ;$$

that is

$$Q(x)+\sum_{i=1}^{n+1} c_i Q(\phi_i(x)+\psi_i(y)) = K \quad .$$

Let $x = y = 0$; then we have $K = 0$. In this case from (6) and (7) we see that f_0 is a bounded solution of (1) and P is a polynomial solution of (2) of degree at most n.

If $c = \sum_{i=1}^{n+1} c_i \neq -1$, then we define $Q(x) = P(x) - K(1+c)^{-1}$ and $g_0(x) = f_0(x) + K(1+c)^{-1}$. From (7) we infer

$$Q(x)+\sum_{i=1}^{n+1} c_i Q(\phi_i(x)+\psi_i(y)) =$$

$$= P(x)-K(1+c)^{-1}+\sum_{i=1}^{n+1} c_i P(\phi_i(x)+\psi_i(y))-\sum_{i=1}^{n+1} c_i K(1+c)^{-1} =$$

$$= P(x)+\sum_{i=1}^{n+1} c_i P(\phi_i(x)+\psi_i(y))-K = 0 \ ;$$

that is, Q is a solution of (2). Of course, Q is a polynomial of degree at most n, and $f = g_0 + Q$. Finally, from (6) we have

$$g_0(x)+\sum_{i=1}^{n+1} c_i g_0(\phi_i(x)+\psi_i(y)) =$$

$$= f_0(x)+\sum_{i=1}^{n+1} c_i f_0(\phi_i(x)+\psi_i(y))+K =$$

$$= F(x,y)-[P(x)+\sum_{i=1}^{n+1} c_i P(\phi_i(x)+\psi_i(y))]+K = F(x,y);$$

that is, g_0 is a bounded solution of (1). Hence the theorem is proved.

REFERENCES

[1] Albert, M., Baker, J.A., Functions with bounded n-th differences. (unpublished manuscript)

[2] Djoković, D.Ž., A representation theorem for $(X_1-1)(X_1-1)\ldots(X_n-1)$ and its applications. Ann. Polon. Math. 22 (1969), 189-198.

[3] Hyers, D.H., Transformations with bounded m-th differences. Pacific J. Math. 11 (1961), 591-602.

[4] McKiernan, M.A., On vanishing n-th ordered differences and Hamel bases. Ann. Polon. Math. 19 (1967), 331-336.

[5] Székelyhidi, L., Remark on a paper of M.A. McKiernan. Ann. Polon. Math. 36 (1979), 245-247.

[6] Székelyhidi, L., On a class of linear functional equations. Publ. Math. Debrecen, to appear.

[7] Székelyhidi, L., Note on a stability theorem. to appear.

Department of Mathematics,
Kossuth Lajos University,
H-4010 Debrecen, Hungary

Received December 20, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

PERIODIC SPLINES AND NILPOTENT HARMONIC ANALYSIS

Walter Schempp

Presented by G.F.D. Duff F.R.S.C.

The Heisenberg groups are remarkably little known objects considering their wide range of applications in pure as well as in applied mathematics (cf. Howe [3]). It is the purpose of the present note to expose various applications of nilpotent harmonic analysis to the theory of periodic spline functions. These include an application of the finite Heisenberg group to periodic spline interpolants and a treatment of the theory of attenuation factors of practical Fourier analysis from the viewpoint of harmonic analysis on the Heisenberg manifold (which is a compact nilmanifold). The final topic is a Fourier analytic viewpoint towards the classical Whittaker-Shannon cardinal series.

1. Group Theoretic Aspects of Splines

The (periodic and cardinal) spline interpolants with respect to "regular" knot sequences may be treated by means of harmonic analysis ([4], [5], [6]). The table displayed below summarizes some of the various spline interpolants, their underlying groups and the associated linear transforms that have been dealt with from the viewpoint towards harmonic analysis. It should be observed that the group theoretic approach to periodic and cardinal splines gives rise to a comprehensive theory which is a desideratum of I.J. Schoenberg.

$\{(n,m,p) \in \widetilde{A}(\mathbb{R}) \mid n,m,p \in \mathbb{Z}\}$. Construct the compact Heisenberg manifold $P\backslash\widetilde{A}(\mathbb{R})$, the complex Hilbert space $L^2(P\backslash\widetilde{A})$ with respect to the natural probability measure on $P\backslash\widetilde{A}(\mathbb{R})$ and the primary decomposition (orthogonal direct sum)

$$L^2(P\backslash\widetilde{A}) = \widehat{\bigoplus_{N \in \mathbb{Z}}} H_N(P)$$

associated with the right regular representation δ of $\widetilde{A}(\mathbb{R})$ in the Hilbert space $L^2(P\backslash\widetilde{A})$. The multiplicity of δ on $H_N(P)$ is $|N|$ when $N \neq 0$. Let γ denote the natural left action of the Heisenberg group $A(\mathbb{Z}/N\mathbb{Z})$ mod N on $L^2(P\backslash\widetilde{A})$. Then the orthogonal sum

$$H_N(P) = \bigoplus_{0 \leq n \leq N-1} \gamma(n,0,1)(H_{N,0}(P)) \qquad (N \geq 1)$$

is a decomposition of $H_N(P)$ into isotypic components for δ where $H_{N,0}(P)$ denotes the image of $H_1(P)$ under the linear mapping $\varphi_N: L^2(P\backslash\widetilde{A}) \to L^2(P\backslash\widetilde{A})$ induced by the automorphism $(x,y,z) \rightsquigarrow (x,Ny,Nz)$ of $\widetilde{A}(\mathbb{R})$. Notice that $H_1(P)$ is unitarily isomorphic to the standard Hilbert space $L^2(\mathbb{R})$ by the Weil-Brezin isomorphism W (Auslander [1]).

<u>Definition.</u> Let $h_0 \in H_{N,0}(P)$ be a fixed function on the compact Heisenberg manifold $P\backslash\widetilde{A}(\mathbb{R})$. The complex numbers

$$\tau_n = N(\varphi_N \circ W)^{-1}(h_0)(n) \qquad (n \in \mathbb{Z})$$

are called the attenuation factors associated with h_0.

Let σ denote the automorphism of $\widetilde{A}(\mathbb{R})$ associated with the skew-symmetric matrix $\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $\sigma(P) = P$ and $s_{2r-1} \in \mathfrak{S}_{2r-1}(\mathbb{R}; N\mathbb{Z})$ the fundamental cardinal spline interpolant of degree $2r-1$ $(r \geq 1)$. Define the function

$$h_0 = (\varphi_N \circ \sigma^{-1} \circ W)(s_{2r-1}).$$

Then we have $h_0 \in H_{N,0}(P) \cap \mathcal{C}^{2r-2}(P\backslash\widetilde{A})$. The Weil factorization formula $W^{-1} \circ \sigma^{-1} \circ W = \mathcal{F}_{\mathbb{R}}$ for the Fourier-Plancherel transform $\mathcal{F}_{\mathbb{R}}$ ([8], [7]) combined with the Schoenberg interpolation formula ([9], [10]) yields the following result (cf. [7]):

Theorem 3. The attenuation factors associated with h_o admit the form

$$\tau_n = \frac{1}{G_{2r-1}(\frac{n}{N})} \qquad (n \in \mathbb{Z})$$

where G_k denotes the meromorphic function $z \rightsquigarrow \sum_{m \in \mathbb{Z}} (\frac{z}{z+m})^{k+1}$, $k \geq 0$.

The case of several attenuation factors can be treated in the same vein. See the forthcoming second part of the paper [7].

4. The Whittaker-Shannon Series

According to the theorem of Stone-von Neumann-Segal there exists, up to unitary equivalence, a unique unitary representation U_o of the reduced Heisenberg group $A(\mathbb{R})$ with the identity on its center as central character. This unitary representation (which is called Schrödinger representation of $A(\mathbb{R})$) is square integrable. It can be proved that the Stone-von Neumann-Segal theorem and the classical Plancherel theorem are equivalent (Howe [3]). Therefore the sampling theorem which covers the classical Whittaker-Shannon cardinal series can be established via nilpotent harmonic analysis. This will be the topic of a forthcoming paper.

References

1. Auslander, L.: Lecture notes on nil-theta functions. CBMS Regional Conference Series in Mathematics, No. 34, Providence, R.I.: American Mathematical Society 1977

2. Auslander, L., Tolimieri, R.: Is computing with the finite Fourier transform pure or applied mathematics? Bull. Amer. Math. Soc. (New Series) 1, 847-897 (1979)

3. Howe, R.: On the role of the Heisenberg group in harmonic analysis, Bull. Amer. Math. Soc. (New Series) 3, 821-843 (1980)

4. Schempp, W.: Contour integral representation of cardinal spline functions. C.R. Math. Rep. Acad. Sci. Canada 2, 165-170 (1980)

5. Schempp, W.: Approximation und Transformationsmethoden III. In: Approximation and functional analysis. P.L. Butzer, E. Görlich, B. Sz.-Nagy editors. ISNM, Vol. 60. Basel-Boston-Stuttgart: Birkhäuser-Verlag 1981

6. Schempp, W.: Complex contour integral representation of cardinal spline functions (to appear)

7. Schempp, W.: Attenuation factors and nilpotent harmonic analysis I (to appear)

8. Schempp, W., Dreseler, B.: Einführung in die harmonische Analyse. Stuttgart: Teubner 1980

9. Schoenberg, I.J.: Cardinal interpolation and spline functions. J. Approx. 2, 167-206 (1969)

10. Schoenberg, I.J.: Cardinal interpolation and spline functions: II. Interpolation of data of power growth. J. Approx. Theory 6, 404-420 (1972)

Lehrstuhl für Mathematik I,
Universität Siegen,
Hölderlinstraße 3,
D-5900 Siegen 21, W. Germany

Received January 21, 1981

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

COMPLEXES EQUILIBRES DU PLAN PROJECTIF

Robert BANTEGNIE

Presented by H.S.M. Coxeter F.R.S.C.

0. Je les détermine tous. En particulier, on obtient à une isomorphie près un seul complexe fini équilibré non strictement sur les sommets dérivé du rhombicosidodécaèdre parabitourné et deux complexes duaux équilibrés non strictement sur les côtés dérivés du 1° pseudoicosidodécaèdre bichapeauté et de son dual.

1. Surfaces compactes connexes. Soit Σ une telle surface de caractéristique χ, $\mathcal{C}$ un complexe fini de Σ, $\mathcal{S}$, $\mathcal{E}$, $\mathcal{F}$ l'ensemble des sommets, des côtés, des faces de $\mathcal{C}$. On suppose ≥ 3 la valence $v(\sigma)$ d'un élément σ de $\mathcal{S}$ ou de $\mathcal{F}$. Deux côtés de $\mathcal{C}$ sont adjacents et forment un angle s'ils appartiennent à une même face et ont un sommet commun. $\mathcal{A}$ est l'ensemble des angles de $\mathcal{C}$. Le cycle, le pseudocycle, la contribution d'un élément de $\mathcal{S}$, $\mathcal{E}$, $\mathcal{F}$ est défini comme pour les complexes polyédraux plans ([1], [2], [3]). Par exemple, pour $S \in \mathcal{S}$ commun aux faces $F_1, \ldots, F_r$ numérotées en tournant autour de S de valences $f_1, \ldots, f_r$, $(f_1, \ldots, f_r)$ modulo toutes les permutations de $[1, r]$ est le pseudocycle de S, modulo les permutations circulaires et le renversement son cycle, sa contribution étant $c(S) = 1 - (r/2) + \Sigma_1^r\, 1/f_i$; si $E \in \mathcal{E}$ d'extrémités S_1, S_2 de valences s_1, s_2 est commun à F_1, F_2 de valences f_1, f_2 le symbole $(s_1, s_2\, ; f_1, f_2)$ modulo les permutations sur les 4 lettres est le pseudocycle de E, modulo les permutations sur les 2 premières et les 2 dernières son cycle, sa contribution étant $c(E) = (1/s_1) + (1/s_2) + (1/f_1) + (1/f_2) - 1$; $c(S) \leq 1 - r/6$, $c(E) \leq 1/3$. Introduisons cycle, pseudocycle et contribution pour un angle. Pour $\alpha \in \mathcal{A}$, S_α et F_α sont le sommet et la face de α, $\big(v(F_\alpha), v(S_\alpha)\big)$ son cycle, $\big(v(F_\alpha), v(S_\alpha)\big)$ ou $\big(v(S_\alpha), v(F_\alpha)\big)$ son pseudocycle, $c(\alpha) = (1/v(F_\alpha)) + (1/v(S_\alpha)) - 1/2$ sa contribution. Pour $S \in \mathcal{S}$, $c(S) = \sum_{S_\alpha = S} c(\alpha)$; pour $F \in \mathcal{F}$, $c(F) = \sum_{F_\alpha = F} c(\alpha)$. "Euler" se traduit par $\sum_{\alpha \in \mathcal{A}} c(\alpha) = \sum_{S \in \mathcal{S}} c(S) = \sum_{F \in \mathcal{F}} c(F) = \sum_{E \in \mathcal{E}} c(E) = \chi$ (1) (Lebesgue [5] et Ore [7] ont considéré $\chi = 2$).

Soit sur Σ, $\mathcal{C}^*$ un complexe dual de $\mathcal{C}$, $\mathcal{S}^*$, $\mathcal{E}^*$, $\mathcal{F}^*$, $\mathcal{A}^*$ l'ensemble des sommets, des côtés, des faces, des angles de $\mathcal{C}^*$. Le dual de $\alpha \in \mathcal{A}$ est $\alpha^* \in \mathcal{A}^*$ tel que $S_{\alpha^*} \in \mathring{F}_\alpha$ et $S_\alpha \in \mathring{F}_{\alpha^*}$. $\mathcal{C}$ est dual de $\mathcal{C}^*$ et α de α^*, $v(S_\alpha) = v\big(F_{\alpha^*}\big)$, $v\big(S_{\alpha^*}\big) = v(F_\alpha)$, $c(\alpha^*) = c(\alpha)$. Si $E \in \mathcal{E}$ est comme ci-dessus et S_1^*, S_2^* les sommets de $\mathcal{C}^*$ éléments de F_1, F_2, le dual de E est $E^* \in \mathcal{E}^*$ d'extrêmités S_1^* et S_2^* ; E est dual de E^*, $c(E) = c(E^*)$. Pour $S \in \mathcal{S}$ le dual S^* est la face de $\mathcal{C}^*$ avec $S \in \overset{\circ}{S^*}$; si $F \in \mathcal{F}$ le dual F^* est le sommet de $\mathcal{C}^*$ avec $F^* \in \overset{\circ}{F}$; S est dual de S^*, F de F^*, $c(S) = c(S^*)$, $c(F) = c(F^*)$.

$\mathcal{C}$ est équilibré (resp. strictement équilibré) sur les angles, les sommets, les côtés, les faces quand le pseudocycle (resp. cycle) d'un angle, ... est toujours le même. La contribution c d'un angle, ... est alors toujours la même. On a $c|X| = \chi$ (2) où X désigne $\mathcal{A}$, $\mathcal{S}$, $\mathcal{E}$ ou $\mathcal{F}$. $\mathcal{C}$ est équilibré (resp. strictement) sur les sommets, resp. les faces, resp. les côtés ssi $\mathcal{C}^*$ est équilibré (resp. strictement) sur les faces, resp. les sommets, resp. les côtés. De plus si $\mathcal{C}$ est équilibré sur les angles, il l'est strictement ; cela a lieu ssi $\mathcal{C}^*$ est strictement équilibré sur les angles et on a $|\mathcal{S}|\, v(S) = |\mathcal{F}|\, v(F) = |\mathcal{A}| = 2\,|\mathcal{E}|$. $\mathcal{C}$ est alors strictement équilibré sur les sommets, sur les faces et sur les côtés. Les équations $c > 0$ (3) et $c = 0$ ont un nombre fini de pseudocycles solutions, $c < 0$ un nombre infini. Pour $\chi > 0$, (2) a un nombre fini de complexes solutions, pour $\chi = 0$ un nombre infini.

On suppose $\chi > 0$. Comme $\chi = 1$, 2 et que, sauf pour les angles, j'ai examiné ([2], [3]) $\chi = 2$ reste $\chi = 1$. Notons que sur la sphère S_2 de centre O ($\chi = 2$) les strictement équilibrés sur les angles sont les polygones réguliers sphériques.

D'autre part, si $\bar{\Sigma}$ est un recouvrement de Σ, $\bar{\mathcal{C}}$ le relèvement de $\mathcal{C}$ sur $\bar{\Sigma}$ $\mathcal{C}$ est équilibré (resp. strictement) sur les angles, ..., ssi $\bar{\mathcal{C}}$ l'est.

2. <u>Plan projectif réel P_2</u> (on le représente par les points d'un disque de $\mathbb{R}^2$ dont on a identifié les points de la frontière diamétralement opposés). On raisonne souvent à isomorphie près. S_2 est recouvrement universel de P_2 ; si un complexe $\mathcal{C}'$ de P_2 est équilibré (resp. strictement) sur les angles, ... son relèvement $\mathcal{C}$ sur S_2 est équilibré sur les angles, ... <u>et</u> admet O comme centre de symétrie ; de plus un tel complexe de S_2 donne un complexe de P_2 ayant les mêmes propriétés d'équilibrage par identification de 2 points diamétralement opposés.

On dit d'un complexe $\mathcal{C}$ de S_2 qu'il est <u>sym</u>. s'il admet une réalisation à centre de symétrie (nécessairement O).

* <u>Cas des angles</u>. Sur S_2 les $\mathcal{C}$ possibles sont $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{3, 5\}$, $\{5, 3\}$, $\{3, 3\}$ n'est pas sym. et on obtient $\{4, 3\}'$ et son dual $\{3, 4\}'$ le cube

et l'octaèdre projectifs, $\{5, 3\}'$ et son dual $\{3, 5\}'$ le dodécaèdre et l'icosaèdre projectifs. Ils forment la collection $\mathcal{R}'$.

* Cas des sommets (et des faces). Pour les sommets $C' \in \mathcal{R}'$ est solution ; si $C' \notin \mathcal{R}'$ employant les notations de [4] et utilisant [2] on voit α) si C est strictement équilibré, c'est un des 13 polyèdres archimédiens, le polyèdre (M) de Miller, un n-prisme ($n \geq 3$) ou un n-antiprisme ($n \geq 4$) β) si C est équilibré non strictement l'orthobicupola triangulaire ou pentagonal ou l'un des 4 rhombicosidodécaèdres tourné, parabitourné, métabitourné, tritourné. Or pour α) des 13 archimédiens seul 10 sont sym. :

$\{{3 \atop 4}\}$, $\{{3 \atop 5}\}$, $t\{3, 4\}$, $t\{4, 3\}$, $t\{3, 5\}$, $t\{5, 3\}$, $r\{{3 \atop 4}\}$, $r\{{3 \atop 5}\}$, $t\{{3 \atop 4}\}$, $t\{{3 \atop 5}\}$, (M) ne l'est pas, le n-prisme est sym. ssi n pair, le n-antiprisme ssi n impair ; pour β) les 2 orthobicupola ne sont pas sym. et des rhombicosidodécaèdres seul le parabitourné est sym. Il existe donc un seul complexe équilibré non strictement sur les sommets le rhombicosidodécaèdre tourné projectif. On le trouve en 1b ; on a représenté en 1a le rhombicosidodécaèdre projectif strictement équilibré. Pour ces 2 complexes $|\mathcal{S}| = 30$, $|\mathcal{C}| = 60$, $|\mathcal{F}| = 31$.

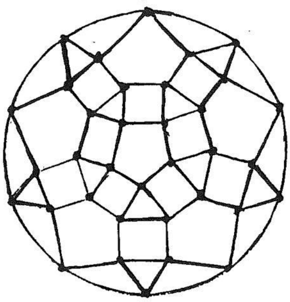
On obtient les équilibrés sur les faces par dualité. On a représenté en 2b le seul équilibré non strictement l'hexacontaèdre quadrangulaire tourné projectif et en 2a le strictement équilibré qu'est l'hexacontaèdre quadrangulaire projectif. Pour ces 2 complexes on a $|\mathcal{S}| = 31$, $|\mathcal{C}| = 60$, $|\mathcal{F}| = 30$.

* Cas des côtés. $C' \in \mathcal{R}'$ est solution. Pour $C' \notin \mathcal{R}'$ α) si C est strictement équilibré c'est $\{{3 \atop 4}\}$, $\{{3 \atop 5}\}$ ou leurs duaux β) si C n'est pas strictement équilibré c'est à la dualité près l'un des 4 polygones décrits p. 187 de [3] et notés là (α), (β), (γ), (δ) ou l'un des 4 pseudoicosidodécaèdres tournés. Des pseudoicosidodécaèdres seul le 1° 2-chapeauté est sym. (α), (β), (γ), (δ) ne le sont pas. On obtient donc comme seuls équilibrés non strictement sur les côtés le pseudoicosidodécaèdre chapeauté et son dual le pseudoicosidodécaèdre rhombohédral creusé projectifs. On les trouve en $3b_1$ et $3b_2$. En $3a_1$ et $3a_2$ sont les strictement équilibrés que sont l'icosidodécaèdre et son dual le tricontaèdre rhombohédral projectifs. •

On a :

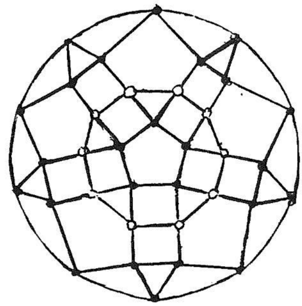
	$\|\mathcal{S}\|$	$\|\mathcal{C}\|$	$\|\mathcal{F}\|$
Icosidodécaèdre projectif	15	30	16
Pseudoicosidodécaèdre chapeauté projectif	16	30	15
Tricontaèdre rhombohédral projectif	16	30	15
Pseudotricontaèdre rhombohédral creusé projectif	15	30	16

1a



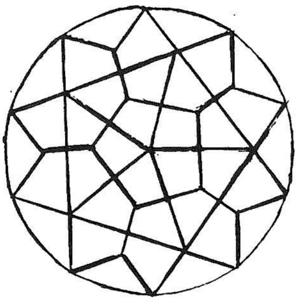
Rhombicosidodécaèdre

1b



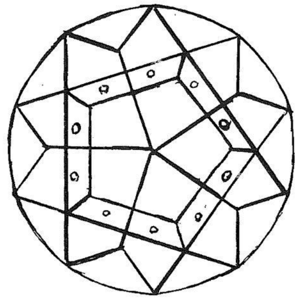
Rhombicosidodécaèdre tourné

2a



Hexacontaèdre quadrangulaire

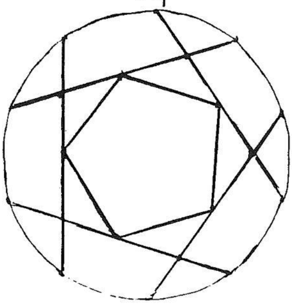
2b



Hexacontaèdre quadrangulaire tourné

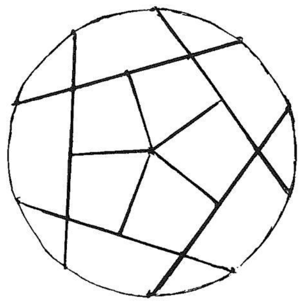
Notons que les résultats de [2], [3] ont été annoncés (parmi d'autres) semble-t-il par T. S. Motzkin dans [6]. On y trouve le seul polyèdre convexe faiblement équilibré sur les côtés (c'est-à-dire tel que la contribution de chaque côté est la même) non équilibré. Cet autodual est formé de deux 6-pyramides, l'une tronquée, l'autre non ayant une face commune. Il a 24 côtés de contribution 1/12, 6 de chacun des cycles (6, 4; 3, 3), (3, 3; 6, 4), (4, 4; 4, 3), (4, 3; 4, 4), 13 sommets et 13 faces. Comme il n'est pas sym. il n'existe pas (dans le plan projectif) de faiblement équilibré non équilibré.

$3a_1$

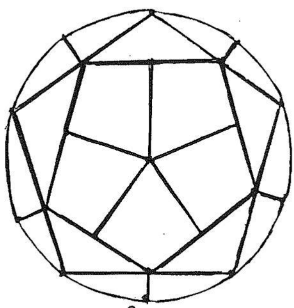


Icosidodécaèdre et son dual

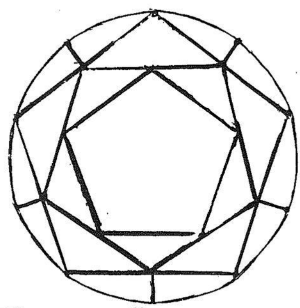
$3b_1$



Pseudoicosidodécaèdre chapeauté et son dual



$3a_2$



$3b_2$

REFERENCES

[1] R. BANTEGNIE, Variations sur des thèmes de B. Grünbaum et G. C. Shephard, C. R. Math. Rep. Acad. Sci. Canada **1** (1979), p. 197-198.

[2] R. BANTEGNIE, Polyèdres équilibrés, Can. Math. Bull. (à paraitre).

[3] R. BANTEGNIE, Complexes polyédraux équilibrés sur les côtés, C. R. Math. Rep. Acad. Sci. Canada **2** (1980), p. 187-192.

[4] N. W. JOHNSON, Convex polyhedra with regular faces, Can. J. of Math. 18 (1966), p. 169-200.

[5] H. LEBESGUE, Quelques conséquences simples de la formule d'Euler, J. de Math. pur. et appl. 19 (1940), p. 27-43.

[6] T. S. MOTZKIN, Homogeneity properties of convex polyhedra, Proc. of a colloquy on convexity, Copenhaghen 1965, p. 205-211.

[7] O. ORE, The four color problem, Academic Press, New York and London, 1967.

Université de Franche-Comté, Mathématiques,
E. R. A. n° 070654
Route de Gray - 25030 BESANCON CEDEX - F -

Received January 29, 1981

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

PRIME IDEALS IN POLYNOMIAL RINGS

A. Bouvier, M. Contessa and P. Ribenboim, F.R.S.C.

We study relations between Spec(A) and Spec(A[X]), where A is a commutative ring (not necessarily noetherian). The proofs of these results will appear elsewhere.

1. If D is a domain, K its field of quotients, $f \in K[X]$, the D-<u>content</u> of f is the D-fractional ideal $C_D(f)$ generated by the coefficients of f.

<u>Proposition 1</u>. Let P be a prime ideal of A, $\bar{A} = A/P$, $\bar{K}$ the field of quotients of $\bar{A}$, $f \in A[X] \setminus P[X]$, $\bar{f} = f \bmod P[X]$, $\deg(f) \geq 1$.

1) If (P,f) is a prime ideal then $\bar{f}$ is irreducible over $\bar{A}$.

2) If $\bar{f}$ is irreducible over $\bar{A}$ and $(C_{\bar{A}}(\bar{f})^r : C_{\bar{A}}(\bar{f})^{r+1}) \subseteq \bar{A}$ for every $r \geq 1$, then (P,f) is a prime ideal.

3) The number of prime ideals Q of A[X] such that $Q \cap A = P$ is at least $\#(\bar{A}) \times \aleph_0$.

The <u>pseudo-radical</u> of $P \in \mathrm{Spec}(A)$, is the ideal $I = \mathrm{PsRad}(P)$ equal to the intersection of all prime ideals P' of A properly containing P.

<u>Proposition 2</u>. Let $P \in \mathrm{Spec}(A)$, $I = \mathrm{PsRad}(P)$, $\bar{A} = A/P$, $\tilde{A} = A/I$, $f \in A[X]$, $\bar{f} = f \bmod P[X]$, $\deg(\bar{f}) \geq 1$, $\tilde{f} = f \bmod I[X]$.

1) If (P,f) is a prime ideal of $A[X]$ and $\tilde{f}$ is a unit in $\tilde{A}[X]$ then (P,f) is a maximal ideal.

2) The number of maximal ideals M of $A[X]$ such that $M \cap A = P$ is at least $\#(I/P)$.

3) If P is not maximal and $P \neq I$ then $\#(I/P) \geq \aleph_0$.

A prime ideal P of A is a <u>Goldman ideal</u> (G-ideal) if there exists a maximal ideal M of $A[X]$ such that $M \cap A[X] = P$. Let $\mathrm{Gold}(A)$ denote the set of Goldman ideals of A .

We are able to give simple proofs of various known results, among which the following one, due to Gilmer [2]:

<u>Proposition 3</u>. If $P \in \mathrm{Spec}(A)$ the following statements are equivalent:

a) there exists $a \in A \setminus P$ such that P is a maximal ideal of A not containing a .

b) $P \subset \mathrm{PsRad}(P)$

c) P is a G-ideal.

As a consequence:

<u>Proposition 4</u>. $\mathrm{Gold}(A) = \mathrm{Spec}(A)$ if and only if every prime ideal P of A is strictly contained in its pseudo-radical.

In particular, if $\mathrm{Spec}(A)$ is totally ordered by inclusion we obtain results of Picavet [4] , Fontana & Maroscia [1] and Ramaswamy & Viswanathan [5] .

Proposition 5. In any ring A we have the implications (1) → (2) → (3) between the following statements:

1) every descending chain of radical ideals is finite.

2) $Gold(A) = Spec(A)$.

3) every descending chain of prime ideals is finite.

If $P \in Spec(A)$ let $U(P)$ be the family of prime ideals P' of A such that $P' \supseteq PsRad(P)$.

The following is a generalization of a result of Picavet [4]:

Proposition 6. Let $A_1, \ldots, A_n$ be pairwise incomparable valuation domains with field of quotients K, let $A = A_1 \cap \ldots \cap A_n$. The following statements are equivalent:

a) for every $i = 1, \ldots, n$, $Gold(A_i) = Spec(A_i)$.

b) if $P \in Spec(A)$, P not maximal, then $U(P)$ is inductive by decreasing inclusion.

c) $Gold(A) = Spec(A)$.

Corollary 1. With the same notations, the following statements are equivalent:

1) every ascending chain of $Spec(A)$ is well-ordered.

2) for every $i = 1, \ldots, n$, every ascending chain of $Spec(A_i)$ is well-ordered.

2. $P\in \mathrm{Spec}(A)$ is a g.c.d. prime if $\bar{A} = A/P$ is a g.c.d. domain. With the notations of propositions 1 and 2:

Proposition 7. Let P be a g.c.d. prime of A.

1) If $f\in A[X]\setminus P[X]$, $\deg(\bar{f})\geq 1$, then (P,f) is a prime ideal (resp. a maximal ideal) if and only if $\bar{f}$ is irreducible over $\bar{A}$ (resp., and $\tilde{f} = f \bmod I[X]$ is a unit of $\tilde{A} = A/I$).

2) If Q is a prime ideal (resp. a maximal ideal) of $A[X]$ and $Q \cap A = P$ then $Q = P[X]$ or $Q = (P,f)$ where $f\in A[X]\setminus P[X]$, $\deg(f)\geq 1$, $\bar{f}$ is irreducible over $\bar{A}$ (resp., and $\tilde{f}$ is a unit of $\tilde{A}$).

3) the number of prime ideals (resp. maximal ideals) Q of $A[X]$ such that $Q \cap A = P$ is equal to $\#(\bar{A}) \times \aleph_0$ (resp. $\#(I/P) \times \aleph_0$).

Let P be a prime ideal of a g.c.d. domain A. P is g.c.d.-closed when: if $a,b \in P$ then each $\gcd(a,b)$ belongs to P. (see Sheldon [6]).

Proposition 8. Let $P \subseteq P_1 \subset P'$ be prime ideals of A, $\bar{A} = A/P$. Assume that P is a g.c.d. prime and that $\bar{P}' = P'/P$ is g.c.d.-closed. If $Q_1 \in \mathrm{Spec}(A[X])$, $Q_1 \cap A = P_1$ and $Q_1 \subset P'[X]$ then $Q_1 = P_1[X]$.

Corollary 1. If $P \subset P'$ are prime ideals of A, if $\bar{A} = A/P$ is a Bézout domain, if $Q \in \mathrm{Spec}(A[X])$ $Q \cap A = P$, and $Q \subset P'[X]$ then $Q = P[X]$.

A is a S-<u>domain</u> whenever if $P \in \mathrm{Spec}(A)$ has height 1 then $P[X]$ has also height 1 .

A is a <u>strong</u> S-<u>ring</u> when A/P is a S-domain for every $P \in \mathrm{Spec}(A)$.

<u>Corollary 2</u>. 1) If A is a g.c.d. domain then A is a S-domain.

2) If A is a Bézout ring then A is a strong S-ring.

Actually, J. L. Mott has communicated in a letter that, in fact, if A is a Prüfer domain then $A[X_1, \ldots, X_n]$ is a strong S-domain. The proof of our weaker result is much simpler.

A is a <u>residually Bézout ring</u> when A/P is a Bézout domain, for every $P \in \mathrm{Spec}(A)$.

<u>Corollary 3</u>. Let A be a residually Bézout ring, let $P \in \mathrm{Spec}(A)$.

1) If $Q \in \mathrm{Spec}(A[X])$, then $Q \subseteq P[X]$ if and only if $Q = P'[X]$ when $P' \in \mathrm{Spec}(A)$, $P' \subseteq P$,

2) If P has height m then $P[X]$ has height m .

References

[1] M. Fontana and P. Maroscia, Sur les anneaux de Goldman. Boll. Unione Mat. Italiana, (5), 13-B, 1976, 743-759.

[2] R. W. Gilmer, The pseudo-radical of a commutative ring. Pac. J. M., 19, 1966, 275-284.

[3] G. Picavet, Autour des idéaux premiers de Goldman d'un anneau commutatif. Ann. Univ. Clermont-Ferrand, 39, 1975, 73-90.

[4] G. Picavet, Sur les anneaux commutatifs dont tout idéal premier est de Goldman. C. R. Acad. Sci. Paris, 280, 1975, A, 1719-1721.

[5] R. Ramaswamy and T. M. Viswanathan, Overring properties of G-domains. Proc. A.M.S., 58, 1976, 59-66.

[6] P. S. Sheldon, Prime ideals in g.c.d.-domains. Can. J. Math., 26, 1974, 98-107.

A. Bouvier: Département de Mathématiques, Université Claude Bernard, 69100 Villeurbanne, France.

M. Contessa: Istituto di Matematica "Guido Castelnuovo" Università di Roma, Roma, Italia.

P. Ribenboim: Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6 Canada.

Received Jan. 30, 1981.

ON CHAINS OF PRIME IDEALS IN POLYNOMIAL RINGS

A. Bouvier, M. Contessa and P. Ribenboim, F.R.S.C.

We indicate some results about chains in Spec(A[X]), where A is a commutative ring (not necessarily noetherian). We also give some results on the noetherian type of A[X] .

1. A prime ideal P of a ring A is a g.c.d. prime if A/P is a g.c.d. domain. P is a g.c.d. closed prime ideal whenever a, b ∈ P then gcd(a,b) ∈ P (see Sheldon [4]).

Proposition 1. Let $P_1 \subset P_2 \subset P_3$ be g.c.d. prime ideals of A such that $\bar{P}_3 = P_3/P_1$ is a g.c.d. closed prime ideal of $\bar{A} = A/P_1$. Let $Q_1 \subset Q_3$ be prime ideals of A[X] such that $Q_1 \cap A = P_1$, $Q_3 \cap A = P_3$. Then there exists a prime ideal Q_2 of A[X] such that $Q_1 \subset Q_2 \subset Q_3$ and $Q_2 \cap A = P_2$.

In particular, the conclusion holds if one assumes only that $\bar{A}$ (or A) is a Bézout domain.

Let P ⊂ P' be prime ideals of A . A is catenary (of length s) between P and P' when there exists a maximal chain of prime ideals between P and P' of length s , and any two such chains have the same length.

A is catenary when for any prime ideals P ⊂ P' it is catenary between P and P' .

Proposition 2. Let A be catenary of length s between the prime ideals $P \subset P'$. Assume that P is a g.c.d. prime and that $\bar{P}' = P'/P$ is a g.c.d.-closed prime of $\bar{A} = A/P$. Let $Q \subset Q'$ be prime ideals of $A[X]$ such that $Q \cap A = P$, $Q' \cap A = P'$. Then:

1) if $Q' = P'[X]$ or $Q \neq P[X]$ then $A[X]$ is catenary of length s between Q and Q'.

2) if $Q' \neq P'[X]$ and $Q = P[X]$ then $A[X]$ is catenary of length $s+1$ between Q and Q'.

Corollary 1. If A is a Bézout domain and every chain of prime ideals is finite then $A[X]$ is catenary.

See also Bouvier, Burq and Germain [2]. Mott communicated in a letter that if A is a Prüfer domain then $A[X_1, \ldots, X_n]$ is catenary.

2. Let A be a ring satisfying the condition

(WOC) Every ascending chain $\mathcal{C}$ of prime ideals is well-ordered. The length $\ell(\mathcal{C})$ is the ordinal number of the well-ordered set $\mathcal{C}$. If $P \in \text{Spec}(A)$ there exists the supremum of $\ell(\mathcal{C})$ for all chains $\mathcal{C}$ (with last member equal $\mathcal{C}$); it is called the height of P denoted by $\text{ht}(P)$. The height of A, denoted by $\text{ht}(A)$, is the supremum of $\text{ht}(P)$, for all $P \in \text{Spec}(A)$.

Proposition 3. Let A be a ring satisfying the condition (WOC). Then:

1) $A[X]$ satisfies (WOC).

2) If $P \in \mathrm{Spec}(A)$ then $ht(P) \leq ht(P[X])$.

If moreover A is a strong S-ring (see [1]) then:

3) If $P \in \mathrm{Spec}(A)$ then $ht(P) = ht(P[X])$; if $Q \in \mathrm{Spec}(A[X])$, $Q \cap A = P$, $Q \neq P[X]$ then $ht(Q) = ht(P) + 1$.

4) $ht(A[X]) = ht(A) + 1$.

5) if $\alpha \leq ht(A) + 1$, α not a limit ordinal, and if $ht(A)$ is not a limit ordinal (when $\alpha = ht(A) + 1$) then there exist infinitely many prime ideals Q of $A[X]$ such that $ht(Q) = \alpha$.

6) if $\alpha \leq ht(A) + 1$, α limit ordinal, if there exists $Q \in \mathrm{Spec}(A[X])$ such that $ht(Q) = \alpha$, then $Q = (Q \cap A)[X]$.

Corollary 1. Let A be a strong S-ring, satisfying condition (WOC). Assume that any set of pairwise incomparable prime ideals of A is finite. Then if $\alpha \leq ht(A)$, α not a limit ordinal, there exist infinitely many maximal ideals Q of $A[X]$ such that $ht(Q) = \alpha$.

The proposition and the corollary are applicable to the ring $A = A_1 \cap \ldots \cap A_n$, where $n \geq 1$ and each A_i is a valuation domain with field of quotients K , and $\mathrm{Spec}(A_i)$ is well-ordered (by increasing inclusion).

References

[1] A. Bouvier, M. Contessa, and P. Ribenboim, Prime ideals in polynomial rings, C. R. Math. Rep. Acad. Sci. Canada, this issue.

[2] A. Bouvier, F. Burq and G. Germain, Un exemple d'anneau caténaire, Publ. Depart. Math., Lyon, vol. 17, (1980), 1-5.

[3] P. Ribenboim, Meta-noetherian rings, Ann. Univ. Ferrara, 23 (1976), 143-164.

[4] P. S. Sheldon, Prime ideals in g.c.d. domains, Can. J. Math., 26 (1974), 98-107.

A. Bouvier: Département de Mathématiques, Université Claude Bernard, 69100 Villeurbanne, France.

M. Contessa: Istituto di Matematica "Guido Castelnuovo" Università di Roma, Roma, Italia.

P. Ribenboim: Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6 Canada.

Received Jan. 30, 1981.

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

THE NOTION OF TORSION AND SECOND FUNDAMENTAL TENSOR REVISITED

E. Binz

Presented by K.B. Ranger F.R.S.C.

Abstract: For a fixed C^∞-immersion i of a simply connected C^∞manifold M into $\mathbb{R}^n$ and any $f \in C^\infty(M, GL(n))$ the pointwise formed composition $f \cdot di$, an $\mathbb{R}^n$-valued one-form, is considered. Associated with it are natural connections $\nabla(i,f)$ and (a Levi-Civita connection) $\bar{\nabla}(i,f)$ on M. We answer the following questions: When is $f \cdot di = dj$ for some immersion $j : M \longrightarrow \mathbb{R}^n$? When is $\nabla(i,f) = \bar{\nabla}(i,f)$? When is a connection ∇ on M of the form $\nabla(i,f)$? In this context the notions of torsion and second fundamental tensor play a central role.

1. Levi-Civita connections associated with immersions:

Let M be a simply connected C^∞-manifold of dimension m, TM its tangent bundle, $i : M \longrightarrow \mathbb{R}^n$ a fixed C^∞ immersion and $\langle , \rangle$ a fixed scalar product on $\mathbb{R}^n$. Call the main part of the tangent map Ti of i by di.

Denote by $L(M \times \mathbb{R}^n, TM)$ the vector bundle on M of which the fibre over $p \in M$ consists of $L(\mathbb{R}^n, T_pM)$, the vector space of all linear maps from $\mathbb{R}^n$ into T_pM. Let, furthermore $P(i) : M \longrightarrow L(M \times \mathbb{R}^n, TM)$ be the C^∞-section for which $di \cdot P(i)(p) : \mathbb{R}^n \longrightarrow diT_pM$ is the orthogonal projection for each $p \in M$. The dot means the pointwise formed composition. On the Grassmanian $G(m,n)$ we have the canonical m-plane bundle ξ of m planes in $\mathbb{R}^n$. The tangential representation $\bar{i}$

of M in $G(m,n)$ assigns to any $p \in M$ the m-plane diT_pM. The map $\bar{i}$ is of class C^∞. Call η the bundle over $G(m,n)$ of $n-m$ planes in $\mathbb{R}^n$ for which $G(m,n)\times\mathbb{R}^n$ splits orthogonally into $\xi \oplus \eta$. Given j in the homotopy class $[i]$ of i, the bundles $TM \oplus \bar{i}^*\eta$ and $TM \oplus \bar{j}^*\eta$ are diffeomorphic. Thus there exists a C^∞-map $f : M \longrightarrow GL(n)$, which satisfies

$$1) \qquad f \cdot di = dj \quad .$$

Moreover, f maps the normal bundle $\nu(i)$ of i into the normal bundle $\nu(j)$ of j. Consult [H] for the study of homotopy classes of immersions. C^∞-vector fields on M are denoted by X,Y,Z etc. The Levi-Civita connection $\nabla(j)$ of the metric $j^*\langle,\rangle$ on M has the form

$$2) \qquad \nabla(j)_X Y = \nabla(i)_X Y + P(i)\cdot f^{-1}\cdot df(X)\cdot diY$$

for any two X,Y. Here $\nabla(i)$ is the Levi-Civita connection of $i^*\langle,\rangle$. By $P(i)\cdot f^{-1}\cdot df(X)\cdot diY(p)$ we mean $P(i)(p)\circ f^{-1}(p)\circ df(p)(X(p))(diY(p))$ for any $p \in M$. As the torsion of $\nabla(j)$ we obtain

$$3) \qquad P(i)\cdot f^{-1}\cdot(df(X)\cdot diY - df(Y)\cdot diX)$$

which is zero.

2. Forms with coefficients in $\mathrm{End}\mathbb{R}^n$:

Denote by $\mathrm{End}\mathbb{R}^n$ the vector space of all endomorphisms of $\mathbb{R}^n$. Let $f : M \longrightarrow \mathrm{End}\mathbb{R}^n$ be a C^∞-map. Associated with it we consider the $\mathbb{R}^n$-valued form $f\cdot di$, mapping any tangent vector w_p of M to $f(p)(diw_p)$. Using a basis in $\mathbb{R}^n$ and the Poincaré Lemma we find a C^∞-map $j : M \longrightarrow \mathbb{R}^n$ for which

$f \cdot di = dj$ iff for any two X, Y

$$4) \qquad \delta(f \cdot di)(X,Y) = 0 \quad .$$

Here $\delta(f \cdot di)(X,Y) := d(f \cdot diY)(X) - d(f \cdot diY)(Y) - f \cdot di[X,Y]$. Thus

$$5) \qquad \delta(f \cdot di)(X,Y) = df(X) \cdot diY - df(Y) \cdot diX \quad .$$

If $f(M) \subset GL(n)$, then j is an immersion. Define a connection $\nabla(i,f)$ by setting

$$6) \qquad \nabla(i,f)_X Y = \nabla(i)_X Y + P(i) \cdot f^{-1} \cdot df(X) \cdot diY \quad .$$

As its torsion we obtain:

$$7) \quad T(i,f)(X,Y) = P(i) \cdot f^{-1} \cdot (df(X) \cdot diY - df(Y) \cdot diX) \quad .$$

Given X and Y we furthermore introduce

$$S(i,f)(X,Y)(p) := (d(diY)(X)(p) + f^{-1} \cdot df(X) \cdot diY(p))^{\perp_i}$$

for any $p \in M$. Here $\perp_i$ means the component in $\nu(i)$. Since $d(diY)(X)^{\perp_i}$ is symmetric in X and Y , we observe, that $S(i,f)$ is symmetric iff

$$8) \qquad (f^{-1} \cdot (df(X) \cdot diY - df(Y) \cdot diX))^{\perp_i} = 0 \quad .$$

In summarizing we state:

<u>Proposition 1:</u> For any C^∞-map $f : M \longrightarrow GL(n)$ and a fixed immersion $i : M \longrightarrow \mathbb{R}^n$ the following are equivalent:

(i) $\quad f \cdot di = dj \quad$ for some immersion $j : M \longrightarrow \mathbb{R}^n$

(ii) $\quad \delta(f \cdot di) = 0$

(iii) $\quad T(i,f) = 0$ and $S(i,f)$ is symmetric.

3. <u>Metric properties of $\nabla(i,f)$:</u>

Let $f \in C^\infty(M, GL(n))$. Denote by $P(i,f) : M \longrightarrow L(M \times \mathbb{R}^n, TM)$ the C^∞-section for which $f \cdot di \cdot P(i,f)(p) : \mathbb{R}^n \longrightarrow f \cdot di(T_pM) \subset \mathbb{R}^n$

is the orthogonal projection (with respect to $<,>$) for each $p \in M$. Obviously $P(i,\iota) = P(i)$, where ι assignes to each $p \in M$ the value $id \in GL(n)$. One easily verifies:

$$9)\qquad P(i,f) = P(i)\cdot f^{-1}\cdot {}^tf^{-1}\cdot di\cdot P(i)\cdot {}^tf$$

and $P(i,f) = P(i)\cdot f^{-1}$ in case $f(\nu(i)) \subset \ker P(i,f)$. Here ${}^tf$ denotes the pointwise formed adjoint of f with respect to $<,>$. We obtain a new connection on M by setting:

$$10)\qquad \bar{\nabla}(i,f)_X Y = P(i,f)\cdot d(f\cdot diY)(X)$$

for any two X,Y. This connection decomposes into

$$11)\quad \bar{\nabla}(i,f)_X Y = \nabla(i)_X Y + P(i,f)\cdot df(X)\cdot diY + P(i,f)\cdot f\cdot (d(diY)(X))^{\perp_i}.$$

The torsion $\bar{T}(i,f)$ of $\bar{\nabla}(i,f)$ is given by

$$12)\qquad \bar{T}(i,f)(X,Y) = P(i,f)\cdot\delta(f\cdot di)(X,Y) \quad .$$

Denote by $m_i(f)$ the Riemannian metric on M assigning to any X,Y the C^∞-function $< f\cdot diX , f\cdot diY >$. The second fundamental tensor $\bar{S}(i,f)(X,Y)$ is given by $(\iota - f\cdot di\cdot P(i,f))(d(f\cdot diY)(X))$. This tensor corresponds to Θ in [G]. We immediately obtain:

<u>Proposition 2:</u> $\bar{\nabla}(i,f)$ is the Levi-Civita connection of $m_i(f)$ iff $\bar{T}(i,f) = 0$. Moreover, $\bar{\nabla}(i,f)$ and $\nabla(i,f)$ agree if $f(\nu(i)) \subset \ker P(i,f)$. In case $f(\nu(i)) \subset \ker P(i,f)$ and $\delta(f\cdot di) = 0$ then $\bar{\nabla}(i,f) = \nabla(i,f) = \nabla(j)$ for some immersion $j : M \longrightarrow \mathbb{R}^n$. Conversely, given an immersion $j \in [i]$, then $dj = f\cdot di$ for some $f \in C^\infty(M,GL(n))$ with $f(\nu(i)) \subset \ker P(j)$ and $\nabla(i,f) = \nabla(j)$.

4. Which connections on M are of the form $\nabla(i,f)$?

Assume first that $\nabla(i,f)$ satisfies (6) . Define $\delta(f^{-1}\cdot df)$ in analogy to $\delta(f\cdot di)$. Then

$$13) \qquad \delta(f^{-1}\cdot df)(X,Y) = -[f^{-1}\cdot df(X)\ ,\ f^{-1}\cdot df(Y)]$$

for any two X,Y . Here the brackets denote the commutator. Given next any C^{∞}-connection ∇ on M and a fixed immersion $i: M \longrightarrow \mathbb{R}^n$, there is a $L(TM,TM)$ - valued one-form F of class C^{∞} such that

$$\nabla_X Y = \nabla(i)_X Y + F(X)\cdot Y$$

for any two X,Y on M . Since $f^{-1}\cdot df$ is $\mathrm{End}\mathbb{R}^n$- valued, we let $\bar{F}$ be an $\mathrm{End}\mathbb{R}^n$-valued one-form such that

$$\bar{F}(X)\cdot diY = di\cdot F(X)\cdot Y\ .$$

Clearly $P(i)\cdot\bar{F}(X)\cdot diY = F(X)\cdot Y$. The equation

$$\bar{F} = f^{-1}\cdot df$$

yields the total differential equation $df = f\cdot\bar{F}$, of which the integrability condition in Frobenius's theorem [D] reads as:

$$14) \qquad \delta\bar{F}(X,Y) = -[\bar{F}(X),\bar{F}(Y)]$$

for any X,Y . Replace $[\bar{F}(X),\bar{F}(Y)]$ by $\bar{F}\wedge\bar{F}\ (X,Y)$. The equation (14) relates immeditately to $\tilde{R} = 0$ in [G] . Thus we have:

<u>Proposition 3:</u> Given a connection ∇ . If $di\cdot(\nabla-\nabla(i))$ allows an extension $\bar{F}$ to an $\mathrm{End}\mathbb{R}^n$ - valued one-form on M , then

$$\nabla = \nabla(i) + P(i)\cdot f^{-1}\cdot df$$

holds iff $\delta\bar{F} = -\bar{F}\wedge\bar{F}$.

In conclusion let us remark that the curvature $R(i,f)$ of $\nabla(i,f)$ depends only on the first derivative of f .

References:

[D] Dieudonné "Foundations of Modern Analysis", Vol.1, Academic Press, New York, London, 1960.

[G] Greub W. "Gauss Godazzi Tensor Fields and the Bonnet Immersion Theorem", Collectanea Mathematica, Vol.XX VIII, Fasc. 2, (1977) pp.148 - 162.

[H] Hirsch M.W. "Immersions of Manifolds", Trans.AMS 93, (1959) pp. 243 - 276.

Department of Mathematics
University of Mannheim
D 68 Mannheim/Germany

Received February 2, 1981

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

AN APPLICATION OF HOOLEY'S METHOD FOR COUNTING SOLUTIONS OF A DIOPHANTINE EQUATION

J.H.H. Chalk F.R.C.S.

1. Introduction. Consider the diophantine equation

$$p(x_1^2 + x_2^2) - q(x_3^2 + x_4^2) = a, \tag{1}$$

where $p > 0$, $q > 0$ and $a \neq 0$ are elements of $\mathbb{Z}$ and suppose that it has at least one solution with $x_i \in \mathbb{Z}$ $(1 \le i \le 4)$. Our object is to produce an upper bound $h = h(p,q,a)$ for which (1) has such a solution with $p(x_1^2 + x_2^2) \le h^2$. For the special case $p = a = 1$, I [1] showed that one could take $h^2(1,q,1) = q^{2+\varepsilon}$, for any $\varepsilon > 0$ and all q sufficiently large. A similar estimate for the general case may be obtained by the circle method of Hardy-Littlewood (cf. [2]), but the resulting bound is poor. Some years ago, C. Hooley showed me how his method could be applied to (1) and indicated the expected result, (it was surprisingly good). So far as I know, this programme has not been carried out in detail and, on looking at it again recently, I noticed some technical obstacles. In particular, I was unable to apply the theory of congruences in restricted resi- systems as developed by Vinogradov, Mordell and others (cf. [3], for references). However, an effective alternative is the following version* of Theorem 9 in the article of R.A. Smith [4] on the circle problem in arithmetic progressions.

"*If* $(b,m) = (c,m) = (c,b) = 1$, $m \ll X^{2/3}$ *and* $r(n)$ *denotes the number of representations of* n *as a sum of two squares of integers,*

---

* The Vinogradov symbol "<<" is used in place of the equivalent big "O"-symbol; an attached suffix indicating dependence upon it in the implied constant.

then, for any $\delta > 0$,

(2) $$\sum_{\substack{n\le X,\ (n,b)=1 \\ cn\equiv b \pmod m}} r(n) - \frac{\pi}{4}\frac{X}{m} M(b,m) \ll_\delta (X^{2/3} m^{-1/2})^{1+\delta} |b|^\delta ,$$

where

(3) $$M(b,m) = \frac{\phi(b)}{b} H_m(1) H_b(1) ,$$

and

(4) $$H_\ell(n) = \sum_{d|\ell} \chi(d) c_d(n) d^{-1} ;$$

$\chi(n)$ *being the non-principal character*, mod 4, $c_d(n)$ *the Ramanujan function and* $\phi(n)$ *the Euler totient function."*

In (1), we may suppose, without loss of generality, that $(p,q) = 1$ and that aq is odd, since otherwise we can divide out any common factors in p and q and a power of 2 can be removed from both sides of the equation by an obvious change in the variables. It will be convenient, though not essential, to limit the size of a. These assumptions are listed as follows:

(5) <u>Hypotheses</u> $(p,q) = 1,\ p > 0,\ q > 0,\ q$ odd, $(2pq,a) = 1$ and $0 \ne |a| = o(h^2)$ as $h \to \infty$.

Our starting point is a formula for the number $4S$ of solutions of (1) with x_1, x_2 both even, $x_3^2 + x_4^2$ prime to a and with $x_1^2 + x_2^2$ bounded by $h^2/4p$:

(6) $$S = \sum_{\substack{4p(x_1^2+x_2^2)-q\mu=a,\ (\mu,a)=1 \\ 4p(x_1^2+x_2^2)\le h^2}} \tfrac{1}{4} r(\mu) = \sum_{\substack{4p(x_1^2+x_2^2)-q\rho\sigma=a,\ (\rho\sigma,a)=1 , \\ 4p(x_1^2+x_2^2)\le h^2}} \chi(\rho)$$

where $r(n) = 4 \sum_{d|n} \chi(d)$. We shall, in fact, prove, for any $\varepsilon > 0$, that $S > 0$, if

(7) $$h^2/p \gg_\varepsilon (pq)^{3+\varepsilon} |a|^\varepsilon$$

and pq is sufficiently large; thereby establishing the existence of a solution of (1) with $x_1^2 + x_2^2 \ll_\varepsilon (pq)^{3+\varepsilon}|a|^\varepsilon$. As we shall see, this is a direct consequence of the asymptotic formula for S stated in our main theorem*.

THEOREM. *For any* $\delta > 0$ *and* $h^2/p \gg (pq)^3$ *as* $pq \to \infty$,

(8) $$S - \frac{\pi}{8}\frac{h^2}{pq} W(p,q,a) \ll_\delta [h^{11/6} p^{-2/3} q^{-3/4}]^{1+\delta}|a|^\delta ,$$

where

(9) $$W(p,q,a) = \frac{\phi(a)}{a} H_a(1) H_q(1) L_{pq|a|}(1) \zeta_{2pq|a|}(2)^{-1}$$

and

(10) $$L_A(1) = \sum_{\substack{1 \le n < \infty \\ (n,A)=1}} \chi(n) n^{-1} , \quad \zeta_A(2) = \sum_{\substack{1 \le n < \infty \\ (n,2A)=1}} n^{-2} ,$$

$$H_A(1) = \sum_{d|A} \chi(d)\mu(d) d^{-1} .$$

Since

(11) $$H_A(1) = \prod_{p|A} \left(1 - \chi(p)p^{-1}\right)$$

(12) $$L_A(1) = \prod_{p\nmid A} \left(1 - \chi(p)p^{-1}\right) \quad \text{and} \quad \zeta_A(2)^{-1} = \prod_{p\nmid 2A} (1 - p^{-2})$$

it is clear that $W(p,q,a) > 0$. Moreover, by elementary estimates for $\phi(a)$, $H_A(1)$, $L_A(1)$, $\zeta_A(2)$, we have

(13) $$W(p,q,a) \gg_\delta (pq|a|)^{-5\delta}, \quad \text{say.}$$

Hence, if δ is sufficiently small compared with $\varepsilon > 0$, the main term $\pi/8\, h^2/pq\, W(p,q,a)$ dominates the error term on the right of (8) if h^2 is chosen large enough to satisfy (7).

---

* to be published later.

Remarks. H. Iwaniec has informed me that the estimate (7) can also be obtained by sieve methods. Moreover, with the aid of an additional hypothesis,* as yet unproved, he can obtain using sieve methods the improved bound

(14) $$(pq)^{2+\varepsilon} + (q|a|)^{\frac{2}{3}+\varepsilon}$$

in place of (7).

On a recent visit to Toronto, E. Bombieri pointed out to me that if $r_Q(n)$ denotes the number of integer solutions (x_1, x_2, x_3, x_4) of

$$Q(x) = \sum_{1\le i\le 4} a_i x_i^2 = n, \quad \left(n \prod_{1\le i\le 4} a_i \ne 0\right)$$

satisfying $|a_i x_i^2| \le N$ $(1 \le i \le 4)$ then the "expected" estimate

(15) $$r_Q(n) \ll_\varepsilon N^{1+\varepsilon} \left|\prod_{1\le i\le 4} a_i\right|^{-\frac{1}{2}}$$

under the additional hypothesis $|n| \le N$, $\left|\prod_{1\le i\le 4} a_i\right| \ll N$, has not been established. I note that some such result should be amenable by Hooley's method using a more general form of Smith's theorem in which the function $r(n)$ in (2) is replaced by $r_f(n)$, the number of representations of n by the form $f = Ax^2 + By^2$ $(A > 0, B > 0)$. As we require only an upper bound in (15), an important technical simplification is achieved by replacing $r_f(n)$ in turn by the number $\psi_D(n)$ of representations of n by a *set* of forms f_i $(1 \le i \le h(D))$ of determinant $D = AB$, representing the $h(D)$ classes of inequivalent primitive forms.

---

* cf., C. Hooley, J. reine angew. Math., 303/4(1978), 21-50 (Hypothesis R^* with $\ell_1 = 0$ on p. 44).

References

[1] J.H.H. Chalk, "An Estimate for the fundamental solutions of a Generalized Pell Equation", Math. Annalen 132 (1956), 263-276.

[2] T. Estermann, "A new application of the Hardy-Littlewood-Kloosterman method", Proc. Lond. Math. Soc. 12 (1962), 425-444 (for explicit constants, see K.S. Williams, Ph.D. Thesis, Toronto, 1965).

[3] L.J. Mordell, "On the number of solutions in incomplete residue sets of quadratic congruences", Archiv. der Math. 8 (1957), 153-157, (for the general case, see J.H.H. Chalk, Canad. J. of Math. 15 (1963), 291-296 and Indag. Math. 4, 42 (1980), 367-374).

[4] R.A. Smith, "The Circle Problem in an Arithmetic Progression", Canad. Math. Bull. 11, no. 2 (1968), 175-184.

University of Toronto,
Toronto, Canada,
M5S 1A1.

Received February 17, 1981

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

BOCHNER'S THEOREM AND THE EXISTENCE OF INVARIANT SET FUNCTIONS[1]

by

J.-M. Belley

Presented by M. Shinbrot F.R.S.C.

Abstract. We provide a result which constitutes a solution to the general moment problem for finitely additive set functions and then show its usefulness in establishing the existence of finitely additive set functions which are invariant under certain transformations.

1. In this paper, X will denote a compact Hausdorff space. Given an algebra A of subsets of a set S dense in X, we shall write C(S,A) for the class of all complex-valued functions on S which can be uniformly approximated by A-measurable step functions, and write I_E for the indicator function of the set $E \subset X$. Let C(X) be the space, with supremum norm, consisting of all continuous complex-valued functions on X. A finitely additive set function $\lambda: A \to \mathbb{C}$ will be said to be regular on A if, given $\epsilon > 0$ and $E \in A$, there exist constants $a,b > 0$ and real-valued functions $f,g \in C(X)$ such that the sets $K=(x \in S: f(x) \leq a)$ and $V=(x \in S: g(x) < b)$ have the properties i) $K,V \in A$, ii) $K \subset E \subset V$, iii) $|\lambda|(V \setminus K) < \epsilon$ where $|\lambda|$ is the total variation of λ. A linear functional $L: C(X) \to \mathbb{C}$ with norm $\|L\|$ is said to be nonnegative if $L(f) \geq 0$ for all nonnegative $f \in C(X)$. The following useful analogue of the

[1]This research was supported by grants from the Natural Sciences and Engineering Research Council of Canada and the ministry of education of Quebec.

Riesz representation theorem in the context of finitely additive set functions can be deduced from Corollary 2.9 and Remark 2.8 in [1,p. 271].

<u>Lemma</u>. Let S be a dense subset of a compact Hausdorff space X and let $L:C(X) \to \mathbb{C}$ be a bounded linear functional. There exists an algebra A of Borel sets in S such that $f|_S \in C(S,A)$ for all $f \in C(X)$, and there exists a regular bounded finitely additive set function $\lambda : A \to \mathbb{C}$ such that $\int_S f|_S \, d\lambda$ exists in the sense of Moore-Smith convergence (see [5, pp. 401-404]) and is equal to L(f) for all $f \in C(X)$. Moreover, $\|L\| = |\lambda|(S)$ and λ is nonnegative whenever L is.

<u>Example 1</u>. When $S=(1,2,3,\ldots)$ or $(0,\infty)$, the lemma can be used to show the existence of an algebra of (Borel) subsets of S on which is defined a regular nonnegative finitely additive set function λ for which $\lambda(S) = 1$ and the integral $\int_S z^s d\lambda(s)$ exists in the sense of Moore-Smith convergence and is equal to 0 for all z in the closed unit disk in $\mathbb{C}$, except for z=1 where it is equal to 1.

2. Given a semigroup G with commutative operation written additively, we consider a family $\pi=(\pi_z : z \in G)$ of continuous functions $\pi_z : X \to \mathbb{C}$ $(z \in G)$ for which $\pi_{u+v}(x) = \pi_u(x)\pi_v(x)$ for all $x \in X$ and all $u,v \in G$. We write $F_\pi(X)$ for the space consisting of all finite linear combinations of elements in π, and write $\overline{F_\pi(X)}$ for its supremum norm closure in C(X). Now, given $f \in F_\pi(X)$, we shall write $\Sigma_{z\in G}\hat{f}(z)\pi_z(x)$ for f, where the complex coefficient $\hat{f}(z)$ vanishes for all but at most finitely many $z \in G$. We say that the function $p:G \to \mathbb{C}$ is <u>positive definite with respect to π</u> if $\Sigma_{z\in G}\hat{f}(z)p(z) \geq 0$ for all nonnegative $f \in F_\pi(X)$. Let M(S) denote the family of tuples (λ, A) where A

is an algebra of Borel sets in S such that $f|_S \in C(S,A)$ for each $f \in C(X)$, and λ is a bounded regular finitely additive complex-valued set function on A. The above lemma permits us to obtain the following solution to the general moment problem for finitely additive set functions (see [3, pp. 310, 311] and the footnote on p. 379 in [2]).

<u>Theorem 1</u>. Suppose that i) G is a commutative semigroup, ii) S is a dense subset of a compact Hausdorff space X, and iii) $\pi=(\pi_z : z \in G)$ is a family of functions $\pi_z : X \to \mathbb{C}$ which are linearly independent in C(X) and for which $\pi_{u+v}(x) = \pi_u(x)\pi_v(x)$ for all $x \in X$ and all $u,v \in G$. Then the function $p:G \to \mathbb{C}$ is positive definite with respect to π if and only if there is a tuple $(\lambda,A) \in M(S)$ for which λ is nonnegative and

$$p(z) = \int_S \pi_z(s)\, d\lambda(s) \qquad (z \in G),$$

where the integral exists in the sense of Moore-Smith convergence.

<u>Example 2</u>. Let G be the multiplicative semigroup consisting of the closed unit disk in $\mathbb{C}$ and let π be that family of complex-valued functions π_z defined on the positive integers by $\pi_z(k) = z^k$ $(z \in G)$. By means of theorem 1 and example 1, it can be shown that the discontinuous function $p:G \to \mathbb{C}$ given by

$$p(z) = \begin{cases} 1 \text{ if } z=1 \\ 0 \text{ otherwise} \end{cases}$$

is positive definite with respect to π.

3. The following is a consequence of theorem 1.

Theorem 2. Given an arbitrary locally compact abelian group S, there exists a tuple $(\lambda_S, A_S) \in M(S)$ with the following properties.

1) $s+E \in A_S$ and $\lambda_S(E) = \lambda_S(s+E)$ for all $E \in A_S$ and all $s \in S$ (i.e. (λ_S, A_S) is translation invariant).

2) λ_S is nonnegative and $\lambda_S(S) = 1$.

3) Given any two locally compact abelian groups G and H, and any (not necessarily continuous) surjective homomorphism $T: G \to H$, then $T^{-1}E \in A_G$ and $\lambda_G(T^{-1}E) = \lambda_H(E)$ for all $E \in A_H$.

Example 3. When $G = \mathbb{R}^n$ $(n > 0)$, theorem 2 says that A_G is closed and λ_G is invariant under invertible affine transformations. So we have a result analogous to that of Mycielcki in [4, p. 317] for a larger class of operators (i.e. operators that include the similarities of G) and a smaller algebra of sets. Furthermore,

$$\int_{\mathbb{R}^n} f \, d\lambda = \lim_{|B| \to \infty} \frac{1}{|B|} \int_B f$$

for all $f \in C(G, A_G)$, where the integral on the left exists in the sense of Moore-Smith convergence, while those on the right are taken in the sense of Lebesgue over balls $B \subset \mathbb{R}^n$ with volume $|B|$.

References

[1] J.-M. Belley, A representation theorem and applications to topological groups, Trans. Amer. Math. Soc. 260 (1980), 267-279.

[2] N. Dunford and J.T. Schwartz, Linear operators, Part I, Interscience, New York, 1958.

[3] E. Hewitt, Linear functionals on almost periodic functions, Trans. Amer. Math. Soc. 74 (1953), 303-322.

[4] J. Mycielski, Finitely additive invariant measures (I), Coll. Math. 42 (1979), 309-318.

[5] A. Taylor, Introduction to functional analysis, Wiley, New York, 1958.

Received 18 February, 1981.

Département de Mathématiques,
Université de Sherbrooke, Quebec, Canada.

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

ON A SYSTEM OF FUNCTIONAL EQUATIONS

I. Fenyö and L. Paganoni

Presented by J. Aczél F.R.S.C.

Abstract. We prove by an explicit example that the following system of functional equations for $d : R \times R \to R$

$$d(x+y,y) = d(x,y); \quad d(x,y) = d(y,x) \quad (x,y \in R)$$

has solutions which are not identically constant. If continuity is assumed, then the system above has no solutions other than constant functions.

1. We start by considering the following functional equation

$$g(x+2y) - 2g(x+y) + g(x) = 0 \quad (x,y \in R) .$$

This is equivalent to

$$(1) \quad g(x+2y) - g(x+y) - g(y) = g(x+y) - g(x) - g(y) .$$

If we define

$$d(x,y) = g(x+y) - g(x) - g(y) \quad (x,y \in R)$$

and observe that d is symmetric with respect to its variables, then $d : R \times R \to R$ is a solution of the following system of functional equations

$$(2) \quad d(x+y,y) = d(x,y); \quad d(x,y) = d(y,x) \quad (x,y \in R)$$

We know (see [1]) that the most general solution of (1) has the form $g(x) = c + f(x)$ $(x \in R)$, where c is an arbitrary constant and f an arbitrary additive function. By this fact $d(x,y) = -c$, i.e. d is a constant function.

On account of this situation the following question arises in a natural way. Is there a non-constant function d which satisfies (2)?

Certainly, if a non-constant solution d of (2) exists, then it does not yield a solution g of (1). This means, in view of the definition of d, that such a non-constant solution of (2) is not in the range of the Cauchy functional operator.

2. A simple example shows that <u>there exist solutions of (2) which are not identically constant functions</u>. Such a solution can be found for example as follows: Let H be a Hamel basis of the reals over the rationals Q and $H_o \subset H$ an arbitrary proper subset of H. Furthermore, let $S_o = V(H_o, Q, +, \cdot)$ be the subspace of reals generated by H_o. We define the function $h : R \to R$ by

$$h(x) = \begin{cases} 1 & \text{if } x \in S_o \\ 0 & \text{if } x \notin S_o \end{cases} .$$

Then the function $d(x,y) = h(x)h(y)$ $(x,y \in R)$ is obviously non-constant and fulfils conditions (2).

3. If we suppose continuity, the situation will be quite different.

<u>Theorem</u>. Under the assumption of continuity, the system (2) has only constant solutions.

<u>Proof</u>. We put into (2) $x = 0$ and write x for y

$$d(x,0) = d(0,x) = d(x,x) \qquad (x \in R) . \tag{3}$$

Equation (2) implies

$$d(nx,x) = d((n-x)x+x,x) = d((n-1)x,x)$$

and so we get by induction

(4) $$d(nx,x) = d(x,nx) = d(x,x) \quad (n=1,2,3,\ldots) \; .$$

Substituting x/n for x, we can write

(5) $$d(x,x/n) = d(x/n,x) = d(x/n,x/n) \; .$$

If we take the limit as $n \to \infty$ we get by continuity

$$d(x,0) = d(0,x) = d(0,0) = K$$

(constant) and on account of (4) and (3)

(6) $$d(nx,x) = d(x,nx) = d(x,x) = d(x,0) = K$$

for every real x. After putting x/n in place of x, we get also

$$d(x,x/n) = d(x/n,x) = K \; .$$

This last relation means that d is equal to the constant K along the lines $y = x/n$ $(n=1,2,3,\ldots)$.

Let us now consider the straight line $y = rx$ with $r \in Q \cap [0,1]$. We will show that $d(x,rx) = K$ for every $x \in R$. Let $r = m/n$ with $0 < m \leq n$ and $t = x/n$. We have $d(x,rx) = d(nt,mt)$. If $n \equiv n_1 \pmod m$ then by (2)

$$d(nt,mt) = d(n_1t+kmt,mt) = d(n_1t+(k-1)mt,mt) = \ldots =$$
$$= d(n_1t,mt) \quad \text{with} \quad 0 \leq n_1 < m \; .$$

If $n_1 = 0$, the statement is proved. For the case $n_1 > 0$ let us consider $m = n_2 \pmod{n_1}$ and we get

$$d(n_1t,mt) = d(mt,n_1t) = d(n_2t,n_1t), \quad 0 \leq n_2 < n_1 \; .$$

We proceed as before and after a finite number of steps we conclude

$$d(x,rx) = d(0,n_1t) = K \quad \text{for every} \quad x \in R .$$

This implies, again by continuity, that $d(x,ax) = K$ for all $x \in R$ and $0 \le a \le 1$. Considering the symmetry of d, $d(ax,x) = K$ also holds, which means that d is equal to the constant K along any straight line $y = bx$ for all non-negative b, from which follows that $d(x,y) = K$ for all x and y with $xy \ge 0$.

$$d(x,-y) = d(x-y,-y) = d(x-2y,-y) = \ldots = d(x-ky,-y) \qquad (k=0,1,2,\ldots)$$

and if k is large enough, $x - ky < 0$, and so for such k the relation $d(x-ky,-y) = K$ $(x\ge 0, y\ge 0)$ holds. Therefore $d(x,-y) = K$ is valid for all non-negative x and y. This completes the proof.

Reference

[1] I. Fenyö, Osservazioni su alcuni teoremi di D.H. Hyers. Istituto Matematico "F. Enriques" dell' Università di Milano. Quaderno 49/s (1980), 1-8.

Università di Milano
Istituto Matematico
Via C. Saldini
I-20133 Milano, Italy.

Received February 23, 1981

C. R. Math. Rep. Acad. Sci. Canada - Vol. III (1981) No. 2

THE INDIVIDUAL ERGODIC THEOREM FOR LAMPERTI CONTRACTIONS

James H. Olsen*

Presented by M. Akcoglu F.R.S.C.

1. Introduction. In what follows we assume p fixed, $1 < p < \infty$. Let T be a contraction of $L_p(X,F,\mu)$ $(\|T\|_p \le 1)$. If T takes functions with disjoint supports to functions of disjoint supports, we say that T is Lamperti. In this paper we prove that if $\{k_i\}_{i=1}^{\infty}$ is a uniform sequence (see Section 2 for definition) and T is a Lamperti contraction of L_p, then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f(x)$ exists and is finite almost everywhere for every $f \in L_p(X,F,\mu)$.

2. Uniform Sequences. We begin with describing the construction of a uniform sequence given in [2]. Let Ω be a compact metric space, B the collection of Borel subsets of Ω, and ϕ a homeomorphism of Ω such that $\{\phi^n\}$, n a positive integer, is an equicontinuous set of mappings. The system (Ω,ϕ) is then called uniformly L-stable. We assume that Ω possesses a dense orbit, and it then follows (see [2]) that there exists a ϕ invariant probability measure on (Ω,B) which we denote by ν, such that for any $w \in \Omega$, and any continuous function f on Ω,

$$\int f d\nu = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(\phi^k(w)) .$$

Such a system will be called strictly L-stable.

* Research supported in part by NSERC Grant No. A3974.

If $Y \in B$ and $y \in \Omega$, we define the i^{th} entry time $k_i(y,Y)$ of y into Y recursively as

$$k_1(y,Y) = \min\{i \geq 1 : \phi^i y \in Y\}$$
$$k_i(y,Y) = \min\{j > k_{i-1}(y,Y) : \phi^j y \in Y\}$$

allowing infinity as a value.

Definition: A sequence $\{k_i\}_{i=1}^{\infty}$ of natural numbers will be called uniform if there exist:

1) a strictly L-stable system (Ω,B,ν,ϕ)

2) a $Y \in B$ such that $\nu(Y) > 0 = \nu(\partial Y)$

and 3) a point $y \in \Omega$ such that $k_i = k_i(y,Y)$ for each $i \geq 1$.

The (Ω,B,ν,ϕ), Y and y in the above definition will be called the apparatus connected with the uniform sequence $\{k_i\}_{i=1}^{\infty}$.

3. Lamperti operators. Recall that a Lamperti operator is one that separates supports. We will use the following characterization of Lamperti operators (see [3]).

Theorem 3.1: $T: L_p(X,F,\mu) \to L_p(X,F,\mu)$, where (X,F,μ) is a Lebesgue space, is Lamperti if and only if there exists an L_∞ function g and a non-singular point transformation τ such that

$$Tf(x) = g(x)f(\tau x).$$

τ is non-singular if $\mu(\tau A) = 0$ if and only if $\mu(A) = 0$. The fact that (X,F,μ) is a Lebesgue space implies that the set transformation mentioned in [2] induces the point transformation τ.

Finally, we will need the ergodic theorem for Lamperti contractions [3].

Theorem 3.2: Let T be a Lamperti contraction of $L_p(X,F,\mu)$, p fixed, $1 < p < \infty$. Then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x)$ exists and is finite a.e. for all $f \in L_p(X,F,\mu)$.

4. Main Result. We now state and prove our main result. The theorem is stated in the case (X,F,μ) is a Lebesgue space to avoid technical difficulties, but is easily extended.

Theorem 4.1: Let T be a Lamperti contraction of $L_p(X,F,\mu)$, p fixed, $1 < p < \infty$, $\{k_i\}_{i=1}^{\infty}$ a uniform sequence, and (X,F,μ) a Lebesgue space. Then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f(x)$ exists and is finite a.e.

Proof: Let $X_1 \in F$, $\mu(X_1) < \infty$, (Ω,B,ν,ϕ), y and Y be the apparatus connected with the uniform sequence. Define the operator T' on $(X_1 \times \Omega, F \times B, \mu \times \nu = \nu')$ by $T'(f\cdot g) = Tf\cdot g\circ\phi$ whenever $f \in L_p(X,F,\mu)$ and $g \in L_p(\Omega,B,\nu)$. We note that T' is Lamperti since if $Tf(x) = g(x)f(\tau x)$, $T'(f(x,y)) = g(x)f(\tau x, \phi y)$. Put

$$S(x) = \limsup_{n} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_Y(\phi^k(y))$$

$$s(x) = \liminf_{n} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_Y \phi^k(y) .$$

We wish to show $S(x) = s(x)$ a.e., or equivalently

$$\int_{X_1} (S(X) - s(x))d\mu = 0 .$$

Let $\varepsilon > 0$. As in the proof of Theorem 1 of [2], choose open subsets Y_1, Y_2 and W of Ω such that

1) $Y_1 \subset Y \subset Y_2$

2) $\nu(Y_2 - Y_1) < \varepsilon$

3) $y \in W$

4) for $w \in W$ and $n \geq 0$,

$$1_{Y_1}(\phi^n w) \leq 1_Y(\phi^n y) \leq 1_{Y_2}(\phi^n w) .$$

Put

$$g_1(x,w) = f(x) 1_{Y_1}(w)$$

$$g_2(x,w) = f(x) 1_{Y_2}(w) .$$

Now from Theorem 3.2 we have

$$\bar{g}_i(x,w) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} T'^k g_i(x,w)$$

exists and is finite a.e.

We now note that

$$\liminf_n \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_Y \phi^k(y) - \liminf_n \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_{Y_1} \phi^k(w)$$

$$= \liminf_n \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) \left(1_Y \phi^k(y) - 1_{Y_1} \phi^k(w)\right)$$

so

$$|s(x) - \bar{g}_1| \leq \liminf_n \frac{1}{n} \sum_{k=0}^{n-1} |T^k f(x)| \left(1_Y \phi^k(y) - 1_{Y_1} \phi^k(w)\right)$$

$$= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} |T^k f| \left(1_Y \phi^k(y) - 1_{Y_1} \phi^k(w)\right)$$

for a.e. $w \in W$ and is finite a.e. $x \in X$ since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |T^k f| 1_Y \phi^k(y)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |T|^k |f| 1_Y \phi^k(y)$$

exists by the result of [4] and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |T^k f|(x) 1_{Y_1} \phi^k(w)$$

exists and is finite a.e. by applying Akcoglu's ergodic theorem [1].

Similarly, $|\bar{g}_2 - S(x)| \le \lim_n \frac{1}{n} \sum_{k=0} |T^k f| \left(1_{Y_2} \phi^k(w) - 1_Y \phi^k(y)\right)$.

Now we have

$$\int_{X_1} \left(S(x) - s(x)\right) d\mu = \frac{1}{\nu(W)} \int_{X_1 \times W} \left(S(x) - s(x)\right) d\nu$$

$$= \frac{1}{\nu(W)} \int_{X_1 \times W} |S(x) - s(x)| d\nu'$$

$$= \frac{1}{\nu(W)} \int_{X_1 \times W} |S(x) - \bar{g}_2 + \bar{g}_2 - \bar{g}_1 + \bar{g}_1 - s(x)| d\nu'$$

$$\le \frac{1}{\nu(W)} \int_{X_1 \times W} \left(|S(x) - \bar{g}_2| + |\bar{g}_2 - \bar{g}_1| + |g_1 - s(x)| d\nu\right)$$

$$\le \frac{1}{\nu(W)} \left(\int_{X_1 \times W} |\bar{g}_2 - \bar{g}_1| d\nu' + \int_{X_1 \times W} \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \left(1_{Y_2} \phi^k(w) - 1_Y \phi^k(y) \right.\right.$$

$$\left.\left. + 1_Y \phi^k(y) - 1_{Y_1} \phi^k(w)\right) d\nu' \right)$$

$$\le \frac{1}{\nu(W)} \left(\int_{X_1 \times W} |\bar{g}_2 - \bar{g}_1| d\nu' \right.$$

$$\left. + \int_{X_1 \times W} \lim_n \frac{1}{n} \sum_{k=0}^{n-1} |T^k f| \left[1_{Y_2} \left(\phi^k(w) - 1_{Y_1} \phi^k(w)\right)\right] d\nu \right)$$

$$\le \frac{1}{\nu(W)} \left(\int_{X_1 \times W} |\bar{g}_2 - \bar{g}_1| d\nu' + \|f\|_p \mu(x_1)^{1/q} \nu(Y_2 - Y_1) \nu(W) \right)$$

$$\le \frac{2}{\nu(W)} \|f\|_p \mu(x_1)^{1/q} \nu(Y_2 - Y_1) \nu(W)$$

by applying the argument of the Theorem of (4) to the operator $|T|$ and the function $|f|$.

Therefore, $\int_{X_1} S(x) - s(x)d\nu' < 2\varepsilon\|f\|\mu(x)^{1/q}$ and $S(x) = s(x)$ a.e. on X_1, and since X is σ-finite, we have

$$s(X) = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x) 1_Y(\phi^k(y))$$

exists and is finite a.e. However,

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) 1_Y(\phi^k(y)) = \lim_n \frac{1}{n}\sum_{i=1}^{n} \chi_{\{i:k_i \le n\}} T^{k_i} f(x)$$

and in [2], it is shown that $\lim_n \frac{1}{|\{i:k_i \le n\}|}$ exists. Hence

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^{k_i} f(x) = \lim_n \frac{1}{|\{i:K_i \le n\}|} \frac{1}{n}\sum_{k=0}^{n} \chi_{\{i:k_i \le n\}} T^{k_i} f(x)$$

exists and is finite a.e. This concludes the proof of the theorem.

References

1. Akcoglu, M.A., A pointwise ergodic theorem in L_p-spaces, Can. J. Math. XXVII, 1975, pp. 1075-1082.

2. Brunel, A.; Keane, M., Ergodic Theorems for Operator Sequences, Z. Wahrscheinlichkeitstheorie verw. Geb. 12 (1969), pp. 231-240.

3. Kan, C.H., Ergodic properties of Lamperti operators, Can. J. Math. 30 (1978), 1206-1214.

4. Olsen, J.H., Akcoglu's Ergodic Theorem for Uniform Sequences, Can. J. Math. 32 (1980), 880-884.

Department of Mathematical Sciences,
North Dakota State University,
Fargo, North Dakota 58105, U.S.A.
and
Department of Mathematics,
University of Toronto,
Toronto, Canada, M5S 1A1.

Received March 2, 1981

MAILING ADDRESSES

1. R. Bantagnie	Université de Franche-Compte, Mathématiques, E.R.A. No. 070654, Route de Gray-25030 Besançon Cedex, France
2. J.-M. Belley	Département de Mathématiques, Université de Sherbrooke, Quebec, Canada
3. E. Binz	Department of Mathematics, University of Mannheim, D 68 Mannheim, W. Germany
4. A. Bouvier	Département de Mathématiques, Université Claude Bernard, 69100 Villeurbanne, France
5. J.H.H. Chalk	Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1
6. M. Contessa	Instituto di Mathematica, "Guido Castelnuovo", Universita di Roma, Roma, Italia
7. I. Fenyö	Università di Milano, Instituto Matematico, Via C. Saldini, I-20133 Milano, Italy
8. J.A. Lester	Math. Seminar der Universität Hamburg, Bundesstr. 55, 2 Hamburg 13, W. Germany
9. J.H. Olsen	Department of Mathematical Sciences, North Dakota State University, Fargo, North Dakota 58105, U.S.A.
10. L. Paganoni	Università di Milano, Instituto Matematico, Via C. Saldini, I-20133 Milano, Italy
11. P. Ribenboim	Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada K7L 3N6
12. W. Schemp	Lehrstuhl für Mathematik I, Universität Siegen, Hölderlinstrasze 3, D-5900 Siegen 21, W. Germany
13. L. Székelyhidi	Department of Mathematics, Kossuth Lajos University, H-4010 Debrecen, Hungary