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REMARKS ON THE HOMOTOPY TYPE OF GROUPS OF GAUGE TRANSFORMATIONS

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Presented by P. Ribenboim F.R.S.C.

1. Generalities and Statements of Results - Let $(E,p,B;G)$ be a principal G-bundle where B and E are k-spaces [3]. We denote by $\mathcal{G}$ (or $\mathcal{G}(p)$) the group of gauge transformations of p, i.e., the group of all bundle automorphisms of p, and by $\mathcal{G}^1$ the subgroup of $\mathcal{G}$ consisting of all base-point preserving gauge transformations. We study the homotopy type (notation: $\simeq$) and homotopy groups of $\mathcal{G}$ and $\mathcal{G}^1$ using only properties of fibrations and their cross-sections. In what follows $M(X,Y)$ (resp. $M(X,Y;f)$) shall denote the space of all maps from X into Y (resp. the path component of $M(X,Y)$ containing a map $f : X \to Y$); if we deal with based spaces and based maps, the analogous function spaces will be denoted by $M_*(X,Y)$ and $M_*(X,Y;f)$, respectively. We use BG and $k : B \to BG$ to denote the classifying space of G and the classifying map for p, respectively.

Theorem 1 - $\mathcal{G} \simeq \Omega M(B,BG;k)$ and $\mathcal{G}^1 \simeq \Omega M_*(B,BG;k)$ (cf.[1,Prop.2.4]).

Theorem 2 - If BG is an H-group, $\mathcal{G} \simeq M(B,G)$ and $\mathcal{G}^1 \simeq M_*(B,G)$.

Theorem 3 - If B is an H-cogroup, $\mathcal{G}^1 \simeq M_*(B,G)$. Furthermore, if G is n-connected, $\pi_j(\mathcal{G}) \cong \pi_j(M_*(B,G))$, $j < n$. (This generalizes the second part of [7,Theorem 5].)

Theorem 4 - If G is $(n-1)$-connected and $\dim B = m < 2n$ then, for every $0 \leq j \leq 2n-m-1$, $\pi_j(\mathcal{G}) \cong \pi_j(M(B,G))$ and $\pi_j(\mathcal{G}^1) \cong \pi_j(M_*(B,G))$.

Theorem 5 (Bencivenga-Booth) - If $(E,p,B;G_1 \times \ldots \times G_n)$ is a principal $(G_1 \times \ldots \times G_n)$-bundle, there exist principal G_i-bundles $(X_i,q_i,B;G_i)$, $i = 1, \ldots, n$, such that $\mathcal{G}(p) \simeq \mathcal{G}(q_1) \times \ldots \times \mathcal{G}(q_n)$ and $\mathcal{G}^1(p) \simeq \mathcal{G}^1(q_1) \times \ldots \times$

$G^1(q_n)$. (Bundles with groups of this type are discussed in [2].)

2. Outline of Proofs - Let $(Y,q,A;G)$ and $(Z,r,B;G)$ be principal G-bundles and let $(Y*Z = \bigcup_{(a,b)\in A\times B} M_G(Y_a,Z_b)$, $q*r$, $A \times B)$ be the functional fibration of [4,page 108]; we recall that $M_G(Y_a,Z_b)$ is the set of all right G-homeomorphisms from the fibre Y_a of q over a into the fibre Z_b of r over b, and that $q*r$ takes a map $f : Y_a \to Z_b$ to the pair (a,b). Denote by $q*_i r$ the composite of $q*r$ with the projection on the i^{th}-factor, $i=1$, 2; in addition, for a principal G-bundle $(E,p,B;G)$ let (pp) be the fibration obtained by restriction of $p*p$ to the diagonal of $B \times B$. As a consequence of the exponential law [3,Theorem 1] we obtain:

Proposition 6 - There is a homeomorphism between G (resp. G^1) and the space $sec(pp)$ (resp. $sec_*(pp)$) of all cross-sections (resp. based cross-sections) to (pp).

Corollary 7 - There is a fibration $(G, \omega_p, G; G^1)$ where ω_p consists of the composite of the evaluation $G \to M_G(G,G)$ and the usual homeomorphism $M_G(G,G) \cong G$.

Let $(EG,p_G,BG;G)$ be Milnor's Universal G-bundle and $b \in B$ fixed; the fibre of $p*_1p_G$ over b is homeomorphic to $G*EG$; this space is, in turn homeomorphic to EG (see [4,page 180]). It follows that $p*_1p_G$ is a homotopy equivalence and therefore, so is the map $(p*_1p_G)^B : M(B,E*EG) \to M(B,B)$. We conclude that the fibre $sec\, p*_1p_G$ of $(p*_1p_G)^B$ is contractible. Consider the map $\phi : sec\, p*_1p_G \to M(B,BG;k)$ defined by $\phi(s) = p*_2p_G \cdot s$. Double pull-back arguments show that ϕ is induced from $(p*p_G)^B : M(B,E*EG) \to M(B, B \times BG)$ and hence, ϕ is a fibration with fibre G (there is a based version of that result with G^1 replacing G as fibre). We now apply the γ-operation of [5] to the fibrations ϕ and ϕ_* to show that $\partial : M(B,BG;k) \simeq G$ and ∂_*

: $M_*(B,BG;k) \simeq G^1$. We acknowledge the influence of D.H.Gottlieb's work on the ideas we just explained.

To prove Theorems 2 and 3 we first observe that if BG is an H-group (e.g. a loop space), both $M(B,BG)$ and $M_*(B,BG)$ are H-groups and if B is an H-cogroup (e.g. a suspension) then $M_*(B,BG)$ is also an H-group; furthermore, under those circumstances, the path-components of $M(B,BG)$ (or of $M_*(B,BG)$) all have the same homotopy type. Now, Theorem 1 and a double application of the exponential law establish Theorem 2 and the first part of Theorem 3. The last part of Theorem 3 is obtained using Corollary 7.

The proof of Theorem 4 uses the fact that the adjoint $s : BG \to \Omega SBG$ of the identity map on SBG is a $(2n+1)$-equivalence and if $\dim B = m < 2n$ the induced map $s_* : \Omega M(B,BG;k) \to \Omega M(B,SBG;k)$ is a $(2n-m)$-equivalence.

Finally, we make some observations about Theorem 5. Since any principal $(G_1 \times \ldots \times G_n)$-bundle is induced from the Universal $(G_1 \times \ldots \times G_n)$-bundle $p_1 \times \ldots \times p_n$, it follows that the former one is, to within a $(G_1 \times \ldots \times G_n)$-isomorphism, a pull-back of a principal G_1-bundle q_1 over B, a principal G_2-bundle q_2 over B, etc.. It is now easy to see that the spaces in question have the same homotopy type, for $G(p) \simeq M(B, \prod_{i=1}^{n} B_i ; k) \cong \prod_{i=1}^{n} (B,B_i;k_i) \simeq \prod_{i=1}^{n} G(q_i)$, where each k_i is the composition of the classifying map for p,i.e., $k : B \to \prod_{i=1}^{n} B_i$ with the corresponding projection onto B_i. We wish to note that the homotopy equivalence $G(p) \simeq G(q_1) \times \ldots \times G(q_n)$ is actually given by the rule $(f_1,\ldots,f_n) \to f_1 \times \ldots \times f_n$, where $f_i \in G(q_i)$, $i=1,\ldots,n$, is a homotopy equivalence (the same rule determines the homotopy in the based case).

3.<u>Examples</u> - We give some computations concerning G .

(i) Consider the case of a principal U-bundle (U = infinite unitary group) over a sphere S^r, $r \geq 1$. Using Theorem 2, Bott periodicity and the fact that $M(S^r,U) \simeq U \times M_*(S^r,U)$ [6,Theorem 2.2] we see that $\pi_i(G\ )$ is isomorphic to: a) 0, if r = even, i = even; b) $Z \oplus Z$, if r = even, i = odd; c) Z, if r = odd.

(ii) Let p be a principal $SU(2) \times SU(2) \times U(1)$-bundle over B (see [2]). It follows from Theorem 5 that $G(p) \simeq G(q_1) \times G(q_2) \times G(q_3)$, where q_1, q_2 are principal $SU(2)$-bundles and q_3 is a pricipal $U(1)$-bundle. It is a known result that if G is abelian and q is a principal G-bundle over B, $G(q) \simeq M(B,G)$; an easy proof of this result using our method involves noticing that (qq) is a principal G-bundle with a cross-section. Hence, in our example, $G(q_3) \simeq M(B,U(1))$. If B has the homotopy type of a 3-manifold, q_1 and q_2 are trivial and $G(p) \simeq M(B,SU(2) \times SU(2) \times U(1))$. If B is a 4-manifold, Theorem 4 shows that $\pi_i(\ G(p)) \cong \pi_i(M(B,SU(2) \times SU(2) \times U(1))$, $i = 0,1$.

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FUNCTIONS OF TWO VARIABLES AND MATRICES INVOLVING FACTORIZATIONS

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Abstract. We give a survey of results concerning characterization of functions and matrices that can be decomposed in the forms

$$h(x,t) = \sum_{k=1}^{n} f_k(x) g_k(t), \quad \text{and}$$

$$(a_{ij}) = \left(\sum_{k=1}^{n} b_k(i) c_k(j)\right) .$$

In the case when h is sufficiently often differentiable we get a characterization and construction of f_k and g_k from h in terms of partial and ordinary differential equations. Without regularity conditions on the function h and for matrices, we give a characterization and even explicit formulas for evaluating f_k, g_k, and $b_k(i)$, $c_k(j)$. These formulas are suitable for computer computations, because the values of $f_k(x)$, $g_k(t)$, as well as $b_k(i)$, $c_k(j)$ can be evaluated parallelly for different (x,t), and (i,j) by pointwise multiplications only. Moreover, if h is continuous or of the class C^d on $I \times J$, then the same kind of regularity holds for f_k on I and g_k on J for all $k=1,\ldots,n$. So we also have explicit formulas for solutions of the partial and ordinary differential equations mentioned above. On the other hand, the set of all functions from $\mathbb{L}_2$, admitting such factorization for any fixed n, does not provide a good approximation in $\mathbb{L}_2$.

1. We write

$$D_m(h) := \begin{pmatrix} h & h_t & h_{tt} & \dots & h_{t^m} \\ h_x & h_{xt} & h_{xtt} & \dots & h_{xt^m} \\ \dots \\ h_{x^m} & h_{x^m t} & h_{x^m tt} & \dots & h_{x^m t^m} \end{pmatrix}$$

for a function h of x and t with continuous $\frac{\partial^{i+j}h}{\partial^i x \partial^j t} = h_{x^i t^j}$ on $I \times J \subset R^2$, $i,j \le m$. (Here I and J are unions of intervals.)

Theorem 1. A function $h : I \times J \to R$, having continuous derivatives $h_{x^i t^j}$ for $i,j \le n$, can be written in the form

(1) $$h(x,t) = \sum_{k=1}^{n} f_k(x) g_k(t) \quad \text{on} \quad I \times J$$

with $f_k \in C^n(I)$, $g_k \in C^n(J)$ $(k=1,2,\dots,n)$ and

(1') $$\det (f_k^{(j)}(x)) \neq 0 \text{ for } x \in I \text{ and } \det (g_k^j(y)) \neq 0 \text{ for } y \in J$$

if and only if

(2) $$\det D_n(h) = 0 \quad \text{and} \quad \det D_{n-1}(h) \neq 0 \quad \text{on} \quad I \times J.$$

If the assumptions (2) are satisfied, then there exist $f_k \in C^n(I)$ and $g_k \in C^n(J)$ $(k=1,\dots,n)$ such that (1) and (1') are satisfied and all decompositions of h of the form

$$h(x,t) = \sum_{k=1}^{n} \bar{f}_k(x) \bar{g}_k(t)$$

are exactly those for which

$$(\bar{f}_1,\dots,\bar{f}_n) = (f_1,\dots,f_n) \cdot C^T, \quad \text{and}$$
$$(\bar{g}_1,\dots,\bar{g}_n) = (g_1,\dots,g_n) \cdot C^{-1} ,$$

where C is an arbitrary n by n regular constant matrix, C^T and C^{-1} being its transpose and inverse, respectively.

Remark 1. If $\det D_{n-1}(h) \neq 0$ for each $(x,t) \in I \times J$, then the functions f_k and g_k in (1) can be constructed from an h satisfying (2) as solutions of two ordinary linear differential equations with coefficients determined by h. For $\det D_{n-1}(h) \not\equiv 0$, see Theorem 3 and Remark 5 below.

2. Theorem 2. A matrix $H = (H_{ij})$, $i=1,\ldots,r$, $j=1,\ldots,s$, can be written in the form

$$(3) \qquad (H_{ij}) = \Big(\sum_{k=1}^{n} F_k(i)G_k(j)\Big)$$

with n independent vectors $F_1(i),\ldots,F_n(i)$, and n independent vectors $G_1(j),\ldots,G_n(j)$, if and only if

$$\operatorname{rank} h = n, \quad n \le \min(r,s) \ .$$

If this assumption is satisfied, then all decompositions of H in the form

$$(H_{ij}) = \Big(\sum_{k=1}^{n} \bar{F}_k(i)\bar{G}_k(j)\Big)$$

are exactly those where $F = F\cdot C$, $G = C^{-1}\cdot G$, C being a regular real n by n matrix

$$\bar{F} = (\bar{F}_{ik}) := (\bar{F}_k(i)), \quad F = (F_{ik}) := (F_k(i)),$$
$$\bar{G} = (\bar{G}_{kj}) := (\bar{G}_k(j)), \quad G = (G_{kj}) := (G_k(j)) \ .$$

Remark 2. If the linear independence of F_k and G_k is not supposed, then $\operatorname{rank} H \le n \le \min(r,s)$.

Remark 3. If the assumption of Theorem 2 is satisfied, then all decompositions (3) can be constructed from H in the following way. Reindex H so that

$$H = \begin{pmatrix} H^* & H^*\cdot M \\ P\cdot H^* & P\cdot H^*\cdot M \end{pmatrix},$$

where H^* is any of the regular n by n submatrices of H. Let C be an arbitrary regular n by n matrix. Then

$$F := \begin{pmatrix} H^*\cdot C \\ P\cdot H^*\cdot C \end{pmatrix} \qquad \text{and} \quad G := (C^{-1}, C^{-1}\cdot M) \ .$$

3. Theorem 3. Let I and J be arbitrary subsets. A function $h : I \times J \to R$ can be written in the form (1) with linearly independent f_k and g_k if and only if the maximum of the rank of the matrices $(h(x_i,t_j))$, $i=1,\dots,r$, $j=1,\dots,s$, is n when $x_i \in I$, $t_j \in J$, r and s being arbitrary integers. If, in addition, I and J are intervals, $h \in C^d(I\times J)$, $d \geq 0$, then $f_k \in C^d(I)$ and $g_k \in C^d(J)$ for all $k=1,\dots,n$.

Remark 4. If the assumption of Theorem 3 is satisfied, then all decompositions (1) can be constructed from h in the following manner. Let

$$H^* := \begin{pmatrix} h(x_1,t_1),\dots,h(x_1,t_n) \\ \dots \\ h(x_n,t_1),\dots,h(x_n,t_n) \end{pmatrix}$$

be any (but fixed) regular n by n matrix, and C be an arbitrary regular n by n matrix. Then, for all $x \in I$, $t \in J$,

$$(4) \quad \begin{cases} (f_1(x),\dots,f_n(x)) = (h(x,t_1),\dots,h(x,t_n))\cdot C, \quad \text{and} \\ \begin{pmatrix} g_1(t) \\ \dots \\ g_n(t) \end{pmatrix} = C^{-1}\cdot H^{*-1}\cdot \begin{pmatrix} h(x_1,t) \\ \dots \\ h(x_n,t) \end{pmatrix}. \end{cases}$$

Remark 5. The formula (4) gives in a constructive way functions f_k and g_k in terms of which solutions h of the nonlinear partial differential equation $\det D_n(h) = 0$ can be decomposed in the sense of (1) without the necessity of solving linear differential equations as mentioned in Remark 1.

4. We write $\Delta = (\alpha,\beta)\times(\gamma,\delta) \subset R^2$, the cases $\alpha = -\infty$, $\beta = \infty$, $\gamma = -\infty$, and $\delta = \infty$ are not excluded. Let $\mathbb{L}_2 = \{k:\Delta\to R, \int_\Delta k^2<\infty\}$. For a fixed positive integer n, let

$$P_n := \{h \in \mathbb{L}_2;\ h(x,t) = \sum_{k=1}^{n} f_k(x) g_k(t)\} .$$

Theorem 4. For every positive integer N, and every positive ε, there exists $k \in \mathbb{L}_2$ such that

$$\| k-h \|_{\mathbb{L}_2} > \varepsilon$$

for all $h \in P_N$.

The proofs of the above results will be published in [1] and [2].

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L-FUNCTIONS OF ARITHMETIC ORDERS

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Let Λ denote a $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A. We consider L-functions $L_\Lambda(M,s,\psi)$ which are generalisations of the classical L-functions in algebraic number theory, and which are closely related to L. Solomon's zeta-functions (see our earlier work [1]).

In discussing left ideals M of Λ, we shall always restrict our attention to ideals of finite index $(\Lambda:M)$. The genus of M, denoted by g(M), consists of all left Λ-lattices X such that, for each rational prime p, $X_p \cong M_p$ as Λ_p-lattices. (The subscript p indicates p-adic completion, not localisation.) For X, Y $\in$ g(M), we say that X is stably isomorphic to Y if $X \oplus M \cong Y \oplus M$; let $[X]$ denote the stable isomorphism class of X. These stable classes, for X $\in$ g(M), form a finite abelian group Cl(M), called the genus group (or class group) of M, with addition given by $[X] + [X'] = [X'']$, where $X \oplus X' \cong X'' \oplus M$.

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Let s be a complex variable. Given a left ideal M of Λ and a character ψ of Cl(M), we define

$$L_\Lambda(M,s,\psi) = \sum_X \psi[X]\,(\Lambda:X)^{-s}, \qquad \mathrm{Re}(s) > 1,$$

where X ranges over all left ideals of Λ of finite index $(\Lambda:X)$, subject to the condition $X \in g(M)$. The series converges and defines a holomorphic function of s for $\mathrm{Re}(s) > 1$.

Our first main result is:

THEOREM 1: <u>The function $L_\Lambda(M,s,\psi)$ can be analytically continued to a meromorphic function on the whole s-plane.</u>

The proof is based on the following ideas: we view ψ as a character of the idele group J(A) of A, and write $\psi = \prod_p \psi_p$, where for each rational prime p, ψ_p is a character of $A_p^\times$, the group of units of A_p. For $\mathrm{Re}(s) > 1$, there is an Euler product

$$L_\Lambda(M,s,\psi) = \prod_p L_{\Lambda_p}(M_p,s,\psi_p),$$

(product over rational prime numbers p) where

$$L_{\Lambda_p}(M_p,s,\psi_p) = \sum_x \psi_p(x)\,(\Lambda_p:M_px)^{-s}.$$

Here, x ranges over the orbits of $(M_p:\Lambda_p)\cap A_p^\times$ under the left action of $\mathrm{Aut}_{\Lambda_p}(M_p)$, where

$$\{M_p : \Lambda_p\} \quad = \quad \{x \in A_p : M_p x \subset \Lambda_p\}.$$

For almost all primes p, Λ_p is a maximal order in A_p, and one can then evaluate the local factor $L_{\Lambda_p}(M_p, s, \psi_p)$ explicitly, in the same manner as in proving K.Hey's formula for the zeta-function of a maximal order (see [1], formula (16)). At each exceptional prime p for which Λ_p is not maximal, we may generalise the arguments of [2], §5.2, and [1], Theorem 1, to show that $L_{\Lambda_p}(M_p, s, \psi_p)$ is a product of functions derived from classical L-functions, multiplied by a polynomial function $f_p(s) \in \mathbb{C}[p^{-s}]$. Theorem 1 then follows readily from known results on continuation of these classical L-functions. For details, see the forthcoming paper [3].

Let h_M be the number of stable isomorphism classes $[M_i]$ in g(M). For each $[M_i]$, we define a <u>partial</u> <u>zeta-function</u>

$$Z_\Lambda([M_i], s) \quad = \quad \sum_X (\Lambda : X)^{-s}, \qquad \mathrm{Re}(s) > 1,$$

where X ranges over all left ideals of Λ such that

$$X \in g(M), \quad \text{and} \quad [X] = [M_i].$$

The orthogonality relations for characters of a finite abelian group imply at once that

$$Z_\Lambda([M_i],s) = h_M^{-1} \sum_\psi \psi[M_i]^{-1} L_\Lambda(M,s,\psi),$$

the sum extending over all characters ψ of $Cl(M)$. Thus we obtain:

THEOREM 2: *Each partial zeta-function $Z_\Lambda([M_i],s)$ can be analytically continued to the whole s-plane.*

Now let Λ' be a maximal order in the semisimple $\mathbb{Q}$-algebra A, and let A have t simple components. Define

$$\zeta_{\Lambda'}(s) = \sum_X (\Lambda':X)^{-s}, \qquad Re(s) > 1,$$

where X ranges over all left ideals of Λ'. Then $\zeta_{\Lambda'}(s)$ can be expressed as a product of various Dedekind zeta-functions, multiplied by known correction factors for the finitely many rational primes p for which A_p is not split semisimple. Using standard facts about Dedekind zeta-functions, we deduce readily that $\zeta_{\Lambda'}(s)$ has a pole of order t at $s = 1$. An easy argument then yields:

THEOREM 3: *Each partial zeta-function $Z_\Lambda([M_i],s)$ has a pole of order t at $s = 1$.*

In order to determine the leading term in the Laurent expansion of $Z_\Lambda([M_i],s)$ in powers of $s-1$, we begin by setting

$$c_0 = \lim_{s \to 1} (s-1)^t \zeta_{\Lambda'}(s),$$

a positive constant which can be computed explicitly. By analyzing the behaviour of the L-functions $L_\Lambda(M,s,\psi)$ at $s = 1$, we obtain:

THEOREM 4: *Let $[M_i]$ be a stable isomorphism class in $g(M)$, where M is a left ideal of Λ. Let h_M be the number of stable classes in $g(M)$, and set*

$$d_M = (\Lambda:M)^{-1}(\{M:\Lambda\}:\Lambda')\prod_p((\Lambda_p')^\times:\mathrm{Aut}_{\Lambda_p}(M_p)),$$

where Λ' is a maximal order in A, and where

$$\{M:\Lambda\} = \{x \in A:\ Mx \subset \Lambda\}.$$

Then
$$\lim_{s\to 1}(s-1)^t Z_\Lambda([M_i],s) = c_0 d_M/h_M.$$

Applying the Delange-Ikehara Tauberian theorem, we obtain our second main result:

THEOREM 5: *Let M be a left ideal of Λ, and $[M_i]$ a stable isomorphism class in $g(M)$. Then, for positive T tending to infinity, the number of left ideals X of Λ satisfying the conditions*

$$(\Lambda:X) \le T,\qquad X \in g(M),\qquad [X] = [M_i],$$

is asymptotically equal to

$$\frac{c_0 d_M}{h_M}\,\frac{T(\log T)^{t-1}}{(t-1)!},$$

where t is the number of simple components of A. In particular, the left ideals of Λ in a given genus g(M) are asymptotically uniformly distributed among the stable isomorphism classes in the genus.

We remark finally that if none of the simple components of A is a totally definite quaternion algebra, then stable isomorphism coincides with ordinary isomorphism.

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PRIME-ELEMENT ALGEBRAS WITH TRANSITIVE AUTOMORPHISM GROUPS

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1. Several authors have discussed the problem establishing conditions for the functional completeness of a finite algebra. Many of these theorems observe that if the automorphism group of an algebra is a sufficiently large subgroup of the permutation group of the underlying set, then this algebra is functionally complete. For example, it was known, that the finite algebras $<A,t>$ and $<A,d>$, $A > 2$, where t and d are the discriminator and the dual discriminator function, respectively, are functionally complete. (See H. Werner [12] and E. Fried and A. Pixley [4].) In these cases aut $(\mathfrak{A})$ is the full symmetric group on A.

A generalisation of this result was given in B. Csákány [2]; he proved that all but six homogeneous algebras that is algebras for which Aut $(\mathfrak{A})$ is the full symmetric group on A, are functonally complete. Analogous results were proved in Á. Szendrei and L. Szabó [10] and P.P. Pálffy, Á. Szendrei and L. Szabó [6].

A finite algebra $\mathfrak{A} = <A,F>$ with triply transitive automorphism group is either functionally complete or equivalent to an affine space over $GF(2)$ ([10].) A finite algebra $\mathfrak{A} = <A,F>$ with doubly transitive automorphism group is either functionally complete or equivalent to an affine space over a finite field ([6]).

The structure of affine spaces or in the terminology of k-valued logic: linear closed classes is well known. (See J. Bagyinszki and J. Demetrovics [1] and Á. Szendrei [11].) Accordingly, the exceptional algebras are sufficiently described.

We present in this work a similar result for algebras with transitive automorphism groups under some constraints. For the basic concepts and notations the reader is referred to G. Grätzer [5] and R. Pöschel and L.A. Kaluznin [7].

2. Our main result is the following:

Theorem. Let $\mathfrak{A} = (A,F)$ be an algebra; let $|A| = p$ where p is a prime number and $p > 3$. Let us assume that Aut$(\mathfrak{A})$ is a transitive subgroup of S_A. Then r is either functionally complete or it is polynomially equivalent to an affine algebra over $GF(p)$.

NOTE 1. In the two element lattice, the dual discriminator is an algebraic function (in fact the median) but this algebra is not functionally complete, since all algebraic functions are isotone. On the other hand, d is a homogeneous function, hence the restriction $p > 3$ is essential.

NOTE 2. It is easily seen that under the conditions of the Theorem Aut $(\mathfrak{A})$ contains a cycle of lenght p; this we shall denote by $\mathfrak{Z}$. Of course, the subgroup $<\mathfrak{Z}>$ itself acts transitively on the set A. To simplify the notation, we shall assume that $A = \{0,1,\dots,p-1\}$

and $\mathfrak{Z} : x \to x + 1$ for all $x \epsilon A$, where $+$ denotes addition mod p.

The proof of the Theorem uses Rosenberg's completeness theorem. (I.G. Rosenberg [9] and R.W. Quackenbush [8].) We show that whenever F preserves none of the linear relations of A, then F preserves none of the six types of relations in Rosenberg's classification. The main difficulty is to show that W is simple.

It is well known that for an arbitrary algebra Con $(\mathfrak{A})$ = Con $(<A,F_1>)$ where F_1 is the set of all unary algebraic functions of $\mathfrak{A}$. This lead us to investigate the unary algebraic functions of $\mathfrak{A}$. A basic tool in the proof of the Theorem is the following

LEMMA 1. Let $\mathfrak{A} = (A,F)$ be an algebra, let $|A| = p$, where p is a prime and $p > 3$. We further assume that not all operations of $\mathfrak{A}$ are projections and that $\mathfrak{Z} \epsilon$ Aut$(\mathfrak{A})$.

Then (at least) one of the following conditions holds:

(a) $F_1 \cap S_A \neq <id_A>$;

(b) There is an integer k with $k \not\equiv 0 \pmod p$ and there are elements f,g of F_1 such that $1 < |\mathrm{im} f| < p$ and $g(y) = y + k$ holds for all $y \epsilon$ im$f$.

An easy consequence of Lemma 1 is

LEMMA 2. Under condition (b) of Lemma 1, for any $k \epsilon A$ there is an $h \epsilon F_1$ satisfying $h(y) = y + k$ for all $y \epsilon$ im$f$.

NOTE 3. $\mathfrak{Z}^{-1} f \mathfrak{Z} \epsilon F_1$ holds for all $f \epsilon F_1$. Under condition (a) of Lemma 1 this means that $F_1 \cap S_A$ is a transitive subgroup of S_A. So in both of the above two cases we have a "large" set of unary algebraic functions.

The following two lemmas facilitate handling central and k-regular relations. (See [7], [8].) Let 0_A denote the set of all finitary operations on A.

LEMMA 3. Let $\rho \subseteq A^k$ be a k-ary totally reflexive relation and let $k > 3$. If $H \subseteq 0_A$ and all operations in H are subjective (or constant), then $\rho \epsilon$ InvH implies that $\rho' \epsilon$ InvH where

$$\rho' = \{(x,y) | (x,y,a_1, \dots, a_{k-2}) \epsilon \rho \text{ for all } a_1, \dots, a_{k-2} \epsilon A\}.$$

LEMMA 4. Let H,ρ,ρ' be as in Lemma 3.

(i) If ρ is a non-trivial central relation, then ρ' is also central with the same center.

(ii) If ρ is a k-regular relation defined by the equivalence relations $\Theta_1, \dots, \Theta_m$, then

$$\rho' = \bigcap_{i=1}^{m} \Theta_i.$$

A subset H of 0_A is called *basic* if $[\{f\} \cup H] = 0_A$ holds for all Slupecki functions f. For any $X \subseteq 0_A$ the closed class generated by X shall be denoted by $[X]$. A group $G \subseteq S_A$ is a *basic group* if G is a basic subset of 0_A. L Szabó has conjectured the following (personal communication):

Let $\mathfrak{A} = \langle A,F \rangle$ be a nontrivial finite algebra and let us assume that $\mathrm{Aut}(\mathfrak{A})$ is a basic group. Then $\mathfrak{A}$ is functionally complete.

From our Theorem follows:

COROLLARY 1. If $\mathfrak{A} = (A,F)$, $|A| = p$ and p is a prime number, then Szabó's conjecture holds.

Let A be a nonempty set and let $\pi \in S_A$. The graph of π is defined by:

$$\rho_\pi = \{(x, \pi(x)) | x \in A\}$$

$\rho \subseteq A^2$ is a binary relation on A. J. Demetrovics and L. Hannák in [3] have proved the following: if $|A| > 5$, then for any $\pi \in S_A$, $\mathrm{Pol}\rho$ contains continuumly many closed classes. In other words, in this case there are continuumly many polynomially non-equivalent algebras $\mathfrak{A}_\beta = \langle A,F_\beta \rangle$ $(\beta < \mathfrak{c})$ for which $\pi \in \mathrm{Aut}(\mathfrak{A}_\beta)$ $(\beta < \mathfrak{c})$. This result and our Theorem imply:

COROLLARY 2. If A is a set, $|A| = p$ where p is a prime number and $p > 5$, then there are continuumly many pairwise polinomially non-equivalent functionally complete algebras $\mathfrak{A}_\beta = \langle A,F_\beta \rangle$ $(\beta < \mathfrak{c})$

For $|A| = 3$, we were unable to determine the number of different algebras with $\mathfrak{Z} \in \mathrm{Aut}(\mathfrak{A})$.

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AN INEQUALITY FOR THE L^p-NORM RELATED TO UNIFORM CONVEXITY

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Presented by L.A. Lorch F.R.S.C.

It is well-known that the L^p (and ℓ^p) spaces are uniformly convex for $1 < p < +\infty$ (see Clarkson [1]; and Köthe [4],pp.355-360). For $p \geq 2$ this is a direct consequence of a known elementary inequality (the case $\lambda = \frac{1}{2}$ of (10) below), but for $1 < p < 2$ the argument is more involved. It is the purpose of this note to provide an analogous inequality for the L^p-norm, when $1 < p < 2$, which is of interest in its own right and which leads immediately to the uniform convexity of the space in this case. The inequality can also be used in one of the proofs of the mean ergodic theorem (see Garsia [3], p.21).

We prove the following inequality:

THEOREM 1. Let $1 < p < +\infty$, $0 < \lambda < 1$, and Ω be a measurable subset of $\mathbb{R}$. Then there is a constant $c_{p,\lambda}$, $0 < c_{p,\lambda} < +\infty$, such that $\forall\, f,g \in L^p(\Omega)$,

$$(1) \quad \|f - g\|_p^{\,p} \leq c_{p,\lambda} \{ (1-\lambda)\|f\|_p^{\,p} + \lambda\|g\|_p^{\,p} - \|(1-\lambda)f + \lambda g\|_p^{\,p} \}^{\min(1,\frac{1}{2}p)} \times$$
$$\times \{ \textstyle\int_\Omega \max(|f|^p, |g|^p) \}^{\max(0,1-\frac{1}{2}p)},$$

where $c_{p,\lambda} = [\lambda(1-\lambda)]^{-\frac{1}{2}p}$ for $p \geq 2$.

Furnished with this theorem, we have immediately the following

COROLLARY 1. $L^p(\Omega)$ is uniformly convex for $1 < p < +\infty$.

Proof. The result will be true (see Köthe [4], p.353) if, whenever $(f_n),(g_n)$ are sequences in $L^p(\Omega)$ such that $\|f_n\| \leq 1$, $\|g_n\| \leq 1$, $\|\frac{1}{2}(f_n + g_n)\| \to 1$, we have $\|f_n - g_n\| \to 0$. Let A_n be the subset of Ω for which $|f_n(t)| > |g_n(t)|$, and $B_n = \Omega \setminus A_n$. Thus

$$\int_\Omega \max(|f_n|^p,|g_n|^p) = \int_{A_n} |f_n|^p + \int_{B_n} |g_n|^p \leqslant \int_\Omega (|f_n|^p + |g_n|^p) \leqslant 2.$$

Putting $\lambda = \frac{1}{2}$ in (1) we then have

$$\|f_n - g_n\|^p < c_{p,\frac{1}{2}} \max(1, 2^{1-\frac{1}{2}p}) \{ \tfrac{1}{2} + \tfrac{1}{2} - \|\tfrac{1}{2}(f_n + g_n)\|^p \} \to 0 \text{ as } n \to \infty.$$

We suppose throughout that $0 < \lambda < 1$, and define

$$F_{p,\lambda}(z) := 1 - \lambda + \lambda|z|^p - |1-\lambda + \lambda z|^p \quad \text{for} \quad z \in \mathbb{C},\ 1 < p < +\infty.$$

A simple calculation shows that for $p = 2$ this reduces to

$$F_{2,\lambda}(z) = \lambda(1-\lambda)|z - 1|^2.$$

LEMMA 1. <u>If</u> $1 < p < +\infty$ <u>and</u> $x > 0$, <u>then</u>

$$\tfrac{1}{2}p(p-1)\lambda(1-\lambda)(x-1)^2 \min(1,x^{p-2}) \leqslant F_{p,\lambda}(x) \leqslant \tfrac{1}{2}p(p-1)\lambda(1-\lambda)(x-1)^2 \max(1,x^{p-2}).$$

<u>Proof</u>. $F_{p,\lambda}(x)/[\lambda(1-\lambda)(x-1)^2]$ is a second divided difference of the function $\phi(t) = t^p$ at the points 1, $1-\lambda+\lambda x$, x; thus for some ξ between 1 and x, we have (e.g. see [2], p.65)

$$\frac{F_{p,\lambda}(x)}{\lambda(1-\lambda)(x-1)^2} = \frac{(1-\lambda)\phi(1) - \phi(1-\lambda + \lambda x) + \lambda\phi(x)}{\lambda(1-\lambda)(x-1)^2} = \frac{\phi''(\xi)}{2!} = \tfrac{1}{2}p(p-1)\,\xi^{p-2}.$$

Since $\min(1,x^{p-2}) \leqslant \xi^{p-2} \leqslant \max(1,x^{p-2})$, the result follows.

LEMMA 2. <u>Let</u> $g_p(r) := F_{p,\lambda}(1+r) - F_{p,\lambda}(1-r)$, $0 < r < 1$. <u>Then</u>

$$g_p(r) < 0 \text{ for } 1 < p < 2,\quad g_2(r) = 0,\quad g_p(r) > 0 \text{ for } p > 2.$$

<u>Proof</u>. By the proof of Lemma 1,

$$g_p(r) = \tfrac{1}{2}p(p-1)\lambda(1-\lambda)\, r^2(\xi_1^{p-2} - \xi_2^{p-2}), \quad 1-r < \xi_2 < 1 < \xi_1 < 1+r.$$

LEMMA 3. <u>Let</u> $|z - 1| = r < 1$. <u>Then for</u> $1 < p < 2$ <u>we have</u>

$$\tfrac{1}{2}p(p-1)\lambda(1-\lambda)\, r^2\, (1+r)^{p-2} \leqslant F_{p,\lambda}(z) \leqslant \tfrac{1}{2}p\lambda(1-\lambda)\, r^2\, (1-\lambda r^2)^{\frac{1}{2}p-1}.$$

<u>The inequalities are reversed for</u> $p \geqslant 2$.

Proof. Note first that for $p = 2$ we have exactly $F_{2,\lambda}(z) = \lambda(1-\lambda)|z-1|^2$, while $F_{p,\lambda}(1) = 0$ for every p.

Suppose that $p \neq 2$ and let $z = 1 + re^{i\theta}$, $0 < r < 1$. Then

$$F_{p,\lambda}(z) = 1 - \lambda + \lambda|1 + re^{i\theta}|^p - |1 + \lambda re^{i\theta}|^p$$

$$= 1 - \lambda + \lambda(1 + 2r\cos\theta + r^2)^{\frac{1}{2}p} - (1 + 2\lambda r\cos\theta + \lambda^2 r^2)^{\frac{1}{2}p} := F(r,\theta).$$

$$\frac{\partial}{\partial\theta}F(r,\theta) = p\lambda r\sin\theta\,[(1 + 2\lambda r\cos\theta + \lambda^2 r^2)^{\frac{1}{2}p-1} - (1 + 2r\cos\theta + r^2)^{\frac{1}{2}p-1}].$$

Thus, for a given r, the only possible stationary values of $F(r,\theta)$ occur when $\theta = \theta_0$ or when $\theta = \theta_1$, where

(2) $$\cos\theta_0 = -\tfrac{1}{2}(1+\lambda)\,r,$$

(3) $$\sin\theta_1 = 0.$$

Here (2) corresponds to the points z_0, $\overline{z}_0$ where the circles $|z-1| = r$ and $|z| = |1-\lambda+\lambda z|$ intersect;and (3) corresponds to points $z_1 = 1 \pm r$ where the circle $|z-1| = r$ intersects the real axis. It is now easy to verify that

(4) $$\frac{\partial^2}{\partial\theta^2}F(r,\theta) = \begin{cases} p(p-2)\lambda(1-\lambda)r^2\sin^2\theta_0\cdot|z_0|^{p-4} & \text{for } z = z_0 \text{ (or } \overline{z}_0) \\ p\lambda r[(1+\lambda r)^{p-2} - (1+r)^{p-2}] & \text{for } z = 1+r \\ -p\lambda r[(1-\lambda r)^{p-2} - (1-r)^{p-2}] & \text{for } z = 1-r. \end{cases}$$

Consider the case $1 < p < 2$. We see from (4) that $\partial^2F/\partial\theta^2$ is then negative for $z = z_0$ (or $\overline{z}_0$) and positive for $z = 1 \pm r$. Hence, for $|z-1| = r$ (and since $F_{p,\lambda}(z) = F_{p,\lambda}(\overline{z})$),

(5) $$\frac{F_{p,\lambda}(z)}{\lambda(1-\lambda)\,r^2} \leq \frac{F_{p,\lambda}(z_0)}{\lambda(1-\lambda)\,r^2} = \frac{1-(1-\lambda r^2)^{\frac{1}{2}p}}{\lambda r^2} = \tfrac{1}{2}p\,\xi^{\frac{1}{2}p-1} \leq p|z_0|^{p-2},$$

where $|z_0|^2 = 1 - \lambda r^2 < \xi < 1$. In the opposite direction,

$$F_{p,\lambda}(z) \geq \min[\,F_{p,\lambda}(1-r),\ F_{p,\lambda}(1+r)\,] = F_{p,\lambda}(1+r) \qquad \text{by Lemma 2}$$

(6) $$\geq \tfrac{1}{2}p(p-1)\lambda(1-\lambda)r^2\min[\,1,\ (1+r)^{p-2}] \qquad \text{by Lemma 1}$$

$$= \tfrac{1}{2}p(p-1)\lambda(1-\lambda)r^2(1+r)^{p-2} \qquad \text{since } 1 < p < 2.$$

The combination of (5) and (6) gives the inequality stated in the enunciation. In the case $p > 2$, the signs of $\partial^2F/\partial\theta^2$ in (4) are reversed, and the inequalities in (5) and (6) are reversed (with max in place of min in (6)).

LEMMA 4. Let $1 < p < 2$. Then $\exists\, a_{p,\lambda}$, $0 < a_{p,\lambda} < +\infty$, such that

(7) $$F_{p,\lambda}(z) \geqslant a_{p,\lambda}|z-1|^2 \min(1, |z|^{p-2}) \quad \text{for all} \quad z \in \mathbf{C}.$$

Proof. Since $|1-\lambda+\lambda z| \leqslant |1-\lambda+\lambda|z||$, we have $F_{p,\lambda}(z) \geqslant F_{p,\lambda}(|z|)$, with equality only if $z \geqslant 0$. By Lemma 1, $F_{p,\lambda}(|z|) \geqslant 0$, with equality only if $|z| = 1$. Hence $F_{p,\lambda}(z) > 0$ if $z \in \mathbf{C}$, $z \neq 1$. Thus, on the compact set $\{ |z| \leqslant 1, |z-1| \geqslant 1 \}$, $F_{p,\lambda}(z)/|z-1|^2$ is positive and continuous, and hence bounded below by a positive constant. Since, by Lemma 3,

$$F_{p,\lambda}(z) \geqslant \tfrac{1}{2}p(p-1)\lambda(1-\lambda)\, 2^{p-2}\, |z-1|^2 \qquad \text{for} \quad |z-1| < 1,$$

$\exists\, a'_{p,\lambda}$, $0 < a'_{p,\lambda} < +\infty$, such that

(8) $$F_{p,\lambda}(z) \geqslant a'_{p,\lambda}\, |z-1|^2 \quad \text{for} \quad |z| \leqslant 1,$$

and (7) reduces to (8) in the case $|z| \leqslant 1$.

If $|z| > 1$ we use $|1/z| < 1$ in (8) to give

$$F_{p,\lambda}(z) = |z|^p F_{p,1-\lambda}(z^{-1}) \geqslant a'_{p,1-\lambda}|z|^p|z^{-1}-1|^2 = a'_{p,1-\lambda}|z|^{p-2}|z-1|^2,$$

which is inequality (7) for $|z| > 1$, where we take $a_{p,\lambda} = \min(a'_{p,\lambda}, a'_{p,1-\lambda})$.

LEMMA 5. (a) Let $1 < p < 2$. Then $\exists\, b_{p,\lambda}$, $0 < b_{p,\lambda} < +\infty$, such that $\forall\, \alpha,\beta \in \mathbf{C}$,

(9) $$|\alpha-\beta|^2 \leqslant b_{p,\lambda}\{(1-\lambda)|\alpha|^p + \lambda|\beta|^p - |(1-\lambda)\alpha+\lambda\beta|^p\} \max(|\alpha|^{2-p}, |\beta|^{2-p}).$$

(b) Let $p \geqslant 2$ and denote $c_{p,\lambda} = [\lambda(1-\lambda)]^{-\frac{1}{2}p}$. Then $\forall\, \alpha,\beta \in \mathbf{C}$,

(10) $$|\alpha-\beta|^p \leqslant c_{p,\lambda}\{(1-\lambda)|\alpha|^p + \lambda|\beta|^p - |(1-\lambda)\alpha+\lambda\beta|^p\}.$$

Proof. (a) Let $1 < p < 2$. Then (9) is obviously true if $a = 0$ or if $b = 0$. Assume that $a \neq 0$ and $b \neq 0$ and substitute $z = b/a$ in (7).

(b) By the monotonicity of the ℓ^p-norm and by Hölder's inequality,

$$\{\lambda^{\frac{1}{2}p}(1-\lambda)^{\frac{1}{2}p}|\alpha-\beta|^p + |(1-\lambda)\alpha+\lambda\beta|^p\}^{1/p}$$

$$\leq \{\lambda(1-\lambda)|\alpha-\beta|^2 + |(1-\lambda)\alpha+\lambda\beta|^2\}^{\frac{1}{2}}$$

$$= \{(1-\lambda)|\alpha|^2 + \lambda|\beta|^2\}^{\frac{1}{2}}$$

$$= \{(1-\lambda)^{(p-2)/p}(1-\lambda)^{2/p}|\alpha|^2 + \lambda^{(p-2)/p}\lambda^{2/p}|\beta|^2\}^{\frac{1}{2}}$$

$$\leq \{1-\lambda+\lambda\}^{(p-2)/2p}\{(1-\lambda)|\alpha|^p + \lambda|\beta|^p\}^{\frac{1}{2}(2/p)}$$

$$= \{(1-\lambda)|\alpha|^p + \lambda|\beta|^p\}^{1/p}.$$

Proof of Theorem 1. Put $\alpha = f(t)$, $\beta = g(t)$ in Lemma 5.

(a) Using (9) when $1 < p < 2$, we raise both sides of the inequality to the power $\frac{1}{2}p$ and integrate over Ω, applying Hölder's inequality to the right hand side, with $\frac{1}{2}p + \frac{1}{2}(2-p) = 1$; and this gives us (1) for $1 < p < 2$, with $c_{p,\lambda} = b_{p,\lambda}^{\frac{1}{2}p} = a_{p,\lambda}^{-\frac{1}{2}p}$.

(b) For $p \geq 2$ we use (10) and integrate directly over Ω.

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NON-SPLITTING EXAMPLES FOR NORMALIZED UNITS IN INTEGRAL GROUP RINGS OF METACYCLIC FROBENIUS GROUPS

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Presented by G. de B. Robinson F.R.S.C.

Let G be a finite group and V(G) the normalized units (those which map to 1 under the augmentation) in the integral group ring $\mathbb{Z}G$. The question has been raised by K. Dennis in 1976 [D] as to when the natural inclusion $G \to V(G)$ is split. This area has been of considerable activity recently, partly because the group ring problem is answered very instructively whenever a splitting $V(G) \to G$ exists with torsion-free kernel. The most far-reaching splitting results of this kind have been described by Cliff, Sehgal and Weiss in a recent preprint [CSW], for G containing an abelian normal subgroup A with G/A abelian of odd order or exponent dividing 4 or 6.

We give here the first examples which show that the map $G \to V(G)$ does not always split. The groups are $G_1 = C_{73} \rtimes C_8$, the metacyclic Frobenius group of order $73 \cdot 2^3$, and $G_2 = C_{241} \rtimes C_{10}$, also a metacyclic Frobenius group, of order $241 \cdot 5 \cdot 2$. These examples suggest that the above results of [CSW] for metabelian groups are best possible.

The construction is based on the following observations: Let $G = H \rtimes C$ be a metacyclic Frobenius group, H cyclic of prime order $p \geq 7$ and C cyclic of order $m > 2$, $m|(p-1)$, with generator c. We denote by $\mathfrak{h}\uparrow^G$ the kernel of the natural map $\mathbb{Z}G \to \mathbb{Z}C$ and put $V_H(G) = \{1+x : x \in \mathfrak{h}\uparrow^G\} \cap V(G)$.

<u>Proposition 1:</u> If $\phi: V(G) \to G$ splits the inclusion, then ϕ renders the following diagram commutative

$$\begin{array}{ccccccccc} 1 & \to & H & \to & G & \to & C & \to & 1 \\ & & \uparrow\phi' & & \uparrow\phi & & \uparrow\phi'' & & \\ 1 & \to & V_H(G) & \to & V(G) & \to & V(C) & \to & 1 \end{array},$$

and $\operatorname{Ker} \phi' \supset \{1+y : y \in \mathfrak{h}^{2+G}\} \cap V(G)$.

The proposition shows that <u>every</u> splitting here must essentially arise in the form considered in [CSW]. The main tool in the proof is Bass-Milnor-Serre's congruence subgroup theorem [BMS], and we are indebted to H. Bass for pointing out that the kernel of ϕ' must contain a congruence subgroup.

Let now μ be a primitive m-th root of unity in $\mathbb{F}_p$, the field with p elements, and let $\nu_i: \mathbb{Z}C \to \mathbb{F}_p$ be the representation induced by $c \mapsto \mu^i$, $1 \le i \le m$. Using the first proposition, one obtains

<u>Proposition 2:</u> There exists a splitting $\phi: V(G) \to G$ if and only if there exists i, $1 \le i \le m$, such that for every unit $u \in V(C)$, the product $\nu_i(u)\nu_{i+1}(u)^{-1}$ has order dividing m in the multiplicative group of $\mathbb{F}_p$. (Read $i+1 = 1$ for $i = m$.)

<u>Example I:</u> Let $G = G_1 = C_{73} \rtimes C_8$. The element

$$u = 2 + c - c^3 - c^4 - c^5 + c^7$$

of $\mathbb{Z}C_8$ is a unit, and, if $\mu = 10$ in $\mathbb{F}_{73}$, then for all $1 \le i \le 8$, $\nu_i(u)\nu_{i+1}(u)^{-1}$ has order 36. Hence by Proposition 2 there cannot exist a splitting.

Example II: Let $G = G_2 = C_{241} \rtimes C_{10}$. Then

$$u = -372099 + 114985c + 301035c^2 -301035c^3 + 114985c^4 + 372100c^5 -114985c^6 - 301035c^7 + 301035c^8 +114985c^9$$

and

$$v = c - c^5 + c^9$$

are units in $\mathbf{Z}C_{10}$, and if $\mu = 36$ in $\mathbf{F}_{241}$, we get the following list

	1	2	3	4	5	6	7	8	9	10
$\omega_i(u)$	233	233	30	1	1	233	30	30	233	30
$\omega_i(v)$	151	-1	83	191	53	151	-1	83	191	53

where $\omega_i(u) = \nu_i(u)\nu_{i+1}(u)^{-1}$ and similarly for v.

Since $|233| = |30| = 8$, $|151| = |83| = 60$, $|191| = |53| = 120$ in $\mathbf{F}_{241}$, this shows that for G_2 there cannot exist a splitting.

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ON A THEOREM OF STERLING BERBERIAN

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In this note x and y denote bounded linear operators in an arbitrary Banach space, satisfying $xy = 1$. If either x or y is compact then 1 must be compact, so the space is finite dimensional and hence $yx = 1$. Thus, if $yx \neq 1$ then for every linear, compact a : neither $x - a$ nor $y - a$ can be compact. On the other hand:

Theorem: If $yx \neq 1$ then for every bounded, linear, invertible a :

(i) $x - a$ is not compact,

(ii) $y - a$ is not compact.

Note: Berberian's original proof, to appear in Proc. Amer. Math. Soc., was for the case of a Hilbert space.

Proof. First suppose that the Theorem holds for the case $a = 1$. Then for any bounded, linear, invertible a, we have: a^{-1} is linear, bounded (by the closed graph theorem); $a^{-1}x$, ya satisfy the hypotheses of the Theorem, so $a^{-1}x-1$ is not compact; since $a^{-1}x-1 = a^{-1}(x-a)$, so $x - a$ is not compact. Similarly, ax, ya^{-1} satisfy the hypotheses of the Theorem; $ya^{-1}-1 = (y-a)a^{-1}$; so $y - a$ is not compact.

Thus, in the proof of the Theorem, we need consider only the case $a = 1$.

Proof of i). Let M_n denote the closed linear null set of x^n. Clearly $M_0 \subset M_1 \subset M_2 \ldots$. We shall show that all M_n are different.

Equality $M_0 = M_1$ would mean that x is $(1,1)$; since the range of x includes the range of xy , i.e. the whole space, it would follow that $zx = 1$ for some linear operator z , hence $z = zxy = y$, so $yx = 1$, contradicting the hypotheses. So $M_0 \neq M_1$.

Now for any $n \geq 1$, equality $M_{n+1} = M_n$ would imply:

$$M_n = xyM_n \subset xM_{n+1} \subset xM_n \subset M_{n-1} \quad , \quad \text{so} \quad M_n = M_{n-1} \quad ;$$

by repetition we would derive $M_1 = M_0$, which is false.

Now for each $n \geq 1$, choose a unit vector v_n in M_n such that $\| v_n + w \| > \frac{1}{2}$ for all w in M_{n-1} (in a Hilbert space it suffices to choose v_n a unit vector in M_n and orthogonal to M_{n-1} ; in a general linear normed space, use the technique of F. Riesz: choose any vector u in M_n but not in M_{n-1} , let $d = \inf \| u+w \|$ for all w in M_{n-1} (then $d > 0$); choose w_0 in M_{n-1} so that $\| u+w_0 \| < 2d$, and take v_n to be $\| u+w_0 \|^{-1}(u+w_0)$) .

With these unit vectors v_n we have, for all $n > m$:

$$\| (x-1)v_n - (x-1)v_m \| = \| v_n + w \| \quad (\text{with } w = xv_m - xv_n - v_m \text{, in } M_{n-1}) > \tfrac{1}{2}$$

so $x - 1$ cannot be compact.

<u>Proof of ii)</u>. Let M_n denote now the range of y^n , i.e. the closed linear null set of $1 - y^n x^n$. Then

$$M_0 \supset M_1 \supset M_2 \ldots$$

and the type of argument used in the proof of (i) applies again.

Remark. Sterling Berberian has pointed out that whenever $xy = 1$ and a is bounded, linear, invertible:

$$x-a^{-1} = -x(y-a)a^{-1} \quad \text{and} \quad y-a = -a(x-a^{-1})y$$

so $y-a$ is compact if and only if $x-a^{-1}$ is compact; hence $y-a$ fails to be compact for every bounded, linear, invertible a if and only if $x-a$ fails to be compact for every such a. Thus (ii) above is an immediate consequence of (i) above.

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INVERSIVE GEOMETRY AND CERTAIN CHAINS OF CIRCLES

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Presented by H.S.M. Coxeter F.R.S.C.

Let Σ be the unit sphere $\sum_{i=1}^{n+1} x_i^2=1$ in $\mathbb{R}^{n+1}$ and let Π be the hyperplane $x_{n+1}=0$ augmented by a single point at ∞. Useful coordinates for the points and caps of Σ and for the points, n-balls and halfspaces of Π can be introduced as follows. Full details are given in [8].

Let C be a cap on Σ with centre $c\in\Sigma$ and angular radius $0<\theta<\pi$. Then a point x of Σ belongs to C iff $x\cdot c\geq\cos\theta$ or, equivalently, iff $(\lambda x,\lambda)*((\csc\theta)c, \cot\theta)\geq 0$ where $\lambda>0$ and $U*V$ is the Lorentz inner product $U_1V_1+U_2V_2+...+U_{n+1}V_{n+1}-U_{n+2}V_{n+2}$. With this motivation we assign positive homogeneous coordinates X satisfying $X*X=0$ and $X_{n+2}>0$ to the points of Σ and coordinates C satisfying $C*C=1$ to the caps of Σ. With these coordinates the point X belongs to the cap C iff $X*C\geq 0$. Inversion in the (n-1)-sphere which bounds the complementary caps C and -C is represented by the linear transformation $U\to U-2(C*U)C$ where U may be either a point or a cap. The Möbius group generated by all inversions is equal to the orthochroneous Lorentz group consisting of all linear transformations of $\mathbb{R}^{n+2}$ which preserve the inner product $U*V$ and the sign of U_{n+2} on the cone $U*U=0$.

If we map Π to Σ by stereographic projection we find that the points, n-balls and halfspaces of Π are described by the following (n+2)-vectors:

point $x = (x_1, x_2, \ldots, x_n)$	$(2x,\ \|x\|^2-1,\ \|x\|^2+1)$
point ∞	$(0,\ 1,\ 1)$
halfspace $x \cdot n \geq d$ $[\|n\|=1]$	$(n,\ d,\ d)$
proper n-ball $\|x-a\| \leq r$	$\frac{1}{2r}(2a,\ \|a\|^2-1-r^2,\ \|a\|^2+1-r^2)$
improper n-ball $\|x-a\| \geq r$	$\frac{-1}{2r}(2a,\ \|a\|^2-1-r^2,\ \|a\|^2+1-r^2)$.

Note that the curvature of the boundary of an "n-ball" in Π is given by $C*E=\frac{1}{r}$ (positive for proper n-balls, negative for improper n-balls and 0 for halfspaces) where $E=(0,\ldots 0,-1,-1)$. Similarly the size of a cap on Σ is given by $C*S=\cot\theta$ where $S=(0,\ldots 0,0,-1)$. Moreover, if we construct a Poincaré model of hyperbolic n-space in an n-ball or halfspace H then it can be shown that the size of a hyperbolic "n-ball" is given by $C*H=\pm\coth\rho, \pm 1, \pm\tanh\beta$ or 0 depending on whether C names a proper n-ball of radius ρ, a region bounded by a horosphere, a region bounded by an equidistant surface a distance β above its hyperplane, a halfspace or, for negative values, the complement of such a region.

Let me sketch one recent application of these coordinates to a rather well known configuration. Soddy's hexlet [4] consists of three mutually tangent balls in Euclidean 3-space and a chain of successively tangent balls each tangent to the original three. The surprising feature of this chain is that, no matter where it is started, it closes to a ring with the sixth ball. The proof of this is easy once one realizes that the hexlet belongs to inversive geometry and can be transformed by a suitable inversion to a very simple Euclidean canonical form. In [7] we used this canonical form to write down a vector equation which describes the hexlet from the point of view of inversive geometry. Because this equation is linear it applies equally well when the hexlet is in general position. The inversive models of the classical geometries show that hexlets exist in 3-dimensional spherical and hyperbolic geometry as well as Euclidean. Moreover scalar equations describing the size of the balls in the Euclidean, spherical and hyperbolic figures can be derived by taking the inner product of our vector equation with the vector E,S or H indexing the model in question.

In this note we shall apply the approach described above to certain chains of circles. René Descartes [2] derived the formula $\left(\sum_{i=1}^{4}\varepsilon_i\right)^2=2\sum_{i=1}^{4}\varepsilon_i^2$ for the curvatures (with suitable sign conventions) of four mutually tangent circles in the Euclidean plane. Jacob Steiner ([5] p. 274) put this in the form $\varepsilon_4=\varepsilon_1+\varepsilon_2+\varepsilon_3+2\sqrt{\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1}$ and H.S.M. Coxeter [1] recognized $\delta=\sqrt{\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1}$ as the curvature of the circle through the points of contact of the first 3 circles. In [6] we wrote $\varepsilon_1=\alpha$, $\varepsilon_2=\beta,\varepsilon_3=\gamma_0,\varepsilon_4=\gamma_1$ and used inversion in the point of contact of α and β [here we use Greek letters for the name of the circle as well as its curvature] to prove the generalized Descartes circle formula $\gamma_n=\gamma_0+2n\delta+n^2(\alpha+\beta)$ for the curvature of the n^{th} circle in a chain of successively tangent circles each tangent to α and β.

We can now extend the meaning of this formula. In the first instance it should really be regarded as a vector equation describing a figure of "disks" in an inversively invariant way. This equation can be verified by checking it for the canonical configuration in which α is the halfplane $y\geq 1$, β is the halfplane $y\leq -1$, δ is the halfplane $x\geq 1$, γ_0 is the proper disk $x^2+y^2\leq 1$ and γ_n is the proper disk $(x-2n)^2+y^2\leq 1$. Because the formula is linear, it applies even when the figure is in general position. Moreover scalar equations (of exactly the same form) describing the size of the "disks" in Euclidean spherical and hyperbolic figures can be obtained by taking the scalar product of our vector equation with E,S or H respectively. A final benefit of this approach is that the sign convention for curvatures is dictated by considering "disks" instead of "circles". In this connection, notice that α and β are tangent but not overlapping, the chain ... $\gamma_{-2},\gamma_{-1},\gamma_0,\gamma_1,\gamma_2$ ... lies in the complement of $\alpha\cup\beta$ and δ contains γ_1.

Recently M.D. Fox [3] has used elementary methods like those in [2] to further generalize the Descartes circle formula in its Euclidean context to the case when α and β are not tangent but either (i) intersecting or (ii) disjoint. In case (i) he considers separately (a) the canonical form in which α and β are straight lines (b) the case when α is a chord of the circle β and (c) the case when α and β are intersecting circles. In case (ii) he considers separately (d) the canonical form in which α and β are concentric circles and then the other cases when both are circles and (e) the case in which α is a line not intersecting the circle β.

By applying the methods used earlier in this paper it is possible to obtain the vector formulae for case (i) and case (ii) and then to specialize these vector formulae first to Eucidean, spherical and hyperbolic scalar equations and then to the various particular instances of these in the three geometries.

In the intersecting case we obtain

$$\gamma_n = (\cosh 2n\Delta)\gamma_0+(\sinh 2n\Delta)(\operatorname{csch} \Delta)\delta+(\sinh n\Delta)^2(\coth \Delta)^2(\alpha+\beta)$$

where $\alpha*\beta=-\cos\psi$ and $\sinh\Delta=\tan\psi/2$. The chain lies in the complement of $\alpha\cup\beta$ in a lune with vertex angle ψ; δ is the "disk" perpendicular to α,β and γ_0 which contains γ_1.

In the non-intersecting case we obtain

$$\gamma_n = (\cos 2n\psi)\gamma_0+(\sin 2n\psi)(\csc\psi)\delta+(\sin n\psi)^2(\cot\psi)^2(\alpha+\beta)$$

where $\alpha*\beta=-\cosh\Delta$ and $\sin\psi=\tanh\Delta/2$. The chain lies in the complement of $\alpha\cup\beta$ in an annulus; Δ is the inversive distance between the boundaries of α and β; δ is the "disk" perpendicular to α,β and γ_0 which contains γ_1.

These new formulae are dual to one another and in the limit $\psi,\Delta\to 0$ they both tend to the original generalized Descartes circle formula $\gamma_n=\gamma_0+2n\delta+n^2(\alpha+\beta)$ where $\alpha*\beta=-1$.

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A CONJECTURE ON COHEN-MACAULAY TYPE

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We discuss the Cohen-Macaulay type of the coordinate ring of s lines through the origin in $A^{n+1}(k)$.

1. Introduction. Let R be the co-ordinate ring of s distinct straight lines in $A^{n+1}(k) = \operatorname{Spec} k[X_0,\ldots,X_n]$, all passing through the origin (k a field). Let $\bar{R}$ be the integral closure of R. Then $R \subset \bar{R} = \prod_{j=1}^{s} k[T_j]$, where $k[T_j]$ is the coordinate ring of the j^{th} line. The ring R is graded (in a manner compatible with the grading on $\bar{R}$) so we can write $R = \oplus_{i\geq 0} R_i$ where $R_0 = k$ and R_i is contained in the s-dimensional i^{th} graded piece of $\bar{R}$. If i is sufficiently large then $\dim_k R_i = s$. A more detailed discussion of the embedding $R \subset \bar{R}$ can be found in [1]§2 or [5]§2. This note is motivated by [1] to which we refer for the definition of Cohen-Macaulay type.

Assume that X_0 is a non-zero-divisor in R. If k is infinite this can always be obtained by a co-ordinate change - take the hyperplane $X_0 = 0$ to be any plane containing none of the lines. Let $M = \oplus_{i\geq 1} R_i$ and let $\operatorname{ann}(M/X_0R)$ be the annihilator of M/X_0R in the graded ring R/X_0R. Since X_0 is a non-zero-divisor in R, the ring R/X_0R is of finite dimension over k. It is observed in [1]§6 that the Cohen-Macaulay type (C-M type) of R is equal to $\dim_k \operatorname{ann}(M/X_0R)$. The ideal $\operatorname{ann}(M/X_0R)$ is a homogeneous ideal of the graded ring R/X_0R.

Let d be the least integer such that $\dim_k R_d = s$. (Since X_0 is a non-zero-divisor we must then have $\dim_k R_i = s$ for $i \geq d$.) Then $M/X_0R = \oplus_{i=1}^{d} R_i/X_0R_{i-1}$ where $R_d/X_0R_{d-1} \neq 0$. (If $i \geq d$ we have $X_0R_i = R_{i+1}$ so higher degree components of R/X_0R are all zero.) Clearly $R_d/X_0R_{d-1} \subseteq \operatorname{ann}(M/X_0R)$ so the problem is to find the lower degree components of $\operatorname{ann}(M/X_0R)$.

2. <u>The conjecture.</u> Suppose that the lines are in generic s-position [1], [4]. This means that $\dim_k(R_i) = \inf(s, \binom{i+n}{n})$, where $\binom{i+n}{n} = (i+n)!/(i!)(n!)$ = number of monomials of degree i in the indeterminates $X_0,\ldots,X_n$. Then d is the smallest integer for which $s \leq \binom{d+n}{n}$. If $1 \leq i < d$ then $R_i \cong i^{th}$ graded piece of $k[X_0,\ldots,X_n]$ and $(R/X_0R)_i \cong i^{th}$ graded piece of $k[X_1,\ldots,X_n]$. Thus $\operatorname{ann}(M/X_0R)$ can have non-zero components only in degrees d and $d-1$. The ring structure of R/X_0R induces a homomorphism $\phi\colon (R/X_0R)_{d-1} \to \operatorname{Hom}_k((R/X_0R)_1, (R/X_0R)_d)$ and the degree $d-1$ component is equal to $\ker\phi$. My conjecture is that if the s lines are "sufficiently generic" then ϕ is as injective as possible (i.e. that $\dim_k \ker\phi = \operatorname{Max}(0, \dim_k \operatorname{domain}\phi - \dim_k \operatorname{codomain}\phi)$). There is no reason to expect $\dim\ker\phi$ to depend only on s.

If $n = 2$ (i.e. the lines are contained in $A^3(k)$) then there is a simple formula for the conjectured C-M type. We have $\binom{d+1}{2} < s \leq \binom{d+2}{2}$. Write $s = \binom{d+1}{2} + i$, $1 \leq i \leq d+1$. Then $\dim_k(R_d/X_0R_{d-1}) = i$, $\dim_k(R_{d-1}/X_0R_{d-2}) = d$, and $\dim(R/X_0R)_1 = 2$. Thus ϕ maps from a d dimensional vector space to a space of dimension $2i$. The conjecture then implies that $\dim\ker\phi = \operatorname{Max}(0, d-2i)$ and that the C-M type $= i + \dim_k \ker\phi$

$= \text{Max}(i, d-i)$. (This formula was first worked out by C. Weibel).

The coefficients of a matrix for ϕ can be obtained by solving a system of linear equations. Furthermore the rank of ϕ is given by the non-vanishing of determinants in these coefficients. Thus there is a Zariski open subset of $(P^n)^s$ whose corresponding lines yield the maximum value of rank ϕ and minimum value of C-M type. In order to verify my conjecture one need only find one set of points where ϕ has the conjectured rank.

3. Examples. As far as I know, the conjecture has not been disproved for any s . The simplest case is $s = \binom{d+n}{n}$. Then $(R/X_0R)_i \cong i^{th}$ graded piece of $k[X_1,\ldots,X_n]$ for $1 \leq i \leq d$ and $\text{ann}(M/X_0R)$ has no non-zero elements in degree $d-1$. This is proposition 12 of [1]. More generally assume that the s lines are in generic s position and that $s = \binom{d+n}{n} - \lambda$. Let $T_i = i^{th}$ graded piece of $k[X_1,\ldots,X_n]$. Then $(R/X_0R)_d \cong T_d/V$ where V is a vector space of dimension λ , and $(R/X_0R)_i \cong T_i$ for $1 \leq i < d$. The homomorphism ϕ: $(R/X_0R)_{d-1} \to \text{Hom}_k((R/X_0R)_1, (R/X_0R)_d)$ lifts in a natural way to an injection $\tilde{\phi}$: $(R/X_0R)_{d-1} \to \text{Hom}_k((R/X_0R)_1, T_d)$, or in the other notation $\tilde{\phi}$: $T_{d-1} \to \text{Hom}_k(T_1,T_d)$. If $w \in (R/X_0R)_{d-1}$ then $\phi(w) = 0$ if and only if $\tilde{\phi}(w)$ maps $(R/X_0R)_1$ into V . The latter is not possible for $\lambda < n$ since $\tilde{\phi}(w)$ is an injection. This generalizes Proposition 14 of [1] .

The case $\lambda = n$ is more difficult. Here I assume that k is algebraically closed. Let $U \subset (P^n)^s$ correspond to s points in generic s-position. Let W be the grassmanian of n dimensional spaces of forms of degree d in $X_0,\ldots,X_n$. Using the formula $s+n = \binom{d+n}{n}$ we see that $\dim W = ns$. There is a morphism $\pi: U \to W$ defined by mapping a set of s points (in generic s-position) in P^n to the n-dimensional space of polynomials of degree d vanishing on these points. There is a Zariski open set $W' \subset W$ such that $W' \subset \text{image } \pi$. To see this consider the d-uple embedding $\rho_d: P^n \to P^N$, where $N = n+s-1$. A form of degree d in $X_0,\ldots,X_n$ corresponds to a hyperplane in P^N and an element of $L \epsilon W$ corresponds to a linear subspace $\bar{L}$ of codimension n (dimension $s-1$) in P^N. Let W' be an open set of W such that if $L \in W'$ then $\bar{L}$ and $\rho_d(P^n)$ intersect transversally in d^n distinct points. (W' exists by a version of Bertini's theorem that can be proved by modifying the proof of Theorem 8.18 page 179 of [3]. Swan [6] has an affine version that we could use also. The number of points is d^n since $\rho_d(P^n)$ has degree d^n.) Let F_i $(1 \leq i \leq n)$ be a basis of $L \in W'$. Think of the F_i again as forms of degree d in P^n. The hypersurfaces $F_i = 0$ intersect transversally in d^n points. This implies that $\operatorname{Proj} k[X_0,\ldots,X_n]/(F_1,\ldots,F_n)$ is the product of d^n copies of $\operatorname{Spec} k$, and in particular is a reduced scheme. Thus if $G^r \in (F_1,\ldots,F_n)k[X_0,\ldots,X_n]$, then for each i there exists an integer n_i such that $X_i^{n_i}G \in (F_1,\ldots,F_n)$. But $S = k[X_0,\ldots,X_n]/(F_1,\ldots,F_n)$ is Cohen-Macaulay of dimension one, so the maximal ideal $(X_0,\ldots,X_n)S$ contains a non-zero-divisor. Thus we conclude that $G = 0$ in S, and that $(F_1,\ldots,F_n)$ is a radical ideal of $k[X_0,\ldots,X_n]$. If H is a form of degree d in $X_0,\ldots,X_n$ vanishing on our d^n points in P^n, then $H = 0$ in S, or equivalently $H \in (F_1,\ldots,F_n)k[X_0,\ldots,X_n]$. Comparing coefficients of degree d we conclude that $H \in L$.

(A. Broughton suggested that I use this analogue of Noether's theorem. We appear to essentially have Theorem 255 of "Topics in Commutative Ring Theory", lecture notes by Kaplansky, U. of Chicago, 1974). This means that $\bar{L}$ is spanned by the d^n intersection points $\rho_d(P^n) \cap \bar{L}$. Some s of these points must span $\bar{L}$. The corresponding s points of P^n are in generic s-position by Lemma 3.2 of [4] , so $L \in$ image π . Let $\tilde{W}$ be the variety of n dimensional subspaces of T_d . The dimension of $\tilde{W}$ is $[\binom{d+n-1}{n-1}-n]n$. There is a surjective rational map $W \to \tilde{W}$ given by setting $X_0 = 0$ in a basis for an element of W . The composition $U \to W \to \tilde{W}$ is generically surjective. The dimension of $(R/X_0R)_{d-1}$ is equal to $\binom{n+d-2}{n-1}$ so the elements of $\tilde{W}$ of the form $\tilde{\phi}(w)\ (R/X_0R)_1$ are of dimension $\binom{n+d-2}{n-1}-1 < \dim(\tilde{W})$. For each choice of $u \in U$ we get a ring R . Of course ϕ is not defined until we choose u , but $\tilde{\phi}$: $T_{d-1} \to \mathrm{Hom}_k(T_1, T_d)$ is and $(R/X_0R)_d = T_d/\tilde{\pi}(u)$. We saw above that one can choose $u \in U$ so that $\tilde{\pi}(u) \neq \tilde{\phi}(w)T_1$ for any $w \in T_{d-1}$. Then ϕ is injective. Thus my conjecture holds for $\lambda = n$. If $n = 2$ (hence $\lambda = 2$) a different geometric proof will appear in [2].

Here is a table for $n = 2$ (lines in A_k^3) . The notation is as above: s = number of lines, d is the smallest integer such that $\binom{d+2}{2} \geq s$ (recall that $\dim_k(R_{d-1}/X_0R_{d-2}) = d$ and that $i = \dim(R_d/X_0R_{d-1})$) , and the conjectured C-M type equals the sum of the previous two columns. All dimensions are over k .

s	d	$\dim(R_d/X_0R_{d-1})$	$\dim \ker \phi$	conjectured C-M type
3	1	2	0	2
4	2	1	0	1
5	2	2	0	2
6	2	3	0	3
7	3	1	1	2
8	3	2	0	2
9	3	3	0	3
10	3	4	0	4
11	4	1	2	3
12	4	2	0	2
13	4	3	0	3
14	4	4	0	4
15	4	5	0	5
16	5	1	3	4
17	5	2	1	3
18	5	3	0	3
19	5	4	0	4
20	5	5	0	5
21	5	6	0	6

The binomial coefficients $\binom{2+d}{2}$ are 3, 6, 10, 15, 21 so the conjecture has been proved for all s in the above table except 7, 11, 12, 16, 17, 18. The case $s = 7$ is in §6 of [1], I believe I have given explicit examples to verify the conjecture for $s = 11, 12$, so the simplest case where the conjecture is not yet verified is $s=16$.

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TRANSFORMATION-PARAMETER/STRUCTURAL MODELS: ASYMPTOTIC CONDITIONAL DISTRIBUTIONS

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This is a preliminary report on the large sample (asymptotic) properties of the conditional distributions that arise in the statistical analysis of transformation-parameter [4]/structural [2,3] models. The almost-sure convergence of the standardized structural distribution to a unit normal is derived from a corresponding convergence of the likelihood function. The statistical literature contains a variety of results on the limiting nature of the likelihood function but these have focussed on specific conditional point-estimates, posterior distributions from proper priors, or the central (local) shape of the likelihood function. In this report we obtain the almost-sure convergence of the structural distribution in the context of the location model. The techniques employed are being extended to handle the general transformation-parameter/ structural model.

TRANSFORMATION-PARAMETER/STRUCTURAL MODELS. The general structural model has the form: $y=\theta z$ with $\theta \epsilon G$, $z \sim (S,A,P)$, where G is a group of measurable transformations acting exactly on S; foundational support may be found in [1]. With an observed y_o, the distribution P is replaced by the conditional distribution P^{Gy_o} given the observed orbit Gy_o. The corresponding transformation parameter model has the form $y \sim (S,A,P\circ\theta^{-1})$, $\theta \epsilon G$ yields the conditional model G-orbit Gy_o.

For a sample $\underset{\sim}{y}_n=(y_1, \ldots, y_n)$ the n-fold independent product is $\underset{\sim}{y}_n=\theta \underset{\sim}{z}_n$ with θ in G, $\underset{\sim}{z}_n \sim (S^n, A^n, P^n)$, where the transformations act coordinate-wise. Under regularity conditions, it is the conditional measure $P^{G\underset{\sim}{y}_n}$, suitably normalized, that converges almost-surely to the unit normal Φ.

THE STRUCTURAL DISTRIBUTION. For what follows we specify some general smoothness assumptions about the underlying structural model. S1: The sample space S and the group G are presented as open sets of R^N and R^L respectively. S2: The functions $g=g_1g_2$ and $y=g_1g_2z$ are continuously differentiable in the arguments g_1 and g_2 in G and z in S. S3: There is a global cross-section Q and the projection $D:S\to Q$ defines an equivariant map [z] by $z=[z]D(z)$ so that [z] and D(z) are continuously differentiable.

Since the asymptotic properties being considered here are essentially "local" ones, these assumptions can be relaxed so that S and G can be taken to be differentiable manifolds and Q can define local cross-sections; the assumptions as stated cover most statistical applications.

Let $J_N(g,x)=\partial gx/\partial x$ and define $J_N(z)=J_N([z],D(z))$; let $J_L(g,h)=\partial gh/\partial h$ and define $J_L(g)=J_L(g,i)$: this gives (locally) the invariant measures $M(dz)=|J_N(z)|^{-1}dz$ and $\mu(dg)=|J_L(g)|^{-1}dg$ standardized with respect to Euclidean volume at Q and i respectively, where $|A|$ is the absolute value of the determinant of A. We make a final assumption S4: The measure P is given by a density p with respect to M, and p is twice continuously differentiable.

The marginal density for the orbit Gz, using coordinates on the cross-section Q is then $k(D)=\int_G p(gD)J(D)\mu(dg)$ relative to Euclidean volume at D orthogonal to GD and $J(D)=|A'A|^{\frac{1}{2}}$ where $A=\partial gD/\partial g$. The conditional distribution then has density $k(D)^{-1}p(gD)J(D)$ relative to the (left) haar measure μ on G. The structural distribution for θ given D is then $k(D)^{-1}p(\theta y)J(D)\Delta([y])\mu(D\theta^{-1})$ where $\Delta(g)=|\partial\mu(g)/\partial\mu(g^{-1})|$ is the modular function of the group. The likelihood function is the equivalence class $L(\theta;y)=\{cp(\theta^{-1}y):c\in R^+\}$. We consider the limiting properties of the normalized likelihood.

<u>ASYMPTOTIC STRUCTURAL DISTRIBUTION</u>. For a sample of size n from the structural model satisfying assumptions S1-S4, the compound model satisfies the same assumptions. Let $\hat\theta_n = \hat\theta_n(\underset{\sim}{y}_n)$ be a value maximizing the likelihood function; it is called the maximum likelihood estimate of θ (MLE(θ)).

<u>S5</u>: $\hat\theta_n(\underset{\sim}{y}_n)$ is uniquely determined and continuously differentiable. It follows that $\hat\theta_n(\underset{\sim}{z}_n) = [\underset{\sim}{z}_n]$ and $D_n(\underset{\sim}{z}_n) = \hat\theta_n^{-1}(\underset{\sim}{z}_n)\underset{\sim}{z}_n$ satisfy <u>S3</u>. We are concerned with the almost-sure limiting normality (non-degenerate) of the normalized likelihood/structural distribution $k_n^{-1}(D_n)s_n(\theta^{-1}y_n)J(D_n)\Delta([\underset{\sim}{y}_n])\mu(d\theta^{-1})$.

<u>LOCATION MODEL; LIMITING NORMALITY</u>. The location model is obtained with $S = R$ and G given by the additive group with application to S by translation. The structural density function with respect to Lebesque measure $\mu(d\theta^{-1})$ is $\pi_n(\theta) = \prod_{i=1}^{n} p\,(y_i - \theta)/\int \prod_{i=1}^{n} p(y_i - t)dt$, and the log-likelihood is

$$\ell_n(\theta) = \log\Big(\prod_{i=1}^{n} p(y_i - \theta)\Big).$$

For the asymptotic behavior of the log-likelihood we record some mild regularity conditions; derivatives are with respect to θ.

A1,2: $\hat\theta_n(\underset{\sim}{y}_n) \xrightarrow{a.s.} \theta$; $\ell_n''(\theta)/\ell_n''(\hat\theta_n(\underset{\sim}{y}_n)) \xrightarrow{a.s.} 1$.

An information function at the maximum likelihood estimate defines the scale σ_n by $\sigma_n^{-2} = -\ell_n''(\hat\theta_n)$.

A3: For each $\tau_o > 0$, $\sup_{|\tau|<\tau_o} \left| \dfrac{\ell_n''(\hat\theta_n + \tau\sigma_n) - \ell_n''(\hat\theta_n)}{\ell_n''(\hat\theta_n)} \right| \xrightarrow{a.s.} 0$;

A4,5: $\ell_n''(\theta)$ is continuous in θ ; $\sigma_n^2 \xrightarrow{a.s.} 0$;

A6: For each $\varepsilon > 0$, there exists a δ such that

$$\lim_{n\to\infty} \sup_{|\theta'-\theta|<\delta} \left| \frac{\ell_n''(\theta') - \ell_n''(\hat\theta_n)}{\ell_n''(\hat\theta_n)} \right| < \varepsilon$$

A7,8: The density p(z) is strictly positive and

$$\sup |z|^{1+\epsilon} p(z) < \infty \text{ for some } \epsilon > 0;$$

There exists an $x_o \geq 0$ and an $M \geq 0$ such that for all $x \leq x_o$, $P(p(z)<x) \leq Mx_o$.

The structural distribution, standardized by location $\hat{\theta}_n$ and scale σ_n, has density $g_n(\tau) = \sigma_n \pi_n(\hat{\theta}_n + \tau\sigma_n)$.

THEOREM: Under the assumptions A1 - A8 $g_n(T)$ converges to $(2\pi)^{-1/2} e^{-\tau^2/2}$ with probability one. From Scheffe's theorem [5] it follows that with probability one the measures Q_n defined by

$$Q_n([\tau_1, \tau_2]) = \int_{\hat{\theta}_n + \tau_1\sigma_n}^{\hat{\theta}_n + \tau_2\sigma_n} \pi_n(\theta)d\theta$$

converge weakly to the unit normal.

<u>LIMITING LIKELIHOOD AND CONDITIONAL DISTRIBUTIONS</u>. There is a close connection between the strong limiting form of the likelihood function structural density and the limiting form of the conditional distribution. This connection is given in the following theorem; for general notation the observed information matrix is the negative Hessian

$$I_n(\hat{\theta}_n, D_n) = - \frac{\partial^2}{\partial\theta^2} \ell_n(\theta^{-1}\hat{\theta}_n D_n)\Big|_{\theta=\hat{\theta}_n} .$$

THEOREM: For a sample from a transformation-parameter/structural model satisfying assumptions S1 - S4, if the structural density function standardized with respect to location $\hat{\theta}_n(y_n)$ and scale matrix $I_n(\hat{\theta}_n, D_n)^{-1}$ converges in probability (almost surely) to the unit normal then the conditional density for $\hat{\theta}_n$ standardized with respect to location θ and scale matrix $I_n(\theta, D_n)^{-1}$ converges in probability (almost surely) to the unit normal.

This theorem with the results from the preceding section gives the almost-sure convergence of the conditional distribution for the location model.

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