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ERROR BOUNDS FOR MIXED FINITE ELEMENT METHODS

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ABSTRACT.

Error bounds for the mixed methods for the finite element approximations of the mildly nonlinear problems are derived.

The main motivation of this paper is to derive the error bounds for the finite element approximations of mildly nonlinear problems of the mixed type. The finite element method, we study, is based on an extended variational principle in which the constraint of interelement continuity has been removed at the expense of introducing a Lagrange multiplier. This type of methods have been introduced and analyzed by Brezzi [1], Raviart and Thomas [5] for linear problems.

Let V, W and H be the Hilbert spaces with their duals V',W', H' and norms $||.||_V$, $||.||_W$ and $||.||_H$ respectively. We assume $V \subset H$ with a continuous embedding. Let a(.,.) and b(.,.) be continuous bilinear forms on $V \times V$ and $V \times W$ respectively and F be a continuous functional on V and $g \in W'$. We denote by <.,.>, the between V' and V or W' and W.

Consider now the functional

$$(1) \qquad I[v,\phi] = a(v,v) + 2b(v,\phi) - 2F(v) - 2\langle g,\phi\rangle.$$

We show that the minimum of $I[v,\phi]$ on $V \times W$ can be character -ized by a class of variational equations. We now state the result, see for proof [4].

Theorem 1.

Let F', the Fréchet differential of a nonlinear continuous functional F on V, be antimonotone. a(.,.) and b(.,.) be contin -uous bilinear forms on $V \times V$ and $V \times W$ respectively. If a(.,.) is a positive symmetric bilinear form on $V \times V$, then for given $g \in W'$, (u,ψ) minimizes the functional $I[v,\phi]$, if and only if $(u,\psi) \in V \times W$ satisfies the variational equations

(2) $a(u,v) + b(v,\psi) = \langle F'(u),v\rangle,$ for all $v \in V$

(3) $b(u,\phi) = \langle g,\phi\rangle,$ for all $\phi \in W$

REMARKS.

Theorem shows that the variational problem 1 and the weak formulations (2) and (3) are equivalent. This equivalnce plays a basic part in deriving the error bounds for the finite element method of the mixed type.

We also note that for $F \in V'$, the result of theorem 1 is exac -tly the same as proved in [1].

In this paper, we consider the problem (P):

Find $(u,\psi) \in V \times W$ such that

(4) $a(u,v) + b(v,\psi) = \langle F'(u),v\rangle,$ for all $v \in V$

(5) $b(u,\phi) = \langle g,\phi\rangle,$ for all $\phi \in W$.

For $(F'(u),g) \in D$, where D is a subclass of $V' \times W'$, we assume

(H1) (P) has a unique solution.

Given $d \in G'$, where G is a Hilbert space satisfying $W \subset G$ with a continuous, find $(y,\lambda) = (y_d,\lambda_d) \in V \times W$ such that

(6) $a(v,y) + b(v,\lambda) = 0,$ for all $v \in V$

(7) $b(y,\phi) = \langle d,\phi\rangle,$ for all $\phi \in W$.

We assume that

(H2) problems (6) and (7) have a unique solution for each $d \in G'$.

For $V_h \subseteq V$ and $W_h \subseteq W$, finite dimensional subspaces, we consider the finite element approximate problem (P_h):

Find $(u_h,\psi_h) \in V_h \times W_h$ such that

(8) $a(u_h,v_h) + b(v_h,\psi_h) = \langle F'(u_h),v_h\rangle,$ for all $v_h \in V_h$

(9) $b(u_h,\phi_h) = \langle g,\phi_h\rangle,$ for all $\phi_h \in W_h$.

In order to derive the error bounds for $u-u_h$ and $\psi-\psi_h$, we make the following hypothesis:

(H3) There exists a constant $\alpha>0$ such that

$$a(v,v) \geq \alpha||v||^2, \qquad \text{for all } v \in Z_h,$$

where

$$Z_h = \{v_h \varepsilon V_h : b(v_h, \phi_h) = 0, \text{ for all } \phi_h \varepsilon W_h\}$$

(H4) There exists a number S(h) such that

$$||v_h||_V \leq S(h)||v_h||_H, \text{ for all } v_h \varepsilon V_h.$$

(H5) There is an operator $\pi_h : \longrightarrow V_h$ sa-isfying $b(y-\pi_h y, \ _h)=0$, for all $y \varepsilon Y$ and $\phi_h \varepsilon W_h$, where $Y = \text{span}(\{y_d\}_{d\varepsilon G'}, u)$, (u,ψ) is the solution of (P) and (y_d, λ_d) is the solution of (6) and (7) correspodning to $d\varepsilon G'$.

(H6) $F'(u)$ is antimonotone on V, that is $\langle F'(u)-F'(v), u-v\rangle \leq 0$, for all $u,v\varepsilon V$.

(H7) $F'(u)$ is required to satisfy the Lipschitz condition, that is there exists a constant $\gamma > 0$ such that

$$||F'(u)-F'(v)||_{V'} \leq \gamma||u-v||_V, \quad \text{for all } u,v\varepsilon V$$

The hypothesis (H1)-(H5) are due to Falk and Osborn [2] and (H6)-(H7) are mainly due to Noor [3]. In addition, let a(.,.) and b(.,.) be continuous bilinear forms on H×H and V×W respectiv -ely, i.e.,

$$a(u,v) \leq \alpha_1 ||u|| \ ||v||, \quad \text{for all } u,v\varepsilon H.$$
$$b(u,\psi) \leq \beta_1 ||u||_V \ ||\psi||_W, \quad \text{for all } u\varepsilon V, \psi\varepsilon W.$$

We now state the main result of this paper.

Theorem 2.

Suppose that the hypothesis (H)-(H7) hold. If (u,ψ) and (u_h,ψ_h) are solutions of (P) and (P_h), then

$$||u-u_h||_H \leq C||u-\pi_h u||_H + C_2||\psi-\phi_h||_W, \quad \text{for all } \phi_h \varepsilon W_h$$
$$||u-u_h||_V \leq ||u-\ _h u||_V + C_3||\pi_h u-u||_H + C_4||\psi-\phi_h||_W.$$

If in addition

$$Z_h \quad Z = \{v\varepsilon V, b(v,\phi) = 0, \text{ for all } \phi\varepsilon W\}, \text{ then}$$

$$||u-u_h||_H \leq C_5||\pi_h u-u||_H,$$
$$||u-u_h||_V \leq ||u-\pi_h u||_V + C_6||u-\pi_h u||_H,$$

where C's are constants independent of u and ψ .

Theorem 3.

Assume that (H1)-(H3) and (H5) hold, (u,ψ) and (u_h,ψ_h) are the solutions of (P) and (P_h), then

$$||\psi-\psi_h||_G = \sup_{d\in G'} \{b(y_d-\pi_h y_d,\psi-\phi_h) + a(u_h-u,\pi_h y_d-y_d) + b(u-u_h,\lambda_d-\eta) + \langle F'(u)-F'(u_h),y_d\rangle\}/||d||_{G'}, \text{ for all } \phi_h,\eta\in W_h.$$

For proof and other details, see Noor [4].

REMARKS.

If F(u) is independent of u, i.e., F(u) = f,(say), then the Lipschitz constant γ is zero. Consequently our results reduce to that of Falk and Osborn [2]. Thus our results are more general than and include the previous one as special case. These results allow us to consider the mildly nonlinear elliptic boundary value problems.

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ON THE MAXIMAL ABELIAN SUBGROUPS OF THE LINEAR CLASSICAL ALGEBRAIC GROUPS

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1. The linear classical algebraic groups (LCAGs) over a field F are defined as the group of units, $U(R)$, of the centrally simple hypercomplex systems R over F. The task of surveying the $U(R)$-conjugacy classes of the maximal abelian subgroups (MASGs) A of $U(R)$ is equivalent to the task of surveying the $U(R)$-conjugacy classes of the maximal abelian subalgebras (MASAs) M of R over F. This is because the module M generated by A is a MASA and for any MASA M of R its unit group $A = U(M)$ is the MASG of $U(R)$ contained in M. The latter task was dealt with first by I. Schur [9] , later by Kravchuk [2-7]. We extend and refine Kravchuk's methods to deal with the task in full generality. We aim at establishing by algebraic methods the requisite 'linear invariants' in order to segregate the multitude of algebraic MASA varieties into irreducible subclasses (cp. [8], [10], [1]).

2. Decomposition. By McLagan-Wedderburn's second structure theorem R is characterized as a ring which is isomorphic to a ring of matrices of finite degree f over a division ring D of finite (square) dimension m^2 over its center, say $R = D^{f \times f}$.

A decomposition of a nonempty subset M of R is defined as a transformation of M by an element δ of $U(R)$ such that for every element X of M we have $\underline{\delta}(X) = \delta X \delta^{-1} = \oplus_{i=1}^{s} X_i$ where $X_i \in R_i = D^{fi \times fi}$; $s, f_i \in \mathbb{Z}^{>0}$ $(1 \leq i \leq s;\ \Sigma_{i=1}^{s} fi = f)$. If s is maximal then we speak of a Remak decomposition. If also $s = 1$ then M is indecomposable. Always the subset M_i of R_i formed by the X_i' s which is said to be the i-th component is indecomposable in case of a Remak decomposition.

If M is a MASA of R then the components M_i are MASAs of R_i and $\delta M \delta^{-1} = \oplus_{i=1}^{s} M_i$ where the M_i's are unique up to $U(R_i)$-conjugacy and component permutation. Conversely, if M_i is a MASA of $D^{f_i \times f_i}$ $(s, f_i \in \mathbb{Z}^{>0},\ 1 \leq i \leq s,\ \Sigma_{i=1}^{s} f_i = f)$ then $\oplus_{i=1}^{s} M_i$ is a MASA of R.

Any indecomposable MASA M of R contains a unique maximal separable extension $S = S(M)$ of F such that M is an indecomposable MASA of the centrally semisimple hypercomplex system over S formed by the R-centralizer

$$C_R(S) = \{Y \mid Y \in R \;\&\; \forall X\ (X \in S \Rightarrow XY = YX)\}$$

of S. If M is an indecomposable MASA of R satisfying $S(M) = F$ then the factor algebra of M over its radical $J(M)$ is F-isomorphic to a finite extension E of F. If $E = F$ then $J(M)$ is a maximal abelian nilpotent subalgebra (MANS) of R over F such that $M = J(M) + F$. Conversely, the R-centralizer of any nonzero MANS N of R is an indecomposable MASA $M = C_R(N) = N + F$.

Note that for any (field) extension E of F and any MASA M of R the tensor product algebra $E \otimes_F R$ is a centrally simple hypercomplex system over E with MASA $E \otimes_F M$.

In case $M/J(M) \underset{F}{\simeq} E$ is purely inseparable over F then we have $E \otimes_F M = J(E \otimes_F M) + E$. Field extension invariance also holds for any nonzero MANS N of R inasmuch $E \otimes_F N$ is a MANS of $E \otimes_F R$. Thus the theory of MASAs is reduced to the theory of MANSs.

3. MANSs. For any nonzero MANS N of R there is a unit δ of R transforming N into a Kravchuk normal form

$$\delta X \delta^{-1} = \begin{pmatrix} 0^\lambda & \Lambda_{12}(X) & \Lambda_{13}(X) \\ & \Lambda_{22}(X) & \Lambda_{23}(X) \\ & & 0^\nu \end{pmatrix} \qquad (1)$$

where $\lambda, \nu \in \mathbb{Z}^{>0}$, $\mu = f - \lambda - \nu \geq 0$, and $\Lambda_{12}: N \to D^{\lambda \times \mu}$, $\Lambda_{13}: N \to D^{\lambda \times \nu}$, $\Lambda_{22}: N \to D^{\mu \times \mu}$, $\Lambda_{23}: N \to D^{\mu \times \nu}$ are F-linear mappings such that

$$N_o = \ker \Lambda_{12} = \ker \Lambda_{23} = \{X | X \in N \;\&\; XN = 0\} \qquad (2)$$

is the annihilator ideal of N,

$$N_o \underset{F.}{\simeq} \Lambda_{13}(N) = D^{\lambda \times \nu}, \; N = N_o + \ker \Lambda_{13}, \qquad (3)$$

$$N_o \subseteq \ker \Lambda_{22}. \qquad (4)$$

The triplet $(\lambda m, \mu m, \nu m)$ is independent of the choice of δ and is said to be the <u>Kravchuk signature</u> of N. It is invariant under extension of F. Note that $\mu = 0 \Leftrightarrow N^2 = 0$ and that the Schur upper estimate of the dimension of N over F generalizes to the sharp upper estimate:

$$\dim_F N \leq m^2 \left[\left(\frac{f}{2}\right)^2\right]. \qquad (5)$$

Now let $\mu>0$. Generalizing Kravchuk's Lemma we show that the right annihilator

$$[\Lambda_{12}/D^{\mu \times 1}] = \{y | y \in D^{\mu \times 1} \quad \forall X(X \in N \Rightarrow \Lambda_{12}(X)\, y = 0)\}$$

of Λ_{12} is zero. Since the transpose $N^T = \{X^T | X \in N\}$ of N is a MANS of Kravchuk signature $(\nu m, \mu m, \lambda m)$ of the dual ring $R^d = (D^d)^{f \times f}$ it follows similarly that the left annihilator

$$[D^{1 \times \mu} \backslash \Lambda_{23}] = \{y | y \in D^{1 \times \mu} \;\&\; \forall X(X \in N \Rightarrow y\Lambda_{23}(X) = 0)\}$$

of Λ_{23} is zero.

Note that $\Lambda_{22} = 0 \Leftrightarrow N^3 = 0$. If $\Lambda_{22} = 0$ then N is said to be <u>corefree</u>.

<u>First fundamental theorem on MANSs:</u>

The matrices

$$\tilde{X} = \begin{pmatrix} 0^{\lambda} & \Lambda_{12}(X) & \Lambda_{13}(X) \\ & 0^{\mu} & \Lambda_{23}(X) \\ & & 0^{\nu} \end{pmatrix} \quad (X \in N)$$

form a corefree MANS $\tilde{N}$ such that the mapping of X on $\tilde{X}$ provides an F-isomorphism of the underlying F-linear spaces of N, $\tilde{N}$. Given $\tilde{N}$ then the corresponding N's are constructed by choosing F-linear mappings Λ_{22}: $\tilde{N} \to D^{\mu \times \mu}$ subject to the requisite commutativity conditions.

The proof of the first fundamental theorem is based on a lemma characterizing the right multiplication mapping $\varphi = \varphi W$: $S_1 \to S_2$, $\varphi W(X) = XW$ $(W \in R)$ of a nonempty subset S_1 of R into another one, say S_2, by the conditions $\forall s (\beta \in \mathbb{Z}^{>0} \,\&\, A_i \in R, X_i \in S_1 \ (1 \leq i \leq s) \,\&\, \Sigma_{i=1}^{s} A_i X_i = 0$

$$\Rightarrow \Sigma_{i=1}^{s} A_i \varphi(X_i) = 0).$$

Denoting the i-th row of the rectangular matrix $\Lambda_{12}(X)$ by $r_i(X)$ the λ F-linear mappings r_i: $N \to D^{1 \times \mu}$ generate a D-left linear space row (N). Similarly the ν F-linear mappings c_k: $N \to D^{\mu \times 1}$ of X of N on the k-th column of $\Lambda_{23}(X)$ generate a D-right linear space $\mathrm{col}(N)$. The MANS N of $D^{f \times f}$ is said to be <u>similar</u> to the MANS N' of $R' = D^{f \times f}$ with annihilator N'_o if there is an F-isomorphism θ: $N/N_o \to N'/N'_o$ and a Kravčuk normal form $\delta' N' \delta'^{-1}$ $(\delta \in U(R'))$ such that $\Lambda'_{22}(X') = \Lambda'_{22}(X)$ if $X'/N'_o = \theta(X|N_o)$ and if $\mathrm{row}(N) = \mathrm{row}(N')$, $\mathrm{col}(N) = \mathrm{col}(N')$. Similarity of MANSs is an equivalence relation. Even broader classes using the concepts of <u>boundary retract</u>, <u>boundarization</u> and <u>saturation</u> permit to deal effectively with the classification problem for MANSs N with fixed representation Δ: $N/N_o \to D^{\mu \times \mu}$,

$$\Delta: N/N_o \to D^{\mu \times \mu}, \ \Delta(X/N_o) = \Lambda_{22}(X) \ \text{for } X \text{ of } N.$$

Second fundamental theorem on MANSs. Let $F = D$ an infinite field. Then there are decompositions

$$N = N_o + \Sigma_{i=o}^{\lambda-1} \Sigma_{k=o}^{i} N_{ik} + \Sigma_{i=1}^{\lambda-1} N_i, \tag{6}$$

$$F^{1 \times \mu} = \Sigma_{i=o}^{\lambda-1} L_i + \Sigma_{i=1}^{\lambda-1} \Sigma_{k=o}^{i} L_{ik} \tag{7}$$

of N, $F^{1 \times \mu}$ into the direct sum of F-linear subspaces such that for an appropriate unit of δ of R the conjugate $\delta N \delta^{-1}$ is in refined Kravchuk normal form for which

$$\mu_{-i} = \dim_F N_{ik} = \dim_F L_i \in \mathbb{Z}^{\geq o}, \tag{8a}$$

$$r_{\lambda-j}(N_{ik}) = \delta_{jk} L_i \quad (0 \leq j \leq i), \tag{8b}$$

$$r_{\lambda-j}(N_{ik}) \subseteq \Sigma_{j<g<h} L_g + \Sigma_{g=1}^{\lambda-1} \Sigma_{o\leq h<j} L_{gh} \quad (i<j<\lambda) \tag{8c}$$

$$(0 \leq k \leq i < \lambda),$$

$$\mu_i = \dim_F N_i = \dim_F L_{ij} \quad (0 \leq j \leq i) \in \mathbb{Z}^{>o}, \tag{9a}$$

$$r_{\lambda-j}(N_i) = L_{ij} \quad (0 \leq j \leq i), \tag{9b}$$

$$r_{\lambda-j}(N_i) \subseteq \Sigma_{o<g<j} L_g + \Sigma_{g=1}^{\lambda-1} \Sigma_{o\leq h<j} L_{gh} \quad (i<j<\lambda) \tag{9c}$$

$$(0 < i < \lambda),$$

and the nonnegative integers μ_i $(|i| < \lambda)$ are independent of the choice of δ and invariant under extension of the field of reference such that

$$\dim_F N = \lambda\nu + \Sigma_{i=o}^{\lambda-1} (i+1)\mu_i + \Sigma_{i=1}^{\lambda-1} \mu_i, \tag{10}$$

$$\mu = \Sigma_{i=o}^{\lambda-1} \mu_{-i} + \Sigma_{i=1}^{\lambda-1} (i+1)\mu_i. \tag{11}$$

Using the refined Kravchuk normal form we find the sharp lower estimates

$$\dim_F N \geq m(f,\lambda), \tag{12a}$$

$$m(f,\lambda) = \min_{\substack{\lambda\leq\nu\in\mathbb{Z}^{>o} \\ \mu=f-\lambda-\nu\geq 0}} (1 + \lambda\nu + [(\mu-1)/\lambda] + \operatorname{sign}(\mu-1-\lambda[(\mu-1)/\lambda]), \tag{12b}$$

$$\dim_F N \geq \min_{o<\lambda\leq f-2} m(f,\lambda) = m(f), \tag{12c}$$

$$m(f) \sim 3(f/2)^{2/3} \text{ for } f \to \infty. \tag{12d}$$

4. Conclusion. The task of finding all MASAs of a linear classical Lie algebra of dimension f^2 has been reduced to the task of classifying all MANS of the linear classical Lie algebras of dimension

$$f_i^2 \text{ with } f_i \leq f.$$

The Kravchuk theory of MANSs has been further developed and refined by introducing the notions of corefree MANSs, boundary retracts, boundarization and saturation of MANSs.

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ON THE MAXIMAL ABELIAN SUBGROUPS OF THE QUADRATIC CLASSICAL ALGEBRAIC GROUPS

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1. The quadratic classical algebraic groups (QCAGs) over a field F are defined as the subgroups

$$O(\alpha/R) = \{\delta \mid \delta \in U(R) \;\&\; \alpha(\delta) = \delta^{-1}\} \tag{1a}$$

of the unit group of a centrally simple hypercomplex system R over F corresponding to the involutory antiautomorphism (involution) α of R acting on the unit group U(R) of R.

The task of surveying the $O(\alpha/R)$-conjugacy classes of the maximal abelian subgroups (MASGs) A of $O(\alpha/R)$ is equivalent to the task of surveying the $O(\alpha/R)$-conjugacy classes of the maximal abelian subalgebras (MASAs) M of the quadratic classical Lie algebra

$$o(\alpha/R) = \{X \mid X \in R \;\&\; \alpha(X) = -X\} \tag{1b}$$

corresponding to the involution α. This is because the Cayley map C of $U(R)-I_f$ on $U(R)-I_f$ which maps X on $(I_f-X)\,(I_f+X)^{-1}$ $O(\alpha/R) \cap (U(R)-I_f)$ is mapping $O(\alpha/R)$ on a generating set of the Lie ring $o(\alpha/R)$. On the other hand the application of C to $o(\alpha/R) \cap (U(R)-I_f)$ yields a generating set of the group

$$SO(\alpha/R) = \{X \mid X \in O(\alpha/R) \;\&\; \det(X) = 1\} \tag{1c}$$

(with slight modification in case the characteristic $\chi(R)$ of R is 2).

Note that $O(\alpha/R)$ is the normalizer of $O(\alpha/R)$ relative to U(R).

For any involution α of R there is an involution α_o of D and a unit K or R such that

$$\alpha(S) = K\alpha_o(X)^T K^{-1} (X \in R) \tag{2}$$

where $K = K(\alpha) = \epsilon\alpha_o(K)^T$, $\epsilon = \epsilon(\alpha) = \pm 1$ and $\epsilon = 1$ in case $\alpha|F \neq \underline{1}_F$.

The involution α is said to be <u>unitary</u> if $\alpha|F \neq \underline{1}_F$. In that case $O(\alpha/R)$ is a <u>unitary Lie algebra</u> of dimension $(fm)^2$ over the α-fixed subfield $F^\alpha = \{\xi | \xi \in F \;\&\; \alpha(\xi) = \xi\}$ of the center of F. Relative to F^α the field F is a separable quadratic extension. For any extension E of F^α the hypercomplex system $\hat{R} = E \otimes R$ with involution $\hat{\alpha} = 1 \otimes_{F^\alpha} \alpha$ either is centrally simple over E such that $O(\hat{\alpha}/\hat{R}) = E \otimes_{F^\alpha} O(\alpha/R)$ is a unitary Lie algebra or it is the algebraic sum of two centrally simple hypercomplex systems R_1, R_2 over E such that $\hat{\alpha}(R_i) = R_{3-i}$ $(i = 1,2)$ and $O(\hat{\alpha}/\hat{R}) = \{X_1 \oplus -\hat{\alpha}(X_1) | X_1 \in R_1\}$ is E-isomorphic to the E-Lie algebra LR_1 attached to R_1.

The involutions α, α' are said to be <u>properly equivalent</u> if there is a unit δ of R for which $\alpha' = \delta\alpha\delta^{-1}$. It follows that

$$O(\alpha'/R) = \delta O(\alpha/R)\delta^{-1},\ O(\alpha'/R) = \delta O(\alpha/R)\delta^{-1},\ K(\alpha') = \delta K(\alpha)\alpha_0(\delta)^T.$$

Let $D = F$, $\alpha|F = \underline{1}_F$. Now we have $K^T = \epsilon K$. If $\epsilon = 1$, $\chi(F) \neq 2$ or if $\chi(F) = 2$ and K has a nonzero diagonal coefficient then $O(\alpha/R) = O(K/R)$ is said to be the <u>generalized orthogonal</u> group. If F is algebraically closed then, after a proper equivalence transformation, $K = I_f$ and $O(\alpha/R)$ is the <u>orthogonal group</u>.

$$O(f,F) = \{X | X \in F^{f\times f} \;\&\; XX^T = I_f\} \tag{3}$$

of degree f over F. Otherwise $\epsilon = -1$, $2|f$ and after proper equivalence $K = \begin{pmatrix} & I_{f/2} \\ -I_{f/2} & \end{pmatrix}$ and $O(\alpha/R) = Sp(f,F)$ is the <u>symplectic</u> group of even degree f over F.

If $\alpha|F = \underline{1}_F$ then for any extension E of F the hypercomplex system $\hat{R} = E \otimes R$ is centrally simple over E with involution $\underline{\alpha} = \underline{1}_F \otimes \alpha$ such that $E \otimes O(\alpha/R) = O(\hat{\alpha}/\hat{R})$ in case E is the algebraic closure of F.

2. An α-<u>orthogonal decompositon</u> of a nonempty subset M of R is defined as a transformation of M by an element δ of $U(R)$ such that $\delta K(\alpha)\alpha_0(\delta)^T = \oplus_{i=1}^s K_i$, $\delta X \delta^{-1} = \oplus_{i=1}^s X_i$;

$$X_i, K_i \in R_i\, D^{f_i \times f_i};\ s, f_i \in \mathbb{Z}^{>0}$$
$$(1 \leq i \leq s);\ \Sigma_{i=1}^s f_i = f.$$

If s is maximal then we speak of an α-<u>Remak decomposition</u>. If also $s = 1$ then M is said to be α-<u>indecomposable</u> Always the subset M_i formed by the X_i's is said to be the i-th <u>component set</u> of the given orthogonal decomposition. Here α_i is the i-th component involution of R_i defined by setting $\alpha_i(Y) = K_i\alpha_0(Y)^T K_i^{-1}$ $(Y \in R_i)$. Note that $K_i = \epsilon(\alpha)\alpha_0(K_i)^T \in GL(f_i, D)$, $\epsilon(\alpha_I) = \epsilon(\alpha)$, $\alpha_i|F = \alpha_0|F$ $(1 \leq i \leq f)$.

<u>The</u> R-<u>centralizer of a</u> MASA <u>of</u> $O(\alpha/R)$ <u>is an</u> α-<u>invariant</u> MASA of R. (see notations of [2])

Not every α-invariant MASA of R intersects $O(\alpha/R)$ in a MASA of $O(\alpha/R)$.

For any α-orthogonal decomposition of an α-invariant MASA M of R the components M_i are α-invariant MASAs of R_i (notations as in [2]. Each M_i is α_i-indecomposable precisely if the given decomposition is α-Remak. The α-Remak decompositions are unique up to the order of the components and up to proper equivalence. Given an α-orthogonal decompositon we know that

$$M \cap O(\alpha/R)\ Y = \delta(\oplus_{i=1}^s (M_i \cap O(\alpha_i/Ri))\ \delta^{-1}$$

is a MASA of $O(\alpha/R)$ precisely if all component intersections $M_i \cap O(\alpha_i/R)$ form a MASA of $O(\alpha_i/R)$ and if at most one of them generates a nilpotent subring of R_i. Conversely, if for a fixed involution α_0 of D, fixed $\epsilon = \pm 1$ such that $\epsilon = 1$ in case $\alpha_0/F \neq 1_F$ and a natural number s we know s natural numbers f_i, matrices $K_i = \epsilon\,\alpha_0(K_i)^T$ of $GL(Fi,D)$ and α_i-invariant MASAs M_i of $R_i = D^{f_i \times f_i}$ where α_i is the involution of R_i mapping the element Y on $K_i\alpha_0(Y)^T K_i^{-1}$ $(1 \leq i \leq s)$ then there is the involution $\alpha = \oplus_{i=1}^{s} \alpha_i$ of $R = D^{f \times f}$ with $f = \Sigma_{i=1}^{s} f_i$ such that $K(\alpha) = \oplus_{i=1}^{s} K_i$, $\epsilon(\alpha) = \epsilon$ and $M = \oplus_{i=1}^{s} M_i$ is an α-invariant MASA of R.

3. Orthogonally indecomposable MASAs. Let M be an α-invariant α-indecomposable MASA of R.

If M is decomposable then f is even and there is a unit δ of R for which $\delta R(\alpha)\alpha_0(\delta)^T = \begin{pmatrix} & I_{fR} \\ \epsilon(\alpha) I_{fk} & \end{pmatrix}$ such that

$\delta M\delta^{-1} = \{X_1 \oplus \theta(X_1) \mid X_1 \in M_1\}$ where M_1 is a MASA of $D^{(f/2)\times(f/2)}$ and θ is an F^{α}-isomorphism of M_1 on $\alpha_0(M_1)^T$, and conversely.

If M is indecomposable and $M = F + J(M)$ then $J(M)$ is an α-invariant MANS. Conversely, for any α-invariant MANS N of R the MASA $N + F$ of R is α-invariant and indecomposable. The intersection $N \cap O(\alpha/R)$ is a MASA of $O/\alpha/R)$ i.e. $C_R(N \cap O(\alpha/R)) = N + F$.

4. Symmetric MANSs. In the previous section the theory of the MASAs of the quadratic classical Lie algebra $O(\alpha/R)$ was reduced to the task of surveying the $O(\alpha/R)$-conjugacy classed of the MANSs N of R for which $N = \alpha(N) = J(C_R(N \cap O(\alpha/R)))$. A MANS N of R is said to be symmetric if it is invariant under an involution α of R. The Kravchuk signature of a symmetric MANS is of the form $(\lambda m, \mu m, \lambda m)$.

After application of a suitable equivalence transformation to α and corresponding conjugation of N the Kravchuk normal form can be so arranged that

$$\Lambda_{23}(N) = K_{22}\alpha_0(\Lambda_{12})^T, \Lambda_{22}(N) = K_{22}\alpha_0(\Lambda_{22})^T K_{22}^{-1},$$

$$K(\alpha) = \begin{pmatrix} & & I_\lambda \\ & K_{22} & \\ \epsilon I_\lambda & & \end{pmatrix}.$$

We form the associative subalgebra $N^+ = \{X | X \in N \,\&\, \alpha(X) = (X)\}$ and the Lie algebra $N^- = N \cap O(\alpha/R)$ over F^α such that $N^+N^+ + N^-N^- \subseteq N^+$, $N^+N^- + N^-N^+ \subseteq N^-$. If $\chi(R) \neq 2$ then $N = N^+ + N^-$.

Now let $\chi(R) \neq 2$, $D = F$. If α is a symplectic type then μ is even and we adopt a normal form for which $K_{22} = \begin{pmatrix} & I_{\mu/2} \\ -I_{\mu/2} & \end{pmatrix}$,

$\Lambda_{12} = (\Lambda_{12}^- \Lambda_{12}^+)$, $\Lambda_{23} = \begin{pmatrix} -(\Lambda_{12}^+)^T \\ -(\Lambda_{12}^-)^T \end{pmatrix}$. We set $\mu_1 = \mu_2 = \mu/2$.

Otherwise we may have $K_{22} = K_1 \oplus K_2$ with

$$K_h = K_h^T \in GL(\mu_h, F^\alpha),\ \mu_h \in \mathbb{Z}^{\geq 0},\ (h = 1,2);\ \mu = \mu_1 + \mu_2;\ \Lambda_{12} = (\Lambda_{12}^+ \Lambda_{12}^-),$$

$$\Lambda_{23} = \begin{pmatrix} K_1\alpha_0(\Lambda_{12}^+)^T \\ -K_2\alpha_0(\Lambda_{12}^-)^T \end{pmatrix},$$

where in either case $\Lambda_{12}^+(N) \subseteq D^{\lambda \times \mu_2}$, $\Lambda^-{}_{12}(N^+) = 0$;

$$\Lambda_{12}^-(N) \subseteq D^{\lambda \times \mu_2},\ \Lambda_{12}^-(^+) = 0;$$

$$\Lambda_{22} = \begin{pmatrix} \Delta_{11}^+ & \Delta_{12}^- \\ \Delta_{21}^- & \Delta_{22}^+ \end{pmatrix};\ \Delta_{hh}^{\ddagger}(N) \subseteq D^{\mu_h \times \mu_h},\ \Delta_{hh}^+(N^-) = 0;$$

$$\Delta_{h,3-h}^-(N) \subseteq D^{\mu_h \times \mu_{3-h}},\ \Delta_{h,3-h}(N^+) = 0;\ (h = 1,2).$$

5. Conclusion. The classification and construction of all MASAs of the linear and quadratic classical Lie algebras has numerous applications in both mathematics and physics. It has enabled us to obtain a new universal normal form for the elements of the classical groups and Lie algebras. It complements the study of normal forms of elements by providing normal forms for certain sets of elements namely the MASAs themselves.

Among applications in mathematical physics we mention the solution of the Schrödinger equations for quadratic Hamiltonians in phase space (using MASAs of $sp(2n,\mathbb{R})$), the construction of superposition principles for certain nonlinear differential equations (using MASAs of $sl(n,\mathbb{R})$ and $o(p,q)$), the separation of variables in Helmholtz and Hamilton-Jacobi equations and many others.

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NEW INTEGRAL BASES FOR SYMMETRIC FUNCTIONS

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Presented by G. de B. Robinson, F.R.S.C.

Abstract

New symmetric functions labelled by partitions are defined. They are related to the hook structure of the Young diagrams specified by these partitions. They form an integral basis of the space of homogeneous symmetric functions, as do their dual functions which are also defined.

1. Introduction

Several integral bases of the space of homogeneous symmetric functions of degree n are well known. These are all labelled by the partitions of n which each specify at the same time a Young diagram. Two such integral bases are associated with the row and column structure of that diagram. In this paper a new integral basis is defined which is associated with the hook structure of the diagram.

2. Partitions.

A partition $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of n into p non-vanishing parts with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$ defines a Young diagram [4; 5], $F^{\underline{\lambda}}$, consisting of n boxes arranged in left-adjusted rows, with the ith row containing λ_i boxes for $i = 1, 2, \ldots, p$. The notation [2] $\underline{\lambda} \vdash n$ is used to signify that $\underline{\lambda}$ is a partition of n, so that $\lambda_1 + \lambda_2 + \ldots + \lambda_p = n$.

†Professor Jahn died in October, 1979 at which time he was collaborating with one of us (N.C. El-Sharkaway) on the work reported here. Calculations found amongst his papers suggested the definition of the new symmetric functions.

The diagram $F^{\underline{\lambda}}$ also consists of top-adjusted columns with the jth column containing $\tilde{\lambda}_j$ boxes for $j = 1, 2, \ldots, q$. Clearly $\tilde{\underline{\lambda}} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_q)$ is also a partition of n.

Furthermore $F^{\underline{\lambda}}$ consists of diagonally-adjusted right-angled hooks with the kth hook containing h_k boxes arranged with one box on the main diagonal of $F^{\underline{\lambda}}$ together with an arm of $a_k = \lambda_k - k$ boxes and a leg of $b_k = \tilde{\lambda}_k - k$ boxes [4; 5]. The number of boxes on the main diagonal of $F^{\underline{\lambda}}$ is the Frobenius rank r. With this notation $h_k = a_k + b_k + 1$ for $k = 1, 2, \ldots, r$, whilst $a_1 > a_2 > \ldots > a_r \geq 0$ and $b_1 > b_2 > \ldots > b_r \geq 0$.

3. Symmetric functions

The vector space, A_n, of all homogeneous symmetric functions of degree n in a set of indeterminates has as a basis the monomial symmetric functions $k_{\underline{\lambda}}$ with $\underline{\lambda} \vdash n$ [2; 6]. Bases are also provided by the Schür functions, $e_{\underline{\lambda}}$, and also by the classical symmetric functions $h_{\underline{\lambda}}$ and $a_{\underline{\lambda}}$ for $\underline{\lambda} \vdash n$.

It is to be noted that $h_m = e_m$ and $a_m = e_{1^m}$, whilst

$$h_{\underline{\lambda}} = \prod_{i=1}^{p} e_{\lambda_i} \quad \text{and} \quad a_{\tilde{\underline{\lambda}}} = \prod_{j=1}^{q} e_{1^{\tilde{\lambda}_j}} \quad . \tag{3.1}$$

These particular functions are thus products of Schür functions associated with the row and column structure of the Young diagram $F^{\underline{\lambda}}$. In exactly the same way it is possible to define $r_{1+a,1^b} = e_{1+a,1^b}$ and, more generally,

$$r_{\underline{\lambda}} = \prod_{k=1}^{r} e_{1+a_k,1^{b_k}} \tag{3.2}$$

in accordance with the hook structure of $F^{\underline{\lambda}}$.

These new symmetric functions may be evaluated in terms of the basis $e_{\underline{\lambda}}$ merely by calculating the appropriate Schür function products using the Littlewood-Richardson rule [4,5]. This yields the coefficients in the expansion:

$$r_{\lambda} = \sum_{\mu \vdash n} H_{\lambda}^{\mu} e_{\mu} . \tag{3.3}$$

This procedure is entirely analagous to that which yields the expansions:

$$h_{\lambda} = \sum_{\mu \vdash n} K_{\lambda}^{\mu} e_{\mu} \quad \text{and} \quad a_{\lambda} = \sum_{\mu \vdash n} K_{\tilde{\lambda}}^{\tilde{\mu}} e_{\mu} . \tag{3.4}$$

4. Integral bases.

A basis of A_n is said to be an integral basis if it is related to the basis k_{λ} by a matrix with integral coefficients and determinant ± 1. It is a remarkable fact [3] that the coefficients of (3.4) also appear in the expansion:

$$e_{\lambda} = \sum_{\mu \vdash n} K_{\mu}^{\lambda} k_{\mu} . \tag{4.1}$$

It follows that the matrices relating e_{λ}, h_{λ}, a_{λ} and r_{λ} to k_{λ} are K^T, KK^T, $\tilde{K}K^T$ and HK^T, where $\tilde{K}$ is obtained from K by permuting its rows and columns suitably. Clearly the coefficients of all these matrices are integers. Furthermore it is well known that if the partitions labelling the rows and column of K are ordered lexicographically then this matrix is lower triangular with all its diagonal elements 1 [3; 4; 5]. It is shown elsewhere [1] that, with a different ordering, the same is true of H. It follows that all the matrices referred to have determinant 1. This then ensures not only that e_{λ}, h_{λ} and a_{λ} each form an integral basis of A_n for $\lambda \vdash n$, but also that the new symmetric functions r_{λ} for $\lambda \vdash n$ do the same.

A further integral basis of A_n is provided by the symmetric functions $g_{\underline{\lambda}}$ for $\underline{\lambda} \vdash n$, defined by:

$$g_{\underline{\lambda}} = \sum_{\underline{\mu} \vdash n} J_{\underline{\mu}}^{\underline{\lambda}} e_{\underline{\mu}} \qquad (4.2)$$

where the matrix of coefficients J^T is such that $J = H^{-1}$. The significance of this new basis is that it is dual to that provided by $r_{\underline{\lambda}}$ with respect to the inner product with which the space A_n may be equipped [2; 6].

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BOUNDED SYMMETRIC INFORMATION FUNCTIONS

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Abstract. The axiomatic characterization of the Shannon entropy $-\sum p_i \log_2 p_i$ leads to the investigation of the functional equation (called the fundamental equation of information)

$$(1) \qquad f(x) + (1-x)f\left(\frac{y}{1-x}\right) = f(y) + (1-y)f\left(\frac{x}{1-y}\right)$$

which holds for all $x,y \in [0,1[$, $x+y \le 1$. If, in addition, $f(0) = f(1)$, then f is symmetric, i.e. $f(x) = f(1-x)$ $(x \in [0,1])$. We suppose also $f(\frac{1}{2}) = 1$ and call real valued functions f defined on $[0,1]$ and satisfying these conditions symmetric information functions (see [1]). Several authors have weakened the regularity conditions (continuity, measurability, integrability, etc.) needed for a symmetric information function (SIF) to coincide with the Shannon function S defined by

$$(2) \qquad S(x) = -x \log_2 x - (1-x)\log_2(1-x), \quad x \in [0,1]$$

(with the convention $0 \log_2 0 = 0$). We give here a new proof of the still unpublished result that every SIF, bounded on a set of positive measure, is of the form (2). We need only the general solution of (1), a lemma of Lee [1], a stability theorem [2] and a remark [7] about boundedness of all real functions, defined on $[0,1]$, on a subset of sufficiently large outer measure.

0. In [3] Diderrich proved that, if a symmetric information function (SIF) f is bounded on $[0,1]$, then $f = S$. The proof is based upon an approximation of $\log_2 N$, in order to determine f

on the rationals, and upon an iteration of (1). Finally, a theorem of Daróczy establishes f = S from the continuity of f at 0. - In [4] and also in [5] Diderrich generalized his result proving that, if a SIF f is bounded on a nonvanishing subinterval of [0,1], then f = S. In his first proof ([4]) he extends the boundedness to every closed subset in]0,1[. Then, he applies his "approximately" locally Lipschitz lemma to make local and global estimates of the boundedness of f. Certain number theoretical results are also employed for determining f on the rationals. Continuity at 0 gives f = S. In the second proof ([5]) some ideas from ergodic theory are used, in order to show that f must be continuous a.e. on [0,1] and, as a consequence, f must be measurable. Then f = S is known. In this proof no number theoretical results are applied. - Supposing that the SIF f is bounded on a subset, of positive Lebesgue measure, of [0,1] Diderrich proved in [6] that f = S. With help of his "density point method" he shows that f is bounded on a Borel subset of [0,1] whose measure is large enough and, imitating a procedure in [1] (section 3.4), he proves that f is bounded on a non-vanishing interval contained in [0,1].

In this paper we give new and, we think, simpler proofs for the above results of Diderrich. We shall denote the Lebesgue measure and the outer measure on R by μ and μ^* respectively.

1. THEOREM 1. Suppose that the symmetric information function f is bounded on a nonvanishing interval contained in [0,1]. Then f = S.

PROOF. It is proved in [1] that there exists a function

$g : [0,1] \to R$ such that $g(\frac{1}{2}) = \frac{1}{2}$,

(3) $$(x+y)f(\frac{x}{x+y}) = g(x) + g(y) - g(x+y) \qquad x,y\in[0,1],\ 0<x+y\le 1$$

and

(4) $$g(xy) = xg(y) + yg(x) \qquad x,y\in[0,1].$$

Define the function g_1 on $[0,1]$ by

(5) $$g_1(x) = 2g(\frac{x}{2}+\frac{1}{4}) + 2x[g(\frac{1}{4})-g(\frac{3}{4})] - 2g(\frac{1}{4}).$$

Then $g_1(0) = g_1(1) = 0$. Let $\Delta = \{(x,y): x,y,x+y\in[0,1]\}$. If $(x,y)\in\Delta$ then, by (5) and (3), we have

$$g_1(x) + g_1(y) - g_1(x+y) = 2g(\frac{x}{2}+\frac{1}{4})+2g(\frac{y}{2}+\frac{1}{4})-2g(\frac{x+y}{2}+\frac{1}{4})-2g(\frac{1}{4}) =$$
$$= 2[g(\frac{x}{2}+\frac{1}{4})+g(\frac{y}{2}+\frac{1}{4})-g(\frac{x+y+1}{2})] - 2[g(\frac{x+y}{2}+\frac{1}{4})+g(\frac{1}{4})-g(\frac{x+y+1}{2})] =$$
$$= (x+y+1)[f(\frac{2x+1}{2(x+y+1)})-f(\frac{2(x+y)+1}{2(x+y+1)})] .$$

Since f is bounded on a nonvanishing interval, f is also bounded on $[\frac{1}{4},\frac{3}{4}]$ (see [1]), thus we get that

(6) $$|g_1(x)+g_1(y)-g_1(x+y)| < \delta$$

for some $\delta\in R$ and for all $(x,y)\in\Delta$. Let g_2 be the periodic extension of g_1 to R with the period 1 and define the function G on R^2 by

$$G(x,y) = g_2(x) + g_2(y) - g_2(x+y) .$$

We will show that G is bounded on R^2. Since $G(x+1,y+1) = G(x,y)$ for all $(x,y)\in R^2$ and, by (6), G is bounded on Δ, it is enough to prove the boundedness of G on the set $\{(x,y): x,y\in[0,1], 1<x+y\}$. This, however, follows from the boundedness of G on Δ and from the following identity

$$G(x,y) = G(x,1-x)+G(y,1-y)-G(1-x,1-y)-G(2-x-y,x+y-1) .$$

Since G is bounded on R^2, applying Theorem 1.2 of [2], we obtain that g_2 is the sum of a function bounded on R and a function

$d: R \to R$ satisfying $d(x+y) = d(x)+d(y)$ for all $x,y \in R$ (i.e. d is an additive function). According to the definition of g_2 and (5), we get

(7) $\qquad g(x) = g^*(x) + d(x) \qquad x \in [\frac{1}{4},\frac{3}{4}]$

where $g^*: [\frac{1}{4},\frac{3}{4}] \to R$ is a bounded function. Obviously, we may suppose here that $d(1) = 0$. Define the function D on R^2 by $D(x,y) = xd(y)+yd(x)-d(xy)$. It follows from (7) and (4) that

(8) $\qquad D(x,y) = g^*(xy) - xg^*(y) - yg^*(x)$

holds for all $x,y \in [\frac{1}{2},\frac{3}{4}]$. Since D is a symmetric biadditive function and, by (8), bounded on $[\frac{1}{2},\frac{3}{4}]^2$, and since $D(1,y) = 0$ for all $y \in R$, we have $D \equiv 0$, i.e. d is a real derivation. Thus the function $g-d|[0,1]$ satisfies (4) and, by (7), it is bounded on $[\frac{1}{4},\frac{3}{4}]$. Therefore

(9) $\qquad g(x) - d(x) = cx \log_2 x$

for some $c \in R$ and for all $x \in [0,1]$. Since $g(\frac{1}{2}) = \frac{1}{2}$ and $d(\frac{1}{2}) = 0$ we have that $c = -1$. Finally (3), (9) and (2) imply that

$$f(x) = g(x)+g(1-x) = S(x)+d(x)+d(1-x) = S(x)+d(1) = S(x)$$

for all $x \in [0,1]$.

<u>2.</u> THEOREM 2. <u>Let</u> f <u>be a symmetric information function. Suppose that there exist</u> $K \in R$ <u>and a set of positive Lebesgue measure</u> E <u>in</u> [0,1] <u>such that</u> $|f(x)| \le K$ <u>for all</u> $x \in E$. <u>Then</u> $f = S$.

PROOF. Since f is symmetric, we may assume that $E \subset [0,\frac{1}{2}]$. Define $\phi(x) = \mu(E \cap (1-x)E)$, $x \in [0,1]$, which is continuous and so $\phi(0) = \mu E > 0$ implies that there exists $\frac{1}{2} \ge \beta > 0$ such that $\phi(x) > 0$ for all $x \in [0,\beta]$. Let $0 < \alpha < \beta$ be fixed and define the sets

$$B_x = E \cap (1-x)E,\ x \in [0,\beta] \text{ and } C_x = \frac{x}{1-B_x},\ x \in [\alpha,\beta] .$$

Then C_x is measurable and $B_x \subset E \subset [0,\frac{1}{2}]$ implies that $C_x \subset [x,2x] \subset [\alpha,1]$. Define ψ on $[\alpha,\beta]$ by $\psi(x) = C_x$. We have for all $x \in [\alpha,\beta]$

$$\psi(x) = \int_x^{2x} \chi_{C_x}(t)dt = \int_x^{2x} \chi_{B_x}(1-\frac{x}{t})dt = \int_0^{\frac{1}{2}} \chi_{B_x}(u)\frac{x}{(1-u)^2}\,du \geq$$

$$\geq x\int_0^{\frac{1}{2}} \chi_{B_x}(\mu)d\mu = x\phi(x)>0.$$

(χ_{C_x} and χ_{B_x} are the characteristic functions of C_x and B_x, respectively). Therefore

$$\gamma = \inf_{x\in[\alpha,\beta]} \psi(x) \geq \inf_{x\in[\alpha,\beta]} x\phi(x) = x_0\phi(x_0)$$

for some $x_0 \in [\alpha,\beta]$. Thus $\gamma>0$. Let $E_n = \{x\in[0,1]: |f(x)|\leq n\}$, $n=1,2,\ldots$. Then $E_n \subset E_{n+1}, n=1,2,\ldots$ and $\bigcup_{n=1}^{\infty} E_n = [0,1]$, and so $\lim_{n\to\infty} \mu^* E_n = 1$. It follows that there is a natural number n_0 such that $\mu^* E_{n_0} > 1-\gamma$. Suppose that $C_x \cap E_{n_0} = \emptyset$ for some $x \in [\alpha,\beta]$. Then $E_{n_0} \subset [0,1]\setminus C_x$ which implies that $1-\gamma<\mu^*E_{n_0} \leq 1-\psi(x)$. This is a contradiction. Consequently, for all $x\in[\alpha,\beta]$ there exists $t_x \in C_x \cap E_{n_0}$, therefore $|f(t_x)|\leq n_0$ and $t_x = \frac{x}{1-y_x}$ where $y_x \in B_x$. It follows that $y_x \in E$ and $\frac{y_x}{1-x} \in E$. Finally, using the equation (1), we have

$$|f(x)| = |f(y_x)+(1-y_x)f(\frac{x}{1-y_x})-(1-x)f(\frac{y_x}{1-x})| \leq$$

$$\leq |f(y_x)|+|f(t_x)|+|f(\frac{y_x}{1-x})| \leq K+n_0+K$$

for all $x\in[\alpha,\beta]$. That is, f is bounded on $[\alpha,\beta]$ and Theorem 1 completes the proof.

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DEGREE POLYNOMIALS FOR THE ORTHOGONAL GROUPS OVER GF(2)

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The degree of each ordinary character of the orthogonal groups $G_n = O_{2n+1}(2)$, and of their maximal subgroups $H_n = O_{2n}(2,+) = G_n^+$ and $K_n = O_{2n}(2,-) = G_n^-$, is a specialization for $z = N = 2^n$ of a degree polynomial $P_\ell{}^j(z)/d_j^\tau$ or $P_\ell{}^{j\sigma}(z)/d_j$ having 2ℓ distinct roots $\pm 2^{r-1}$, where $\tau, \sigma = +$ or $-$ and d_j^τ, d_j are respectively the codegrees (group order/degree) of a parent character $\phi^{j\tau}$ or ϕ^j from which the monic polynomial $P_\ell{}^j(z)$ or $P_\ell{}^{j\sigma}(z)$ is derived. Such "level ℓ" characters of type ℓ_m^τ have sign τ on the transposition class C_t and are derived from parents of level ℓ-m in G_ℓ^τ for G_n or in G_ℓ for G_n^σ, with monic polynomials $p_{\ell-m}{}^{j\tau}(z)$ and $p_{\ell-m}{}^j(z)$ respectively. Parent polynomials and generic degree formulas, together with numerical degrees and values on the class C_t for n = 4 are given below for $\ell < 5$, $\ell + m \leq 5$. Tables 1 and 2 list generic degrees of G_n-characters of types ℓ_m^+ and ℓ_m^-, while Table 3 lists degrees for H_n and K_n, and for H_4 (the central factor group of the Weyl group E_8) and K_4.

The monic polynomials for G_n characters are coded by a word of $w = \ell + m$ letters, having r^{th} letter h, k, g, or i according as $z-2^{r-1}$, $z+2^{r-1}$, both, or neither is a factor. The five level 1 characters of G_n, their degree symbols, and degrees for G_n, G_4 and G_5 are:

G_n character :	3_1	U	V	X	Y
Degree symbol:	g/3	hh/6	kk/6	kh/2	hk/2
G_n degree:	$(N^2-1)/3$	$(N-1)(N-2)/6$	$(N+1)(N+2)/6$	$\frac{1}{2}(N+1)(N-2)$	$\frac{1}{2}(N-1)(N+2)$
G_4 degree:	85	35	51	119	135
G_5 degree:	341	155	187	495	527

Replacing z by -z, N by -N, or h by k and k by h, changes a character formula to that of its mate. Thus U and V, X and Y are mates in G_n, and character 3_1, labeled for a corresponding class, is self paired. Each generic character of $H_n = G_n^+$ of degree $P_\ell^{j+}(N)/d_j$ has a mate in $K_n = G_n^-$ of degree $P_\ell^{j-}(N)/d_j = P_\ell^{j+}(-N)/d_j$. However, degeneracies occur when n = w-1. Then a generic character with degree factor z+N but not z-N yields for $z = N = 2^n$ an ordinary character called a "widow" whose mate is a "ghost" of degree 0. The same widow may arise from two distinct generic characters whose parents are mates. For example, s in G_1 has degree hk/2 and also kk/6, and represents for $n = 1$ the equal widows Y and V whose mates vanish.

Each ordinary character $\phi^{j\tau}$ of G_ℓ^τ (H_ℓ or K_ℓ) of level ℓ-m generates a unique generic character of G_n of type ℓ_m^τ, and if $\phi^{j\tau}$ is a widow (like 4_r^+ of H_3 in Table 1) both of its formulas yield the same pair of mated characters. Each ordinary character ϕ^j of G_ℓ of level ℓ-m generates a tetrad of generic characters with word length $w = \ell+m+1$ of which one pair yields for $z = N$ the associated characters χ^{j+} and $\bar{1}\chi^{j+}$ of $G_n^+ = H_n$ and the other pair yields their mates in $G_n^- = K_n$ (see Table 3). The three pairs of level 1 associated characters of H_n have degrees hh/6, ig/3 and hik/6, while their mates in K_n have degrees kk/6, ig/3 and kih/6.

We define $P_m^*(z) = \prod_{r=1}^{m} (z^2-4^{r-1})$, and $L = 2^\ell$. Then from the polynomials $p_{\ell-m}^{j\tau}(z)$ of parents of C_t sign σ in G_ℓ^τ, and from the polynomials $p_{\ell-m}^{j}(z)$ of parents of C_t sign τ in G_ℓ, we can construct the generic polynomials $P_\ell^{j}(z)$ for G_n and $P_\ell^{j\sigma}(z)$ for G_n^σ, as described in [1].

$$G_n: \quad P_\ell^j(z) = P_m^*(z)\, M^{2\ell-2m}\, p_{\ell-m}^{j\tau}(z/M)\ (z-LM)/(z-\tau M), \quad M = -\sigma 2^{m-1}.$$

$$G_n: \quad P_\ell^{j\sigma}(z) = P_m^*(z)\, M^{2\ell-2m}\, p_{\ell-m}^{j}(z/M)\ (z-LM)/(z+\sigma), \quad M = -\sigma\tau 2^{m}.$$

Formulas for the values χ_t^j and $\chi_t^{j\sigma}$ of these characters on class C_t, given in right hand columns of the tables, are expressed in terms of the sums s and s^σ of the roots of the degree polynomials by the following formulas derived in [1].

$$C_t \text{ of } G_n: \quad \chi_t^j = \chi_1^j(\tau(N+s)N/L - 1)/(N^2 - 1)$$

$$C_t \text{ of } G_n^\sigma: \quad \chi_t^{j\sigma} = \chi_1^{j\sigma}\tau(1+ s^\sigma/(N-\sigma))/L$$

Generic degree polynomials have also been constructed for level with word lengths $w \leq 6$, from which degrees were found for the 64 characters of H_5, 155 of K_5 and 198 of G_5. Those for H_5 check ith the character table completed using other techniques.

Table 1. Degrees for $G_n = O_{2n+1}(2)$ and G_4 of characters positive on C_t.

Parent in $O^+_{2\ell}(2)$ Label	Degree	Word	ℓ^+_m Type	Generic degree Word/d^+_j	Mate Word	G_4-character Label	Degree	χ^j_t	G_4-mate Label	Degree	χ^j_t
H_0 1	1	i	0^+_0	i/1	-	1	1	1	-	-	-
H_1 1	1	i	1^+_1	hk/2	kh	Y	135	63	X	119	63
H_2 3_{11}	2	hgk	2^+_0	gg/36	-	3_{11}	1785	441	-	-	-
3_2	4	hgk	2^+_0	gg/18	-	3_2	3570	882	-	-	-
r	4	hik	2^+_1	gig/18	-	r	3400	840	-	-	-
3_1	1	hh	2^+_1	gkk/72	ghh	3_1V	1275	195	3_1U	595	203
s	4	ig	2^+_1	hgk/18	kgh	Y_2	4200	840	X_2	2856	840
1	1	i	2^+_2	ghik/72	gkih	Y'	1190	182	X'	510	174
H_3 3_15_1	42	hggk	3^+_0	$ggg/2^6 15$	-	3_15_1	16065	1953	-	-	-
7_1	90	hggk	3^+_0	$ggg/2^6 7$	-	7_1	34425	4185	-	-	-
3_{11}	35	hgk	3^+_1	$gghk/2^7 3^2$	ggkh	$3_{11}Y$	16065	1449	$3_{11}X$	8925	1365
3_2	70	hgk	3^+_1	$gghk/2^6 3^2$	ggkh	3_2Y	32130	2898	3_2X	17850	2730
5_1	21	hgh	3^+_1	$ggkk/2^7 15$	gghh	5_1V	16065	441	5_1U	3213	693
3_1s	28	hhg	3^+_1	$gkgk/2^5 45$	ghgh	3_1V_2	18360	792	3_1U_2	4760	952
4_st	64	igg	3^+_1	hggk/630	kggh	Y_3	34560	2304	X_3	13056	2304
4^+_r	56	$h^{hk}_{kh}k$	3^+_1	$ghkg/2^4 45$	gkhg	r'Y	19040	2016	r'X	14688	2016
r	14	hik	3^+_2	$ggihk/2^6 45$	ggikh	riY	5712	336	riX	0	0
s	20	ig	3^+_2	$ghgik/2^5 63$	gkgih	Y_2'	13600	160	X_2'	0	0
H_4 3_{1111}	70	hgggk	4^+_0	$gggg/2^{12}3^5 5$	-	3_{1111}	595	35	-	-	-
3_{211}	420	hgggk	4^+_0	$gggg/2^{11}3^4 5$	-	3_{211}	3570	210	-	-	-
3_{22}	1400	hgggk	4^+_0	$gggg/2^{10}3^5$	-	3_{22}	11900	700	-	-	-
3_{31}	1680	hgggk	4^+_0	$gggg/2^9 3^4 5$	-	3_{31}	14280	840	-	-	-
3_4	4480	hgggk	4^+_0	$gggg/2^6 3^5 5$	-	3_4	38080	2240	-	-	-
3_19_1	3150	hgggk	4^+_0	$gggg/2^{12}3^3$	-	3_19_1	26775	1575	-	-	-
5_{11}	1134	hgggk	4^+_0	$gggg/2^{12}75$	-	5_{11}	9639	567	-	-	-
5_2	4536	hgggk	4^+_0	$gggg/2^{10}75$	-	5_2	38556	2268	-	-	-
15_1	5670	hgggk	4^+_0	$gggg/2^{12}15$	-	15_1	48195	2835	-	-	-
3_15_1	2835	hggk	4^+_1	$ggghk/2^{13}15$	gggkh	3_15_1Y	32130	882	3_15_1X	0	0
7_1	6075	hggk	4^+_1	$ggghk/2^{13}7$	gggkh	7_1Y	68850	1890	7_1X	0	0
$3_{11}s$	2100	hgkg	4^+_1	$gghgk/2^{11}3^4$	ggkgh	$3_{11}Y_2$	28560	336	$3_{11}X_2$	0	0
3_2s	4200	hgkg	4^+_1	$gghgk/2^{10}3^4$	ggkgh	3_2Y_2	57120	672	3_2X_2	0	0

Table 2. Degrees for $G_n = O_{2n+1}(2)$ and G_4 Of characters <u>negative</u> on C_t.

Parent in $O^-_{2\ell}(2)$				ℓ^-_m	Generic degree	Mate	G_4-character χ^j_t			G_4-mate χ^j_t		
	Label	Degree	Word	Type	Word/d^-_j	Word	Label	Degree		Label	Degree	
K_1	3_1	2	kk	1^-_0	g/3	-	3_1	85	-43	-	-	-
	1	1	i	1^-_1	kk/6	hh	V	51	-21	U	35	-21
K_2	5_1	6	kgk	2^-_0	gg/20	-	5_1	3213	-819	-	-	-
	3_1	5	kk	2^-_1	ghk/24	gkh	3_1Y	2975	-665	3_1X	2295	-657
	s	4	ig	2^-_1	kgk/30	hgh	V_2	2856	-504	U_2	1512	-504
	1	1	i	2^-_2	gkik/120	ghih	V'	918	-90	U'	238	-98
K_3	3_{111}	10	kggk	3^-_0	$ggg/2^{6}3^{4}$	-	3_{111}	2975	-385	-	-	-
	3_{21}	20	kggk	3^-_0	$ggg/2^{5}3^{4}$	-	3_{21}	5950	-770	-	-	-
	3_3	80	kggk	3^-_0	$ggg/2^{3}3^{4}$	-	3_3	23800	-3080	-	-	-
	9_1	90	kggk	3^-_0	$ggg/2^{6}3^{2}$	-	9_1	26775	-3465	-	-	-
	3_1r	60	kgik	3^-_1	$ggig/2^{5}3^{3}$	-	3_1r	14280	-1848	-	-	-
	3_{11}	15	kgh	3^-_1	$ggkk/2^{7}3^{3}$	gghh	$3_{11}V$	8925	-315	$3_{11}U$	1785	-399
	3_2	30	kgh	3^-_1	$ggkk/2^{6}3^{3}$	gghh	3_2V	17850	-630	3_2U	3570	-798
	5_1	81	kgk	3^-_1	$gghk/2^{7}5$	ggkh	5_1Y	28917	-2835	5_1X	16065	-2583
	3_1s	60	kkg	3^-_1	$ghgk/2^{5}3^{3}$	gkgh	3_1Y_2	23800	-1960	3_1X_2	10200	-1800
	$4_s t$	64	igg	3^-_1	kggk/810	hggh	V_3	30464	-1792	U_3	8960	-1792
	4^-_r	24	$k^{kk}_{hh}k$	3^-_1	$gkkg/2^{4}3^{3}5$	ghhg	r'V	8160	-672	r'U	3808	-672
	3_1	15	kk	3^-_2	$gghik/2^{7}3^{3}$	ggkih	$3_1Y'$	7140	-252	$3_1X'$	0	0
K_4	$3_{11}5_1$	2142	kgggk	4^-_0	$gggg/2^{12}45$	-	$3_{11}5_1$	16065	-1071	-	-	-
	$3_2 5_1$	4284	kgggk	4^-_0	$gggg/2^{11}45$	-	$3_2 5_1$	32130	-2142	-	-	-
	$3_1 7_1$	4590	kgggk	4^-_0	$gggg/2^{12}21$	-	$3_1 7_1$	34425	-2295	-	-	-
	17_1	5670	kgggk	4^-_0	$gggg/2^{12}17$	-	17_1	42525	-2385	-	-	-
	$17'_1$	5670	kgggk	4^-_0	$gggg/2^{12}17$	-	$17'_1$	42525	-2385	-	-	-
	3_{111}	595	kggk	4^-_1	$ggghk/2^{13}3^{4}$	gggkh	$3_{111}Y$	5950	-210	$3_{111}X$	0	0
	3_{21}	1190	kggk	4^-_1	$ggghk/2^{12}3^{4}$	gggkh	$3_{21}Y$	11900	-420	$3_{21}X$	0	0
	3_3	4760	kggk	4^-_1	$ggghk/2^{10}3^{4}$	gggkh	3_3Y	47600	-1680	3_3X	0	0
	9_1	5355	kggk	4^-_1	$ggghk/2^{13}3^{2}$	gggkh	9_1Y	53550	-1890	9_1X	0	0
	5_1s	4284	kgkg	4^-_1	$gghgk/2^{11}45$	ggkgh	5_1Y_2	51408	-1008	5_1X_2	0	0
	$3_1 4_s t$	3264	kkgg	4^-_1	$ghggk/2^{7}945$	gkggh	3_1Y_3	43520	-512	3_1X_3	0	0
	z=ss	4096	iggg	4^-_1	kgggk/96390	hgggh	V_4	65536	0	U_4	0	0

Table 3. Degrees for $H_n = O^+_{2n}(2)$ and $K_n = O^-_{2n}(2)$. Values for $n = 4$.

Parent in $O_{2\ell+1}(2)$				ℓ^τ_m Generic H_n-degree			K_n	O^+_8 degree	O^-_8	Character
	Label	Degree	Word	Type	Label	H-word/d_j	K-word	H_4	K_4	on C_t
G_0	1	1	i+	C^+_o	i	i/1	i	1	1	1
G_1	3_1	1	g-	1^-_0	3_1	hh/6	kk	35	51	21
	s	2	${}^{h}_{k}k^{+}_{-}$	1_o	s	ig/3	ig	84	84	42
	1	1	i+	1^+_1	r	hik/6	kih	50	34	20
G_2	3_{11}	5	gg+	2^+_o	3_{11}	hgk/144	kgh	525	357	105
	3_2	10	gg+	2^+_o	3_2	hgk/72	kgh	1050	714	210
	5_1	9	gg-	2^-_o	5_1	hgh/80	kgk	567	1071	189
	3_1s	10	$g^{k}_{h}k^{+}_{-}$	2_o	3_1s	hhg/72	kkg	700	1020	210
	s_2	16	${}^{h}_{k}gk^{+}_{-}$	2_o	4_st	igg/45	igg	1344	1344	336
	3_1	5	g-	2^-_1	3_1r	hgih/144	kgik	210	714	84
	U	1	hh-	2^-_1	8^-_rt	hhhh/720	kkkk	28	204	10 ± 4
	V	5	kk-	2^-_1	4^-_r	hkkh/144	khhk	300	476	94 ∓ 4
	X	5	kh+	2^+_1	8^+_rt	hhkk/144	kkhh	700	204	74 ∓ 4
	Y	9	hk+	2^+_1	4^+_r	hkhk/80	khkh	972	476	158 ± 4
	1	1	i+	2^+_2	rr	hgiik/720	kgiih	168	0	0
G_3	3_{111}	35	ggg-	3^-_o	3_{111}	hggh/$2^9 3^4$	kggk	175	595	35
	3_{21}	70	ggg-	3^-_o	3_{21}	hggh/$2^8 3^4$	kggk	350	1190	70
	3_3	280	ggg-	3^-_o	3_3	hggh/$2^6 3^4$	kggk	1400	4760	280
	9_1	315	ggg-	3^-_o	9_1	hggh/$2^9 3^2$	kggk	1575	5355	315
	$3_1 5_1$	189	ggg+	3^+_o	$3_1 5_1$	hggk/$2^9 15$	kggh	2835	1071	189
	7_1	405	ggg+	3^+_o	7_1	hggk/$2^9 7$	kggh	6075	2295	405
	$3_{11}s$	210	$gg^{h}_{k}k^{+}_{-}$	3_o	$3_{11}s$	hgkg/$2^8 3^3$	kghg	2100	1428	210
	3_2s	420	$gg^{h}_{k}k^{+}_{-}$	3_o	3_2s	hgkg/$2^7 3^3$	kghg	4200	2856	420
	5_1s	378	$gg^{k}_{h}k^{+}_{-}$	3_o	5_1s	hghg/$2^8 15$	kgkg	2268	4284	378
	3_1s_2	336	$g^{k}_{h}gk^{+}_{-}$	3_o	$3_1 4_s t$	hhgg/$2^5 3^3 5$	kkgg	2240	3264	336
	s_3	512	${}^{h}_{k}ggk^{+}_{-}$	3_o	z=ss	iggg/2835	iggg	4096	4096	512
	3_{11}	105	gg+	3^+_1	$3_{11}r$	hggik/$2^9 3^3$	kggih	2100	0	0
	3_2	210	gg+	3^+_1	3_2r	hggik/$2^8 3^3$	kggih	4200	0	0
	5_1	189	gg-	3^-_1	5_1r	hggih/$2^9 15$	kggik	0	4284	0
	r	168	gig+	3^+_1	4^+_{rr}	hgigk/$2^6 3^3 5$	kgigh	2688	0	0

Table 3 continued. Degrees for H_n and K_n with values for n = 4.

Parent in G_ℓ				ℓ_m^τ	Generic H_n-degree		K_n	O_8^+ degree	O_8^-	Character
	Label	Degree	Word	Type	Label	H-word/d_j	K-word	H_4	K_4	on C_t
G_3	3_1U	21	ghh+	3_1^+	$3_1 8_r^- t$	hgkkk/$2^9 3^3 5$	kghhh	840	0	0
	3_1V	105	gkk+	3_1^+	$3_1 4_r^-$	hghhk/$2^9 3^3$	kgkkh			
	3_1X	105	gkh-	3_1^-	$3_1 8_r^+ t$	hgkhh/$2^9 3^3$	kghkk	0	2856	0
	3_1Y	189	ghk-	3_1^-	$3_1 4_r^+$	hghkh/$2^9 15$	kgkhk			
	U_2	56	hgh-	3_1^-	$8_z^- t$	hhghh/$2^6 3^4 5$	kkgkk	0	2176	0
	V_2	216	kgk-	3_1^-	4_z^-	hkgkh/$2^6 105$	khghk			
	X_2	120	kgh+	3_1^+	$8_z^+ t$	hhgkk/$2^6 3^3 7$	kkghh	3200	0	0
	Y_2	280	hgk+	3_1^+	4_z^+	hkghk/$2^6 3^4$	khgkh			
	s'	84	$g_k^h ik_-^+$	3_1	$4_r s$	hgkig/$2^7 3^3 5$	kghig	0	0	0
G_4	3_{1111}	595	gggg+	4_o^+	3_{1111}	hgggk/$2^{16} 3^5 5$	kgggh	70	0	0
	3_{211}	3570	gggg+	4_o^+	3_{211}	hgggk/$2^{15} 3^4 5$	kgggh	420	0	0
	3_{22}	11900	gggg+	4_o^+	3_{22}	hgggk/$2^{14} 3^5$	kgggh	1400	0	0
	3_{31}	14280	gggg+	4_o^+	3_{31}	hgggk/$2^{13} 3^4 5$	kgggh	1680	0	0
	3_4	38080	gggg+	4_o^+	3_4	hgggk/$2^{10} 3^5 5$	kgggh	4480	0	0
	$3_1 9_1$	26775	gggg+	4_o^+	$3_1 9_1$	hgggk/$2^{16} 3^3$	kgggh	3150	0	0
	5_{11}	9639	gggg+	4_o^+	5_{11}	hgggk/$2^{16} 75$	kgggh	1134	0	0
	5_2	38556	gggg+	4_o^+	5_2	hgggk/$2^{14} 75$	kgggh	4536	0	0
	15_1	48195	gggg+	4_o^+	15_1	hgggk/$2^{16} 15$	kgggh	5670	0	0
	$3_{11} 5_1$	16065	gggg-	4_o^-	$3_{11} 5_1$	hgggh/$2^{16} 45$	kgggk	0	2142	0
	$3_2 5_1$	32130	gggg-	4_o^-	$3_2 5_1$	hgggh/$2^{15} 45$	kgggk	0	4284	0
	$3_1 7_1$	34425	gggg-	4_o^-	$3_1 7_1$	hgggh/$2^{16} 21$	kgggk	0	4590	0
	17_1	42525	gggg-	4_o^-	17_1	hgggh/$2^{16} 17$	kgggk	0	5670	0
	$17_1'$	42525	gggg-	4_o^-	$17_1'$	hgggh/$2^{16} 17$	kgggk	0	5670	0
	$3_{111} s$	5950	$ggg_h^k k_-^+$	4_o	$3_{111} s$	hgghg/$2^{15} 3^5$	kggkg	0	0	0
	$3_{21} s$	11900	"	4_o	$3_{21} s$	hgghg/$2^{14} 3^5$	kggkg	0	0	0
	$3_3 s$	47600	"	4_o	$3_3 s$	hgghg/$2^{12} 3^5$	kggkg	0	0	0
	$9_1 s$	53550	"	4_o	$9_1 s$	hgghg/$2^{15} 3^3$	kggkg	0	0	0
	$3_1 5_1 s$	32130	$ggg_k^h k_-^+$	4_o	$3_1 5_1 s$	hggkg/$2^{15} 45$	kgghg	0	0	0
	$7_1 s$	68850	"	4_o	$7_1 s$	hggkg/$2^{15} 21$	kgghg	0	0	0

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LIE ALGEBRA SUBJOINING

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Abstract: In [6] Patera, Sharp, and Slansky introduced a new relation between semi-simple Lie algebras, subjoining, which appears as a generalization of the subalgebra relation. Already in [5] Patera and Sharp have used this concept to considerably facilitate the computations of generating functions which appear in their work on group representations. Up to now this relation has had a somewhat empirical flavour and has lacked a precise definition. In this paper we outline a theoretical basis for understanding this notion, give a precise definition, and announce a classification of the equal rank maximal subjoinings.

1. Introduction: Throughout, $\underline{g}$ and $\underline{g}'$ are semi-simple Lie algebras over the complex field $\mathbb{C}$, $\underline{h}$ and $\underline{h}'$ are Cartan subalgebras, and P and P' are the corresponding weight lattices.

If $\underline{g}'$ is a subalgebra of $\underline{g}$ then any representation of $\underline{g}$ is by restriction a representation of $\underline{g}'$. In general a finite-dimensional irreducible representation of $\underline{g}$ will split or branch when restricted to $\underline{g}'$, so that it becomes a certain sum of irreducible representations of $\underline{g}'$. What was observed in [6] is that even when $\underline{g}'$ is not a subalgebra of $\underline{g}$, the $\underline{g}$-representations may appear to split into sums and differences of irreducible $\underline{g}'$-representations. We will not stop to indicate how such a phenomenon was observed but pass directly to the algebraic apparatus in which subtraction of modules makes sense, namely the Grothendieck or representation ring $R(\underline{g})$ of $\underline{g}$.

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2. <u>The Representation Ring</u>: [2, 3]. For $\underline{g}$ semi-simple, the representation ring has the following simple description. Let $\{E_i \mid i \in I\}$ be the set of all isomorphism classes of finite-dimensional irreducible $\underline{g}$-modules. Let $R(\underline{g})$ be the free abelian group on the E_i as generators. Each finite-dimensional $\underline{g}$-module is uniquely isomorphic to some finite sum $\Sigma n_i E_i$, for some positive integers n_μ, whereby it is represented in $R(\underline{g})$. Multiplication is defined by $M . N = M \otimes N$.

Let $\mathbb{Z}[P]$ be the ring of finite formal exponential sums $\{\Sigma n_\lambda e(\lambda) \mid \lambda \in P, n_\lambda \in \mathbb{Z}\}$ with component-wise addition and multiplication $e(\lambda)\, e(\mu) = e(\lambda + \mu)$. The Weyl group W of $\underline{g}$ acts on P, hence $\mathbb{Z}[P]$. $\mathbb{Z}[P]^W$, the subring of W-invariants, is isomorphic to the ring of polynomials in rank ($\underline{g}$) variables. Each finite-dimensional $\underline{g}$-module E has a weight space decomposition ΣE^μ and the character mapping $ch : R(\underline{g}) \to \mathbb{Z}[P]^W$, $ch : E \to \Sigma \dim (E^\mu)\, e(\mu)$, is an isomorphism of rings [2, Ch. 7, §7]. Furthermore this is an isomorphism of λ-rings.

We recall that a λ-structure on a commutative ring A is a homomorphism λ of $(A, +)$ into the multiplicative group of all formal power series $1 + a_1 t + a_2 t^2 + \ldots$ with coefficients in A, such that $\lambda(a) = 1 + at + \ldots$ for all $a \in A$. $R(\underline{g})$ and $\mathbb{Z}[P]^W$ have canonical λ-structures [3]:

$$\lambda(E) = 1 + (E)t + (\Lambda^2 E)\, t^2 + (\Lambda^3 E)\, t^3 + \ldots.$$

$$\lambda(e(\mu)) = 1 + e(\mu)t .$$

3. <u>Subjoining</u>: Let $K(\underline{g})$ denote the field of quotients of $R(\underline{g})$. Suppose that $\underline{g}'$ is a subalgebra of $\underline{g}$. Then, by restriction, we have a mapping $res : R(\underline{g}) \to R(\underline{g}')$ which is a λ-homomorphism of rings. Furthermore the field $K(\underline{g}')$ is a finite extension of

the subfield generated by the image of res . Now this type of situation can occur even if $\underline{g}'$ is not a subalgebra of $\underline{g}$.

Definition: The semi-simple Lie algebra $\underline{g}'$ is subjoined to the semi-simple Lie algebra $\underline{g}$ if there is a λ-homomorphism $f : R(\underline{g}) \to R(\underline{g}')$ such that $K(\underline{g}')$ is a finite extension of the subfield of $R(\underline{g}')$ generated by $f(R(\underline{g}))$.

Theorem 1: (A) Let $f : R(\underline{g}) \to R(\underline{g}')$ be a subjoining. Then

(1) there exists a group homomorphism $f_o : P \to P'$ with $[P' : f_o(P)] < \infty$ such that f_o induces f through the isomorphisms $R(\underline{g}) \cong \mathbb{Z}[P]^W$, $R(\underline{g}') \cong \mathbb{Z}[P']^{W'}$. Furthermore, if $\underline{j} := \ker(f_o)$ and we set $W_D = \{w \in W \mid w\underline{j} = \underline{j}\}$ and $W_I = \{w \in W_D \mid wx \equiv x \bmod \underline{j}$ for all $x \in P\}$ then

(2) there exists an injective homomorphism $\phi : W' \to W_D/W_I$ such that

(3) for all pairs $w \in W_D$, $w' \in W'$ with $\bar{w} = \phi(w')$, $w'f_o = f_o w$ (here $\bar{\ } : W_D \to W_D/W_I$ is the natural quotient map).

(B) Conversely if (1) (2) (3) hold for some pair of maps f_o, ϕ then there exists a unique extension of f_o to a λ-homomorphism $f : \mathbb{Z}[P]^W \to \mathbb{Z}[P']^{W'}$ which, by transfer to the representation rings, is a subjoining.

(C) Processes (A) and (B) are inverse to one another.

The choice of the word subjoining is clearer if we consider the $\mathbb{Z}$-duals $\underline{h}_{\mathbb{Z}}$ and $\underline{h}'_{\mathbb{Z}}$ of P and P' . These are the coroot lattices and may be considered as subsets of $\underline{h}$ and $\underline{h}'$. The transpose map $f_o^* : \underline{h}'_{\mathbb{Z}} \to \underline{h}_{\mathbb{Z}}$ is an injection, and can be taken to be the inclusion map in the case of a subalgebra relationship.

A subjoining is proper if f_o^* (equivalently f) is not an isomorphism. A subjoining is maximal if it is not the composition of two proper subjoinings.

Here is an example: Let $\underline{g}$ be of type B_2 and $\underline{g}'$ of type $A_1 \times A_1$. The coroot systems Σ, Σ' are of types C_2 and $A_1 \times A_1$. Now C_2 contains $A_1 \times A_1$ in two ways: as the four short roots and as the four long roots. In this way we have two embeddings of $\underline{h}'_{\mathbb{Z}}$ into $\underline{h}_{\mathbb{Z}}$. The first is the standard embedding of $A_1 \times A_1$ as a subalgebra of B_2, the second is a maximal subjoining but not a subalgebra (see [6] for examples of branchings).

4. Equal Rank Maximal Subjoinings: We describe a classification of all maximal subjoinings $f : R(\underline{g}) \rightarrow R(\underline{g}')$ when rank $\underline{g}$ = rank $\underline{g}'$. The results are stated in terms of $f_o^* : \underline{h}'_{\mathbb{Z}} \rightarrow \underline{h}_{\mathbb{Z}}$. Let Σ', Σ be the coroot systems in $\underline{h}'_{\mathbb{Z}}, \underline{h}_{\mathbb{Z}}$. Thus $\Sigma := \{\frac{2\alpha}{(\alpha,\alpha)} \mid \alpha$ is a root of $\underline{g}$ relative to $\underline{h}\}$. The bilinear form will always be assumed to be scaled so that $(\alpha, \alpha) = 2$ for long roots. In particular when we dualize Σ to get $\Sigma^\vee$, $\Sigma^\vee$ will only be the original root system if there is only one root length. Otherwise a scaling factor (long, long)/(short, short) appears.

Since here $\underline{j} = 0$, W_D/W_I is W and $\phi : W' \rightarrow W$. It is a simple consequence that $\alpha' \in \Sigma' \xmapsto{f_o^*} k_{\alpha'}\alpha$ for some $\alpha \in \Sigma$, $k_{\alpha'} \in \mathbb{N}$. Identifying $\underline{h}'_{\mathbb{Z}}$ with its image in $\underline{h}_{\mathbb{Z}}$ we are reduced to the study of the relationship of Σ' to Σ. If Σ is decomposable then Σ' decomposes concordantly and all the components of Σ' coincide with components of Σ save one, which is maximally subjoined to the corresponding component of Σ'. Assume then that Σ is indecomposable.

We recall that a subroot system Δ is *closed* if $(\Delta + \Delta) \cap \Sigma = \Delta$. By the results of Borel - de Siebenthal [1] all maximal closed subroot systems of the same rank as Σ are known.

Theorem 2: *Suppose that* f *is an equal rank maximal subjoining and* Σ *is indecomposable*. *Then one of the following occurs*:

(1) Σ' *is a maximal closed subroot system of* Σ ;

(2) $\Sigma'^{\vee}$ *is a maximal closed subroot system of* $\Sigma^{\vee}$;

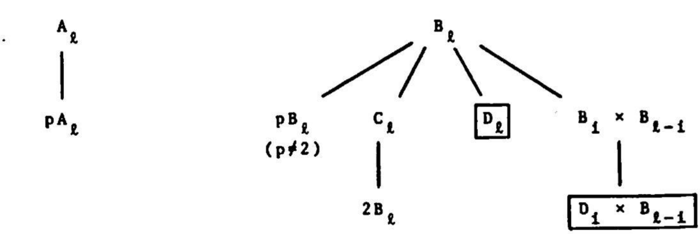
(3) $\Sigma' = \Sigma^{\vee}$;

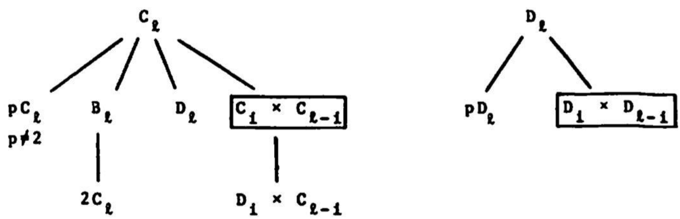
(4) $\Sigma' = p\Sigma$ *where* p *is a prime and*

$p \neq 2$ *if* Σ *is of type* B_ℓ, C_ℓ, *or* F_4

$p \neq 3$ *if* Σ *is of type* G_2 .

In any particular case some work is required to see which of these actually occur. Here is a list of the schemes of maximal subjoinings for the classical Lie algebras. The box $\square$ indicates a subalgebra relationship, p indicates an arbitrary prime number, and i varies over all possibilities which give different algebras.





Details of this work will appear elsewhere.

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AN EXAMPLE OF A DECREASING HILBERT FUNCTION

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Presented by P. Ribenboim, F.R.S.C.

We give an example of a local ring of dimension one whose hilbert function decreases temporarily. Only a few examples of decreasing hilbert functions are known ([4] p. 40).

1. Introduction. Let A be the co-ordinate ring of a reduced curve over a field k, and let M be a maximal ideal of A. Then $G(A)$ (the graded ring of A relative to M) is defined to be $\oplus_{i=0}^{\infty} M^i/M^{i+1}$. We get the same graded ring if we first localize at M, and then form the graded ring of A_M relative to the maximal ideal MA_M.

Let $\bar{A}$ be the integral closure of A, and $M_1,\ldots,M_n$ those maximal ideals of $\bar{A}$ that lie over M. Let $J = M_1 \cap M_2 \cap \ldots \cap M_n$, and $G(\bar{A}) = \oplus_{i=0}^{\infty} J^i/J^{i+1}$. If $\bar{A}/M_i \approx k_i$, $1 \leq i \leq n$, then $G(\bar{A}) = \Pi_{i=1}^{n} k_i[t_i]$. (In general k_i will be a finite algebraic extension of k. We will assume for simplicity that $k_i = k$ for all i.)

The maximal ideal M is an ordinary singularity if Proj $G(A)$ is reduced, or equivalently, if $M\bar{A} = M_1M_2\ldots M_n$ (each M_i having exponent one) and the tangent directions at the branches M_i are distinct. (See [2] for a discussion of this equivalence.)

The M^i/M^{i+1} are finite dimensional vector spaces over k. The hilbert function of M is defined by $f(i) = \dim_k(M^i/M^{i+1})$. The embedding dimension is $f(1) = \dim_k(M/M^2)$. The function $f(i)$ becomes constant if i is sufficiently large, this constant value being the multiplicity of M (in the situation described in the preceeding paragraph the multiplicity is n). Always $f(i) \leq$ multiplicity.

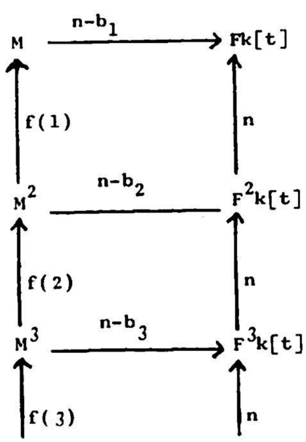
2. The example. Let $x_1,\dots,x_n$ be distinct elements of k, $1 \le r \le n$, and let $A = \{f \in k[t] \mid f(x_1) = f(x_2) = \dots = f(x_n)$ and $f'(x_j) = \Sigma_{i=1}^{r} c_{i,j} f'(x_i)$ $(r+1 \le j \le n)\}$. Let $C = (c_{i,j})$ be the $r \times n$ matrix where $c_{ij} = \delta_{ij}$ if $1 \le j \le r$ and c_{ij} is as defined above for $r+1 \le j \le n$. One checks that A is a sub-k-algebra of $k[t]$, A is of finite type over k, $\bar{A} = k[t]$, and if $M = \{f \in A \mid f(x_i) = 0\}$ then $M_i = (t-x_i)k[t]$ $(1 \le i \le n)$. Geometrically we have identified the points $t = x_i$ $(1 \le i \le n)$ of the affine line to one point M, in such a way that the tangent directions at $x_1,\dots,x_r$ are linearly independent and the tangent directions at $x_{r+1},\dots,x_n$ are linear combinations of those at $x_1,\dots,x_r$. If we work in the subspace of the Zariski tangent space $\mathrm{Hom}_k(M/M^2,k)$ spanned by the tangents at $x_1,\dots,x_r$, then the tangent directions at x_i $(1 \le i \le n)$ are the columns of C. If the columns of C are pairwise independent over k then M is an ordinary singularity. We will make this assumption throughout the rest of this paper.

Let $F = \Pi_{i=1}^{n}(t-x_i)$. Then $J = Fk[t]$ and there exist $h_i \in A$ $(1 \le i \le r)$ such that $h_i'(x_j) = \delta_{ij}$ $(1 \le i, j \le r)$.

Lemma $F^{i+1}k[t] \subset M^i$ and every element of M^i is a k-linear combination of monomials of degree i in the h_j $(1 \le j \le r)$ and an element of $F^{i+1}k[t]$.

3. The calculation of the hilbert function.

We consider the homomorphism $G(A) \to G(\bar{A}) = \Pi_{i=1}^{n} k[t]$. The image of h_i in $Fk[t]/F^2k[t] \cong k^n$ is the i^{th} row of the matrix C . Thus image $G(A)$ is the co-ordinate ring of n straight lines in r-space with directions the columns of C . (For a discussion of this ring see [1] §2.) If i is large enough then M^i/M^{i+1} maps onto $J^i/J^{i+1} = F^ik[t]/F^{i+1}k[t] \cong k^n$. But $\dim M^i/M^{i+1} \leq n$, so for large i $M^i/M^{i+1} \xrightarrow{\cong} J^i/J^{i+1}$. Thus $\operatorname{Proj} G(A) = \operatorname{Proj} G(\bar{A})$. The latter is just n copies of $\operatorname{Spec} k$, so $\operatorname{Proj} G(A)$ is reduced, verifying again that M is an ordinary singularity. Let b_i = k-dimension of the image of M^i/M^{i+1} in $F^ik[t]/F^{i+1}k[t]$. Since $F^{i+1}k[t] \subset M^i$, the codimension of M^i in $F^ik[t]$ is $n-b_i$. We now have a diagram (where the numbers denote codimension)



Thus $f(i) + n-b_i = n-b_{i+1}+n$, and $f(i) = b_i-b_{i+1}+n$. Finally $f(i+1) - f(i) = -b_{i+2} + 2b_{i+1} - b_i$, which is the negative second difference of the sequence $\{b_i\}$. The sequence $\{b_i\}$ eventually stabilizes at n , but the hilbert function will temporarily decrease if $\{b_i\}$ is concave up at the beginning. If $r \geq 3$ this is easily arranged. For if the columns of C are in generic position ([1], [3]) then $b_i = \min[\binom{i+r-1}{r-1}, n]$ and the binomial coefficient is a polynomial of degree $r-1$ in i. The simplest case of a decreasing hilbert function that can be obtained by this method appears to be the generic position case with $r = 3, n = 10$. Then $b_1 = 3$, $b_2 = 6$, $b_3 = 10$ so $f(1) = 7$, $f(2) = 6$, $f(3) = 10$.

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SOME NEW IDENTITIES IN MIXED EXTERIOR ALGEBRA

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Presented by P.G. Rooney, F.R.S.C.

1. We employ the notation and results of [1] throughout this note. The mixed exterior algebra over an n-dimensional vector space E is $\wedge E^* \otimes \wedge E = \wedge(E^*, E)$. It has a natural inner product, a bigradation and two algebraic structures. One, denoted by $(u, v) \mapsto u \cdot v$, arises from the fact that we have a tensor product of algebras and the other, denoted by $(u, v) \mapsto u \circ v$, arises from the fact that L(E) is isomorphic to $E^* \otimes E$.

Many of the important results of [1] stem from relations between the two products. In this note we shall present some new relations and use them to give an explicit representation of the Poincaré isomorphism and an orthogonal decomposition of the diagonal subalgebra.

2. The dot product makes $\Delta(E) = \sum_{p=1}^{n} \Delta_p(E)$, where $\Delta_p(E) = \wedge^p E^* \otimes \wedge^p E$, into a commutative algebra, called the diagonal subalgebra. The unit tensor $t \in \Delta_1(E)$ satisfies $t \circ z = z \circ t = z$, $z \in \Delta_1(E)$, and we write t^p for $\frac{1}{p!}$ times the dot product of t with itself p times. The linear map $u \mapsto a \cdot u$ is written $\mu(a)$ and its dual is written $i(a)$. The Poincaré isomorphism D is the map $u \mapsto i(u) t^n$.

3. For each $a \in \Delta_1(E)$, let θ_a be the derivation of $\wedge E$ induced by regarding a as an element of L(E). Then $\iota \otimes \theta_a$ is a linear map of $\wedge(E^*, E)$ whose restriction to $\Delta(E)$ we denote by λ_a. It satisfies

$$\lambda_a(z_1 \cdots z_p) = \sum_{\nu=1}^{p} z_1 \cdots a \circ z_\nu \cdots z_p \qquad z_1, \ldots, z_p \in \Delta_1(E)$$

The dual of λ_a is written ρ_a and it satisfies a similar relation with $a \circ z_\nu$ replaced by $z_\nu \circ a$.

The lemma of [1], p. 156, may be written in terms of these operators as

$$i(a) \circ \mu(b) - \mu(b) \circ i(a) = \langle a, b \rangle \iota - \lambda_{b \circ a} - \rho_{a \circ b} \qquad a, b \in \Delta_1(E)$$

If we define $\theta(a) = \lambda_a + \rho_a$, then $\theta(a)$ is self-dual and this implies

$$i(a) \circ \mu(t) - \mu(t) \circ i(a) = \Gamma(a),$$

where

$$\Gamma(a) = \langle t, a \rangle \iota - \theta(a).$$

4. The following commutation formulae hold for all $z \in \Delta_1(E)$:-

$$\lambda_z \circ \mu(t) - \mu(t) \circ \lambda_z = \mu(z) = \rho_z \circ \mu(t) - \mu(t) \circ \rho_z$$

$$\theta(z) \circ \mu(t) - \mu(t) \circ \theta(z) = 2\mu(z)$$

$$i(z) \circ \mu(t^p) - \mu(t^p) \circ i(z) = \mu(t^{p-1}) \circ \Gamma(z) - \mu(t^{p-2}) \circ \mu(z)$$

$$i(t^p) \circ \mu(z) - \mu(z) \circ i(t^p) = \Gamma(z) \circ i(t^{p-1}) - i(z) \circ i(t^{p-2}).$$

If we introduce the operators T_k, $k = 0, 1, \ldots$, defined by

$$T_k = \sum_{\nu=0}^{\infty} (-1)^\nu \mu(t^{k+\nu}) \circ i(t^\nu)$$

then the latter two formulae imply that

$$i(z) \circ T_{k+1} + T_{k-1} \circ \mu(z) = 0.$$

5. From the definition of T_k, we find that $T_k(1) = t^k$. Hence $T_k(z_1) = [T_k \circ \mu(z_1)](1) = -\, i(z_1)\, t^{k+2}$ and, by induction,

$$T_k(z_1 \cdots z_p) = (-1)^p\, i(z_1 \cdots z_p)\, t^{k+2p}, \quad z_1, \ldots, z_p \in \Delta_1(E)$$

It follows that

$$\Big[\sum_{\nu=0}^{\infty} (-1)^{\nu} \mu(t^{k+\nu}) \circ i(t^{\nu})\Big](u) = (-1)^p\, i(u)\, t^{k+2p}, \quad u \in \Delta_p(E)$$

In particular, if $k = n-2p$, we have

$$(-1)^p D u = \sum_{\nu=0}^{\infty} (-1)^{\nu} \mu(t^{n+\nu-2p})\, i(t^{\nu}) u, \quad u \in \Delta_p(E).$$

6. Let $F_p = \ker i(t) \cap \Delta_p(E)$ and $G_p = \ker \mu(t) \cap \Delta_p(E)$.

Then

$$\Delta_p(E) = \mu(t^p) F_0 \oplus \mu(t^{p-1}) F_1 \oplus \cdots \oplus F_p = i(t^{n-p}) G_n \oplus \cdots \oplus G_p,$$

where the direct sum is orthogonal, $F_p = 0$ for $p > n/2$, $G_p = 0$ for $p < n/2$ and

$$\dim F_p = \dim G_{n-p} = \binom{n}{p}^2 - \binom{n}{p-1}^2.$$

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