C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

INFORMATION FUNCTIONS ON OPEN DOMAINS II

C.T. Ng

Presented by J. Aczél, F.R.S.C.

Abstract. The n-dimensional fundamental equation of information of degree α is considered in the interior of its usual domain. Its general solution is obtained for all α different from the basic unit vectors.

Introduction. Let $D^o = \{(x,y) \mid x,y \in I \text{ with } x+y \in I\}$ where $I =]0,1[^n$. We call a function $f : I \to R$ an n-place information function of degree $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ if it satisfies the functional equation

$$(1) \qquad f(x) + (1-x)^\alpha f(\tfrac{y}{1-x}) = f(y) + (1-y)^\alpha f(\tfrac{x}{1-y})$$

on the open domain D^o. The general solution of this equation has been obtained in [1] under the assumption that $\Sigma \alpha_i \neq 1$. In this note we report the following improvement covering all α not equal to the basic unit vectors.

Theorem. Under the assumption that $\alpha \neq (1,0,0,\ldots,0)$, $(0,1,0,0,\ldots,0),\ldots,(0,0,\ldots,0,1)$, the general solution of the equation (1) on D^o is given by

$$(2) \qquad \begin{cases} f(x) = ax^\alpha + b(1-x)^\alpha - b & \text{if } \alpha \neq 0 \\ f(x) = a + L(1-x) & \text{if } \alpha = 0 \end{cases}$$

on I, where a, b are constants and L is a solution of the Cauchy equation

$$(3) \qquad L(xy) = L(x) + L(y) \quad \text{for all } x,y \in I.$$

Proof of the theorem. We refer to the proof given in [1] for fine details, but we do include the essential steps here. The function $\Delta : D^o \to R$, defined by

$$(4) \qquad \Delta(x,y) = f(x) + (1-x)f\left(\frac{y}{1-x}\right) - f(x+y),$$

can be extended uniquely to $\bar{\Delta} :]0,\infty[^n \times]0,\infty[^n \to R$ satisfying the following properties:

$$(5) \qquad \bar{\Delta}(x,y) = \bar{\Delta}(y,x)$$

$$(6) \qquad \bar{\Delta}(x,y) + \bar{\Delta}(x+y,z) = \bar{\Delta}(x,y+z) + \bar{\Delta}(y,z)$$

$$(7) \qquad \bar{\Delta}(tx,ty) = t^{\alpha}\bar{\Delta}(x,y)$$

for all x,y,z,t in $]0,\infty[^n$. We obtained in [1] the general form of $\bar{\Delta}$ by means of a diagonalization procedure under the assumption that $\Sigma\alpha_i \neq 1$. This method is now improved. By using (5) and (6) repeatedly, the following calculation

$$\begin{aligned}
&\bar{\Delta}(px,py) + \bar{\Delta}(px+py,qx+qy) \\
&= \bar{\Delta}(px,qx+(p+q)y) + \bar{\Delta}(py,qx+qy) \\
&= [\bar{\Delta}(px,qx+(p+q)y)+\bar{\Delta}(qx,(p+q)y)] - \bar{\Delta}(qx,(p+q)y) \\
&\quad + [\bar{\Delta}(qx,qy)+\bar{\Delta}(qx+qy,py)] - \bar{\Delta}(qx,qy) \\
&= [\bar{\Delta}(px,qx)+\bar{\Delta}((p+q)x,(p+q)y)] - \bar{\Delta}(qx,(p+q)y) \\
&\quad + [\bar{\Delta}(qx,(p+q)y)+\bar{\Delta}(qy,py)] - \bar{\Delta}(qx,qy) \\
&= \bar{\Delta}(px,qx) + \bar{\Delta}((p+q)x,(p+q)y) + \bar{\Delta}(qy,py) - \bar{\Delta}(qx,qy)
\end{aligned}$$

leads, for all x,y,p,q in $]0,\infty[^n$, to

$$\begin{aligned}
&\bar{\Delta}(px,py) + \bar{\Delta}(qx,qy) - \bar{\Delta}((p+q)x,(p+q)y) \\
&= \bar{\Delta}(px,qx) + \bar{\Delta}(py,qy) - \bar{\Delta}(p(x+y),q(x+y)) \quad .
\end{aligned}$$

Together with (7) we get

$$(8) \qquad [p^\alpha+q^\alpha-(p+q)^\alpha]\bar{\Delta}(x,y) = [x^\alpha+y^\alpha-(x+y)^\alpha]\bar{\Delta}(p,q)$$

for all x,y,p,q in $]0,\infty[^n$. There are two cases.

If there exist p_0,q_0 in $]0,\infty[^n$ such that $p_0^\alpha+q_0^\alpha-(p_0+q_0)^\alpha \neq 0$, then, by putting into (8) $p = p_0$, $q = q_0$, we obtain

$$(9) \qquad \bar{\Delta}(x,y) = ax^\alpha + ay^\alpha - a(x+y)^\alpha \quad \text{for all } x,y\in]0,\infty[^n$$

where $a = \bar{\Delta}(p_0,q_0)/(p_0^\alpha+q_0^\alpha-(p_0+q_o)^\alpha)$ is a constant. In particular Δ is of the form (9) on D^o. With this solution for Δ, the function f can be solved from (4) (see [1]) leading to the asserted form (2). Else $p^\alpha+q^\alpha-(p+q)^\alpha = 0$ for all p,q in $]0,\infty[^n$. In this event the function $h(p) = p^\alpha = \Pi p_i^{\alpha_i}$ is a continuous solution of the Cauchy equation $h(p)+h(q) = h(p+q)$ and therefore it must also be of the form $h(p) = \Sigma c_i p_i$, where c_i's are constants. By comparison of these two forms of h we see that both are valid if, and only if, α is a basic unit vector.

The verification that functions defined by (2) indeed satisfy (1) is straightforward. □

<u>A Generalization</u>. A function $\alpha :]0,\infty[^n \to R$ is said to be multiplicative if it satisfies

$$\alpha(xy) = \alpha(x)\alpha(y) \quad \text{for all } x,y \text{ in }]0,\infty[^n .$$

The functional equation

$$(1G) \qquad f(x) + \alpha(1-x)f\left(\frac{y}{1-x}\right) = f(y) + \alpha(1-y)f\left(\frac{x}{1-y}\right) \quad \text{on } D^o$$

with a multiplicative α is more general than (1).

By following similar proof lines and observing that wherever the power function appeared, only its multiplicative property was essential, we obtain the following generalization.

Theorem G. Under the assumption that the multiplicative function α is not additive, the general solution of (1G) is given by

$$(2G)\qquad \begin{cases} f(x) = a\alpha(x) + b\alpha(1-x) - b & \text{if } \alpha(x) \not\equiv 1 \\ f(x) = a + L(1-x) & \text{if } \alpha(x) \equiv 1 \end{cases}$$

on I, where a,b are constants and L is a solution of (3).

REFERENCE

[1] C.T. Ng, Information functions on open domain. C.R. Math. Rep. Acad. Sci. Canada 2 (1980).

Research supported by NSERC grant A-8212

Faculty of Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1

Received May 8, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

FUNCTIONS PARTIALLY CONSTANT ON RINGS OF SETS

J. Aczél, F.R.S.C.

Abstract*. A ring of sets [6] is a collection $\mathbb{B}$ of sets which contains, with any two sets, also their union and difference, thus also their intersection and the empty set 0. - In this paper we prove that certain conditions are sufficient to guarantee that a function is (partially) constant on $\mathbb{B}$ and apply the results to the mixed theory of information.

1. Theorem. Suppose

(1) $$K(E_1,E_3) = K(E_2,E_3), \text{ whenever } E_j \neq 0 \text{ are in } \mathbb{B} \text{ and } E_i \cap E_j = 0 \ (i \neq j=1,2,3).$$

Then K is partially constant on $\mathbb{B}$ in the sense that

(2) $$K(E_1,E_3) = K(E_2,E_3) =: \phi(E_3) \text{ if } E_j \neq 0,\ E_3 \cap E_k = 0 \text{ and } \exists E_k^c \neq 0 \text{ such that } E_k \cap E_k^c = E_3 \cap E_k^c = 0 \ (E_j, E_k^c \in \mathbb{B};\ j=1,2,3; k=1,2)$$

(whether $E_1 \cap E_2 = 0$ or not). The same result holds if (1) is replaced by

(3) $$K(E_1 \cup E_2, E_3) = K(E_1,E_3) \text{ if } E_j \neq 0 \text{ are in } \mathbb{B} \text{ and } E_i \cap E_j = 0 \ (i \neq j=1,2,3) .$$

(Cf. [1] for a somewhat more special result.)

Proof. If (3) holds then, interchanging E_1 and E_2, we also have $K(E_1 \cup E_2, E_3) = K(E_2,E_3)$, so (1) holds too.

Now we prove (2) from (1):

* This research has been supported in part by an NSERC grant.

$$K(E_1,E_3) = K(E_2\setminus E_1,E_3) = K(E_1\setminus E_2,E_3) = K(E_2,E_3)$$

The only cases when this does not work are when $E_1\setminus E_2 = 0$ $(E_1\subset E_2)$ or $E_2\setminus E_1 = 0$ $(E_2\subset E_1)$. Take, for instance, the first. Then

$$K(E_1,E_3) = K(E_2^c,E_3) = K(E_2,E_3). \quad \square$$

<u>Corollary</u>. <u>If</u> $K(E_1\cup E_2) = K(E_1)$ <u>for</u> $E_k \neq 0$, $E_k \in \mathbb{B}$ $(k=1,2)$, $E_1\cap E_2 = 0$, <u>then</u> $K(E_1) = K(E_2) =: \gamma$ <u>for</u> $E_k \neq 0$, $E_k \in \mathbb{B}$, $k=1,2$ (<u>whether</u> $E_1\cap E_2 = 0$ <u>or not</u>), <u>provided that there exist</u> $E_1^c \neq 0$, $E_2^c \neq 0$ <u>in</u> $\mathbb{B}$ <u>such that</u> $E_1\cap E_1^c = E_2\cap E_2^c = 0$.

<u>Notes</u>. Since no "universal set" S is supposed to be contained in $\mathbb{B}$ [such that $(E\in\mathbb{B}) \Rightarrow (E\setminus S=0)$], E_k^c is therefore, in general, not the complement of E_k, but a generalized complement. If $S\in\mathbb{B}$, then we can choose $E_k^c = (S\setminus E_3)\setminus E_k \neq 0$, if $E_k \neq S\setminus E_3$ (that is, E_k^c <u>is</u> a complement of E_k, with respect to $S\setminus E_3$ $(k=1,2)$; replace $S\setminus E_3$ by S for the Corollary). - If $E_1 = 0$ were permissible in (1) or (3), then (2) would follow immediately. The difficulty lies in the exclusion of 0 (empty sets) from (1) and (3) and is increased by the restrictions $E_1\cap E_3 = E_2\cap E_3 = 0$.

2. In the "mixed theory of information" [4, 5, 1, 2, 3] (real) numbers ("inset" measures of information) are associated to "randomized systems of events":

$$M_n\begin{pmatrix}E_1,\dots,E_n\\ p_1,\dots,p_n\end{pmatrix} \quad (E_j\in\mathbb{B},\ E_i\cap E_j=0);\ i\neq j=1,\dots,n;\ p_j>0,\ \textstyle\sum p_j=1).$$

Here the E_j are the events, the p_j the probabilities (measures may also depend upon more than one set of probabilities). The

E_j and p_j may be chosen independently, but it stands to reason to suppose at least that

(4) $E_j = 0$ <u>implies</u> $p_j = 0$ $(j=1,\dots,n)$.

This proved to be important for applications [2].

A measure is <u>recursive of degree</u> α if

$$M_n\begin{pmatrix}E_1, E_2, E_3, \dots, E_n\\ p_1, p_2, p_3, \dots, p_n\end{pmatrix} = M_{n-1}\begin{pmatrix}E_1\cup E_2, E_3, \dots, E_n\\ p_1+p_2, p_3, \dots, p_n\end{pmatrix}$$

$$+ (p_1+p_2)^\alpha M_2\begin{pmatrix}E_1 & , & E_2\\ p_1/(p_1+p_2) & , & p_2/(p_1+p_2)\end{pmatrix}$$

under the above conditions. By writing

$$f(E_1,E_2;t) = M_2\begin{pmatrix}E_1 & ,E_2\\ 1-t, & t\end{pmatrix}$$

one gets, if M_3 is also <u>partially symmetric</u>,

$$(5)\quad\begin{cases} f(E_1\cup E_2,E_3;x) + (1-x)^\alpha f(E_1,E_2;\frac{y}{1-x}) = M_3\begin{pmatrix}E_1, & E_2, E_3\\ 1-x-y, & y\ , x\end{pmatrix}\\ = M_3\begin{pmatrix}E_1, & E_3, E_2\\ 1-x-y, & x\ , y\end{pmatrix} = f(E_1\cup E_3,E_2;y) + (1-y)^\alpha f(E_1,E_3;\frac{x}{1-y})\end{cases}$$

whenever $E_j\in\mathbb{B}$, $E_i\cap E_j = 0$ $(i\neq j=1,2,3)$; $x,y\in[0,1[$, $x+y\le 1$.

In [3], equation (5) has been completely solved for all α, <u>but without (4)</u>. Of course, one has to be careful not to fix both x and y in (5), or else there would be no real variable left. If (4) is taken into consideration, $E_j = 0$ for a j restrains one real variable too (to $x=0$, $y=0$ or $x+y=1$). In [3] we have found without any restrictions (except $E_1\cap E_2=0$)

(6) $f(E_1,E_2;t) = c(E_1)F(t) + A(E_1,E_2)t^\alpha + B(E_1,E_2),$
$(t\in[0,1], 0^\alpha=0)$ for $\alpha\neq 0$

(7) $$f(E_1,E_2;t) = \begin{cases} A(E_1,E_2) & (t=0) \\ B(E_1,E_2) & (t\in]0,1[) \\ C(E_1,E_2) & (t=1) \end{cases}$$ for $\alpha=0$,

where $F(t)$ is linearly independent of 1 and t^α. (Already in proving that c is independent of E_2 we may apply our above Theorem.) Also, F satisfies

(8) $F(x) + (1-x)^\alpha F(\frac{x}{1-x}) = F(y) + (1-y)^\alpha F(\frac{x}{1-y})$ $(x,y\in[0,1[,x+y\leq 1)$

for $\alpha \neq 0$. In particular, for $\alpha \neq 0,1$,

(9) $F(x) = x^\alpha + (1-x)^\alpha - 1$ $(x\in[0,1], 0^\alpha=0)$.

We have substituted (6) into (5) with $y = 0$, compared coefficients, got equations for c, A and B, and solved them by substituting once $E_1 = 0$ and two times $E_3 = 0$. If (4) is to hold, then the latter substitution would give $x = 0$, so $x = y = 0$, and the former $1-x-y = 0$ which, with $y = 0$ would lead to $x = 1$, which is not even in the domain of (5).

However, if we substitute (6) into (5) without putting $y = 0$, but taking (8) into consideration, we get

$$c(E_1\cup E_2)F(x)+A(E_1\cup E_2,E_3)x^\alpha+B(E_1\cup E_2,E_3)$$
$$- c(E_1)F(x)+A(E_1,E_2)y^\alpha+B(E_1,E_2)(1-x)^\alpha$$
$$= c(E_1\cup E_3)F(y)+A(E_1\cup E_3,E_2)y^\alpha+B(E_1\cup E_3,E_2)$$
$$- c(E_1)F(y)+A(E_1,E_3)x^\alpha+B(E_1,E_3)(1-y)^\alpha$$
$(E_j\in\mathbb{B}, E_i\cap E_j=0;\ i\neq j=1,2,3;\ x,y\in[0,1[,\ x+y<1)$.

For $\alpha \neq 1$ we also use (9) and for all $\alpha \neq 0$ we compare the coefficients of x^{α} and also those of $F(x)$ in the above equation. Then we get exactly the same equations as in [3], but the substitutions $E_1 = 0$ or $E_3 = 0$ are now permissible (one at a time), and so the results are the same.

The situation is somewhat different and we need the above Theorem when, as in [3], we substitute (7) into (5), separately for $x,y\in]0,1[$, for $x = 0$, $y\in]0,1[$ and for $y = 1 - x$. There again $E_1 = 0$ was substituted into the equations thus obtained. From (5) it is clear now that only the combination of $x = 0$ and $E_1 = 0$, thus $1-x-y = 0$, is non-permissible by (4). This makes a new solution of $D(E_1 \cup E_2, E_3) = D(E_1, E_3)$ ($E_i \cap E_j = 0$ for $i \neq j=1,2,3$) for $D = A - B$ necessary, and this is yielded by our Theorem as $D(E_1, E_3) = \phi(E_3)$, the same result as obtained in [3]. (In (5) there are always three sets, any two of which have no common element. So, when applying the Theorem to get (2), in the cases $E_1 \subset E_2$ and $E_2 \subset E_1$ there exist E_1^c, E_2^c. In the remaining (first) case of the proof of the Theorem, no generalized complement is needed, but $E_1^c = E_2 \setminus E_1$, $E_2^c = E_1 \setminus E_2$ would do.)

So <u>the inset information functions found in [3] and only these are solutions of (5) even if (4) is supposed</u>.

REFERENCES

[1] Aczél, J., A mixed theory of information. V. How to keep the (inset) expert honest. J. Math. Anal. Appl. 75 (1980).

[2] Aczél, J., A mixed theory of information. VI. An example at last: A proper discrete analogue of the continuous Shannon measure of information (and its characterization). Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978-80), 65-72.

[3] Aczél, J., A mixed theory of information. VII. Inset information functions of all degrees. C.R. Math. Reports Acad. Sci. Canada 2 (1980).

[4] Aczél, J.; Daróczy, Z., A mixed theory of information. I. Symmetric, recursive and measurable entropies of randomized systems of events. RAIRO Inform. Théor. 12 (1978), 149-155.

[5] Aczél, J.; Kannappan, Pl., A mixed theory of information.III. Inset entropies of degree β. Inform. and Control 39 (1978), 315-322.

[6] Halmos, P.R., Measure theory. Van Nostrand, Princeton, N.J.; Toronto;London;New York, 1950.

Faculty of Mathematics,
University of Waterloo,
Waterloo, Ontario N2L 3G1, Canada

Received May 8, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

CONTOUR INTEGRAL REPRESENTATION OF CARDINAL SPLINE FUNCTIONS

Walter Schempp

Presented by P. Scherk, F.R.S.C.

The aspects of local bases and minimal properties (variational methods) in the theory of spline functions are well-known. However, this theory has also some complex analytic features. We can prove that the cardinal splines admit contour integral representations that allow an investigation of their convergence behaviour in the exponential case as well as in the logarithmic case. Moreover, a contour integral representation of the Euler-Frobenius polynomials will be established.

1. Cardinal Exponential Splines

For any integer $m \geq 1$ let $\mathfrak{S}_m(\mathbb{R};\mathbb{Z})$ denote the vector space over the field $\mathbb{C}$ of all complex cardinal spline functions of degree m on $\mathbb{R}$ having (equidistant) knots at the integer points. Moreover, let $U = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the compact unit circle.

Theorem 1. The cardinal exponential splines $s_m \in \mathfrak{S}_m(\mathbb{R};\mathbb{Z})$ of degree $m \geq 1$ and weight $h \in \mathbb{C}^\times - U$ admit the contour integral representation with transcendental meromorphic integrand

$$s_m(x) = C_m(1-\frac{1}{h})^{m+1} \frac{1}{2\pi i}\int_P \frac{e^{(x+1)z}}{(e^z-h)z^{m+1}} dz \qquad (x \in \mathbb{R}),$$

where $C_m \in \mathbb{C}$ denotes an arbitrary constant and P stands for the positively oriented boundary of any closed vertical strip in the open complex right resp. left half-plane according to the cases $|h|>1$ resp. $0<|h|<1$.

The proof follows via the inverse Laplace transform by means of a line integral representation of the basis splines (cf. [4]). As a consequence, Theorem 1 implies

Theorem 2. Let $h \in \mathbb{C}^{\times}-U$ and $x \in [-1,0]$ be given. For all $z \in \mathbb{C}$ so that $|z|<|\log|h||$, the cardinal exponential splines $s_m \neq 0$ $(m\geq 1)$ give rise to the power series expansion

$$\frac{e^{(x+1)z}}{h-e^{z}} = \sum_{m\geq o} \frac{h^{m+1}}{C_m(h-1)^{m+1}} s_m(x)z^m,$$

where $C_o = 1$ and $s_o = \frac{1}{h}$.

Another consequence of Theorem 1 and the Cauchy residue theorem is the following result (cf. Schoenberg [7], [8]):

Theorem 3. If $(z_k(h))_{k \in \mathbb{Z}}$ denotes the bi-infinite sequence of zeros of the function $z \rightsquigarrow e^{z}-h$ $(h \in \mathbb{C}^{\times}-U)$ then

$$s_m(x) = C_m(1-\frac{1}{h})^{m+1} \sum_{k \in \mathbb{Z}} \frac{e^{xz_k(h)}}{z_k^{m+1}(h)} \qquad (m\geq 1)$$

holds for all $x \in \mathbb{R}$.

2. Euler-Frobenius Polynomials

In the case when the weight $h \in \mathbb{C}^{\times}-U$ is not a root of the m-th Euler-Frobenius polynomial p_m $(m\geq 1)$ there exists one and only one cardinal exponential spline interpolator $(s_m)_{m\geq 1}$ of degree m with respect to the bilateral geometric sequence $(h^n)_{n\in\mathbb{Z}}$. For these remarkable polynomials we can prove (cf. [6]):

Theorem 4. For any $h \in \mathbb{C}^{\times}-U$ the Euler-Frobenius polynomials $(p_m)_{m\geq 1}$ admit the contour integral representation

$$p_m(h) = \frac{(h-1)^{m+1}}{h}\frac{m!}{2\pi i}\int_P \frac{e^z}{(e^z-h)z^{m+1}}\,dz \qquad (m\geq 1),$$

where the path P is defined as in Theorem 1.

Corollary 1. For all $h \in \mathbb{C}^{\times}$ the reciprocal identity

$$h^{m-1}\,p_m\!\left(\frac{1}{h}\right) = p_m(h) \qquad (m\geq 1)$$

holds.

Corollary 2. For any $h \in \mathbb{C}^{\times}-U$ the Euler-Frobenius polynomials $(p_m)_{m\geq 1}$ are generated by the power series expansion

$$\frac{(h-1)e^z}{h(h-e^z)} = \sum_{m\geq 0}\frac{p_m(h)}{(h-1)^m}\frac{z^m}{m!} \quad (|z|<|\log|h||),$$

where $p_o(h) = \frac{1}{h}$.

Corollary 3. For any $h\neq 1$ the Euler-Frobenius polynomials $(p_m)_{m\geq 1}$ admit the formal power series expansion

$$\frac{p_m(h)}{(1-h)^{m+1}} = \sum_{n\geq o} (n+1)^m h^n \qquad (m\geq 1).$$

Corollary 4. The Euler-Frobenius polynomials $(p_m)_{m\geq 1}$ satisfy the three-term recurrence relation

$$p_m(h) = (mh+1)p_m(h) - h(h-1)p_m'(h) \qquad (m\geq 1).$$

It follows that for any integer $m\geq 2$ all the roots of the Euler-Frobenius polynomial p_m are located on the open negative real half-line $\mathbb{R}_-^\times$. Therefore, Theorem 3 implies (cf. Schoenberg [7], [8])

Theorem 5. Let the weight $h \in \mathbb{C}-(U \cup \mathbb{R}_-)$ be given. Then the cardinal exponential spline interpolators $(s_m)_{m\geq 1}$ of degree m with respect to the bilateral geometric sequence $(h^n)_{n \in \mathbb{Z}}$ satisfy the pointwise convergence property

$$\lim_{m\to\infty} s_m(x) = h^x$$

for $x \in \mathbb{R}$.

In the logarithmic case the convergence behaviour of the interpolating cardinal spline functions is quite different.

3. Cardinal Logarithmic Splines

Let $h_o>1$ denote a fixed step width and $\mathfrak{h}_o$ the knot sequence $\mathfrak{h}_o = (h_o^n)_{n \in \mathbb{Z}}$ in the open positive real half-line $\mathbb{R}_+^\times$. Consider the function

$$f_o\colon \mathbb{R}_+^\times \ni x \rightsquigarrow \frac{\log x}{\log h_o} \in \mathbb{R}.$$

If $(\Gamma_m)_{m\geq 1}$ denotes the sequence of partial products in Euler's limit formula for the gamma function, we can prove

Theorem 6. The cardinal logarithmic splines $S_m \in \mathfrak{S}_m(\mathbb{R}_+^\times; \mathfrak{h}_o)$ of degree $m\geq 1$ with step width $h_o > 1$ admit the contour integral representation with transcendental meromorphic integrand

$$S_m(x) = \frac{1}{2\pi i} \int_L \Gamma_m(z) h_o^{-zf_o(m)} \frac{1-x^{-z}}{1-h_o^{-z}} \quad (x \in \mathbb{R}_+^\times),$$

where $L=L_1 \vee L_2$ denotes the positively oriented boundary of a closed vertical strip in the complex plane $\mathbb{C}$ delimited by the lines $L_1 = \{z \in \mathbb{C} \mid \operatorname{Re} z = c\}$ with $c>0$ and $L_2 = \{z \in \mathbb{C} \mid \operatorname{Re} z = d\}$ with $d \in]-1,0[$. The contour integral is independent of the particular choices of the real constants c,d.

The proof follows via the inverse Mellin transform by means of a line integral representation of the elements of the truncated power basis (cf. [5]). In this connection also see [3].

An application of Cauchy's residue theorem entails the following striking fact ("Newman-Schoenberg phenomenon" [1], [3]).

Theorem 7. The condition $\lim_{m\to\infty} S_m(x) = f_o(x)$ is satisfied at the point $x \in \mathbb{R}_+^\times$ if and only if $x \in \mathfrak{h}_o$.

In other words: The cardinal logarithmic splines $(S_m)_{m\geq 1}$ are pointwise convergent only at those points $x \in \mathbb{R}_+^\times$ where the convergence holds trivially. At all the other points of $\mathbb{R}_+^\times$ we have divergence. - For an investigation of the Newman-Schoenberg phenomenon by real transformation methods the reader is referred to the paper [2].

References

1. Newman, D.J., Schoenberg, I.J.: Splines and the logarithmic function. Pacific J. Math. 61 (1975), 241-258

2. Schempp, W.: On the convergence of cardinal logarithmic splines. J. Approx. Theory 23 (1978), 108-112

3. Schempp, W.: A note on the Newman-Schoenberg phenomenon Math. Z. 167 (1979), 1-6

4. Schempp, W.: Cardinal exponential splines and Laplace transform, to appear

5. Schempp, W.: Cardinal logarithmic splines and Mellin transform, to appear

6. Schempp, W.: A contour integral representation of Euler-Frobenius polynomials, to appear

7. Schoenberg, I.J.: Cardinal interpolation and spline functions IV. The exponential Euler splines. In: Linear Operators and Approximation I, P.L. Butzer, J.P. Kahane, B.-SZ. Nagy editors. Basel-Boston-Stuttgart: Birkhäuser-Verlag 1972

8. Schoenberg, I.J.: Cardinal spline interpolation. Regional Conference Series in Applied Mathematics, Vol. 12. Philadelphia, Pennsylvania: Society for Industrial and Applied Mathematics 1973

Lehrstuhl für Mathematik I,
Universität Siegen,
Hölderlinstraße 3,
D-5900 Siegen 21, W. Germany

Received June 6, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

A REMARK ON SCHWARZ' INEQUALITY

E. Fischer and P. Scherk, F.R.S.C.

The classical Schwarz inequality reads: Let V denote a vector space over the real field $\mathbb{R}$. Let

(1) $$\langle\ ,\ \rangle : V \to \mathbb{R}$$

denote a symmetric positive definite bilinear form. Then

$$\begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{vmatrix} \geq 0$$

for all $x_1, x_2 \in V$. Equality holds if and only if x_1 and x_2 are linearly dependent.

It is well known that this theorem can be generalized as follows: Let V be a vector space over the complex field $\mathbb{C}$. Let

(2) $$\langle\ ,\ \rangle : V \to \mathbb{C}$$

be a positive definite hermitian form [thus $\langle x,x\rangle > 0$ for all $x \in V \setminus \{0\}$. Then

$$\det(\langle x_j, x_k\rangle)_{j,k=1,\ldots,n} \geq 0$$

for all $x_1, \ldots, x_n$ in V, equality holding if and only if

$x_1, \ldots, x_n$ are linearly dependent.

Various authors proved an analogue of the classical Schwarz inequality for non-singular symmetric bilinear forms of index one over $\mathbb{R}$ (In [1], pp. 57f. it has been referred to as "Aczel's inequality"): Let $\langle\ ,\ \rangle : V \to \mathbb{R}$ be a symmetric bilinear form. The vector x is orthogonal to the subspace W of V, $x \perp W$, if $\langle x,y\rangle = 0$ for every $y \in W$. Our form is non-singular if $x \perp V$ implies $x = 0$. It has the index (of inertia) one if it is positive definite in a subspace of codimension one and if $\langle x,x\rangle < 0$ for some $x \in V$. Non-singular hermitian forms over $\mathbb{C}$ are defined in an analogous fashion.

"Aczel's inequality" is the following one: Suppose (1) is a non-singular symmetric bilinear form of index one. Let $\langle x_1, x_1\rangle < 0$. Then

$$\begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{vmatrix} \le 0 \tag{3}$$

for every $x_2 \in V$, equality holding if and only if x_1 and x_2 are linearly dependent.

This analogue can be generalized to one of the general Schwarz inequality. Actually, we shall prove an even stronger result:

Theorem: Suppose (2) is a non-singular hermitian form of index m (Thus it is positive definite in a subspace P of codimension m and negative definite in a subspace N of dimension m; thus $V = P \oplus N$). Let $n > m$. Let $a_1, \ldots, a_n$ be linearly independent;

(4) $$\{a_1, \ldots, a_m\} \subset N .$$

Put

$$\Delta = \det(\langle a_j, a_k \rangle)_{j,k=1,2,\ldots,n} .$$

Then $(-1)^m \Delta > 0$.

Obviously, P and N are not unique. The case $m = 1$, $n = 2$ is the inequality (3).

We first verify the following

Lemma: Let $0 \neq b \perp N$. Then $\langle b,b \rangle > 0$.

Proof. The subspace spanned by b and N has the dimension $m+1$. Since $\operatorname{codim} P = m$, it intersects P in a one-space. Let c be a vector in the latter; $c \neq 0$. Thus c is a linear combination $c = \lambda b + D$ where $d \in N$. As $c \in P$, we have

$$0 < \langle c,c \rangle = |\lambda|^2 \cdot \langle b,b \rangle + \langle d,d \rangle$$

or

$$|\lambda|^2 \langle b,b \rangle = \langle c,c \rangle - \langle d,d \rangle .$$

Since $c \in P \setminus \{0\}$ and $d \in N$, this number is positive.

Proof of the theorem. Applying Gram-Schmidt orthogonalization to the sequence (4), we obtain an orthonormal sequence

$$(5) \qquad b_1, \ldots, b_n .$$

Thus $b_1, \ldots, b_m \in N$ while $b_{m+1}, \ldots, b_n \perp N$. Hence by our lemma

$$(6) \qquad \langle b_j, b_j \rangle \begin{cases} <0 & \text{if } j=1,\ldots,m \\ >0 & \phantom{\text{if }} j=m+1,\ldots,n . \end{cases}$$

Every vector a_j is a linear combination

$$(7) \qquad a_j = \sum_1^j \alpha_{j\lambda} b_\lambda ; \qquad j=1,\ldots,n .$$

Since the a_j's are linearly independent, we have $\alpha_{jj} \neq 0$ for all j. The vectors (5) being orthonormal, (7) yields

$$\Delta = |\alpha_{11} \cdots \alpha_{nn}|^2 \cdot \langle b_1, b_1 \rangle \cdots \langle b_{n,n} \rangle .$$

Thus (6) yields our assertion.

REFERENCE [1]: D.S. Mitrinovic: Analytic inequalities. Springer Verlag, New York -Heidelberg -Berlin (1970).

Adelphi University
Garden City, Long Island
N.Y. 11530

University of Toronto
Toronto, Canada
M5S 1A1

Received June 6, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

ERGODIC THEOREMS FOR SUPERADDITIVE PROCESSES

M. A. Akcoglu, F.R.S.C. and U. Krengel

Introduction*. The purpose of this note is to announce a maximal ergodic lemma and some of its applications. In the one-parameter case this lemma gives a simple proof of Kingman's subadditive ergodic theorem [2] not depending on his decomposition theorem. In the multiparameter case the same lemma is used to show that the pointwise ergodic theorems hold for general subadditive processes, without the "strong subadditivity" conditions of Smythe [4] or Nguyen [3]. It can be shown by an example that if the strong subadditivity is not assumed then the decomposition theorem fails for subadditive processes with at least two parameters.

Formulation of the results. Let $d \geq 1$ be a fixed integer and let A_d stand for either one of the following two d-dimensional additive semi-groups of $\mathbb{R}_d^+ = [0,\infty)^d$ or $\mathbb{N}_d = \{0,1,2,\ldots\}^d \subset \mathbb{R}_d^+$. The cases of $A_d = \mathbb{N}_d$ and $A_d = \mathbb{R}_d^+$ will be referred to as the discrete and continuous cases, respectively. Let I_1 denote, in the discrete case, the

* This research is partially supported by an NSERC Grant and has been done during a visit of the second named author at the University of Toronto.

class of subsets of N_1 consisting of finitely many consecutive integers, and, in the continuous case, the class of bounded intervals in $\mathbb{R}_1^+$. If $d \geq 2$, then let I_d be the class of cartesian products of sets from the corresponding I_1 classes. The members of I_d will be called the intervals of A_d. If $I \in I_d$ then $|I|$ denotes, in the discrete case, the number of points in I, and, in the continuous case, the d-dimensional Lebesque measure of I.

Now let $\tau = \{\tau_u\}_{u \in A_d}$ be a d-parameter semi-group of measure preserving transformations on a measure space (X, F, μ). In the continuous case it is also assumed that τ defines a measurable (semi-) flow on X.

A superadditive process (with respect to τ) is a set function F that assigns to each interval $I \in I_d$ a real valued integrable function F_I on X such that the following conditions are satisfied:

(1) $F_I \circ \tau_u = F_{u+I}$ for all $u \in A_d$ and $I \in I_d$.

(2) If $I_1, \ldots, I_n$ are disjoint intervals in I_d and if $I = \bigcup_{i=1}^{n} I_i$ is also in I_d then $F_I \geq \Sigma_{i=1}^{n} F_{I_i}$.

(3) $\sup \frac{1}{|I|} \int F_I \, d\mu = \gamma < \infty$, where the supremum is taken over $I \in I_d$ with $|I| \neq 0$.

If F is a superadditive process then $-F$ is called a subadditive process.

To simplify the notation we will write $F_n = F_{\{0,1,\ldots,n\}^d}$,

$n \geq 0$, in the discrete case, and $F_t = F_{[0,t]^d}$, $t \geq 0$ in the continuous case.

Maximal Lemma. Let F be a non-negative discrete superadditive process, $0 < \alpha$ a real number, $1 \leq K < N$ integers and let

$$E_K(\alpha) = \{x \mid \sup_{1\leq k\leq K} \frac{1}{k^d} F_{k-1}(x) > \alpha\}.$$

Then

$$\mu(E_K(\alpha)) \leq \frac{c}{\alpha(N-K)^d} \int F_N d\mu \tag{4}$$

and

$$\mu(E_K(\alpha)) \leq \frac{c\gamma}{\alpha}, \tag{5}$$

with a constant $c = c(d)$ that depends only on the dimension d.

It may be added that if $d = 1$ then c can be taken as 1 and the lemma has a simpler proof.

The conclusion in (5) is sufficient to prove a pointwise ergodic theorem for (not necessarily non-negative) superadditive processes. We state the discrete and continuous versions separately.

Theorem 1. If F is a discrete superadditive process then $\lim_{n\to\infty} \frac{1}{n^d} F_n$ exists a.e.

Theorem 2. Let F be a continuous superadditive process and let $\Omega = \sup|F_I|$, where the supremum is taken over the subintervals I of $[0,1]^d$ with rational end points. If Ω is an integrable function then $\lim \frac{1}{t^d} F_t$ exists a.e. as $t \to \infty$ taking rational values only.

The conclusion in (4) is necessary to prove a local ergodic theorem in the continuous case.

Theorem 3. Let F be a continuous super additive process and assume that $\sup_{t>0} \frac{1}{t^d} \int |F_t| d\mu < \infty$. Then $\lim \frac{1}{t^d} F_t$ exists a.e. as $t \to 0^+$ taking rational values only.

The convergence along the rationals in Theorems 2 and 3 can be replaced by convergence along any other countable dense subset of $\mathbb{R}_1^+$. Under the usual separability condition this implies the convergence along all real t.

The proofs of these results together with some other related material will appear in [1].

References

[1] M.A. Akcoglu and U. Krengel: Ergodic theorems for superadditive processes. In preparation.

[2] J.F.C. Kingman: Subadditive ergodic theory, The Annals of Probability 1, 883-909, 1973.

[3] X.X. Nguyen: Ergodic Theorems for Subadditive Spatial Processes, Z. Wahrsch. verw. Geb. 48, 159-176, 1979.

[4] R.T. Smythe: Multiparameter subadditive processes, The Annals of Probability 4, 772-782, 1976.

The University of Toronto,
Toronto, Canada M5S 1A1.

The University of Göttingen,
Göttingen, Federal Republic of Germany.

Received June 12, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

THE DEGREE OF SUBVARIETIES CONTAINING "CUBES" OF GROUP VARIETIES, III

Gregory V. Chudnovsky

Presented by P. Ribenboim, F.R.S.C.

Let $\mathfrak{A}$ be a quasi-projective group variety and let Γ be a finitely generated subgroup of $\mathfrak{A}$ of rank r over $\mathbb{Z}$ with generators $v_1,\dots,v_r$. Let $N \geq 1$ and $\Gamma_N = \{w: w = \sum_{i=1}^{r} n_i v_i, n_i \in \mathbb{Z}, |n_i| \leq N : i=1,\dots,r\}$. Let $\Omega(\Gamma_N)$ be the minimal degree of subvariety $X \subset \mathfrak{A}$ of codim one, containing Γ_N. Then we prove $\Omega(\Gamma_N) \gg_{\mathfrak{A},r} N^{\mu(\Gamma)}$ where $\mu(\Gamma) = \min_{\mathfrak{b}\subset\mathfrak{A},\mathfrak{b}\neq\mathfrak{A}} \frac{r-\mathrm{rank}_{\mathbb{Z}}(\Gamma\cap\mathfrak{b})}{\dim(\mathfrak{A})-\dim(\mathfrak{b})}$; $\mathfrak{b}$ is a proper groupsubvariety of $\mathfrak{A}$.

0. Our main result is the estimation of the degree of the "cube" lying in an arbitrary group variety $\mathfrak{A}$. By a "cube" in $\mathfrak{A}$ we understand the set Γ_N of linear combinations $\sum_{i=1}^{r} n_i \vec{v}_i$ with integer coefficients n_i bounded in the absolute value by N. Here Γ stands for a subgroup (not a group variety!) of $\mathfrak{A}$ generated by r linearly independent over $\mathbb{Z}$ elements $\vec{v}_i$ from $\mathfrak{A}$: $i=1,\dots,r$. Naturally $r = \mathrm{rank}_{\mathbb{Z}}\Gamma$ and the "cube" Γ_N contains $(2N)^r$ elements. We want to estimate the smallest degree $\Omega(\Gamma_N)$ of the subvariety $X \subset \mathfrak{A}$ containing Γ_N. The main theorem of the paper shows that $\Omega(\Gamma_N)$ may differ by a multiplicative factor only from a natural upper bound on $\Omega(\Gamma_N)$ given by the Dirichlet's box principle. This result is extremely important in applications to Transcendental Number Theory since it gives immediately the bound for the number of zeroes of the function $P(f_1(z),\dots,f_d(z))$ at $z \in \Gamma_N$, where $f_1(z),\dots,f_d(z)$ constitute an analytic uniformization of $\mathfrak{A}$. Already the case of $\mathfrak{A}$ splitting into k one dimensional factors is the most important. In this case for

$\mathfrak{A} = \mathbb{A}^k$ the corresponding statement as a conjecture was formulated by M. Waldschmidt [3]. The answer (with the best value of constant) was given by the author in [2] for $k = 2$; and was generalized using commutative algebra for any k by D. Masser [4]. Similar results for $\mathfrak{A}$ being a product of elliptic curves (or degenerate elliptic curves $\mathbb{A}$ or $\mathbb{A}^*$) were proved by D. Masser and Wustholtz [6].

1. Let $\mathfrak{A}$ be a nonsingular quasi-projective group variety of the dimension n and let $v_1,\dots,v_r$ be r $\mathbb{Z}$-linearly independent elements of $\mathfrak{A}$. We define $\Gamma = \{\Sigma_{i=1}^r n_i v_i : n_i \in \mathbb{Z} ; i=1,\dots,r\}$ and $\Gamma_N = \{\Sigma_{i=1}^r n_i v_i : n_i \in \mathbb{Z} , |n_i| \le N ; i=1,\dots,r\}$ for $N \ge 1$. Following the example of [2] for $\mathfrak{A} = \mathbb{A}^n$ we define an exponent $\mu(\Gamma)$ or $\mu_{\mathfrak{A}}(\Gamma)$ in the

Definition 1.1. We set $\mu(\Gamma) = \inf_{\mathfrak{b}} \frac{r - \mathrm{rank}_{\mathbb{Z}}(\Gamma \cap \mathfrak{b})}{n - \dim \mathfrak{b}}$, where the infimum is taken over all proper group varieties $\mathfrak{b} \subset \mathfrak{A}$ (i.e. $\mathfrak{b} \ne \mathfrak{A}$).

Simple arguments show the following

Lemma 1.2. Always $\mu(\Gamma) \le \mathrm{rank}_{\mathbb{Z}}(\Gamma)/n$ and for a generic $v_1,\dots,v_r$ in $\mathfrak{A}$ we have $\mu(\Gamma) = \mathrm{rank}_{\mathbb{Z}}(\Gamma)/n$. If $\mathfrak{A}$ is a simple variety, then for any lattice Γ, $\mu(\Gamma) = \mathrm{rank}_{\mathbb{Z}}(\Gamma)/n$.

The exponent $\mu(\Gamma)$ is connected with the degree of subvariety of $\mathfrak{A}$, containing Γ_N for $N \gg 1$. Let's give the most general definition:

Definition 1.3. For finite set $S \subset \mathfrak{A}$ and integer $K \ge 1$ we denote by $\Omega_{\mathfrak{A}}(S;K)$ (or simply $\Omega(S;K)$) the minimal degree of the subvariety X of $\mathfrak{A}$ of codimension one, containing S and having all points of S as K-fold points. The number $\Omega_{\mathfrak{A}}(S) = \Omega_{\mathfrak{A}}(S;1)$ is called a degree of S in $\mathfrak{A}$ and the number

$$(1.4) \qquad \lim_{K\to\infty} \frac{\Omega_{\mathfrak{A}}(S;K)}{K} = \Omega^{o}_{\mathfrak{A}}(S)$$

is called a singular degree of S in $\mathfrak{A}$.

The upper bound for $\Omega_{\mathfrak{A}}(S;K)$ can be deduced from the Dirichlet's box principle and for sets of the form Γ_N we have [1],[2] :

<u>Proposition 1.5</u>. For $N \geq 1$ and $K \geq 1$ we have $\Omega_{\mathfrak{A}}(\Gamma_N;K) \leq cK\cdot N^{\mu(\Gamma)}$, where $c = c(\mathfrak{A},\Gamma) > 0$ depends on $\mathfrak{A}$ and Γ.

It was conjectured by M. Waldschmidt that for $\mathfrak{A} = \mathbb{A}^n$, $\Omega(\Gamma_N) \gg N^{\mu(\Gamma)}$. This was proved for $n = 2$ in [2] and for $n > 2$ in [4]. Moreover we can show for $\mathfrak{A} = \mathbb{A}^2$ that $\Omega(\Gamma_N;K) \geq N^{\mu(\Gamma)} + o(N^{\mu(\Gamma)})$ as $N \to \infty$ and for $\mu(\Gamma) = \mathrm{rank}_{\mathbb{Z}}\Gamma/2 > 1$ we have proved $\Omega(\Gamma_N) \geq \sqrt{2}|\Gamma_N|^{1/2} + o(|\Gamma_N|^{1/2})$. [2]. The last results are evidently the best possible.

2. The method of the proof in all cases is the same. We act on $\mathfrak{A}$ by translations by elements of Γ. We use the result [3], [5], that the map $T_g: \mathfrak{A} \to \mathfrak{A}$ defined by $T_g(x) = x + g$ is a regular map on $\mathfrak{A}$ for $g \in \mathfrak{A}$. Using the same arguments together with the intersection theory from [5] we obtain the main result of the paper.

<u>Theorem 2.1</u>. For any subgroup Γ of $\mathfrak{A}$ of the $\mathbb{Z}$-rank r, and $N \geq 1$, $K \geq 1$ we have

(2.2) $$\Omega_{\mathfrak{A}}(\Gamma_N;K) \geq c_1 \cdot N^{\mu_{\mathfrak{A}}(\Gamma)} \cdot K$$

for $c_1 = c_1(\mathfrak{A},r) > 0$.

The constant $c_1 = c_1(\mathfrak{A},r) > 0$ depends not on $\mathfrak{A}$ explicitly, but on the degree of the imbedding $\mathfrak{A} \subset \mathbb{P}^\nu$ and on $n = \dim \mathfrak{A}$ only. For $\mathfrak{A} = \mathbb{A}^n$ c_1 is an absolute constant (that can be taken $c_1 = 1/n$ simply). We expect in the last case $c_1 = 1$ for large K and $c_1 = \sqrt[n]{n!} - \epsilon$ for $\mu_{\mathfrak{A}}(\Gamma) = \mathrm{rank}_{\mathbb{Z}}\Gamma/n > 1 + \epsilon$, $\epsilon > 0$.

In the proof we take the subvariety X of $\mathfrak{A}$ of the codimension one and apply operators $\Pi_{i=1}^r T_{v_i}^{n_i} = T_{\Sigma_{i=1}^r n_i v_i} = T_{(\vec{n},\vec{v})}$ to X . Then we construct such n sequences $\vec{n}_j \in \mathbb{Z}$, $j=1,\ldots,n$ that $T_{(\vec{n}_1,\vec{v})}X \cap \ldots \cap T_{(\vec{n}_n,\vec{v})}X$ has dimension zero and we remark that $\|\vec{n}\| \leq M$ and $\Gamma_N \subset X$ implies $\Gamma_{N-M} \subset T_{(\vec{n},\vec{v})}X$. Then the intersection theory of [5] is applied.

3. This type of argument shows that we can prove a more strong statement, without requirement that T_g is a translation operator. We present below the corresponding generalization and now we can be more precise with the degree of the subvariety in the case of nonsimple $\mathfrak{A}$, when $\mathfrak{A} \subset \mathbb{P}^{\nu_1} \times \ldots \times \mathbb{P}^{\nu_k}$ and we can count degrees of Γ_N more precisely. Here is the first result.

<u>Theorem 3.1.</u> Let $\mathfrak{A} = \mathbb{A}^n$ or $\mathfrak{A} = \mathbb{A}^m \times \mathbb{A}^{n-m}$, and we consider $\mathfrak{A} \subset \mathbb{P}^{\nu_1} \times \ldots \times \mathbb{P}^{\nu_k}$, $\nu_1 + \ldots + \nu_k = n$. Let X be a subvariety of $\mathbb{P}^{\nu_1} \times \ldots \times \mathbb{P}^{\nu_k}$ of codimension one containing the Γ_N . Then X has "homogenuous" degrees $d_1,\ldots,d_k$ as in [5].

We have for $\mathrm{rank}_{\mathbb{Z}}(\Gamma) \geq n+1$,

$$(d_1+1)^{\nu_1} \cdots (d_k+1)^{\nu_k} \geq c_2 N^{n \cdot \mu(\Gamma)}$$

for $\mu(\Gamma) = \mu_{\mathfrak{n}}(\Gamma)$.

Similarly it's possible to bound below the degree of the set $\Gamma(N_1,\ldots,N_r) = \{\Sigma_{i=1}^{r} n_i v_i : n_i \in \mathbb{Z},\ |n_i| \leq N_i,\ i=1,\ldots,r\}$ for different N_i of the form $N_i = N^{a_i}$.

References

1. G.V. Chudnovsky, C.R. Acad. Sci. Paris, v. 286A (1978), pp. A-1133-A-1135; Séminaire Delange-Pisot-Poitou, 19e année 1977/78, January 1978, pp. 1-40.

2. G.V. Chudnovsky, The degree of hypersurfaces containing lattices in C^n. I. Planar case (to appear).

3. M. Waldschmidt, Nombres transcendants et groupes algébriques, Astérisque 67-70, SMF, 1980.

4. D. Masser, (to appear).

5. J.R. Shafarevich, Basic algebraic geometry, Springer, 1974.

6. D. Masser, K. Wustholtz, (to appear).

Gregory V. Chudnovsky
Department of Mathematics
Columbia University
New York, New York 10027

Received June 23, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

COMPLEXES POLYEDRAUX EQUILIBRES SUR LES COTES

Robert BANTEGNIE

Presented by H.S.M. Coxeter, F.R.S.C.

1. Soit $\mathcal{C}$ un complexe polyédral plan, $\mathcal{E}$ l'ensemble de ses côtés. Si $E \in \mathcal{E}$ est commun aux faces F_1, F_2 de valences f_1, f_2 et a pour extrémités les sommets S_1, S_2 de valences s_1, s_2 le symbole $(s_1, s_2 ; f_1, f_2)$ pris modulo les permutations sur les 2 premières lettres et sur les 2 dernières est appelé cycle de E, pris modulo les permutations sur les 4 lettres pseudocycle de E. La contribution de E est $c(E)=(1/s_1)+(1/s_2)+(1+f_1)+(1/f_2)-1$. Elle est la même pour 2 côtés de même pseudocycle; "Euler" se traduit par $\sum_{E\in\mathcal{E}} c(E) = 2$ (1) ([3]). $\mathcal{C}$ est faiblement équilibré sur les côtés quand la contribution d'un côté est toujours la même. $\mathcal{C}$ est équilibré (resp strictement équilibré) sur les côtés quand le pseudocycle (resp cycle) d'un côté est toujours le même. Les $\mathcal{C}$ considérés sont toujours faiblement équilibrés sur les côtés et on supprime cette dernière mention. La contribution d'un côté de $\mathcal{C}$ est $c(\mathcal{C}) = c$ et (1) donne $c|\mathcal{E}| = 2$ (2). Le pseudocycle de $E \in \mathcal{E}$ est appelé pseudocycle de $\mathcal{C}$ lorsque $\mathcal{C}$ est équilibré ; de même pour $\mathcal{C}$ strictement équilibré on parle de cycle de $\mathcal{C}$. De plus si $\mathcal{C}^*$ est le complexe dual de $\mathcal{C}$, $\mathcal{C}^*$ est équilibré, resp faiblement, resp strictement ssi il en est ainsi pour $\mathcal{C}$.

Les diagrammes de Schlegel des polyèdres convexes réguliers, du cuboctaèdre $\{{3 \atop 4}\}$, de l'icosidodécaèdre $\{{3 \atop 5}\}$ et de leurs duaux, le dodécaèdre rhombohédral et le tricontaèdre rhombohédral sont strictement équilibrés. Il n'y a pas d'autres complexes strictement équilibrés. Ce résultat est probablement connu.

J'établis ici la liste exhaustive des complexes équilibrés. Ceux équilibrés non strictement sont (α) la pyramide autoduale ayant pour base un n-gone de pseudocycle $(n. 3^3)$ $(n > 3)$
β) un tétratrièdre autodual de pseudocycle $(4^2. 3^2)$
γ) un décahéxaèdre autodual de pseudocycle $(5^2. 3^2)$
δ) un tétraennéaèdre autodual de pseudocycle $(4^3. 3)$ et

ε) pour le pseudocycle $(5.\,4^2.\,3)$ les 4 pseudoicosidodécaèdres le 1-chapeauté, deux 2-chapeautés, le 3-chapeauté et leurs 4 duaux.

2. On profite du théorème de Steinitz pour assimiler un complexe polyédral et un des polyèdres dont il est diagramme de Schlegel.

Soit P l'ensemble des complexes équilibrés, Q celui de ceux strictement équilibrés.

* Pour $C \in P$, le pseudocycle de C est $(n.\,3^3)$ $(n \geq 3)$, $(4^2.\,3^2)$, $(5^2.\,3^2)$, $(4^3.\,3)$ ou $(5.\,4^2.\,3)$. Pour $C \in Q$ le cycle de C est (3^4), $(4^2;3^2)$, $(3^2;4^2)$, $(5^2;3^2)$, $(3^2;5^2)$, $(4^2;4.\,3)$, $(4.\,3;4^2)$, $(5.\,3;4^2)$ ou $(4^2;5.\,3)$.

Puisque $c > 0$ les pseudocycles maximaux (cf [3] p. 56) sont $(n.\,3^3)$ $(n \geq 3)$, $(11.\,4.\,3^2)$, $(7.\,5.\,3^2)$ et $(5.\,4^2.\,3)$. Si le pseudocycle est $(x.\,4.\,3^2)$ avec $4 \leq x \leq 11$, $x \neq 4$ contredit $C \in P$ et $C \in Q$ est exclu si le cycle est $(4.\,3;4.\,3)$. Si le pseudocycle est $(x.\,5.\,3^2)$ avec $5 \leq x \leq 7$, $x \neq 5$ contredit $C \in P$ et $C \in Q$ est exclu si le cycle est $(5.\,3;5.\,3)$. De plus si le pseudocycle est $(n.\,3^3)$, $n \neq 3$ est exclu pour $C \in Q$ et les cycles $(5.\,4;4.\,3)$ et $(4.\,3;5.\,4)$ sont exclus pour $C \in Q$.

* pseudocycles $(n.\,3^3)$, $(4^2.\,3^2)$, $(5^2.\,3^2)$ et $(4^3.\,3)$.

Pour $(n.\,3^2)$ si $n > 3$ on obtient la pyramide de α) pour $n = 3$ le tétraèdre. Pour $(4^2.\,3^2)$ dans $P \setminus Q$ on obtient β) donné par fig. 1a. Ce tétratrièdre a 12 côtés, 6 de cycle $(4.\,3;4.\,3)$, 3 de chacun des cycles $(4^2;3^2)$ et $(3^2;4^2)$, 3 sommets et 3 faces de valence 4, 4 sommets et 4 faces de valence 3. Sans Q on a le cube et l'octaèdre. Pour $(5^2.\,3^2)$ dans $P \setminus Q$ on obtient γ) donné par fig. 1c. Ce décahexaèdre a 30 côtés, 10 de chacun des cycles $(5,\,3;5,\,3)$, $(5^2;3^2)$ et $(3^2;5^2)$, 6 sommets et 6 faces de valence 5, 10 sommets et 10 faces de valence 3. Dans Q on a le dodécaèdre et l'icosaèdre. Pour $(4^3.\,3)$ dans $P \setminus Q$ on obtient δ) donné par fig. 1b. Ce tétraennéaèdre a 24 côtés, 12 de chacun des cycles $(4.\,4;4.\,3)$ et $(4.\,3;4.\,4)$, 9 sommets et 9 faces de valence 4, 4 sommets et 4 faces de valence 3. Dans Q on obtient $\{{3 \atop 4}\}$ et son dual.

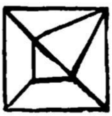


fig. 1a

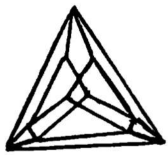


fig. 1b

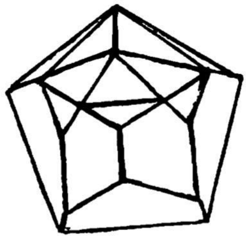


fig. 1c

* pseudocycle $(4^2.\ 5.\ 3)$.

Pour ce pseudocycle c'est plus compliqué. Dans Q on obtient l'icosidodécaèdre $\{\substack{3\\5}\}$ et son dual. Soit un des 12 pentagones de $\{\substack{3\\5}\}$. Il forme avec ses 5 voisins triangulaires un pentagramme fermé . Chapeauter ce pentagramme est le remplacer par . Si dans $\{\substack{3\\5}\}$ on chapeaute un pentagramme on obtient le pseudoicosidodécaèdre 1-chapeauté. Pour un pentagone de $\{\substack{3\\5}\}$, un seul autre pentagone est tel que les 2 pentagrammes associés sont disjoints. Si on chapeaute les 2 pentagrammes on obtient le 1° pseudoicosidodécaèdre 2-chapeauté. Si on chapeaute un des pentagrammes et un de ceux n'ayant en commun avec le premier qu'un point commun à leurs frontières, on obtient le 2° pseudoicosidodécaèdre 2-chapeauté. On peut trouver 3 pentagrammes qui deux à deux n'ont en commun qu'un point. En les chapeautant on obtient le pseudoicosidodécaèdre 3-chapeauté. Ces pseudoicosidodécaèdres et leurs duaux sont dans $P\setminus Q$. Pour prouver que la liste obtenue est complète, on raisonne à l'aide de la figure 2.

Les complexes cherchés ont 60 côtés $(c = 1/30)$. Soit p, q, r, s le nombre de cycles $(4^2;5.\ 3)$, $(5.\ 3;4^2)$, $(4.\ 3;5.\ 4)$, $(5.\ 4;4.\ 3)$, x, y, z le nombre des pentagones ayant 4, 2, 0 quadrilatères adjacents. Pour $x > 0$ on a nécessairement la situation de la fig. 2b. Au centre se trouve un 14-gone (I) qui se décompose suivant une des 3 formes (a), (b), (c)

(a) (b) (c)

Les pseudoicosidodécaèdres

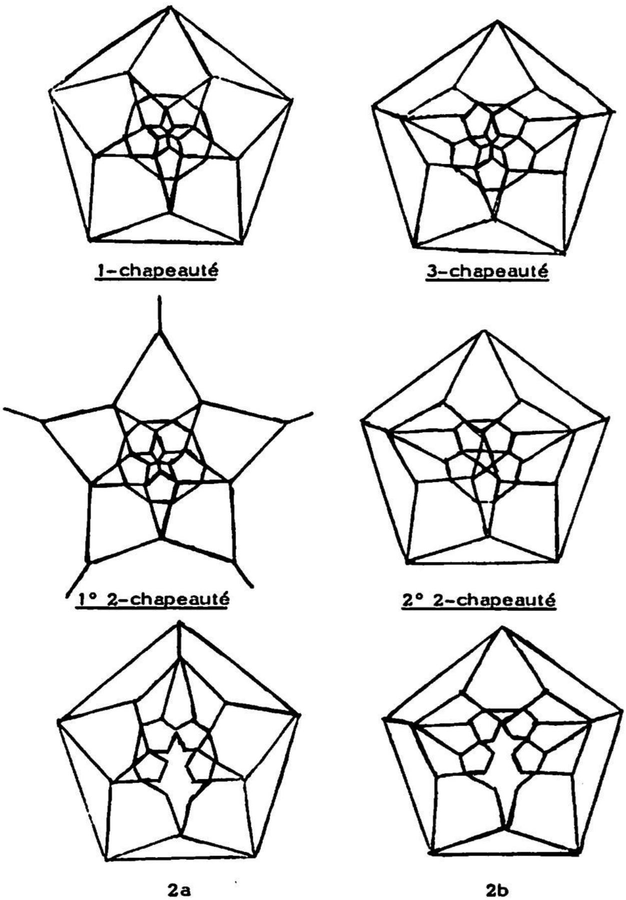


Figure 2

Pour (a) on obtient le 2° 2-chapeauté et pour (b) ou (c) le 3-chapeauté. Pour $x=0$, $y>0$ on a nécessairement fig. 2a où au centre on trouve encore (I). Pour (a) on obtient le 1-chapeauté et pour (b) ou (c) le 1° 2-chapeauté. Comme, à la dualité près, on peut supposer que le nombre total de pentagones $x+y+z$ est $\neq 0$, reste à étudier $x=y=0$, $z\neq 0$. On obtient alors l'icosidodécaèdre dit 0-chapeauté (fig. 3).

Donnons un tableau concernant les chapeautés ; $\mathcal{S}, \mathcal{F}$ y désignent l'ensemble des sommets et des faces de $\mathcal{C}$, x', y', z' le nombre de sommets de valence 5 ayant 4, 2, 0 sommets adjacents de valence 4. On y a $x+y+z+x'+y'+z'=12$, $s=0$, $|\mathcal{A}|=60$.

	(p, q, r, s)	(x, y, z)	(x', y', z')	$(\|\mathcal{S}\|, \|\mathcal{A}\|, \|\mathcal{F}\|)$
0-chapeauté	(60, 0, 0, 0)	(0, 0, 12)	(0, 0, 0)	(30, 60, 32)
1-chapeauté	(45, 5, 10, 0)	(0, 5, 6)	(0, 0, 1)	(31, 60, 31)
1° 2-chapeauté	(30, 10, 20, 0)	(0, 10, 0)	(0, 0, 2)	(32, 60, 30)
2° 2-chapeauté	(30, 10, 20, 0)	(2, 8, 0)	(0, 0, 2)	(32, 60, 30)
3-chapeauté	(15, 15, 30, 0)	(9, 0, 0)	(0, 0, 3)	(33, 60, 29)

Quand on passe de $\mathcal{C}$ à $\mathcal{C}^*$, p et q, r et s, x et x', y et y', z et z', $|\mathcal{S}|$ et $|\mathcal{F}|$ sont permutés. J'ai étudié ailleurs ([1], [2]) les complexes polyédraux équilibrés sur les sommets (ou sur les faces).

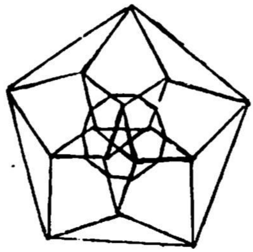


fig. 3 : l'icosidodécaèdre

REFERENCES

[1] R. BANTEGNIE
Variations sur des thèmes de B. Grünbaum et G. C. Shephard. C. R. Math. Rep. Acad. Sci. Canada 1 (1979), p. 197-198.

[2] R. BANTEGNIE
Polyèdres équilibrés. Can. Math. Bull. (à paraitre).

[3] O. ORE
The four color problem. Academic Press, New-York and London, 1967.

Robert BANTEGNIE
Université de Franche-Comté
Mathématiques
E. R. A. n° 070654
Route de Gray
25030 BESANCON CEDEX
FRANCE

Received June 24, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

USING DIRECTED GRAPHS FOR TEXT COMPRESSION

G. V. Cormack and G. Grätzer, F.R.S.C.

Abstract*: We define a class of directed graphs and we show how any such graph G defines a coding, which we shall call G-code. The well-known H(uffman)-code is a simple special case of the G-code. We shall show that with a suitable choice of the graph G, we can achieve much better compression ratios without increasing CPU time.

The main disadvantage of any variable-length code is that the compressed text cannot be scanned without decompression. Another choice of G yields a G-code with a more modest compression ratio that, however, can be scanned in its compressed form.

There exist graphs G whose associated G-code compresses close to the theoretical limit found by Shannon [4] (for a 27 letter alphabet); however, the compression uses excessive amount of core and CPU time.

*The research of the second author was supported by the N.S.E.R.C. of Canada.

1. Compression graphs. We are given two finite alphabets A and C, the source alphabet and the code alphabet, respectively. A member of an alphabet is called a character. A compression graph G is a directed graph some of whose nodes are labeled by a source character and some of whose arrows (directed edges) are labeled by a code character. A node or arrow not labeled will be called unlabeled. The following axioms are required:

1.1. G has a distinguished node called START.

1.2. If $X \to Y$ and $X \to Z$ are both labeled by the same $c \in C$, then $Y = Z$.

1.3. If $X \to Y$ is unlabeled, then X is labeled, Y is unlabeled, and there is no other arrow leaving X.

1.4. If X is a node of G labeled by $a \in A$, or if X is START, then for any $b \in A$ there is a node Y of G labeled by b and there are arrows: $X = X_1 \to X_2, X_2 \to X_3, \ldots, X_{n-1} \to X_n = Y$, $n \geq 1$, such that all X_i with $1 < i < n$ are unlabeled.

2. Coding a text. A text T is a finite sequence of source characters $a_1a_2 \ldots a_n$. We now define the coded text, $G(T)$, and the terminal node of the coding, $t(T)$, which is a labeled node of G or the START.

If T is the empty string, then $G(T)$ is the empty string and $t(T)$ is the START node.

If $k \le n$, $G(a_1 \ldots a_{k-1})$ and $t(a_1 \ldots a_{k-1}) = X$ are defined and the latter is labeled by a (or it is the START), then by Axiom 1.4 we can choose a node Y labeled by a_k such that there exist unlabeled nodes $X_1, \ldots, X_m$ satisfying : $X \to X_1$, $X_1 \to X_2, \ldots, X_m \to Y$. Since $X_1, \ldots, X_m$ are unlabeled, it follows from Axiom 1.3 that $X_1 \to X_2, \ldots, X_m \to Y$ are all labeled; let $c_1, \ldots, c_m$ be the labels of $X_1 \to X_2, \ldots, X_m \to Y$, respectively. We define $t(a_1 \ldots a_k) = Y$. Further,

$$G(a_1 \ldots a_k) = G(a_1 \ldots a_{k-1})c_1 \ldots c_m$$

if $X \to X_1$ is unlabeled; if $X \to X_1$ has label c, then

$$G(a_1 \ldots a_k) = G(a_1 \ldots a_{k-1})cc_1 \ldots c_m.$$

This completes the definition of $G(T)$. Observe that from the same text T we may get many coded versions.

3. <u>Decoding</u>. Given a coded text T' we can find a unique text T with $G(T) = T'$.

Let $T' = c_1 \ldots c_m$. For each i with $0 \le i \le m$, we define the node $N(i)$ and the text $T(i)$. For $i = 0$, $N(0) =$ START and $T(0)$ is the empty sequence. Let $N(k)$ and $T(k)$ be defined, $k < m$. If $N(k)$ is an unlabeled node, then there must be a unique node (by the assumption that T' is a coded text and by Axiom 1.2), Y, such that $N(k) \to Y$ is labeled by c_{k+1}. We set $N(k+1) = Y$. If Y is labeled by $a \in A$ we define $T(k+1) = T(k)a$, otherwise $T(k+1) = T(k)$. If $N(k)$ is labeled and there is an unlabeled arrow $N(k) \rightarrow Z$, then proceed with Z and $T(k)$ as before. Otherwise, proceed with $N(k)$ and $T(k)$ as before. We find T as $T(m)$.

4. Observations. 4.1. The alphabet A would normally include the usual English alphabet, numerals, punctuation marks, control characters, etc. In addition, in some applications it is useful to consider some combination of symbols such as pairs of letters, words, repetition symbols, etc, as part of A. The larger the A the better is the compression; the price to be paid is in table storage, core space, and CPU time.

4.2. Even in the simplest schemes it would be harmful to assume that the map from A to the nodes of G be one-to-one.

4.3. Any coding scheme has to deal with the ambiguity of the choice of Y in Axiom 1.4. If there are, say, two choices of Y : Y_1 and Y_2, usually it is sufficient then to pick the one which together with the next step or next two steps yields the shortest code.

4.4. Not every text in the alphabet C can be decoded.

5. Some examples. The graphs we use are obviously too large to be reproduced here. Let G_1 denote the graph of a coding scheme

using 4, 8, and 12 bit codes ($C = \{0, 1, \ldots, E, F\}$) having the property that $G_1(T)$ can be scanned. Let H denote the graph (really, tree) of the Huffman code. Let G_2 denote the graph obtained by applying the Huffman algorithm to the relative frequency of the source characters after a given source character (using also shift characters for upper case and data input.) Finally, let G_3 and G_4 be the same as G_2 using relative frequencies after pairs and triples of source characters. The graphs G_i', $i = 1, 2, 3$ use words instead of letters. The table below summarizes our results.

Graph	Table size (in K)	Packing (bits/character)
G_1	0.12	4.78
H	0.2	4.1
G_2	2	3.27
G_3	12	2.45
G_4	40	1.8
G_1'	24	1.46
G_2'	64	.74
G_3'	112	.27

6. <u>References</u>. An excellent collection of papers (including Huffman's) is reprinted in [1]. The necessary algorithms are described in [2]. The significance of text compression to database management is analyzed in [3].

References

[1] L. D. Davisson and R. M. Gray (editors): Data Compression. Benchmark Papers in Electrical Engineering and Computer Science. Vol. 14. Dowden, Hutchinson, and Ross, 1976.

[2] D. E. Knuth: The Art of Computer Programming. Volume 1. Fundamental Algorithms. Addison-Wesley, 1973.

[3] J. Martin: Computer Data-base Organization. Prentice Hall, Englewood Cliffs, N. J., 1977.

[4] C. E. Shannon: Prediction and Entropy of Printed English. Bell System Tech. J., 1951, 50-64.

University of Manitoba
Winnipeg, Manitoba
R3T 2N2

Received June 26, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

DEGREE FORMULAS FOR THE ORTHOGONAL GROUPS OVER GF(2)

J. S. Frame

Presented by G. de B. Robinson, F.R.S.C.

1. GENERIC DEGREE POLYNOMIALS. For each of the orthogonal groups $G_n = O_{2n+1}(2)$, isomorphic with $Sp_{2n}(2)$, and for their maximal subgroups $G_n^{\sigma} = O_{2n}^{\sigma}(2)$, ($\sigma$ = + or -) of index $N(N+\sigma)/2$ where $N = 2^n$, each absolutely irreducible complex (AIC) character χ^j or $\chi^{j\sigma}$ is a specialization for $z = N$ of a generic character γ^j or $\gamma^{j\tau}$ of level $\ell \leq n$, whose value on each class is a polynomial in z. [2]. The value on class C_1 is a monic "degree polynomial" $P_\ell^j(z)$ or $P_\ell^{j\sigma}(z)$ of degree 2ℓ, divided by an integer d_j^{τ} or d_j independent of n. This integer is computable as the codegree (group order/degree) of a "parent" character $\phi^{j\tau}$ of G_ℓ^{τ} or ϕ^j of G_ℓ, from which the corresponding generic degree polynomial $P_\ell^j(z)/d_j^{\tau}$ or $P_\ell^{j\sigma}(z)/d_j$ will be constructed in §2.

The 2ℓ distinct integer roots of every degree polynomial of level ℓ each divide the numerically largest root $\pm W$, $W = 2^{w-1}$. This width w (called length L in [2]) is ℓ+m for G_n and ℓ+m+1 for G_n^{σ}, where $0 \leq m \leq \ell$. To a character of level ℓ and width w we assign the "type" ℓ_m^{τ}, if τ = + or - is the sign (for $z > W$) of the generic character value γ_t^j or $\gamma_t^{j\tau}$ on the transposition class C_t. Values of these characters on C_t are obtained in §3.

A generic character is changed into its mate by replacing z by -z, except for self-paired characters whose degree polynomials are even functions of z.

2. DEGREE FORMULAS. As explained for levels 1 and 2 in [1] and [2], each AIC character $\phi^{j\tau}$ of the group G_ℓ^τ generates an extended level ℓ generic character of G_n that contains just one level ℓ irreducible generic character γ^j. The leading coefficient of $z^{2\ell}$ in the degree polynomial of γ^j (coefficient of α^ℓ in [2]) is the parent degree $\phi_1^{j\tau}$ divided by the order of G_ℓ^τ, namely $1/d_j^\tau$, the reciprocal of the parent codegree.

For $m = 0$, all G_n characters of type ℓ_0^τ, called primary, have $W = L/2$, $(L=2^\ell)$, and the monic degree polynomial is

$$P_\ell^*(z) = \prod_{r=1}^{\ell} (z^2 - 4^{r-1}). \tag{2.1}$$

Each primary character may be labeled by a primary class to which it corresponds, such as 3_{111}, 3_{21}, 3_3, 9_1, $3_1 5_1$, or 7_1 for level 3. Their parent codegrees $2^{12}3\cdot3\cdot9$, $2^{11}3^4$, $2^{9}3^4$, $2^{12}15$, $2^{12}7$, and those of higher levels are computed in [3] from modified hook graphs of Young diagrams derived from the subscripts which represent the multiplicities of eigenvalues of orders 3, 5, 7, or 9, etc. in indecomposable constituents of a decomposed matrix similar to the matrices of the class.

<u>Theorem 2.1</u> For $m > 0$, the degree γ_1^j of the generic G_n character γ^j of type ℓ_m^τ is expressible as follows in terms of the monic polynomial $p_{\ell-m}^{j\tau}(z)$ and codegree d_j^τ of its parent AIC character $\phi^{j\tau}$ of G_ℓ^τ of type $(\ell-m)_\mu^\sigma$, where we set $M = -\tau 2^{m-1}$:

$$\gamma_1^j = P_\ell^j(z)/d_j^\tau = P_m^*(z)\, M^{2\ell-2m} p_{\ell-m}^{j\tau}(z/M)(z-LM)/(z-\sigma M)\, d_j \tag{2.2}$$

For $m > 0$ and $n \geq \ell+m$, each AIC character χ^j of G_n is the value for $z = N = 2^n$ of a unique generic character γ^j generated by a lower level parent $\phi^{j\tau}$ in G_ℓ^τ. However, if $n = \ell+m-1$, then $L = N$, and the generic character γ^j or its mate or both may become a "ghost" AIC character of degree 0 which vanishes. A non-vanishing mate of a ghost is a "widow" character obtained for $z = M$ from two distinct generic characters whose parent AIC characters in G_ℓ^+ and G_ℓ^- are mates.

Each AIC character of G_ℓ of type $(\ell-m)_\mu^\tau$ generates a tetrad of generic characters of type $\ell_m^\pm$ including a pair of associated characters (one the product of the other by the alternating character $\bar{1}$) in G_n^+ and their mates in G_n^-.

<u>Theorem 2.2</u> The degree $\chi_1^{j\sigma}$ of the generic G_n^σ character $\chi^{j\sigma}$ of type ℓ_m^τ is expressible as follows in terms of the monic polynomial $p_{\ell-m}^j(z)$ and codegree d_j of its parent AIC character ϕ^j of G_ℓ of type $(\ell-m)_\mu^\tau$, where we set $M = -\sigma\tau 2^m$:

$$\chi_1^{j\sigma} = P_\ell^{j\sigma}(z)/d_j = P_m^*(z)\, M^{2\ell-2m}\, p_{\ell-m}^j(z/M)(z-LM)/(z+\sigma)d_j \quad (2.3)$$

Here again, when $n = \ell+m$ and $N = M$, certain widow characters of G_n^+ or G_n^- may have ghost mates of degree 0 in the other group. They are obtained for $z = M$ from two distinct generic characters whose parents are mates in G_ℓ, or from a generic character whose parent is a self paired character of G_ℓ of type ℓ_m^τ.

These formulas have been tested by computing the degrees of all characters of G_n and G_n^σ for $n \leq 5$, including 155 characters of $O_{10}^+(2)$, 164 of $O_{10}^-(2)$ and 198 of $O_{11}(2)$.

3. CHARACTER VALUES ON CLASS C_t. Coefficients of $z^{2\ell}$ and $z^{2\ell-1}$ in the degree polynomial $P_\ell^{j+}(z)/d_j$ generated by ϕ^j of type $(\ell-m)_\mu^\tau$ of G_ℓ are seen from [2] to be ϕ_1^j and ${}^oC_t\phi_t^j$, divided by the order of G_ℓ, if the transposition class C_t has ${}^oC_t$ elements. Hence the sum of the roots of P_ℓ^{j+} is $-{}^oC_t\phi_t^j/\phi_1^j$.

Theorem 2.2 expresses this sum in terms of the sum s of the roots of $p_{\ell-m}^{j}(z)$ as follows , taking $\sigma = +$:

$$-{}^{o}C_t\, \phi_t^j / \phi_1^j = M(s + L) + 1 = -\tau\, 2^m(L+s) + 1. \qquad (3.1)$$

Replacing ϕ^j of G_ℓ by χ^j of G_n, we replace L by N and 2^m by N/L. The transposition class size in G_n is ${}^{o}C_t = N^2 - 1$.

Theorem 3.1 The value on class C_t of the AIC character χ^j of G_n of type ℓ_m^τ is given by the formula

$$\chi_t^j = \chi_1^j(\tau(N+s)N/L - 1)/(N^2 - 1), \quad L = 2^\ell . \qquad (3.2)$$

where s is the sum of the roots of the degree polynomial.

Using Theorem 2.1, and ${}^{o}C_t = \frac{1}{2}N(N-\sigma)$ for G_n^σ, we obtain

Theorem 3.2 The value on class C_t of the AIC character $\chi^{j\sigma}$ of G_n^σ of type ℓ_m^τ is given by the formula

$$\chi_t^{j\sigma} = \chi_1^{j\sigma}\tau\,(1 + s^\sigma/(N-\sigma\cdot 1))/L, \quad L = 2^\ell. \qquad (3.3)$$

where s^σ is the sum of the roots of the polynomial $P_\ell^{j\sigma}(z)$.

REFERENCES

1.J.S.Frame, Character formulas for the orthogonal groups over GF(2), Mathematical Reports, vol 1, no.2 (1979),99-102.

2.J.S.Frame,Character formulas for the orthogonal groups over GF(2) II. Mathematical Reports, vol 1,no.3 (1979), 153-156.

3.J.S.Frame, Generalized hook graphs and degree formulas for $O_n(2)$, Combinatoric Representation Theory and Statistical Methods in Groups - Young Day Proceedings: Marcel Dekker, 1980.

Michigan State University
East Lansing, Michigan

Received July 1, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

CONVERGENCE SPACES AND CLOSED GRAPHS

Robert A. Herrmann

Presented by F. V. Atkinson, F.R.S.C.

ABSTRACT*. This is a fourth report in a series of investigations answering various unsolved problems relative to convergence spaces. The major goal of this present research is not only to investigate convergence space closed graph theory but also to show that many recent results relative to generalizations of the topological concept of the closed graph such as the strongly closed graph or maps with property (P) are in reality simple corollaries to the appropriate convergence space propositions.

1. Preliminaries. Prior to our basic definitions it should be mentioned that an important typographical error appears in [2]. In Theorem 3 on page 267 the conclusion should read "g is weakly-admissible" not that g is weakly-continuous.

For a set X, let F(X) (resp. U(X)) denote the set of all filters (resp. ultrafilters) on X. As usual if $\mathcal{F} \in F(X)$ and $f: X \to Y$, then $f(\mathcal{F})$ is the filter generated by $\{f(F) \mid F \in \mathcal{F}\}$. Moreover, we let $G(f) = \{(x, f(x)) \mid (x \in X) \wedge (f(x) \in Y)\}$ denote the graph of f and for $A \subset X$, the set $cl_q(A) = \{x \mid (x \in X) \wedge \exists \mathcal{U}((\mathcal{U} \in U(X)) \wedge (A \in \mathcal{U}) \wedge (\mathcal{U} \underset{q}{\to} x))\}$ is the q-closure of A in the preconvergence space [2, p.265] (X,q).

---

*This research was partially supported by a grant from the U. S. Naval Academy Research Council. 54A05

2. Main results. We now give two main results and those propositions which follow from them in a straightforward manner. Throughout assume that (X,q) and (Y,p) are preconvergence spaces.

THEOREM 1. Let $f: X \to Y$.

(i). If $G(f)$ is closed in the product preconvergence space $(X \times Y, q \times p)$, then whenever $\mathcal{F} \in F(X)$, $\mathcal{F} \underset{q}{\to} x \in X$ and $f(\mathcal{F}) \underset{p}{\to} y \in Y$, it follows that $f(x) = y$.

(ii). If whenever $\mathcal{U} \in U(X)$, $\mathcal{U} \underset{q}{\to} x \in X$ and $f(\mathcal{U}) \underset{p}{\to} y \in Y$ imply that $f(x) = y$, then $G(f)$ is closed in $X \times Y$.

Sketch of proof. (i) This follows in a simple and direct manner from the definition of the $q \times p$-closure and the product preconvergence function $q \times p$.

(ii). Assuming the hypothesis this result is somewhat more difficult to obtain. Let $(w,z) \in X \times Y - G(f)$. Then assume that $(w,z) \in cl_{\pi}(G(f))$, where $\pi = q \times p$. It follows that there exists some $\mathcal{U} \in U(X \times Y)$ such that $G(f) \in \mathcal{U}$ and $\mathcal{U} \to (w,z) \nrightarrow (w,f(w))$ and the first projection $P_1(\mathcal{U}) = \mathcal{V} \in U(X)$, $\mathcal{V} \to w$. Letting $(I,f): X \to X \times Y$ be $(I,f)(x) = (x,f(x))$ and using the result that $(I,f)^{-1}(A) \subset P_1(A) \cap f^{-1}(P_2(A))$ for each $A \subset X \times Y$ it can be established that $P_2(\mathcal{U}) \subset f(P_1(\mathcal{U}))$. Hence $f(P_1(\mathcal{U})) = f(\mathcal{V}) \to f(w)$ since $P_2(\mathcal{U})$ is an ultrafilter. From this contradiction and the general fact that $G(f) \subset cl_{\pi}(G(f))$ the result follows.

Application of Theorem 1 yields the following propositions.

THEOREM 2. If $f: X \to Y$ has a closed graph and Y is compact [3, p.456], then f is weakly-continuous [3, p.455].

THEOREM 3. *Let* Y *be compact and Hausdorff* [3, p.459]. *Then* $f: X \to Y$ *has a closed graph if and only if* f *is weakly-continuous*.

THEOREM 4. *Let* $f: X \to Y$ *and* X,Y *be compact*. *Moreover, let* Y *be Hausdorff and* f(X) *closed in* Y. *If* G(f) *is closed in* $X \times Y$, *then* f *is perfect* [3, p.450].

For preconvergence spaces the existence of functions with a closed graph is somewhat more critical than for topological spaces. Let q be a convergence function on X. The *pretopological modification* $\hat{q}$ is defined as follows: $x \in \hat{q}(\mathcal{F})$ for $\mathcal{F} \in F(X)$ if and only if $\mathcal{N}_q(x) = \bigwedge\{\mathcal{U} \mid (\mathcal{U} \in U(X)) \wedge (\mathcal{U} \to x)\} \subset \mathcal{F}$. Theorem 2.8 (iv) in [3] yields the following result.

THEOREM 5. *Let* X *and* Y *be compact and* Y *be Hausdorff*. *If the pretopological modification of* q *is a topology on* X *and* $\hat{p}$ *is not a topology on* Y, *then there does not exist a surjection* $f: X \to Y$ *with a closed graph*.

The second main result shows that important mapping properties such as described in [4] and [5] are in reality simple corollaries to a similar but much more general preconvergence space result. First, let $\mathcal{U} \in U(X)$ be a free ultrafilter on X and $z \in X$. Let $\mathcal{G}_z = \{U \cup \{z\} \mid U \in \mathcal{U}\} \cup \{\{x\} \mid x \in X - \{z\}\}$. Then $\mathcal{G}_z$ is a base for a topology T_z on X. It can be shown that (X, T_z) is a Hausdorff completely normal, fully normal, door space.

THEOREM 6. *Let* (X,q) *be* T_1. *Assume that there exists some* $z \in X$ *such that for every free ultrafilter* $\mathcal{U} \in U(X)$ *the fact that the identity map* $I: (X, T_z) \to (X, q)$ *has a closed graph implies that* I *is weakly-continuous*. *Then* (X,q) *is compact*.

Sketch of proof. Assuming that X is not compact implies that there exists a non-q-convergent $\mathcal{U} \in U(X)$. Thus $\mathcal{U}$ is free. Let z be as hypothesized. Consider (X,T_z) and $\mathcal{F} \in F(X)$ such that $\mathcal{F}$ is T_z-convergent to $x \in X - \{z\}$ and for the identity map $I:(X,T_z) \to (X,q)$, $I(\mathcal{F}) \underset{q}{\to} y \in X$. Now $\mathcal{F} \in U(X)$ and is principal. Hence $I(x) = y$. Now let $x = z$, $\mathcal{F}$ be T_z-convergent to x and $I(\mathcal{F}) \underset{q}{\to} y \in X$. Then consider the neighborhood filter $\mathcal{M}(z)$ for z. It follows that there exists some $\mathcal{V} \in U(\mathcal{F})$ [2, p.447] such that $\mathcal{V} \underset{q}{\to} y$ and $\mathcal{M}(z) \subset \mathcal{V}$. This implies that $\mathcal{V} = \dot{z}$. The T_1 property implies that $z = y$. Hence $I(z) = y$. It follows from Theorem 1 that G(I) is closed and thus that I is weakly--continuous. Since $\mathcal{U}$ is T_z-convergent to z, then $I(\mathcal{U}) = \mathcal{U}$ is q-convergent to $I(z) = z$. This contradiction yields the result.

COROLLARY 6.1. *Let* $\mathcal{S}$ *be the class of all Hausdorff completely normal, fully normal door topological spaces and let* (Y,p) *be a* T_1 *preconvergence space*. *If for every* $X \in \mathcal{S}$ *every bijection* $f:X \to (Y,p)$ *with a closed graph is weakly-continuous, then* Y *is compact*.

3. *Applications*. A filter $\mathcal{F} \in F(X)$ is δ-*convergent* to $x \in X$, where (X,T) is a topological space, if for each regular--open $G \in T$ containing x there exists some $F \in \mathcal{F}$ such that $F \subset G$. A filter $\mathcal{F}$ is θ-*convergent* to $x \in X$ if for each open neighborhood G of $x \in X$ there exists some $F \in \mathcal{F}$ such that $F \subset cl_T(G)$. Mathematicians have extensively studied these convergence structures without realizing that they are at least pretopological in character.

The following characterizations are useful when applying the results from the previous section. A map $f:(X,T) \to (Y,\tau)$, where τ is a topology, has a strongly closed graph [1, p.470] (resp. property (P) [7, p.380]) if and only if $f:(X,T) \to (Y,\theta)$ (resp. $f:(X,T) \to (Y,\delta)$) has a convergence space closed graph. The map $f:(X,T) \to (Y,\tau)$ is θ-perfect [8, p.542] (resp. δ-perfect [8, p.450]) if and only if $f:(X,\theta) \to (Y,\theta)$ (resp. $f:(X,\delta) \to (Y,\delta)$) is convergence space perfect. The map $f:(X,T) \to (Y,\tau)$ is almost-continuous [7, p.379] (resp. weakly-θ-continuous (i.e. weakly-continuous [7, p. 379])) if and only if $f:(X,T) \to (Y,\delta)$ (resp. $f:(X,T) \to (Y,\theta)$) is convergence space continuous.

It is now possible to easily translate some of the results from section two into the corresponding topological language. For example, let (X,T) be regular compact and (Y,τ) Hausdorff N-closed [7, p.379]. If a surjection $f:X \to Y$ has a strongly closed graph or property (P), then f is θ-perfect and δ-perfect. Let $\mathcal{S}$ be the class of all Hausdorff completely normal, fully normal door topological spaces and let (Y,τ) be Hausdorff (resp. weakly-Hausdorff). If for every $X \in \mathcal{S}$ every bijection $f:X \to (Y,\tau)$ with a strongly closed graph (resp. with property (P)) is weakly-θ-continuous (resp. almost-continuous), then Y is H-closed (resp. N-closed).

REFERENCES

[1] Herrington, L.L. and P.E. Long, Characterizations of H-closed spaces, Proc. Amer. Math. Soc., 48(1975), 469-475.

[2] Herrmann, R. A., Convergence spaces and extensions of maps, C.R. Math. Rep. 1(5)(1979), 265-268.

[3] __________, Perfect maps on convergence spaces, Bull. Austral. Math. Soc., 20(1979), 447-466.

[4] Joseph, J.E., On H-closed and minimal Hausdorff spaces, Proc. Amer. Math. Soc., 60(1976), 321-326.

[5] Kasahara, S., Characterization of compactness and countable compactness, Proc. Japan Acad., 49(1973), 523-524.

[6] Kent, D., Convergence functions and their related topologies, Fund. Math. 54(k964), 125-135.

[7] Noiri, T., Weak-continuity and closed graphs, Časpois Pěst. Mat., 101(1976), 379-382.

[8] __________, A generalization of perfect functions, J. London Math. Soc., 17(1978), 540-544.

MATHEMATICS DEPARTMENT
U.S. NAVAL ACADEMY
ANNAPOLIS, MD.21402
U.S.A.

Received in revised form
July 23, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

THE STRUCTURE OF THE MAXIMAL LINEAR CLASSES IN PRIME-VALUED LOGICS

J. Bagyinszki and J. Demetrovics

Presented by G. Grätzer, F.R.S.C.

1. Introduction. The complete lattice of closed classes for 2-valued logic was given by Post [6]. All maximal classes in k-valued logics were given for $k = 3$ and $k > 3$ Jablonskii [3] and Rosenberg [8], respectively. According to a result of Janov and Mučnik [4], there are closed subsets infinitely generated (without base) and a continuum of closed subsets as well in k-valued logics (for $k > 2$). Due to this fact, it may be an interesting question, what the maximal classes having countable many closed subsets are and how we can describe the lattice of closed subsets of these classes. Partial answers about cardinality are given by Salomaa [10], Bagyinszki and Demetrovics [1], Lau [5], Demetrovics and Hannák [2]. The authors gave the complete lattice of closed subsets of linear functions and it was proved to be finite [1]. The finiteness and all closed subsets of linear functions having n-ary functions with $n > 1$ – are presented in [10], too. The maximal and minimal length of chains of closed subset-lattice of k-valued logics was investigated by Pöschel [7]. A survey on closed sets of k-valued logics is given by Rosenberg [9].

In this paper our earlier results on the complete lattice of the maximal linear classes in prime-valued logics (proofs are in [1]) are presented. Having this complete lattice we give

1.) the exact (finite) number of closed subsets;

2.) the length of maximal and minimal chains of the lattice;

3.) all bases with minimal number of elements and the arity of each linear class.

2. Definitions and notations. Let $0_k^{(n)}$ denote the set of n-ary functions over the set $K = \{0,1,2,\ldots,k-1\}$, $k > 2$; $0_k^{(n)} = \{f : K^n \to K\}$. Set $0_k = \cup_{n=0}^{\infty} 0_k^{(n)}$. The set $0_k^{(0)}$ is considered as the set of unary constant functions. For a subset $A \subseteq 0_k$, $[A]$ denotes the closure of A under the following operations:

$$f * (f_1,\ldots,f_n) = f(f_1,\ldots,f_n), \text{ where } f = f(\tilde{x}) \in A,$$

$$f_i = f(\tilde{x}^i) \in A \cup \{x_1,\ldots,x_n\},\ \tilde{x} = (x_1,\ldots,x_n),\ \tilde{x}^i = (x_{i1},\ldots,x_{im_i}),\ i = 1,\ldots,n.$$

A set $0 \subseteq 0_k$ is said to be closed set if $0 = [0]$. A subset A of a closed set 0 is called complete in 0, if $[A] = 0$. Let $0 \subseteq 0_k$ be a closed set further on. A set $A \subseteq 0$ is precomplete in 0, if $[A] \subset B \subseteq 0$ implies $[B] = 0$. A closed precomplete set is called maximal. Let $S(0)$ denote the set of closed subsets of 0. A set $B \subseteq 0$ is said to be a base in 0, if $[B] = 0$ and $[B'] \neq 0$ for $B' \subset B$.

For the remaining part of this paper let $k = p (p > 2)$ be a fixed prime number, $P = \{0,1,\ldots,p-1\}$ (although, some of the following results are valid under more general conditions). L denote the set of linear polynomial functions over $GF(p) = \langle P; + ; \cdot \rangle$;

$L = \{a_0 + \Sigma_{i=1}^n a_i x_i | a_0 \epsilon P, a_i \epsilon P \setminus \{0\}, i = 1, \ldots, n;\ n \geq 0\}$, and

$f(\tilde{x}) = a_0 + \Sigma_{i=1}^n a_i x_i$, $\Sigma_{i=1}^n a_i = a$ will be used.

It can be verified, that the following sets are closed subsets of $L (\subset 0_k)$:

1.) $L^{(0)} = 0_p^{(0)}$, $L^{(1)} = \{a_0 + a_1 x | a_0 \epsilon P, a_1 \epsilon P \setminus \{0\}\}$, $L^{(1)} \setminus L^{(0)}$;

2.) $L(\beta) = \{f \epsilon L | a_0 = \beta(1 - a)\} \cup \{\beta\}$, $\beta = 0, 1, \ldots, p-1$;

3.) $L_\delta = \{f \epsilon L | a = 1\}$;

4.) $L_\delta(0) = \{f \epsilon L | a = 1, a_0 = 0\} = L_\delta \cap L(\beta) = L_\delta(\beta) (= L(\beta_1) \cap L(\beta_2), \beta_1 \neq \beta_2)$, for $\beta \epsilon P$;

5.) $L^{(1)}(\beta) = L^{(1)} \cap L(\beta)$, $L^{(1)}(\beta) \setminus \cdot\beta\cdot$, $\beta = 0, 1, \ldots, p-1$;

6.) $L^{(0)}(\beta) = L^{(0)} \cap L(\beta) = \{\beta\}$, $\beta = 0, 1, \ldots, p-1$;

7.) $L_\delta^{(1)} = L^{(1)} \cap L_\delta = \{a_0 + x | a_0 \epsilon P\}$;

8.) $\{x\} = L_\delta^{(1)} \cap L(\beta)$, for $\beta \epsilon P$;

9.) $L^{(1)}(\beta) \cup L^{(0)}$.

It is easy to check that $< L^{(1)} \setminus L^{(0)}; * >$ is a group of order $p(p-1)$. Denote $r(a)$ the multiplicative order of $a \epsilon GF(p)$ and let q be a divisor of $p - 1$,
$G[q] = \{a_0 + ax | a \neq 0, r(a) \text{ divides } q\}$.

A proper divisor $\bar{q}$ of q is said to be a maximal divisor of q, if $(\bar{q} | q'$ and $q' | q)$ implines $q' \epsilon \{\bar{q}, q\}$. It was proved in [1] that there are three types of closed subsets of $L^{(1)}$, namely: $G[q]$, $G[q] \cup L^{(0)}$ and $G[q](\beta) \cup L^{(0)}$.

Let $p - 1 = q_1^{\epsilon_1} q_2^{\epsilon_2} \ldots\ldots q_u^{\epsilon_u}$ be the prime-power decomposition of $p - 1$ with $q_1 = 2 < q_2 < \ldots . < q_u$ primes, $\epsilon_i > 0$ integers, $i = 1, 2, \ldots, u$.

3. Results. The results about structure of the complete lattice of $S(L)$ are presented on Fig 1. Table 1. contains minimal bases and the arity of each linear class. Chains with minimal and maximal length are:

$$(L) \to (L_\delta) \to L_\delta(0) \to (\{x\}) \to (\emptyset),$$

$$(L) \to (L^{(1)}) \to \ldots\ldots \to (G[q] \cup L^{(0)}) \to (G[\bar{q}] \cup L^{(0)}) \to \ldots\ldots \to (L_\delta^{(1)} \cup L^{(0)}) \to$$
$$\to (\{x\} \cup L^{(0)}) \to (\{x\} \cup L^{(0)} \setminus \{0\}) \to \ldots\ldots \to (\{x\}) \to (\emptyset).$$

The minimal and maximal lengths are 4 and $p + 3 + \Sigma_{i=1}^u \epsilon_i$, respectively.

Theorem 3.2.3 of [7] and theorem 5.1. of [1] can be combined to obtain a stronger upper bound for minimal lenght $l(k)$ of k-valued logic for arbitrary k: $l(k) < 7 \cdot \sum_{i=1} \alpha_i - 2$, where $k = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_t^{\alpha_t}$.

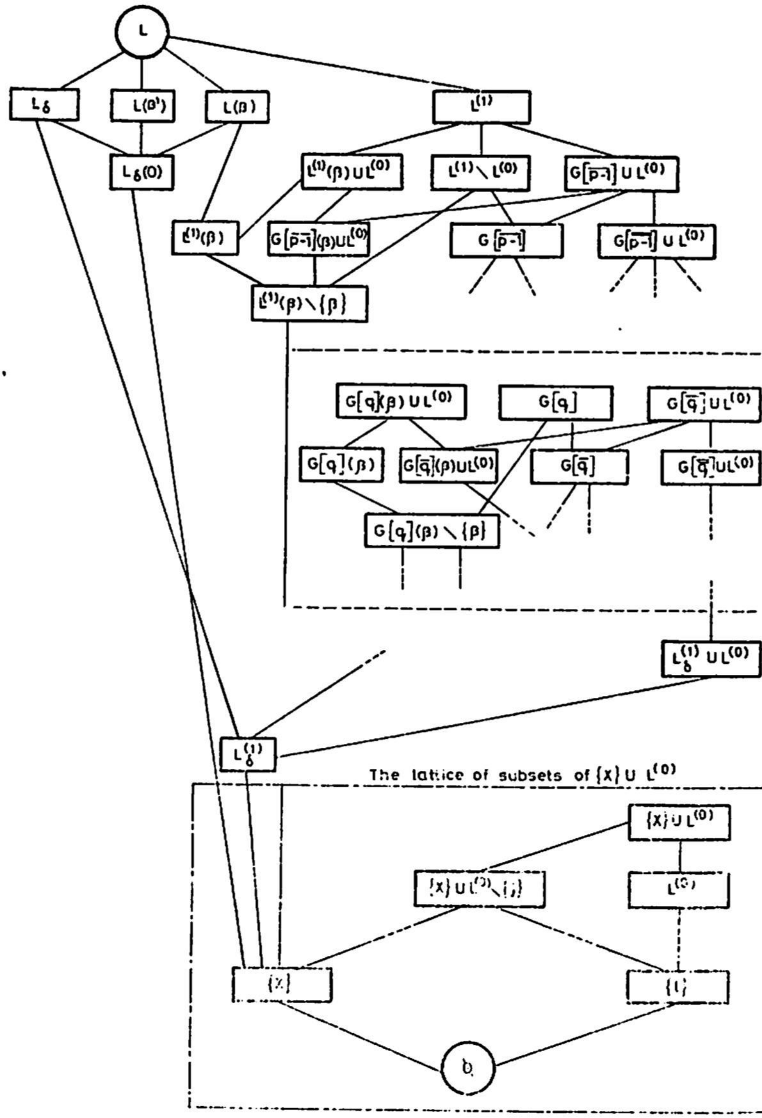

L
L_δ
$L(\beta')$
$L(\beta)$
$L^{(1)}$
$L_\delta(0)$
$L^{(1)}(\beta) \cup L^{(0)}$
$L^{(1)} \setminus L^{(0)}$
$G[\overline{p-1}] \cup L^{(0)}$
$L^{(1)}(\beta)$
$G[\overline{p-1}](\beta) \cup L^{(0)}$
$G[\overline{p-1}]$
$G[\overline{p-1}] \cup L^{(0)}$
$L^{(1)}(\beta) \setminus \{\beta\}$
$G[q](\beta) \cup L^{(0)}$
$G[q]$
$G[\overline{q}] \cup L^{(0)}$
$G[q](\beta)$
$G[\overline{q}](\beta) \cup L^{(0)}$
$G[\overline{q}]$
$G[\overline{q}] \cup L^{(0)}$
$G[q](\beta) \setminus \{\beta\}$
$L_\delta^{(1)} \cup L^{(0)}$
$L_\delta^{(1)}$
The lattice of subsets of $\{X\} \cup L^{(0)}$
$\{X\} \cup L^{(0)}$
$L^{(0)}$
0

Fig 1

Class	Base	Rank
L	$\{x+1, x+y\}$	2
$L(\beta)$	$\{x+y+(p-1)\beta\}$	2
L_δ	$\{2x+(p-1)y+1\}$	2
$L^{(1)}$	$\{0, x+1, ax\}$, with $r(a) = p-1$	1
$L_\delta(0)$	$\{2x+(p-1)y\}$	2
$L^{(1)}(\beta) \cup L^{(0)}$	$\{\beta, \beta', ax+(1-a)\beta\}$, with $r(a) = p$	1 1
$L^{(1)} \setminus L^{(0)}$	$\{x+1, ax\}$, with $r(a) = p-1$	1
$L^{(1)}(\beta)$	$\{\beta, ax+(1-a)\beta\}$, with $r(a) = p-1$	1
$L^{(1)}(\beta) \setminus \{\beta\}$	$[ax+(1-a)\beta]$, with $r(a) = p-1$	1
$L_\delta^{(1)}$	$\{x+1\}$	1
$G[q]$	$\{x+1, ax\}$, with $r(a) = q$	1
$G[q] \cup L^{(0)}$	$\{0, x+1, ax\}$, with $r(a) = q$	1
$G[q](\beta) \cup L^{(0)}$	$\{\beta, \beta', ax+(1-a)\beta\}$, with $r(a) = q$	1

Table 1.

References

[1] Bagyinszki, J., Demetrovics, J.: The structure of linear classes in prime valued logics, (Hungarian), MTA SzTAKI Közlemények. 16 (1976). 25-52.

[2] Demetrovics J., Hannák.: The cardinality of closed sets in pre- complete classes in k-valued logics, Acta Cybernet. 4 (1979), 3, 273-277.

[3] Jablonskii, S.V.: Functional constructions in k-valued logics, (Russian), Trudy Mat. Inst. Steklov 51 (1958), 5-142.

[4] Janov, Ju.I., Mučnik, A.A.: Existence of $k$valued closed classes without a finite basis, (Russian), Dokl. Akad. Nauk. SSSR 127 (1959). 44-46.

[5] Lau, D.: Über die Anzahl abgeschlossenen Mengen von linearen Funktionen der n-wertigen Logik, Elektron. Informationverarb. Kybernet. 14(1978). 567-569.

[6] Post, E., The two-valued iterative systems of mathematical logic, Annals of Math. Studies 5(1941).

[7] Pöschel, R., Kaluznin, L.A.: Funktionen - und Relationen algebren, Birkhasuser Verlag Basel und Stuttgart (1979).

[8] Rosenberg, I.G.: La structure des fonctions de plusieurs variables sur un ensemble fini, C.R. Acad. Sci. Paris Ser. A.B. 260(1965), 3817-19.

[9] Rosenberg, I.G.: Comleteness properties of multiple - valued logic algebras, in "Computer Science and Multiple - valued Logic" (Ed. D.S. Rine), North-Holland (1977), 144-186.

[10] Salomaa, A.: On infinitely generated sets of operations in finite algebras, Ann Univ. Turkuensis, ser.A. I. 74(1964), 1-13.

Authors's adress:

J. Bagyinszki: Institute for Mathematics,
University of Gödöllö,
H– 2103 Gödöllö, Hungary

J. Demetrovics: Computer and Automation
Institute Hungarian Academy of Sciences
H– 1502 Budapest XI. Kende u. 13-17.

Received July 10, 1980

C. R. Math. Rep. Acad. Sci. Canada - Vol. II (1980) No. 4

SOME REMARKS ON THE STRUCTURE OF P_3

J. Demetrovics – L. Hannák – S.S. Marchenkov

Presented by G. Grätzer, F.R.S.C.

E. Post [10] succeeded in describing the structure of the closed classes in the two-valued logics (P_2) in 1941. He also determined their minimal bases and their order. In 1959 Yu I. Yanov and A.A. Muchnik proved [13], that no results similar to Post's can be expected in many-valued logics $P_k (k > 2)$ as these P_k have closed classes of cardinality continuum. It makes studies to determine (instead of the structure of P_k) the maximal elements of P_k and their subclasses natural.

The present paper deals with the structure S of the self-dual by $x + 1 (\mathrm{mod}\ 3)$ functions of P_3. In the rest of the cases the structure of the dual atoms (maximal elements) has either been determined [1, 7, 11] or has been proved unsolvable on the basis that the atoms hold closed classes with cardinality continuum [4, 12].

Special self-dual closed classes [8] and self-dual functions [6,9], homogeneous classes and functions [2, 3] have been dealt with in several papers on universal algebras.

B. Csákány [3] described closed classes in P_3 and S.S. Marchenkov proved [12] them to be all the homogeneous classes in P_3.

In the present paper we describe the structures of all closed classes self-dual by $x + 1 (\mathrm{mod} 3)$ except those in two substructures and we give their definition, order ect., like Post [9]. We determine classes of S that are invariant to any automorphism of the lattice $\mathcal{L}(S)$. We describe the classes that determine the atoms in $\mathcal{L}(S)$ and ones that contain the ternary $(t(x,y,z))$ or the dual $(d(x,y,z))$ descriminator.

Let $E_3 = \{0,1,2\}$ and P_3 be the set of all functions $f(x_1, x_2, \ldots, x_n)$ $n = 1,2, \ldots$ with $f : E_3 \times E_3 \times \ldots \times E_3 \to E_3$. We call the restriction of f to $\varepsilon = \{0,1\}$ its Boole restriction and denote it with $Bf(x_1, \ldots, x_n)$ if its range is within ε. Boole restrictions are also Boole functions.

Let S be the set of all self-dual functions by $x + 1 (\mathrm{mod}\ 3)$ in P_3, i.e. those for which

$$f(x_1, x_2, \ldots, x_n) = f(x_1 + 1, x_2 + 1, \ldots, x_n + 1) + 2$$

holds.

We introduce the following notations:

Let $L = \{f : f \in S$ and f is linear$\}$; $L_1 = \{f : f \in S$ and f is linear and preserves $\tilde{0}\}$;

$L_2 = \{f : f$ equals either x, or $x + 1$, or $x + 2\}$;

$S_1 = \{f : f \in S$ and f preserves $0\}$; $0 = \{f : f$ equals $x\}$;

$S_2 = \{f : f \in S$ and f is self-dual by the permutation $p(x) = 2x\}$, Now, let K be any of the following Boolean classes:

$C_4 = [\{x \vee y, x(y+z+1)\}]$; $A_4 = [\{xy, x \vee y\}]$; $D_1 = [\{xy \vee x\bar{z} \vee y\bar{z}\}]$;

$D_2 = [\{xy \vee xz \vee yz\}]$; $L_4 = [\{x+y+z\}]$; $0_1 = [\{f : f(x_1, \ldots, x_n) \text{ equals } x\}]$;

$F_1^\mu = [\{x \vee y\bar{z}, h_\mu^*(\tilde{x})\}]$, $\mu > 2$; $F_1^\infty = [\{x \vee y\bar{z}\}]$;

$F_2^2 = [\{x \vee yz, h_2^*(\tilde{x})\}]$; $F_2^\mu = [\{h_\mu^*(\tilde{x})\}]$, if $\mu > 3$;

$F_2^\infty = [\{x \vee yz\}]$; $S_1 = [\{x \vee y\}]$; $F_5^\mu = [\{x(y \vee \bar{z}), h_\mu(\tilde{x})\}]$, $\mu > 2$;

$F_5^\infty = [\{x(y \vee \bar{z})\}]$; $F_6^2 = [\{x(y \vee z), h_2(\tilde{x})\}]$; $F_6^\mu = [\{h_\mu(\tilde{x})\}]$, $\mu > 3$;

$F_6^\infty = [\{x(y \vee z)\}]$; $P_1 = [\{xy\}]$.

Where [ ] denotes superposition. $h_\mu(x_1, x_2, \ldots, x_{\mu+1})$ is $\bigvee_{i=1}^{\mu+1} x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_{\mu+1}$ and $h_\mu^*(\tilde{x})$ is the dual of $h_\mu(\tilde{x})$. Once this choice of K is made, let $SK = \{f : f \in S$ and f preserves $\{0,1\}$ and $Bf \in K\}$. For another choice of K let K be any of the following Boolean classes:

$D_1 = [\{xy \vee x\bar{z} \vee y\bar{z}\}]$; $D_2 = [\{xy \vee xz \vee yz\}]$;

$L_4 = [\{x+y+z\}]$; $0_1 = [\{f : f(\tilde{x}) \equiv x\}]$.

For this choice of K let $S_2K = \{f : f \in S_2$ and f preserves $\{0,1\}$ and $Bf \in K\}$.

With proper choises of K, SK may equal to any of the classes SD_2, SF_2^μ, SF_2^∞, SS_1, SF_6^μ, SP_1 ($\mu > 2$) and S_2K may equal to S_2D_2. S^*K and $S_2^*D_2$ denote the class of functions of SK with the property:
if a Boole restriction of the function $g(x_{i_1}, \ldots, x_{i_m})$ obtained from $f(x_1, \ldots, x_n)$ by making the values of variable pairs identical equals an x_{i_j}, this implies $g(x_{i_1}, \ldots, x_{i_m}) \equiv x_{i_j}$.

For $K \in \{S, P_1\}$ let $RK = \{f : f \in S$ and f preserves $\{0,1\}$ and $Bf \in K$, f and Bf have the same set of variables on which they essentially depend$\}$.

The set $\mathcal{L}(S)$ of the closed classes in S is a poset by the set theoretical inclusion by which a lattice structure is induced on it : $\wedge$ is the set theoretical intersection and $\vee$ the smallest closed class in $\mathcal{L}(S)$ containing both operands. The lattice has S as maximal and the void class as minimal element.

Theorem 1. The classes $S, L, S_1, S_2, SC_4, SA_4, SD_1, SD_2, S^*D_2, S_2D_1, SL_4, S0_1, S_2D_2, S_2^*D_2$, $S_2L_4, S_20_1, L_1, L_2, 0, SF_1^\mu, SF_2^\mu, S^*F_2^\mu, SF_5^\mu, SF_6^\mu, S^*F_6^\mu$ ($\mu = 2, 3, \ldots, \infty$), SS_1, S^*S_1, RS_1, SP_1, S^*P_1, RP_1 have finite bases. The classes $L_2, 0$ have order 1, the $S, L, S_1, SA_4, L_1, RS_1, RP_1$ have order 2, the $S_2, SC_4, SD_1, SD_2, S^*D_2, S_2D_1, SL_4, S0_1, S_2D_2, S_2^*D_2, S_2L_4, S_20_1, SF_1^\infty, SF_2^\infty$, $S^*F_2^\infty, SS_1, S^*S_1, SF_5^\infty, SF_6^\infty, S^*F_6^\infty, SP_1, S^*P_1$ have order 3, the $SF_1^\mu, SF_2^\mu, S^*F_2^\mu, SF_5^\mu, SF_6^\mu, S^*F_6^\mu$ ($\mu = 2, 3, \ldots$) have order $\mu + 1$.

Theorem 2. The morphism induced by the function $2x + 1 \pmod 3$ is the only nontrivial automorphism of the lattice $\mathcal{L}(S)$. By this automorphism exactly the classes S, L, S_1, S_2, $SC_4, SA_4, SD_1, SD_2, S^*D_2, S_2D_1, SL_4, S0_1, S_2D_2, S_2^*D_2, S_2L_4, S_20_1, L_1, L_2, 0$ are stationary.

Theorem 3. The closed classes holding the ternary discriminator $(t(x,y,z))$ are exactly

S_2D_1 SD_1 SC_4 S_2 S_1 S. The closed classes holding the dual discriminator $(d(x,y,z))$ are exactly $S_2^*D_2$ S_2D_2 S_2D_1. S^*D_2 $S^*F_2^3$ $S^*F_6^2$ SF_2^3 SF_6^2 SF_1^3 SF_5^3 SD_2 SD_1 SA_4 SC_4 S_2 S_1 S.

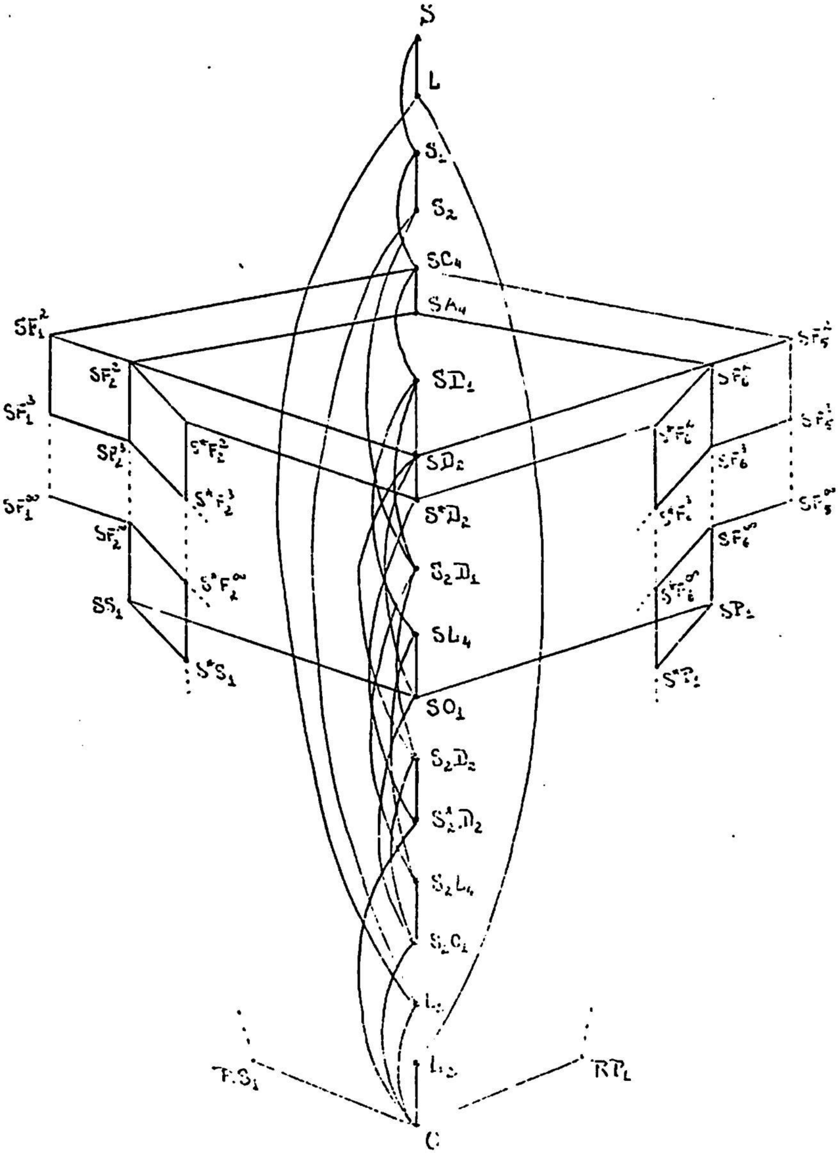
We remark here that S holds some more closed classes between RS_1 and $S^*F_2^3$ and between RP_1 and $S^*P_6^3$. There are no further closed classes in S.

In the following we give a base for every self-dual closed class discussed in the paper. Let $\varphi(x,y,z)$, $\psi(x,y,z)$, $\lambda(x,y,z)$, $\omega(x,y,z)$, $\tau(x,y,z)$, $\alpha(x,y)$, $\beta(x,y)$, $\alpha_1(x,y,z)$, $\gamma(x,y,z)$, $\delta_\mu(x_1,x_2,\ldots$ $\ldots,x_{\mu+1})(\mu > 3)$ be functions of S preserving the set $\{0,1\}$ and for which

$$B\varphi(x,y,z) = x,\ \varphi(0,1,2) = 1,\ \varphi(0,2,1) = 2,\ B\psi(x,y,z) = xy \vee xz \vee yz,$$
$$\psi(0,1,2) = \psi(0,2,1) = 0,\ B\lambda(x,y,z) = x + y + z \pmod 2,\ \lambda(0,1,2) = \lambda(0,2,1) = 0,$$
$$B\omega(x,y,z) = x,\ \omega(0,1,2) = 0,\ \omega(0,2,1) = 1,\ B\tau(x,y,z) = xy \vee xz \vee yz,\ \tau(0,1,2) = 0,$$
$$\tau(0,2,1) = 1,\ B\alpha(x,y) = x \vee y,\ B\beta(x,y) = xy,\ B\alpha_1(x,y,z) = x \vee y,\ \alpha_1(0,1,2) = 0,$$
$$\alpha_1(0,2,1) = 1,\ B\gamma(x,y,z) = x \vee yz,\ \gamma(0,1,2) = \gamma(0,2,1) = 2,\ B\delta_\mu(x_1,x_2,\ldots,x_{\mu+1}) =$$
$$= \mathop{\&}_{i=1}^{\mu+1}(x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_{\mu+1}),\ \delta_\mu(0,\ldots,0,1,2) = \delta_\mu(0,\ldots,0,2,1) = 2$$

holds.

$L_2 = [\{x + 1 \pmod 3\}]$; $SO_1 = [\{\omega(x,y,z)\}]$; $SC_4 = [\{\omega(x,y,z), \alpha(x,y)$ and any $f(x,y,z)$ with $Bf(x,y,z) = x(y + z + 1) \pmod 2\}]$; $SA_4 = [\{\omega(x,y,z), \alpha(x,y), \beta(x,y)\}]$; $SD_1 = [\{\omega(x,y,z), \psi(x,y,\bar{z})\}]$; $SD_2 = [\{\omega(x,y,z), \psi(x,y,z)\}]$; $SL_4 = [\{\omega(x,y,z)\ \lambda(x,y,z)\}]$; $SF_1^\mu = [\{\omega(x,y,z), \gamma(x,y,\bar{z})$; and any $f(\tilde{x})$, with $Bf(\tilde{x}) = h_\mu^*(\tilde{x})$, $\mu > 2\}]$; $SF_1^{\infty} = [\{\omega(x,y,z), \gamma(x,y,\bar{z})\}]$, $SF_2^3 = [\{\omega(x,y,z), \gamma(x,y,z)$, and any $f(x,y,z)$, with $Bf(x,y,z) = h_2^*(x,y,z)\}]$; $SF_2^\mu = [\{\omega(x,y,z)$, and any $f(\tilde{x})$, with $Bf(\tilde{x}) = h_\mu^*(\tilde{x}), \mu > 3\}]$; $SF_2^{\infty} = [\{\omega(x,y,z), \gamma(x,y,z)\}]$; $SS_1 = [\{\omega(x,y,z), \alpha(x,y)\}]$; $S_1 = [\{\alpha(x,y)$, $2x + 2y \pmod 3\}]$, $L = [\{2x + 2y + 1 \pmod 3\}]$; $S = [\{\alpha(x,y), x + 1 \pmod 3\}]$; $RS_1 = [\{\alpha(x,y)\}]$; $S^*S_1 = [\{\alpha_1(x,y,z)\}]$; $S^*F_2^{\infty} = [\{\gamma(x,y,z)\}]$, $S^*F_2^3 = [\{\alpha(x,y)$, $\tau(x,y,z)\}]$, $S^*F_2^\mu = [\{\delta_\mu(x_1,\ldots,x_{\mu+1}), \mu > 3\}]$, $S_2D_2 = [\{\varphi(x,y,z), \psi(x,y,z)\}]$; $S_2^*D_2 = [\{\psi(x,y,z)\}]$, $S_2L_4 = [\{\lambda(x,y,z)\}]$, $S_2O_1 = [\{\varphi(x,y,z)\}]$; $L_1 = [\{2x + 2y \pmod 3\}]$; $S_2D_1 = [\{\psi(x,y,z), \lambda(x,y,z)\}]$; $S_2 = [\{\psi(x,y,z), 2x + 2y \pmod 3\}]$; $0 = [\{x\}]$; $S^*D_2 = [\{\tau(x,y,z)\}]$.


S
L
SC4
SA4
SD1
SD2
S*D2
S2D1
SL4
SO1
S2D2
S2L4
C

References

[1] Bagyinszki, J., Demetrovics, J., Lineáris osztályok szerkezete primszám értékü logikában, MTA SzTAKI Közlemények, 16 (1976), 25-52.

[2] Csákány, B., Homogeneous algebras are functionally complete, Algebra Univ., to appear.

[3] Csákány, B., Gavalcová, T., Finite homogeneous algebras I, to appear.

[4] Demetrovics, J., Hannák L., On the cardinality of self-dual closed classes in k-valued logics, MTA SzTAKI Közlemények 23 (1979), 7-17.

[5] Fried E., Pixley A.F., The dual discriminator function in universal algebra, Acta Sci. Math., 41 (1979), 83-100.

[6] Gratzer, G. Universal Algebra, Van Nonstrand, Princeton 1968.

[7] Lau D., Über die Anzahl von abgeschlossen Mengen von lineazen Funktionen der n-vertigen Logik, EIK, 14, N11, (1978), 567-569.

[8] Pálfy, P.P., Szabó L., Szendrei Á., Algebras with doubly transitive automorphism groups Colloquium soc. J. Bolyai 28 (1979).

[9] Pixley, A.F., The Ternary Discriminator Function in Universal Algebra, Math. Ann., 191, N3, (1971), 167-180.

[10] Post, E., The two-valued iterative systems of mathematical logic, Ann.Math. Studies, 5, (1941).

[11] Szendrei, A., On closed sets of linear operations, Colloquim soc. J. Bolyai 28 (1980). to appear.

[12] Марченков, С.С., О замкнутых классах самодвойственных функций многозначной логики, сб. "Проблемы кибернетики", вып. 36, 1979, 5-22.

[13] Янов, Ю.И., Мучник А.А., О существовании k-значных замкнутых классов, не имеющих конечного базиса, ДАН СССР, 127, №1, 1959, 44-46.

Authors' Addresses:

J. Demetrovics, L. Hannák: Computer and Automation Institute Hungarian Academy of Sciences H 1502 Budapest XI. Kende u. 13-17. Phone: 665-644

S.S. Marchenkov: Moscow, A-47 Miusskajá square 4, Institute of Applied Mathematics

Received July 10, 1980

Mailing Addresses

1. Aczél — Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1

2. Akcoglu, M.A. — Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1

3. Bagyinszki, J. — Institute for Mathematics, University of Gödöllö, H-2103 Gödöllö, Hungary

4. Bantegnie, R. — Université Franche-Comté, Mathématiques, E.R.A. n° 070654, Route de Gray 25030 Besancon Cedex, France

5. Cormack, G.V. — Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2

6. Chudnovsky, G.V. — Department of Mathematics, Columbia University, New York, N.Y. 10027, U.S.A.

7. Demetrovics, J. — Computer and Automation Institute, Hungarian Academy of Sciences H-1502 Budapest XI. Kende u. 13-17

8. Fischer, E. — Adelphi University, Garden City, Long Island, N.Y. 11530, U.S.A.

9. Frame, J.S. — 136 Oakland Drive, East Lansing, Michigan 48823 U.S.A.

10. Grätzer, G. — Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2

11. Herrmann, R.A. — Mathematics Department, U. S. Naval Academy, Annapolis, MD 21402, U.S.A.

12. Hannák, L. — Computer and Automation Institute, Hungarian Academy of Sciences, H-1502 Budapest XI. Kende u. 13-17.

13. Krengel, U. — Lotzestrasze 13, (34) Göttingen, Fed. Republic of Germany

14. Marchenkov, S.S. — Institute of Applied Mathematics, A-47 Miusskajá square 4, Moscow, U.S.S.R.

15. Ng, C.T. — Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1

16. Schempp, W. — Lehrstuhl für Mathematik I, Universität Siegen, Hölderlinstrasze 3, D-5900 Siegen 21, W. Germany

17. Scherk, P. — Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1