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Co-Higgs Bundles of Schwarzenberger Type and the Determinant Morphism

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Abstract. We identify images of the determinant morphism of trace-free co-Higgs bundles modelled on rank 2 Schwarzenberger bundles.

Résumé. Nous identifions les images du morphisme de déterminant des fibrés co-Higgs à trace zéro, modelés sur les fibrés Schwarzenberger de rang 2.

1. Introduction

1.1. Overview Let X be a complex algebraic curve of genus $g_X \geq 2$. The Hitchin morphism $\mathcal{H}$ of Higgs bundles on X is proper and surjective onto the Hitchin base $\mathcal{B}$ while a generic fiber is an abelian variety and a completely integrable system. When X is a variety of a higher dimension $\mathcal{H}$ is not surjective though it is proper. The conjectured image of the Hitchin morphism is a nonlinear closed subscheme $\mathcal{B}^{\heartsuit} \subset \mathcal{B}$ such that $\mathcal{H}$ factors through the inclusion of $\mathcal{B}^{\heartsuit} \subset \mathcal{B}$ and there exists an open dense subscheme $\mathcal{B}^{\diamondsuit} \subset \mathcal{B}^{\heartsuit}$ that is contained inside the image of $\mathcal{H}$ ([2]). In particular, the spectral covering map $\pi_s : X_s \to X$ for $s \in \mathcal{B}^{\heartsuit}$ is finite and the spectral correspondence sets a bijective correspondence between the twisted vector bundles with spectral cover X_s and Cohen-Macaulay sheaves of generic rank 1 on X_s. The conjecture is verified for $\dim X = 2$ on vector bundles ([13]) and for principal Higgs bundles of certain classical Lie groups ([6]). A conjectured image of $\mathcal{H}$ is proposed naturally for V-twisted Higgs bundles of rank 2 on algebraic curves ([4]). Thus far no description more explicit than these conjectured images, where the twist is not a line bundle, is available in the literature. To overcome this difficulty we aim at exploring the Hitchin morphism restricted on twisted Higgs bundles whose underlying vector bundles are well-selected and fixed. Throughout this paper we will call such twisted Higgs bundles *modelled* on their underlying bundles. For example, we can consider co-Higgs bundles or $T_{\mathbb{P}^2}$-twisted Higgs bundles modelled on rank 2 Schwarzenberger bundles. Note that in the rank 2 case $\mathcal{H}$ consists of a linear component — the *trace* component, and a nonlinear component — the *determinant* component. Moreover, $\mathcal{H}$ is represented by the determinant

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component if the trace component is shifted to 0. So we focus on trace-free twisted Higgs bundles and study their determinants to identify the image of $\mathcal{H}$. The main results in this paper (Theorem 2.2.4, Theorem 2.2.6, Theorem 2.2.8, and Theorem 2.2.10) compute images of the determinant morphism of Gieseker semistable traceless co-Higgs bundles modelled on Schwarzenberger bundles V_k^ρ for $k \neq 3$.

Co-Higgs bundles are important geometric objects on multiprojective spaces and stable co-Higgs bundles vanish on algebraic surfaces of general type and so on [9]. For a fixed nonsingular conic ρ inside $\mathbb{P}^2$, Schwarzenberger bundles are a sequence of holomorphic vector bundles $\{V_k^\rho\}_{k=1}^{\infty}$ of rank 2 on $\mathbb{CP}^2$ coming from a $2:1$ covering map $f^\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ branched over ρ ([10], [3]). Let $k \neq 3$. If ϕ is a nonzero stable traceless co-Higgs bundle modelled on V_k^ρ then $\phi = \phi_0 \otimes C$ for some $\phi_0 \in H^0(\mathrm{End}_0(E) \otimes \mathcal{O}(1))$ and $C \in H^0(T_{\mathbb{P}^2}(-1))$. This characterization of stable co-Higgs bundles along with the standard Euler sequence for $T_{\mathbb{P}^2}$ is utilized to construct moduli spaces and to find dimensions of spaces of their infinitesimal deformations ([9]). In particular, $\det(\phi) = \det(\phi_0) \otimes \mathrm{Sym}^2(C)$ where $\mathrm{Sym}^2(C) \in H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2}) \otimes \mathcal{O}(-2))$ and $\det(\phi_0) \in H^0(\mathcal{O}(2))$. This description enables us to write down complex schemes that parametrize the images of the determinant morphism. On V_3^ρ there are stable co-Higgs bundles which are not of such type. So we exclude the case $k = 3$ completely from our analysis.

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2. The Determinant Morphism on Co-Higgs Bundles of Schwarzenberger Type

2.1. Preliminaries Let X be an n-dimensional smooth connected complex projective algebraic variety with structure sheaf $\mathcal{O}_X$ and H be a fixed ample line bundle on X. Let E be a coherent sheaf of pure dimension $d = \dim(E)$ and rank r on X. There are unique integers $\alpha_0(E), \ldots, \alpha_d(E)$ with $\alpha_d(E) > 0$ such that the Hilbert polynomial of E with respect to H, namely,

$$P(E,m) := \chi\left(E \otimes H^m\right), \tag{2.1}$$

satisfies

$$P(E,m) = \sum_{i=0}^{d} \frac{\alpha_i(E)}{i!} m^i, \quad \forall m \geq 0. \tag{2.2}$$

A coherent sheaf E is said to be *Gieseker (semi)stable* if for each nonzero proper coherent subsheaf $F \subset E$,

$$p(F) = \frac{P(F,m)}{r(F)} (\leq) < \frac{P(E,m)}{r(E)} = p(E) \quad \forall m >> 0. \tag{2.3}$$

If E is torsion free then the *degree* of E is defined as the integer $\deg(E) := \alpha_{n-1}(E) - r(E) \cdot \alpha_{n-1}(E)$. It follows that

$$\deg(E) = c_1(E) \cdot c_1(H)^{n-1} \in H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}.$$

Furthermore, E is said to be *slope (semi)stable* if

$$\frac{\deg(F)}{r(F)} (\leq) < \frac{\deg(E)}{r(E)}. \tag{2.4}$$

Note that E is slope-stable $\implies$ E is Gieseker stable $\implies$ E is Gieseker semistable $\implies$ E is slope semistable.

Let E_1, E_2 be Gieseker semistable. If $p(E_1) > p(E_2)$ then $H^0(\mathrm{Hom}(E_1, E_2)) = 0$. If E_1 is Gieseker stable and $p(E_1) = p(E_2)$ then $H^0(\mathrm{Hom}(E_1, E_2)) = 0$ or $E_1 \cong E_2$. If E is Gieseker stable then E is simple i.e. $H^0(\mathrm{End}(E)) \cong \mathbb{C}$.

A pure coherent sheaf E of dimension d admits a filtration of coherent subsheaves

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E \tag{2.5}$$

such that E_i/E_{i-1} is stable, pure of dimension d and $p(E_i/E_{i-1}) = p(E)$. This is said to be a Jordan-Hölder filtration of E. If E is Gieseker semistable then E admits a Jordan-Hölder filtration. The *grading* of E is the direct sum $\mathrm{gr}(E) = \oplus_i \, (E_i/E_{i-1})$ unique up to an isomorphism. Two Gieseker semistable sheaves E and F are *strongly equivalent* if $\mathrm{gr}(E) \cong \mathrm{gr}(F)$. In particular, if E and F are Gieseker stable then E and F are strongly equivalent if and only if isomorphic as sheaves.

Gieseker (semi)stable coherent sheaves form a moduli scheme ([11], [12]). Appealing to a projective embedding of X, it is shown at first that the Gieseker semistable sheaves with a fixed Hilbert polynomial P are bounded ([11] p. 56). Therefore, the functor $Sch/\mathbb{C} \to Sets$ sending a $\mathbb{C}$-scheme S to the set of strong equivalence classes of families of Gieseker semistable sheaves with Hilbert polynomial P parametrized by S is universally corepresented by a projective scheme $\mathcal{M}^{ss}_{X,H}(P)$ ([11], p. 65). There is an open quasi-projective subscheme $\mathcal{M}^{s}_{X,H}(P) \subset \mathcal{M}^{ss}_{X,H}(P)$ that parametrizes isomorphism classes of Gieseker stable sheaves. Unless X is a curve the scheme $\mathcal{M}^{s}_{X,H}(P)$ is not necessarily smooth. At a Gieseker stable point E if the space of obstructions vanishes i.e. $H^2(X, \mathrm{End}(E)) = 0$ then E is a smooth point and the Zariski tangent space is $H^1(X, \mathrm{End}(E))$.

Twisted Higgs sheaves are coherent sheaves with a new module structure ([7], [11], [12], [4]). Let V be a vector bundle. By a *V-twisted Higgs sheaf* (E, ϕ) we mean a pure coherent sheaf E and a morphism of sheaves $\phi : E \to E \otimes V$, in other words, $\phi \in H^0(\mathrm{Hom}(E, E \otimes V))$ which satisfies $\phi \wedge \phi = 0$. If E is locally free then (E, ϕ) is said to be a *V-twisted Higgs bundle*. The morphism ϕ is said to be a *V-twisted Higgs field*. The last condition on ϕ is said to be the *integrability condition* on ϕ (Lemma 2.12, Lemma 2.13 [11]). If E is locally free of rank r one can write

$$\phi = \sum_{i=1}^{r} \phi_i \otimes u_i, \quad \phi_i \in \mathcal{O}(\mathrm{End}E)(\mathcal{U}), \tag{2.6}$$

where $\mathcal{U}$ is a common trivializing neighborhood of E and V and $\{u_i\}$ is a local basis of V. Then integrability can be phrased as

$$\phi \wedge \phi = \sum_{i<j} [\phi_i, \phi_j] \otimes (u_i \wedge u_j) = 0. \tag{2.7}$$

The equation 2.7 reads: ϕ is integrable if and only if the ϕ_i's commute with each other. If V or E is a line bundle then 2.7 is immediate.

A homomorphism between V-twisted Higgs sheaves (E_1, ϕ_1) and (E_2, ϕ_2) is a commutative diagram as follows

$$\begin{tikzcd} E_1 \arrow[r, "\phi_1"] \arrow[d, "\psi"] & E_1 \otimes V \arrow[d, "\psi\otimes 1=\psi'"] \\ E_2 \arrow[r, "\phi_2"] & E_2 \otimes V \end{tikzcd} . \tag{2.8}$$

In particular, (E_1, ϕ_1) and (E_2, ϕ_2) are said to be *isomorphic* if ψ is an isomorphism. In that case we write $(E_1, \phi_1) \cong (E_2, \phi_2)$.

A coherent subsheaf $F \subset E$ is said to be *invariant* under ϕ if $\phi(F) \subseteq F \otimes V$. A coherent twisted sheaf E is said to be Gieseker (semi)stable if for each invariant coherent subsheaf $F \subsetneq E$, it holds that $p(F)(\leq) < p(E)$. A Gieseker semistable V-twisted sheaf (E, ϕ) admits a Jordan-Hölder filtration of V-twisted subsheaves. Let (E, ϕ) be a Gieseker semistable pure coherent sheaf of dimension d. Then there exists a strict filtration of coherent V-twisted subsheaves

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E \tag{2.9}$$

such that $(E_i/E_{i-1}, \phi : E_i/E_{i-1} \to E_i/E_{i-1} \otimes V)$ is stable, pure of dimension d and $p(E_i/E_{i-1}) = p(E)$. Two semistable twisted sheaves are said to be *strongly equivalent* if their grading sheaves are isomorphic to each other.

An invariant theoretic quotient of twisted Higgs sheaves is given with an extended construction of a quotient of pure semistable sheaves. There is a complex quasi-projective scheme $\mathcal{M}^{ss}_{X,H}(P,V)$ ([11]) that consists of the strongly equivalent semistable $\mathrm{Sym}(V^*)$-modules and there exists an open subscheme $\mathcal{M}^{s}_{X,H}(P,V) \subset \mathcal{M}^{ss}_{X,H}(P,V)$ which parametrizes stable $\mathrm{Sym}(V^*)$-modules.

There is another useful construction for twisted Higgs sheaves in terms of coherent sheaves on a projective variety. First, let $\pi : V \to X$ be the bundle projection map and (E,ϕ) be a V-twisted torsion free Higgs sheaf. Then $(E,\phi) \cong \pi_*(\mathcal{E},\eta)$ where $\mathcal{E}$ is a coherent sheaf on $\mathrm{Tot}(V)$ of dimension n and η is the tautological section of π^*V. Let $Z = \mathbb{P}(V^* \oplus \mathcal{O}) = \overline{\mathrm{Tot}(V)}$. Denote the extension map of π also with $\pi : Z \to X$ and with $D \subset Z$ the hyperplane divisor. Then $\exists k \in \mathbb{N}$ such that $H' = \pi^*H^k \otimes \mathcal{O}_Z(D)$ is ample on Z. Then $H'|_{Z\setminus D} = \pi^*H^k$. The sheaf $\mathcal{E}$ is a coherent sheaf on Z supported outside D and $\chi(\mathcal{E}\otimes(\pi^*H^k)^m) = \chi(\pi_*\mathcal{E}\otimes(H^k)^m)$ for all integers m. In particular, (E,ϕ) is Gieseker (semi)stable with respect to H^k if and only if $\mathcal{E}$ is Gieseker (semi)stable with respect to π^*H^k. Here $\mathcal{M}^{ss}_{X,H^k}(P,V)$ is an open subscheme of $\mathcal{M}^{ss}_{Z,H'}(P)$ consisting of Gieseker semistable sheaves with dimension $\dim(E)$ supported outside D.

An integrable V-twisted Higgs field ϕ defines a chain complex of sheaves ([12], [9])

$$C^\bullet : \mathrm{End}E \xrightarrow{\wedge\phi} \mathrm{End}E \otimes V \xrightarrow{\wedge\phi} \mathrm{End}E \otimes \wedge^2 V. \tag{2.10}$$

which fits into a specific exact sequence

$$0 \to \mathcal{E}^{1,0} \to T_{(E,\phi)} = \mathbb{H}^1(C^\bullet) \to \mathcal{E}^{0,1} \to \mathcal{E}^{2,0} \to \mathbb{H}^2(C^\bullet) \tag{2.11}$$

where

$$\mathcal{E}^{p,q} = \frac{\ker\left(H^q(\mathrm{End}E\otimes\wedge^p T_X) \xrightarrow{\wedge\phi} H^q(\mathrm{End}E\otimes\wedge^{p+1}T_X)\right)}{\mathrm{Im}\left(H^q(\mathrm{End}E\otimes\wedge^{p-1} T_X) \xrightarrow{\wedge\phi} H^q(\mathrm{End}E\otimes\wedge^{p}T_X)\right)}. \tag{2.12}$$

Let P be a Hilbert polynomial with degree $d = n$ and (E,ϕ) is a torsion free V-twisted Higgs sheaf with $P(E) = P$. There exists an open dense subset $\mathcal{U} \subseteq X$ such that $E_{|\mathcal{U}}$ is a vector bundle and $\phi_{|\mathcal{U}} : E_{|\mathcal{U}} \to E_{|\mathcal{U}} \otimes V_{|\mathcal{U}}$ is a homomorphism of vector bundles and the characteristic coefficients are global sections of $\mathrm{Sym}^i V_{|\mathcal{U}}$ for $i = 1,\ldots,r$. By Hartogs' theorem the characteristic coefficients extend to global sections of $\mathrm{Sym}^i V$ for each i. These sections are said to be *the characteristic coefficients* of (E,ϕ).

DEFINITION 2.1.1. The Hitchin morphism

$$\mathcal{H} : \mathcal{M}^{ss}_{X,H}(P,V) \to \mathcal{B}(r) = \oplus_{i=1}^r H^0(X, \mathrm{Sym}^i V)$$

computes the characteristic coefficients. The determinant morphism det is the composition of the natural projection morphism $\mathcal{B}(r) \to H^0(X, \mathrm{Sym}^r V)$ with $\mathcal{H}$. In particular, if $V = L$ where L is a line bundle and $(E, \phi) \in \mathcal{M}^{ss}_{X,H}(P, L)$ then $\det(\phi) \in H^0(L^r)$.

In 2.1.1 it suffices to assume that E is a vector bundle on X. If $(E, \phi : E \to E \otimes V)$ is a twisted torsion free sheaf then choose $(E^{**}, \phi^{**} : E^{**} \to E^{**} \otimes V)$, so that $P(E) = P(E^{**})$ and $\mathcal{H}(E, \phi) = \mathcal{H}(E^{**}, \phi^{**})$ and there is a commutative diagram as follows

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes V \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota\otimes 1} \\ E^{**} & \xrightarrow{\phi^{**}} & E^{**} \otimes V \end{array} . \tag{2.13}$$

THEOREM 2.1.2. *([5], [12]) Hitchin morphism* $\mathcal{H} : \mathcal{M}^{ss}_{X,H^k}(P, V) \to \mathcal{B}(r) = \oplus_{i=1}^r H^0(X, \mathrm{Sym}^i V)$ *is proper.*

2.2. Schwarzenberger bundles and the determinant morphism

DEFINITION 2.2.1. For each nonzero irreducible polynomial $\rho \in H^0(\mathbb{P}^2, \mathcal{O}(2))$, there exists a holomorphic cover $f^\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ branched over the nonsingular conic $\rho = 0$. The bundle $V^\rho_k = f^\rho_* \mathcal{O}(0, k)$ where $\mathcal{O}(0, k) = \mathcal{O} \boxtimes \mathcal{O}(k)$ is said to be the k-th Schwarzenberger bundle.

For $k = 0, 1, 2$ the bundle V^ρ_k is rigid, that is, $H^1(\mathrm{End}(V^\rho_k)) = 0$. For $k \geq 3$, it follows that $H^1(\mathrm{End}_0(V^\rho_k)) \cong \mathbb{C}^{k^2-4}$. For every ρ, it follows that

$$V^\rho_k = \begin{cases} \mathcal{O} \oplus \mathcal{O}(-1); \ k = 0 \\ \mathcal{O} \oplus \mathcal{O}; \ k = 1 \\ T_{\mathbb{P}^2}; \ k = 2. \end{cases} \tag{2.14}$$

On the other hand, if $V^\rho_3 \cong V^{\rho'}_3$ then the zero schemes $\rho = 0$ and $\rho' = 0$ coincide. Furthermore, $c_1(V^\rho_k) = (k-1)H$ and $c_2(V^\rho_k) = \frac{k(k-1)}{2} H^2$. Co-Higgs bundles on V^ρ_k are stable under infinitesimal deformations in a sense that the underlying bundle of a deformed co-Higgs bundle is another Schwarzenberger bundle. The map f^ρ also offers a spectral viewpoint on $\mathbb{P}^2$ which is briefly highlighted in [9].

Let $\mathcal{M}_{\mathbb{P}^2}(V^\rho_k, T_{\mathbb{P}^2})$ denote the space of trace-free Gieseker semistable co-Higgs bundles modelled on V^ρ_k and $\det|_{\mathcal{M}_{\mathbb{P}^2}(V^\rho_k, T_{\mathbb{P}^2})} : \mathcal{M}_{\mathbb{P}^2}(V^\rho_k, T_{\mathbb{P}^2}) \to H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2}))$ be the determinant morphism. We will identify $\mathrm{Im}(\det|_{\mathcal{M}_{\mathbb{P}^2}(V^\rho_k, T_{\mathbb{P}^2})})$ with known spaces. The chosen polarization for our computations is $H = \mathcal{O}_{\mathbb{P}^2}(1)$. First we prove some elementary lemmas which will be necessary in our analysis.

LEMMA 2.2.2. *Let* $C, C' \in H^0(T_{\mathbb{P}^2}(-1))$ *be such that* $C \otimes C = C' \otimes C' \in H^0(T_{\mathbb{P}^2} \otimes T_{\mathbb{P}^2} \otimes \mathcal{O}(-2))$. *Then* $C = \pm C'$.

PROOF. The strategy of the proof makes sense for any section $C \in H^0(V)\backslash\{0\}$ where V is a bundle of rank 2. Let $U_0 = \mathbb{C}^2$ be a standard coordinate chart on $\mathbb{P}^2$. On U_0 write C and C' as $C = s_1 e_1 + s_2 e_2$ and $C' = s_1' e_1 + s_2' e_2$. Here s_1, s_2, s_1', s_2' are holomorphic functions on U_0 and $\{e_1, e_2\}$ is a local basis of sections of $T_{\mathbb{P}^2}(-1)$. Then $C \otimes C = C' \otimes C'$ implies that

$$s_1^2 = s_1'^2;\ s_1 s_2 = s_1' s_2';\ s_2^2 = s_2'^2. \tag{2.15}$$

These three equations together imply that either $C = C'$ or $C = -C'$ throughout on U_0. Either of these equalities extends to $C = C'$ or $C = -C'$ throughout $\mathbb{P}^2$. □

LEMMA 2.2.3. *Every section $C \in H^0(T_{\mathbb{P}^2}(-1))$ defines a global holomorphic section* $\mathrm{Sym}^2(C) \in H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2}) \otimes \mathcal{O}(-2))$. *Therefore,* $\mathrm{Sym}^2(C) = \mathrm{Sym}^2(C')$ *if and only if $C \otimes C = C' \otimes C'$.*

PROOF. The strategy of the proof makes sense for any section $C \in H^0(V)$ where V is a bundle of rank 2. The section $\mathrm{Sym}^2(C) \in H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2}) \otimes \mathcal{O}(-2))$ is defined locally as $s_1^2 e_1^2 + 2 s_1 s_2 e_1 e_2 + s_2^2 e_2^2$. Let $g_{\alpha\beta}$ be a transition function of $T_{\mathbb{P}^2}(-1)$ given by

$$g_{\alpha\beta} = \begin{bmatrix} g_{\alpha\beta}^{11} & g_{\alpha\beta}^{12} \\ g_{\alpha\beta}^{21} & g_{\alpha\beta}^{22} \end{bmatrix}. \tag{2.16}$$

Then

$$\mathrm{Sym}^2(g_{\alpha\beta}) = \begin{bmatrix} {g_{\alpha\beta}^{11}}^2 & g_{\alpha\beta}^{11} g_{\alpha\beta}^{12} & {g_{\alpha\beta}^{12}}^2 \\ 2 g_{\alpha\beta}^{21} g_{\alpha\beta}^{11} & g_{\alpha\beta}^{22} g_{\alpha\beta}^{11} + g_{\alpha\beta}^{21} g_{\alpha\beta}^{12} & 2 g_{\alpha\beta}^{22} g_{\alpha\beta}^{12} \\ {g_{\alpha\beta}^{21}}^2 & g_{\alpha\beta}^{21} g_{\alpha\beta}^{22} & {g_{\alpha\beta}^{22}}^2 \end{bmatrix} \tag{2.17}$$

with respect to the ordered basis $\{e_1^2, e_1 e_2, e_2^2\}$. The local representatives of $\mathrm{Sym}^2(C)$ on trivializing neighborhoods agree with the effect of the transition data $\mathrm{Sym}^2(g_{\alpha\beta})$ of $\mathrm{Sym}^2(T_{\mathbb{P}^2}) \otimes \mathcal{O}(-2)$. The second part follows immediately from the definition of $\mathrm{Sym}^2(C)$. □

THEOREM 2.2.4. *There is a bijection*

$$\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_0^\rho, T_{\mathbb{P}^2})}\right) \to \frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q, C) \sim (\alpha^2 \cdot q, \alpha^{-1} \cdot C)} \sqcup \{0\}$$

given by $q \otimes Sym^2(C) \mapsto [(q, C)]$.

PROOF. Since $V_0^\rho = \mathcal{O} \oplus \mathcal{O}(-1)$ has coprime rank and degree a traceless co-Higgs field $\phi : \mathcal{O} \oplus \mathcal{O}(-1) \to (\mathcal{O} \oplus \mathcal{O}(-1)) \otimes T_{\mathbb{P}^2}$ is Gieseker semistable if and only if Gieseker stable. It follows from integrability of ϕ and Hartogs' theorem that ([8])

$$\phi = \begin{bmatrix} \lambda & \mu \\ 1 & -\lambda \end{bmatrix} \otimes C, \quad \lambda \in H^0(\mathcal{O}(1)), \mu \in H^0(\mathcal{O}(2)), C \in H^0(T_{\mathbb{P}^2}(-1)). \tag{2.18}$$

Thus $\det(\phi) = -(\lambda^2+\mu)\otimes \mathrm{Sym}^2(C)$. So,

$$\begin{aligned}&\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_0^{\rho},T_{\mathbb{P}^2})}\right)\\ &= \left\{(\lambda^2+\mu)\otimes \mathrm{Sym}^2(C) : \lambda \in H^0(\mathcal{O}(1)),\ \mu\in H^0(\mathcal{O}(2)),\ C\in H^0(T_{\mathbb{P}^2}(-1))\right\}.\end{aligned}$$

Now

$$\phi_0 = \begin{bmatrix}\lambda & \mu\\ 1 & -\lambda\end{bmatrix} \mapsto -(\lambda^2+\mu) \tag{2.19}$$

is a surjection onto $H^0(\mathcal{O}(2))$. A global holomorphic section of $\mathcal{O}(1)$ is presented by a polynomial $\lambda = az+bw+c$ of total degree ≤ 1 and a global holomorphic section of $\mathcal{O}(2)$ is presented by a complex polynomial $s = az^2+bzw+cw^2+dz+ew+f$ of total degree ≤ 2. It follows that $s=\lambda^2+\mu$. This is an elementary verification via adjusting the coefficients of s.

(i) Let $a\neq 0$. Then

$$\begin{aligned}az^2+bzw+cw^2+dz+ew+f &= \left(\underbrace{\sqrt{a}z+\frac{b}{2\sqrt{a}}w+\frac{d}{2\sqrt{a}}}_{\lambda}\right)^2\\ &+\underbrace{\left(c-\frac{b^2}{4a}\right)w^2+\left(e-\frac{bd}{2a}\right)w+\left(f-\frac{d^2}{4a}\right)}_{\mu}.\end{aligned}$$

(ii) Let $a=0$ and $c\neq 0$. Then

$$\begin{aligned}cw^2+bzw+dz+ew+f &= \left(\underbrace{\sqrt{c}w+\frac{b}{2\sqrt{c}}z+\frac{e}{2\sqrt{c}}}_{\lambda}\right)^2\\ &+\left(\underbrace{-\frac{b^2}{4c}z^2+\left(d-\frac{be}{2c}\right)z+\left(f-\frac{e^2}{4c}\right)}_{\mu}\right).\end{aligned}$$

(iii) Let $a=0,\ c=0,\ b\neq 0$. It follows after application of a rotation

$\begin{cases} z = z' + w' \\ w = z' - w' \end{cases}$ that

$$bzw + dz + ew + f = \Bigg(\underbrace{\sqrt{b}z' + \frac{d+e}{2\sqrt{b}}}_{\lambda}\Bigg)^2 + \Bigg(\underbrace{-bw'^2 + (d-e)w' + \left(f - \frac{(d+e)^2}{4b}\right)}_{\mu}\Bigg).$$

(iv) Let $a = b = c = 0$. Then $\lambda = 0$ and $\mu = dz + ew + f$.

Thus the determinant section is $\det(\phi) = q \otimes \mathrm{Sym}^2(C)$ for some $q \in H^0(\mathcal{O}(2))$ and $C \in H^0(T_{\mathbb{P}^2}(-1))$. So the zero section lies in the image. Now let suppose that $q \otimes \mathrm{Sym}^2(C) = q' \otimes \mathrm{Sym}^2(C')$ for some nonzero global sections q, q' and C, C'. This assumption is equivalent to $q \otimes C \otimes C = q' \otimes C' \otimes C'$. The zero schemes $\mathcal{Z}(q), \mathcal{Z}(q')$ are complex projective curves and the zero schemes $\mathcal{Z}(C), \mathcal{Z}(C')$ are two points on $\mathbb{P}^2$. Thus $\mathcal{Z}(q) = \mathcal{Z}(q')$, that is, $q' = \alpha^2 \cdot q$ for some $\alpha \neq 0$. Thus $C \otimes C = (\alpha \cdot C') \otimes (\alpha \cdot C')$, i.e., $C = \pm\alpha \cdot C'$. Consider the action of $\mathbb{C}^\times$ on $H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times$ by $\alpha \cdot (q, C) = (\alpha^2 q, \alpha^{-1} C)$. The quotient space characterizes the points in the image for which both q and C are nonzero. The zero section is obtained as a determinant if $\phi_0 = \begin{bmatrix} \lambda & \mu \\ 1 & -\lambda \end{bmatrix}$ is nilpotent or $C = 0$. The quotient space along with the zero section is in bijection with the image of the determinant morphism by $q \otimes \mathrm{Sym}^2(C) \mapsto [(q, C)]$. □

Remark 2.2.5. Note that

$$\dim \mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_0^\rho, T_{\mathbb{P}^2})}\right) = \dim\left(\frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q, C) \sim (\alpha^2 \cdot q, \alpha^{-1} \cdot C)} \sqcup \{0\}\right)$$
$$< \dim\left(H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times\right) = 6 + 3 = 9 < 27 = \dim H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2})).$$

Thus $\det|_{\mathcal{M}_{\mathbb{P}^2}(V_0^\rho, T_{\mathbb{P}^2})}$ is not surjective onto $H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2}))$.

THEOREM 2.2.6. *There is a bijection*

$$\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_1^\rho, T_{\mathbb{P}^2})}\right) \to$$
$$\left(\frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q, C) \sim (\alpha^2 \cdot q, \alpha^{-1} \cdot C)} \sqcup \frac{H^0(T_{\mathbb{P}^2})}{\{\pm 1\}}\right) \Big/ [(q, C)] \sim [A] \iff q \otimes \mathrm{Sym}^2(C) = \mathrm{Sym}^2(A)$$

given by $q \otimes \mathrm{Sym}^2(C) \mapsto [(q, C)]$ *and* $\mathrm{Sym}^2(A) \mapsto [A]$.

PROOF. Since $V_1^\rho = \mathcal{O} \oplus \mathcal{O}$ is Gieseker semistable every traceless co-Higgs bundle ϕ modelled on V_1^ρ is Gieseker semistable. Write

$$\phi = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}, \quad A, B, C \in H^0(T_{\mathbb{P}^2}). \tag{2.20}$$

Let $B, C \neq 0$. The integrability condition of ϕ implies that either

$$\phi = \begin{bmatrix} \lambda & \mu \\ \mu' & -\lambda \end{bmatrix} \otimes C', \quad \lambda, \mu, \mu' \in H^0(\mathcal{O}(1)), C \in H^0(T_{\mathbb{P}^1}(-1)) \tag{2.21}$$

or

$$\phi = \begin{bmatrix} \lambda & \mu \\ 1 & -\lambda \end{bmatrix} \otimes C, \quad \mu \neq 0, \lambda \in \mathbb{C}. \tag{2.22}$$

In the second case, $\det(\phi) = -(\lambda^2 + \mu)\mathrm{Sym}^2(C)$.

On the other hand let $B = 0$ or $C = 0$. Without loss of generality $C = 0$. So $\phi = \begin{bmatrix} A & B \\ 0 & -A \end{bmatrix}$. The integrability condition of ϕ is that $A \wedge B = 0$. Irrespective of $B = 0$ or $B \neq 0$ it follows that $\det(\phi) = -\mathrm{Sym}^2(A)$.

Considering all possible cases of the determinants, it follows that

$$\begin{aligned}
&\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_1^\rho, T_{\mathbb{P}^2})}\right) \\
&= \left\{q \otimes \mathrm{Sym}^2(C) : q \in H^0(\mathcal{O}(2)),\ C \in H^0(T_{\mathbb{P}^2}(-1))\right\} \bigcup \left\{\mathrm{Sym}^2(A) : A \in H^0(T_{\mathbb{P}^2})\right\} \\
&= \left\{q \otimes \mathrm{Sym}^2(C) : q \in H^0(\mathcal{O}(2))^\times,\ C \in H^0(T_{\mathbb{P}^2}(-1))^\times\right\} \bigcup \left\{\mathrm{Sym}^2(A) : A \in H^0(T_{\mathbb{P}^2})\right\}.
\end{aligned}$$

However, the intersection of these two collections of determinant sections in the last step is nonempty. There are sections $q \in H^0(\mathcal{O}(2)),\ C \in H^0(T_{\mathbb{P}^2}(-1))$ and $A \in H^0(T_{\mathbb{P}^2})$ such that

$$q \otimes \mathrm{Sym}^2(C) = \mathrm{Sym}^2(A).$$

Such sections should be identified as the same. On the other hand sections of the form $q \otimes \mathrm{Sym}^2(C)$ for $q \neq 0,\ C \neq 0$ are identified with points in $\frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q,C)\sim(\alpha^2 \cdot q, \alpha^{-1} \cdot C)}$ by $q \otimes \mathrm{Sym}^2(C) \mapsto [(q, C)]$ and the sections of the form $\mathrm{Sym}^2(A)$ are identified with points in $H^0(T_{\mathbb{P}^2})$. Under the identification $q \otimes \mathrm{Sym}^2(C) = \mathrm{Sym}^2(A)$ we arrive at a bijective correspondence

$$\begin{aligned}
&\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_1^\rho, T_{\mathbb{P}^2})}\right) \to \\
&\left(\frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q, C) \sim (\alpha^2 \cdot q, \alpha^{-1} \cdot C)} \sqcup \frac{H^0(T_{\mathbb{P}^2})}{\{\pm 1\}}\right) \Big/ [(q, C)] \sim [A] \iff q \otimes \mathrm{Sym}^2(C) = \mathrm{Sym}^2(A).
\end{aligned}$$

via the correspondence $q \otimes \mathrm{Sym}^2(C) \mapsto [(q, C)]; \quad \mathrm{Sym}^2(A) \mapsto [A]$.

Remark 2.2.7. Now we have

$$\begin{aligned}
&\dim \operatorname{Im}\Big(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_1^\rho, T_{\mathbb{P}^2})}\Big)\\
&= \max\left\{\dim \frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q,C)\sim(\alpha^2\cdot q, \alpha^{-1}\cdot C)};\ \dim \frac{H^0(T_{\mathbb{P}^2})}{\{\pm 1\}}\right\}\\
&< \max\{9,8\} = 9 < 27 = \dim H^0(\operatorname{Sym}^2(T_{\mathbb{P}^2})).
\end{aligned}$$

Thus $\det|_{\mathcal{M}_{\mathbb{P}^2}(V_1^\rho, T_{\mathbb{P}^2})}$ is not surjective onto $H^0(\operatorname{Sym}^2(T_{\mathbb{P}^2}))$.

□

THEOREM 2.2.8. *There is a bijection*

$$\operatorname{Im}\Big(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_2^\rho, T_{\mathbb{P}^2})}\Big) \to \frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q,C)\sim(\alpha^2 q, \alpha^{-1} C)} \sqcup \{0\}$$

given by $q \otimes \operatorname{Sym}^2(C) \mapsto [(q,C)]$.

PROOF. Since $V_2^\rho = T_{\mathbb{P}^2}$ is stable any co-Higgs bundle $(T_{\mathbb{P}^2}, \phi)$ is Gieseker stable. It follows from the Euler sequence of $T_{\mathbb{P}^2}$ that there is an isomorphism of complex vector spaces $H^0(\operatorname{End}_0 T_{\mathbb{P}^2} \otimes T_{\mathbb{P}^2}) \cong H^0(\operatorname{End}_0 T_{\mathbb{P}^2} \otimes \mathcal{O}(1)) \otimes H^0(T_{\mathbb{P}^2} \otimes \mathcal{O}(-1))$ ([8], [9]). Thus an algebraic tensor product of nonzero sections $\phi_0 \in H^0(\operatorname{End}_0 T_{\mathbb{P}^2} \otimes \mathcal{O}(1))$ and $C \in H^0(T_{\mathbb{P}^2} \otimes \mathcal{O}(-1))$ is a section of $\operatorname{End}_0 T_{\mathbb{P}^2} \otimes T_{\mathbb{P}^2}$. This algebraic tensor product coincides with the tensor product section $\phi_0 \otimes C$. To verify this fact let us choose a global section $\phi = \phi_0 \otimes C$. Then $\det(\phi) = \det(\phi_0) \otimes \operatorname{Sym}^2(C)$. The zero scheme of $\mathcal{Z}(\det(\phi_0))$ is either $\mathbb{P}^2$ or a conic and $\mathcal{Z}(C)$ is a point. In all cases, C is unique up to a $\mathbb{C}^\times$ action, so is ϕ_0 and their algebraic tensor product remains an invariant.

On the other hand, a co-Higgs field $\phi = \sum c_{ij}\phi_i \otimes C_j$ is integrable if and only if $\phi = \phi_0 \otimes C$. Thus every integrable co-Higgs field modelled on $T_{\mathbb{P}^2}$ is $\phi = \phi_0 \otimes C$. Moreover it follows that the restricted determinant morphism $\det : H^0(\operatorname{End}_0 T_{\mathbb{P}^2} \otimes \mathcal{O}(1)) \to H^0(\mathcal{O}(2))$ is surjective. It suffices to verify this fact on trivializing charts. As a complex manifold $\mathbb{P}^2$ is union of three open sets while on each of these open sets exactly one complex component is nonzero. The trivializing charts of $\mathbb{P}^2$ are given by

$$\begin{cases} \psi_1([z:w:1]) = (z,w)\\ \psi_2([z:1:w]) = (z,w)\\ \psi_3([1:z:w]) = (z,w). \end{cases} \tag{2.23}$$

Letting (z,w) be coordinates on the first trivializing chart ψ_1 the trivialization maps on $T_{\mathbb{P}^2}$ are given by

$$g'_{12} = \begin{bmatrix} \frac{1}{w} & -\frac{z}{w^2}\\ 0 & -\frac{1}{w^2}\end{bmatrix};\ g'_{23} = \begin{bmatrix} -\frac{1}{z^2} & 0\\ -\frac{w}{z^2} & \frac{1}{z}\end{bmatrix};\ g'_{31} = \begin{bmatrix} -\frac{w}{z^2} & \frac{1}{z}\\ -\frac{1}{z^2} & 0\end{bmatrix} \tag{2.24}$$

The local equation of a global section $\phi_0 \in H^0(\mathrm{End}_0(T_{\mathbb{P}^2}) \otimes \mathcal{O}(1))$ is given by

$$\begin{bmatrix} F(z,w) & G(z,w) \\ H(z,w) & -F(z,w) \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{w} & -\frac{z}{w^2} \\ 0 & -\frac{1}{w^2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{z}{w} \\ 0 & -\frac{1}{w} \end{bmatrix} \cdot \begin{bmatrix} f(\frac{z}{w}, \frac{1}{w}) & g(\frac{z}{w}, \frac{1}{w}) \\ h(\frac{z}{w}, \frac{1}{w}) & -f(\frac{z}{w}, \frac{1}{w}) \end{bmatrix}. \tag{2.25}$$

In particular, the determinant of the matrix $\begin{bmatrix} F(z,w) & G(z,w) \\ H(z,w) & -F(z,w) \end{bmatrix}$ is a polynomial in z, w of total degree ≤ 2 which represents $\det(\phi_0) \in H^0(\mathcal{O}(2))$ on one coordinate neighbourhood.

A comparison of the coefficients in the power series of F, G, H around $(0,0)$ with the respective power series of f, g, h suggests that H, h are constants, F, f are polynomials in z, w of degree ≤ 1 and G, g are polynomials in z, w of total degree ≤ 2. It is already observed that every complex polynomial of total degree ≤ 2 is of the form $-F^2 - GH$. So the determinant morphism is surjective onto $H^0(\mathcal{O}(2))$. The same reasoning in the proof of Theorem 2.2.4 concludes that $\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_2^\rho, T_{\mathbb{P}^2})}\right)$ is in bijective correspondence with the space

$$\frac{H^0(\mathcal{O}(2))^\times \times H^0(T_{\mathbb{P}^2}(-1))^\times}{(q, C) \sim (\alpha^2 q, \alpha^{-1} C)} \sqcup \{0\}$$

via the correspondence $q \otimes \mathrm{Sym}^2(C) \mapsto [(q, C)]$. ☐

Remark 2.2.9. An estimate of dimensions similar to the one given in Remark 2.2.5 suggests that $\det|_{\mathcal{M}_{\mathbb{P}^2}(V_2^\rho, T_{\mathbb{P}^2})}$ is not surjective onto $H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2}))$.

THEOREM 2.2.10. *For $k > 3$ there is a bijection*

$$\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_k^\rho, T_{\mathbb{P}^2})}\right) \to \frac{H^0(T_{\mathbb{P}^2}(-1))}{\{\pm 1\}}$$

given by $\mathrm{Sym}^2(A) \mapsto [A]$.

PROOF. From an Euler sequence of V_k^ρ it follows that there is an isomorphism of complex vector spaces $H^0(\mathrm{End}_0(V_k^\rho) \otimes \mathcal{O}(1)) \otimes H^0(T_{\mathbb{P}^2}(-1)) \cong H^0(\mathrm{End}_0(V_k^\rho) \otimes T_{\mathbb{P}^2})$. Here every ϕ is integrable and is of the form $\phi = \phi_0 \otimes C$ where $\det(\phi_0) = \lambda \cdot \rho$ for some $\lambda \in \mathbb{C}^\times$. Hence $\det(\phi) = \lambda\rho \otimes \mathrm{Sym}^2(C)$ and

$$\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_k^\rho, T_{\mathbb{P}^2})}\right) = \left\{\rho \otimes \mathrm{Sym}^2(C) : C \in H^0(T_{\mathbb{P}^2}(-1))\right\}.$$

Since $\rho \neq 0$ it follows that there is a bijection $\mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_k^\rho, T_{\mathbb{P}^2})}\right) \to \frac{H^0(T_{\mathbb{P}^2}(-1))}{\{\pm 1\}}$ given by $\mathrm{Sym}^2(A) \mapsto [A]$. ☐

Remark 2.2.11. Finally, we have

$$\dim \mathrm{Im}\left(\det|_{\mathcal{M}_{\mathbb{P}^2}(V_k^\rho, T_{\mathbb{P}^2})}\right) = \dim \frac{H^0(T_{\mathbb{P}^2}(-1))}{\{\pm 1\}} \leq 3 < 27 = \dim H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2})).$$

Thus $\det|_{\mathcal{M}_{\mathbb{P}^2}(V_k^\rho, T_{\mathbb{P}^2})}$ is not surjective onto $H^0(\mathrm{Sym}^2(T_{\mathbb{P}^2}))$ for $k > 3$.

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