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Equilibria for Set-Valued Maps on Star-Shaped Domains

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Abstract. We generalize the Ky Fan–Halpern equilibrium theorem for a Kakutani type (upper semicontinuous with non-empty closed convex values) set-valued map defined on convex compact domains in locally convex topological linear spaces [5] by replacing the convexity of the domain by its star-shapedness. The proof is simple and relies on an extension of the Ky Fan–Browder fixed point theorem to star-shaped compact domains due to the author [2].

Résumé. Nous généralisons le théorème de Ky Fan–Halpern [5] sur l'existence d'un équilibre pour une application multivoque du type Kakutani (semi-continue supérieurement à valeurs d'ensembles non-vides, convexes et fermés) définie sur un domaine convexe et compact dans un espace vectoriel topologique localement convexe en remplaçant la convexité du domaine par la condition plus générale qu'il soit étoilé. La preuve est simple et basée sur une généralisation du théorème de point fixe de Ky Fan–Browder aux sous ensembles compacts et étoilés d'espaces topologiques vectoriels [2].

1. **Introduction** It is assumed that vector spaces are real and topological spaces are Hausdorff. The interior, closure, and boundary of a subset A of a topological space are denoted as usual by $int(A)$, $\overline{A}$ and ∂A. The convex hull (closed convex hull) of a subset A in a (topological) vector space is denoted by $conv\{A\}$ ($\overline{conv}\{A\}$, respectively). Set-valued maps, simply called *maps*, are denoted by capital Greek letters and double arrows $\Phi, \Psi : X \rightrightarrows E$. The reader is referred to [1] for set-valued and non-smooth analysis concepts used here.

The Ky Fan–Halpern equilibrium theorem (Theorem 2 in [5]) is a far reaching generalization of the celebrated intermediate value theorem of B. Bolzano and of its $n-$dimensional extension due to H. Poincaré. It reads as follows.

Theorem 1. *Let X be a non-empty convex compact subset in a locally convex topological vector space E and let $\Phi : X \rightrightarrows E$ be an upper semicontinuous tangential map with non-empty closed convex values. Then, there exists $x^* \in X$ such that $0 \in \Phi(x^*)$.*

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Such an element $x^* \in X$ with $0 \in \Phi(x^*)$ is known as an *equilibrium* (or a *zero*) for the map Φ. Clearly, if $X \subseteq E$, an equilibrium for Φ is a fixed point for the field $\Psi := I|_X - \Phi$ (I being the identity mapping). In this case of a closed convex domain X, the map Φ is said to be *tangential* on X if it satisfies the tangency condition:

$$(\tau) \qquad \forall x \in \partial X, \Phi(x) \cap T_X(x) \neq \varnothing$$

where $T_X(x) := \overline{\bigcup_{t>0} \frac{1}{t}(X - x)}$ is the *tangent cone* of convex analysis to the set X at the point x[1].

The common proofs of Theorem 1 rely on a partition of unity argument combined with the celebrated Ky Fan infsup inequality (see e.g., Theorem 3.2.1 in [1]) or one of its equivalent formulations (see e.g., Theorem 2 in [5]) and on the geometric Hahn-Banach separation theorem. The aim of this note is to extend Theorem 1 to a star-shaped compact domain X in a locally convex space by using alternatives for systems of nonlinear functional inequalities resulting from a generalization of the Ky Fan–Browder fixed point theorem to star-shaped compact domains [2].

Two classes of maps important in topological set-valued fixed point theory are under consideration.

DEFINITION 2. ([3]) Let X be a topological space.

(a) A map $\Phi : X \rightrightarrows Y$ with values in a convex subset Y of a vector space E is said to be a **Ky Fan map** whenever:

(i) for every $y \in Y$, the set $\Phi^{-1}(y)$ is open in X;

(ii) for every $x \in X$, the set $\Phi(x)$ is non-empty and convex in Y.

We denote by $\mathbf{F}^*(X,Y)$ the class of Ky Fan maps from X into Y and, when appropriate, $\mathbf{F}^*(X) := \mathbf{F}^*(X,X)$.

(b) A map $\Psi : X \rightrightarrows Y$ with values in a convex subset Y of a topological vector space E is said to be a **Kakutani map** whenever:

(i) Ψ is upper semicontinuous on X;

(ii) for every $x \in X$, the set $\Psi(x)$ is non-empty closed and convex in Y.

We denote by $\mathbf{K}^*(X,Y)$ the class of Kakutani maps from X into Y. $\mathbf{K}^*(X) := \mathbf{K}^*(X,X)$.‘

Recall that *upper semicontinuity* at a given point $x \in X$ for a map $\Psi : X \rightrightarrows Y$ amounts to the openess of the upper inverse $\Psi^{-1}_+(V) := \{x' \in X : \Psi(x') \subset V\}$ of any open neighbourhood V of $\Psi(x)$ in Y. Upper semicontinuity on the set X is upper semicontinuity at each point in X.

[1] $T_X(x)$ is a non-empty closed convex cone which, in the case where $x \in int(X)$ is interior to X, amounts to the whole space E. Condition (τ) is thus meaningful only at boundary points of X. In the simplest case of a continuous function $f : X = [a,b] \longrightarrow E = \mathbb{R}$, the boundary conditions ($f(a) \geq 0, f(b) \leq 0$) are precisely the tangency condition (τ) : ($f(a) \in T_X(a) = [a,+\infty)$ and $f(b) \in T_X(b) = (-\infty, b]$) yielding the elementary *intermediate value theorem* of B. Bolzano.

REMARK 3. (1) All results involving $\mathbf{F}^*$ maps (in this note) extend to the larger class of $\mathbf{\Phi}^*$ maps. In effect, every $\mathbf{\Phi}^*$ maps admits an $\mathbf{F}^*$ selection (see Remark 2 in [2] and references there).

(2) The strong regularity condition (ii) on the map Φ in definition (a) above is a strong form of lower semicontinuity[2].

(3) Given a Ky Fan map $\Phi \in \mathbf{F}^*(X, Y)$ where X is a compact topological space and Y is a convex subset of a topological vector space E, there exists a single-valued continuous mapping $s : X \longrightarrow Y$ with:

(i) $s(x) \in \Phi(x)$ for all $x \in X$, and

(ii) $s(X) \subset conv\{y_1, \cdots, y_n\} \subset Y$ for some finite subset $\{y_1, \cdots, y_n\} \subset Y$.

(Such a function s is referred to as a finite-type continuous selection of Φ.)

The starting point for this note is a generalization to compact star-shaped domains (due to the author [2]) of the Ky Fan–Browder fixed point theorem for $\mathbf{F}^*$ maps. Recall that given a non-empty subset X of a vector space E, the *star* of a given element $\hat{x} \in E$ is the set $St(\hat{x}, X) := \bigcup_{x \in X} [\hat{x}, x]$ of all line segments $[\hat{x}, x] := \{\hat{x} + t(x - \hat{x}) : 0 \leq t \leq 1\}$. The set X is said to be *star-shaped at a point* $\hat{x} \in X$ if $St(\hat{x}, X) = X$, that is, for every $x \in X$, the line segment $[\hat{x}, x]$ is contained in X; the point $\hat{x}$ is said to be a *centre* for X. The set X is said to be *star-shaped* if it has at least one centre. The subset K_X of all centres of a star-shaped set X is known as the *kernel* of X. Obviously, every point of a convex set X is a centre for X, that is X is convex if and only if $X = K_X$.

THEOREM 4. *[2] Every Ky Fan map $\Phi \in \mathbf{F}^*(X)$ of a non-empty star-shaped compact subset X of a topological vector space E has a fixed point $x_0 \in \Phi(x_0)$.*

The Ky Fan–Browder fixed point theorem corresponds to the special case where X is convex. As in [4] this "geometric" fixed point theorem has an equivalent and convenient analytical formulation in the form of an alternative for systems of nonlinear inequalities.

COROLLARY 5. *Let X be a non-empty star-shaped compact subset X of a topological vector space E and $f : X \times X \longrightarrow \mathbb{R}$ be real function satisfying the following conditions.*

(i) for every $y \in X$, the function $x \mapsto f(x, y)$ is lower semicontinuous on X.

(ii) for every $x \in X$, the function $y \mapsto f(x, y)$ is quasiconcave on X.

Then, for any given $\lambda \in \mathbb{R}$, the following alternative holds:

(A) there exists $x_0 \in X$ with $f(x_0, x_0) > \lambda$; or

(B) there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq \lambda$ for all $y \in X$.

PROOF. The map $\Phi : X \rightrightarrows X$ given by $\Phi(x) := \{y \in X : f(x, y) > \lambda\}, x \in X$, has open pre-images and convex values. If the conclusion (B) fails, then $\Phi(x) \neq \varnothing$ for all $x \in X$, that is Φ is an $\mathbf{F}^*$ map. A fixed point x_0 of Φ follows from Theorem 4, thus establishing (A). $\square$

[2]The map Φ is *lower semicontinuous* at $x \in X$ whenever the lower inverse $\Phi_-^{-1}(V) := \{x' \in X : \Phi(x') \cap V \neq \varnothing\}$ of any open subset V in Y is an open neighbourhood of x in X.

2. The Main Result We start with a preliminary result on the existence of maximizable upper semicontinuous quasiconcave functionals.

PROPOSITION 6. *Let X be a non-empty star-shaped compact subset of a topological vector space E, let Y be a set of functions in $\{p : X \longrightarrow \mathbb{R} : p$ is upper semicontinuous and quasiconcave$\}$*[3] *and let $s : X \longrightarrow Y$ be a continuous function. Then,*

$$\textit{there exists } \bar{x} \in X \textit{ such that for } \bar{p} = s(\bar{x}) \in Y \textit{ we have } \bar{p}(\bar{x}) = \max_{y \in X} \bar{p}(y).$$

PROOF. Consider the function $f : X \times X \longrightarrow \mathbb{R}$ defined as

$$f(x, y) = s(x)(y) - s(x)(x) \text{ for every pair } (x, y) \in X \times X.$$

Clearly, $x \mapsto f(x, y)$ is lower semicontinuous on X and $y \mapsto f(x, y)$ is quasiconcave on X. Obviously, for $\lambda = 0$, the alternative (A) in the conclusion of Corollary 5:

$$f(x_0, x_0) = s(x_0)(x_0) - s(x_0)(x_0) = 0 > 0, \text{ for some } x_0 \in X,$$

is ruled out. Thus, the alternative (B) of that corollary holds, i.e.,

$$\text{there exists } \bar{x} \in X \text{ such that } f(\bar{x}, y) = s(\bar{x})(y) - s(\bar{x})(\bar{x}) \leq 0, \text{ for every } y \in X.$$

Hence, $\bar{p} = s(\bar{x})$ satisfies $\bar{p}(y) \leq \bar{p}(\bar{x}), \forall y \in X$. □

A particular instance of this proposition corresponds to the case where s is a continuous selection of an $\mathbf{F}^*$ map $\Gamma : X \rightrightarrows Y$ as per Remark 3 (3).

COROLLARY 7. *Let X be a non-empty star-shaped compact subset of a topological vector space E, let Y be a convex subset of functions in $\{p : X \longrightarrow \mathbb{R} : p$ is upper semicontinuous and quasiconcave$\}$, and let $f : X \times Y \longrightarrow \mathbb{R}$ be a function verifying the following hypotheses.*

(i) for every $p \in Y$, $x \mapsto f(x, p)$ is lower semicontinuous on X;

(ii) for every $x \in X$, $p \longmapsto f(x, p)$ is quasiconcave on Y.

Then, for any $\lambda \in \mathbb{R}$, one of the following holds:

(A) there exists $x^* \in X$ with $f(x^*, p) \leq \lambda$ for all $p \in Y$; or

(B) there exist $\bar{x} \in X$ and $\bar{p} \in Y$ with $\begin{cases} f(\bar{x}, \bar{p}) > \lambda \\ \bar{p}(\bar{x}) = \max_{y \in X} \bar{p}(y) \end{cases}$

PROOF. The map $\Gamma : X \rightrightarrows Y$ defined by $\Gamma(x) := \{p \in Y : f(x, p) > \lambda\}$, $x \in X$, has open pre-images by (i) and convex values by (ii). If the conclusion (A) fails, then $\Gamma(x) \neq \varnothing$ for all $x \in X$ and Γ is therefore an $\mathbf{F}^*$ map. Having a compact domain X, Γ admits a continuous selection $s : X \longrightarrow Y$ (see Remark 3 (3) above). Proposition 6 yields the conclusion (B). □

[3] Equipped with a suitable topology.

As mentioned earlier, equilibrium theorems require the mappings to satisfy a tangency condition. A discussion of tangency on star-shaped domains is clearly in order. But first, some basic and trivial facts about star-shaped sets are worth mentioning.

PROPOSITION 8. *The kernel K_X of a star-shaped set X in a topological vector space is:*
(i) convex.
(ii) closed, whenever X is closed.

As mentioned in the introduction, in the classical Ky Fan–Halpern equilibrium result (Theorem 1 above; stated in fixed point form in [5]) whereby the domain X is a compact convex set, the tangency condition (τ) is expressed in terms of the tangent cone $T_X(x) := \overline{S_X(x)}$ of convex analysis ($S_X(x) := \bigcup_{t>0} \frac{1}{t}(X - x)$). While convex for any given convex set X, the closed cone $T_X(x)$ may not be convex for non-convex domains; in which cases, suitable tangency concepts are required. Recall some known concepts of tangent cones from non-smooth analysis (see [1]).

DEFINITION 9. Given a non-empty set X in a real topological vector space E and an element $x \in \overline{X}$, define:
(i) the tangent cone to X at x as

$$T_X(x) := \overline{S_X(x)} \text{ where } S_X(x) := \bigcup_{t>0} \frac{1}{t}(X - x).$$

(ii) The adjacent cone to X at x as

$$T_X^A(x) := \lim\inf_{t\downarrow 0^+} \{\frac{1}{t}(X - x)\}.$$

(iii) The *Bouligand–Severi contingent cone to* X at x as

$$T_X^B(x) := \lim\sup_{t\downarrow 0^+} \{\frac{1}{t}(X - x)\}.$$

(iv) The Clarke circatangent cone to X at x as

$$T_X^C(x) := \lim \inf_{t\downarrow 0^+, x'\to_X x} \{\frac{1}{t}(X - x')\}.$$

One readily sees that for any subset X in a topological vector space and any $x \in \overline{X}$, we have:

- Always, $T_X^C(x) \subseteq T_X^A(x) \subseteq T_X^B(x) \subseteq T_X(x)$.
- If $x \in int(X)$, then $T_X^C(x) = T_X^A(x) = T_X^B(x) = T_X(x) = E$.

- $T_X^A(x)$ is the set of all limit points of generalized sequences $\{x_t\}_{t>0}$ with $x_t \in \frac{1}{t}(X-x)$. $T_X^B(x)$ is the set of all cluster points of generalized sequences $\{x_t\}_{t>0}$ with $x_t \in \frac{1}{t}(X-x)$.
- In the particular case where E is metrizable, we have the simpler sequential characterizations:

$$\begin{aligned} T_X^A(x) &= \{v \in E : \forall t_n \to 0^+, \exists v_n \to v \text{ such that } x + t_n v_n \in X, \forall n\}. \\ T_X^B(x) &= \{v \in E : \exists t_n \to 0^+, \exists v_n \to v \text{ such that } x + t_n v_n \in X, \forall n\}. \\ T_X^C(x) &= \{v \in E : \forall t_n \to 0^+, \forall x_n \to_X x, \exists v_n \to v \text{ such that } x_n + t_n v_n \in X, \forall n\}. \end{aligned}$$

- If X is locally convex at x, that is, there exists an open neighbourhood of x in E such that $X \cap U$ is convex, then

$$T_X^C(x) = T_X^A(x) = T_X^B(x) = T_X(x).$$

  Obviously, a convex subset of a locally convex space is locally convex at each of its points.
- If X is star-shaped and $x \in K_X$ is a center of X then $T_X^A(x) = T_X^B(x) = T_X(x)$. Indeed, note first that given $x \in K_X$, and given any real $0 \le \lambda \le 1$, for any vector $u \in X$, we have $\lambda u + (1-\lambda)x \in X \Leftrightarrow \lambda(u-x) \in X - x$. Thus, $\lambda(X-x) = X - x$ for all $0 \le \lambda \le 1$. Consequently, $S_X(x) = \bigcup_{1 \le \lambda} \lambda(X-x) = \bigcup_{0<t\le 1} \frac{1}{t}(X-x)$.
  For the sake of simplicity, let us assume that E is metrizable to show the inclusion $T_X(x) = \overline{S_X(x)} \subseteq T_X^A(x)$. To do this, let $v \in S_X(x)$ be arbitrary, that is, $v = \lambda(u-x)$ for some $u \in X$ and some $\lambda \ge 1$. Now, let $\{t_n\}$ be any real decreasing sequence convergent to 0^+ and let $\{x_n\}$ be any sequence of elements of X converging to x. We may assume with no loss of generality that $0 < t_n \le t_n\lambda < 1$ for all n. Consider, for each $n \in \mathbb{N}$, the vector $v_n := (1-t_n)\lambda(u-x) \in E$. Clearly, $\lim_{n\to+\infty} v_n = v$. In addition,

$$\begin{aligned} x + t_n v_n &= x + \mu_n(u-x) \text{ with } 0 < \mu_n = (1-t_n)t_n\lambda \le 1 \\ &\Longrightarrow x + t_n v_n \in X \text{ for all } n \in \mathbb{N}. \\ &\therefore \quad v \in T_X^A(x). \end{aligned}$$

  Therefore, $T_X(x) := \overline{S_X(x)} \subseteq \overline{T_X^A(x)} = T_X^A(x)$, yielding $T_X^A(x) = T_X^B(x) = T_X(x)$. This remark describes the well-known convex analysis fact that every centre to a star-shaped set is a point of so-called *pseudo-convexity.* Note that in general, $T_X^C(x) \subsetneq T_X(x)$ for $x \in K_X$.

In this note, we shall call *normal cone* to X at x the negative polar cone to $T_X(x)$:

$$\begin{aligned} N_X(x) &:= T_X(x)^- = \overline{S_X(x)}^- = S_X(x)^- = (X-x)^- \\ &= \{p \in E' : \langle p, x\rangle = \max_{y\in X}\langle p, y\rangle\}. \end{aligned}$$

The *Clarke normal cone* to X at x is the negative polar cone to $T_X^C(x)$:

$$N_X^C(x) = T_X^C(x)^- := \{p \in E' : \langle p, v\rangle \leq 0, \forall v \in T_X^C(x)\}.$$

Note that if X is locally convex at $x \in \overline{X}$, then $N_X^C(x)$ coincides with the closed convex cone $N_X(x)$ of convex analysis.

Definition 10. Let X be a non-empty subset of a topological vector space E. A map $\Phi : \overline{X} \rightrightarrows E$ is said to be tangential on X if it satisfies the condition

$$(\tau) \qquad \forall x \in \partial X, \Phi(x) \cap T_X(x) \neq \varnothing$$

In case of certain non-convex domains (e.g., bi-lipschitzian homeomorphic deformation of convex sets, proximate retracts, lipschitzian retracts, etc., see [3]) tangency has been expressed in terms of the Clarke circatangent cone $T_X^C(x)$. Since $T_X^C(x) \subseteq T_X(x)$ always, in the case of star-shaped domain, tangency as in Definition 10 is less restrictive.

We are now ready to state and prove the main result of this note.

Theorem 11. *Let X be a non-empty star-shaped compact subset in a locally convex topological vector space E. Then every tangential map $\Phi \in \mathbf{K}^*(X, E)$ has an equilibrium.*

Proof. Let $Y = \{p|_X : p \in E'\}$ (a convex set) and define the function $f : X \times Y \longrightarrow \mathbb{R}$ by $f(x, p) := \inf_{y\in\Phi(x)}\langle p, y\rangle$ for all $(x, p) \in X \times Y$.

As the support functional $x \mapsto \sigma_\Phi(x, p) := \sup_{y\in\Phi(x)}\langle p, y\rangle$ associated to Φ is upper semicontinuous, then $x \mapsto f(x, p)$ is lower semicontinuous on X. In addition, as the infimum of linear forms, $p \mapsto f(x, p)$ is concave on Y.

Note that condition (τ) implies the normality condition:

$$\forall x \in \partial X,\ p \in N_X(x) \Longrightarrow f(x, p) \leq 0.$$

This normality condition opposes the alternative (B) of Corollary 7 with $\lambda = 0$. Hence, the alternative (A) of that corollary holds: there exists $x^* \in X$ with $\inf_{y\in\Phi(x^*)}\langle p, y\rangle \leq \lambda = 0$ for all $p \in E'$.

If $0 \notin \Phi(x^*)$, by the Hahn-Banach separation theorem, there exists $p \in E'$, there exists $\lambda \in \mathbb{R}$ with $p(0) = 0 < \alpha < p(y)$, for all $y \in \Phi(x^*)$. This implies $0 < \alpha \leq \inf_{y\in\Phi(x^*)}\langle p, y\rangle \leq 0$, a contradiction. Thus, $0 \in \Phi(x^*)$. □

Note that in view of the facts that a star-shaped set is pseudo-convex at each of its centres and that the kernel K_X is a convex and closed set (by Proposition 8 above, as X is closed), the tangency condition (τ) takes a simpler form on $\partial X \cap K_X$. Also, if X is an epi-lipschitz compact domain in a metrizable topological vector space, or more generally a compact $L-$retract (see [3] and references there), tangency condition (τ) must be expressed in terms of the Clarke circatangent cone $T_X^C(x)$; moreover, in this case, X must also have a non-trivial Euler characteristic.

As an immediate consequence of Theorem 11, we obtain an extension of Theorem 2 in [5] which is itself a generalization of the Kakutani–Ky Fan–Himmelberg fixed point theorem. First, following Halpern [5], given a subset X in a vector space E and an element $x \in E$, define the *inward set* and the *outward set* of x with respect to X as

$$I_X(x) := \{y \in E : [y, x) \cap X \neq \varnothing\} \text{ and } O_X(x) := \{y \in E : x - (y - x) \in I_X(x)\}$$

respectively; $[y, x) := \{(1-t)y + tx : 0 \leq t < 1\}$. A map $\Psi : X \rightrightarrows E$ is said to be *inward* (*outward,* respectively) on X if, for all $x \in X$, $\Psi(x) \cap \overline{I_X(x)} \neq \varnothing$ ($\Psi(x) \cap \overline{O_X(x)} \neq \varnothing$ respectively). It is readily seen that $\Psi : X \rightrightarrows E$ is inward on X if and only if the field $\Phi := \Psi - I : X \rightrightarrows E$ satisfies $\Phi(x) \cap T_X(x) \neq \varnothing$ for all $x \in X$. Also, $\Psi : X \rightrightarrows E$ is outward if and only if $\Gamma : X \rightrightarrows E$ given by $\Gamma(x) := x - \Phi(x) = x - (\Psi(x) - x), x \in X$, is inward. In addition, $x_0 \in \Gamma(x_0)$ if and only if $x_0 \in \Psi(x_0)$.

COROLLARY 12. *Every inward (or outward) map $\Psi \in \mathbf{K}^*(X, E)$ where X is a non-empty star-shaped compact subset X in a locally convex topological vector space E has a fixed point.*

PROOF. The set-valued field $\Phi : X \rightrightarrows E$ defined as $\Phi(x) := \Psi(x) - x$, $x \in X$, is in $\mathbf{K}^*(X, E)$. Moreover, for all $x \in X$, the stronger tangency condition

$$\Phi(x) \subseteq X - x \subseteq T_X(x),$$

holds. An equilibrium for Φ is a fixed point for Ψ. □

This result extends the main result (Theorem 2) of Park [6] to Kakutani set-valued maps as inwardness rules out the existence of an invariant direction (Birkhoff–Kellog condition).

Noteworthy equivalences between various fundamental mathematical results hold as described by the loop: Brouwer fixed point theorem ⇒ Knaster–Kuratowsk–Mazurkiewicz principle ⇒ Ky Fan–Browder fixed point theorem ⇒ Ky Fan infsup inequality ⇒ Theorem 1 (Ky Fan–Halpern; see [1]) ⇒ Kakutani–Ky Fan–Himmelberg fixed point theorem (see [1]) ⇒ Ky Fan matching theorem for open covers of convex sets (see [2]) ⇒ Theorem 4 (Ky Fan–Browder Theorem for star-shaped domains) ⇒ Corollary 12 ⇒ Kakutani fixed point theorem ⇒ Brouwer fixed point theorem.

In conclusion, we note that Theorem 11 implies the existence of stationary solutions to set-valued dynamical systems. Indeed, let us recall that a subset X of some Banach space E is said to be *locally viable* with respect to (w.r.t.) a differential inclusion $x'(t) \in \Phi(x(t))$, where $\Phi \in \mathbf{K}^*(X, E)$, if for any given $x_0 \in X$, there exist $T > 0$ and $x \in C^1$ such that $x'(t) \in \Phi(x(t))$, $0 \leq t \leq T$, $x(0) = x_0$, and $x(t) \in X$ for all $t \in [0, T]$. In this context, the dynamical nature of the tangency condition (τ) on X is strikingly expressed by the set-valued extension of the *Nagumo viability theorem.* Namely, if $E \supseteq X$ is locally compact and $\Phi \in \mathbf{K}^*(X, E)$, then:

$$X \text{ is locally viable w.r.t. } x'(t) \in \Phi(x(t)) \text{ if and only if,}$$
$$\text{for every } x \in X,\ \Phi(x) \cap T_X^B(x) \neq \varnothing,$$

with the Bouligand–Severi contingent cone $T_X^B(x)$ (see [3] and references there). If X is compact, then it is *viable (globally)* w.r.t. $x'(t) \in \Phi(x(t))$, that is, there exists a trajectory of the differential inclusion such that $x(t) \in X$, for all $t \in [0, \infty)$. Theorem 11 contains an extension of Theorem 3.2.1 of [1] to star-shaped (instead of convex) viability domains.

Corollary 13. *Given a compact star-shaped subset X of a Banach space E, if X is a viability domain with respect to a differential inclusion $x'(t) \in \Phi(x(t))$, $\Phi \in \mathbf{K}^*(X, E)$, then X contains a stationary solution.*

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