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Actions of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ on Lattice Ordered Dimension Groups

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Abstract. It is shown that if G is a lattice ordered countable group, then every action of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ on G arises as an inductive limit of actions of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ on simplicial groups. Some parts of the argument work in greater generality, and are proved for general finite abelian groups. A template is given for proving similar results for other such groups.

Résumé. On montre que si G est un groupe dénombrable treillis-ordonné, alors toute action de $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ sur G provient d'une limite inductive d'actions de $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ sur des groupes simpliciaux. Des parties de cet argument fonctionnent dans une généralité plus grande et sont prouvées pour des groupes abéliens finis en général. Un modèle est donné pour prouver des résultats similaires pour d'autres groupes de ce type.

1. Introduction After the classification of AF C^*-algebras via the K_0 group by Elliott in [4], Effros, Handelman, and Shen completed the classification picture for these algebras by characterizing those partially ordered abelian groups that arise as inductive limits of simplicial groups ([3], see also [6]), thus solving the range of invariant problem for this classification. Following this, attention turned to inductive limit type actions on AF algebras. In [7] and [8], locally representable actions were classified using the K-theory of the crossed product algebras. In [5], the restriction of local representability was removed, but the group was restricted to $\mathbb{Z}/2\mathbb{Z}$. The range of invariant problem for the classification in [5] remains open. In particular, the question of when a $\mathbb{Z}/2\mathbb{Z}$ action on a dimension group necessarily arises as an inductive limit of $\mathbb{Z}/2\mathbb{Z}$ actions on simplicial groups remains open. It was shown in [2] that a $\mathbb{Z}/2\mathbb{Z}$ action on a countable lattice ordered group can always be written as an inductive limit of $\mathbb{Z}/2\mathbb{Z}$ actions on simplicial groups. The present work extends this to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, but keeps the hypothesis of lattice ordering.

The organization of this paper is as follows. In the next section, we state the main theorem and prove some parts of the argument that hold for general finite abelian group actions. In the following section, we carry out the remaining steps

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for the special case of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. There is then an appendix with some useful basic results.

2. The Main Theorem

Our main result is the following:

THEOREM 2.1. *(Main Theorem) Let G be a countable lattice ordered group, and let γ be an action of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ on G. Then (G, γ) is an inductive limit of actions of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ on simplicial groups.*

The main tool in proving this will be the following factorization property, which we shall prove for $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$:

(Factorization Property for a finite abelian group Γ) Let G_1 be a simplicial group, G a lattice ordered dimension group, γ_1 an action of Γ on G_1, γ an action of Γ on G, and $g_1 : G_1 \to G$ an equivariant positive homomorphism. Then there exists a simplicial group G_2, an action γ_2 of Γ on G_2, and equivariant positive homomorphisms $h : G_1 \to G_2$ and $g_2 : G_2 \to G$ such that $g_1 = g_2 \circ h$ and $ker(g_1) = ker(h)$.

For a group Γ with this property, the proof of the main theorem is essentially the same as the proof of the main theorem in [2] with the above property in place of lemma 2.2 of [2]. See also [6].

We show that the factorization property for a particular group follows if one can carry out two steps: an orthogonalization process, and a decomposition property. If G is a lattice ordered group and $x, y \in G^+$, we shall write $x \perp y$, and say x is orthogonal to y, if $x \wedge y = 0$. The first step is then to show Γ has the following property:

(Orthogonalization Property for a finite abelian group Γ) Let G be a lattice-ordered group, and let γ be an action of Γ on G. Let $\mathbb{Z}^n$ be a simplicial group with simplicial basis $\{e_1, \dots, e_n\}$. Let γ_n be an action of Γ implemented by an action σ_n on the simplicial basis elements. Suppose that $\varphi : \mathbb{Z}^n \to G$ is an equivariant positive homomorphism and that $x_i = \varphi(e_i)$ for $i = 1, \dots, n$, so γ also acts on $\{x_1, \dots, x_n\}$ by the action σ_n. Then there exists an integer t, an action γ_t of Γ on the simplicial group $\mathbb{Z}^t$ implemented on the simplicial basis by an action σ_t, and equivariant positive homomorphisms $\psi : \mathbb{Z}^n \to \mathbb{Z}^t$ and $\varphi_2 : \mathbb{Z}^t \to G$ such that $\varphi = \varphi_2 \circ \psi$, and if $y_1, \dots, y_t$ denote the images of the simplicial basis elements of $\mathbb{Z}^t$ under φ_2, the elements within each orbit of σ_t on $\{y_1, \dots, y_t\}$ are pairwise orthogonal.

The next step is to show the following decomposition property:

(Decomposition Property with Orthogonalized Orbits for a finite abelian group Γ) Let G be a lattice-ordered dimension group, and γ be an action of Γ on G. Let $\mathbb{Z}^n$ be a simplicial group with simplicial basis $\{e_1, \dots, e_n\}$. Let γ_n be an action of Γ implemented by an action σ_n of the simplicial basis elements. Suppose that $\varphi : \mathbb{Z}^n \to G$ is an equivariant positive homomorphism and that $x_i = \varphi(e_i)$ for $i = 1, \dots, n$. Then γ acts on $\{x_1, \dots, x_n\}$ by the action σ_n. Suppose further

that for each orbit of σ_n, on $\{x_1, \ldots, x_n\}$ the elements of the orbit are pairwise orthogonal. Let $p = p_1 e_1 + \cdots + p_n e_n \in \mathbb{Z}^n$ be an element of the kernel of φ. Then there exists an integer t, an action γ_t of Γ on the simplicial group $\mathbb{Z}^t$ implemented on the simplicial basis by an action σ_t, and equivariant positive homomorphisms $\psi : \mathbb{Z}^n \to \mathbb{Z}^t$ and $\varphi_2 : \mathbb{Z}^t \to G$ having the following properties: If $y_1, \ldots, y_t$ are the images of the simplicial basis elements of $\mathbb{Z}^t$ under φ_2, then the orbits of σ_t on $\{y_1, \ldots, y_t\}$ are each pairwise orthogonal; $\psi(p) = 0$; and $\varphi = \varphi_2 \circ \psi$.

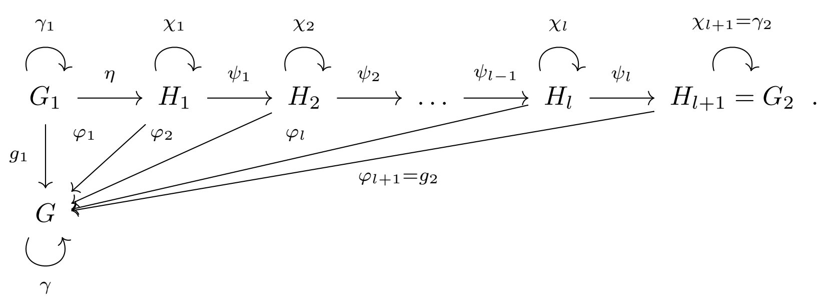
Given these two properties for a finite abelian group Γ, we can prove it has the factorization property as follows.

Proof. (Factorization Property, given Orthogonalization and Decomposition Properties) Let Γ, G_1, γ_1, G, γ and $g_1 : G_1 \to G$ be as in the statement of the property. First, we apply the orthogonalization property to get a simplicial group H_1, an action χ_1 of Γ on H_1, and equivariant positive homomorphisms $\eta : G_1 \to H_1$ and $\varphi_1 : H_1 \to G$ such that $\varphi_1 \circ \eta = g_1$ and if χ_1 is implemented by σ_t on the simplicial basis of H_1 and $y_1 \ldots y_t$ are the images of this basis under φ_1, then the elements of each orbit of σ_t on $\{y_1 \ldots y_t\}$ are pairwise orthogonal.

Now consider $ker(g_1)$. Since G_1 is a finitely generated free abelian group, $ker(g_1)$ is finitely generated. Let $p_1 \ldots p_l$ be a set of generators of $ker(g_1)$. Notice that $\eta(p_i) \in ker(\varphi_1)$ for each i.

Next, we apply the decomposition property with orthogonal variables with $\varphi_2 : H_1 \to G$, χ_1, γ, and $\eta(p_1) = p$ to get a simplicial group H_2, action χ_2 of Γ on H_2, and equivariant positive homomorphisms $\psi_1 : H_1 \to H_2$, $\varphi_2 : H_2 \to G$ such that $\varphi_1 = \varphi_2 \circ \psi_1$, $\psi_1(\eta(p_1)) = 0$, and the orbits of Γ on the image of the simplicial basis of H_2 under φ_2 have pairwise orthogonal elements. Now we repeat the process with the decomposition lemma with H_2, χ_2, and $(\psi_1 \circ \eta)(p_2)$, and so on, until we have killed off all of the ps.

The argument can be summarized in a commutative diagram:



The required h is $h = \psi_l \circ \psi_{l-1} \circ \cdots \circ \psi_1 \circ \eta$. ☐

The proof that a finite abelian group has the decomposition property with orthogonalized orbits involves a reduction to special cases, as follows. We consider the map $\Theta_H : H \to H$, for any dimension group H with group action γ of finite group Γ given by $\Theta_H(x) = \sum_{g\in\Gamma} \gamma_g(x)$. We have $\gamma_g(\Theta_H(x)) = \Theta_H(x)$ for any $x \in H$ and $g \in \Gamma$. If $x \in H^\gamma$, the fixed point subgroup, then $\Theta_H(x) = |\Gamma|x$. Also, if H_1 and H_2 are two dimension groups with Γ actions γ_1 and γ_2, and $\varphi : H_1 \to H_2$ is equivariant, then $\varphi \circ \Theta_{H_1} = \Theta_{H_2} \circ \varphi$. Now suppose $p \in H_1$ and $\varphi(p) = 0$. Then $0 = \Theta_{H_2}(\varphi(p)) = \varphi(\Theta_{H_1}(p))$. Let $q = |\Gamma|p - \Theta_{H_1}(p)$ Then $\Theta_{H_1}(q) = 0$. If we let $r = \Theta_{H_1}(p)$, we have $|\Gamma|p = r + q$, where $r \in H_1^{\gamma_1}$, and $\Theta_{H_1}(q) = 0$. As H_2 is torsion free, $\varphi(|\Gamma|p) = 0$ if, and only if, $\varphi(p) = 0$. We thus have the following two special cases to consider:

Case 1: The relation $p \in H_1$ is fixed by the action γ

Case 2: The relation $p \in H_1$ satisfies $\Theta_{H_1}(p) = 0$.

Since both conditions are preserved by equivariant maps, if we can solve the problem in each special case, we can solve the general problem in two steps.

Theorem 2.2. *For any finite abelian group Γ, the decomposition property with orthogonalized orbits holds for a relation p in Case 1.*

Proof. (Decomposition property with a relation p in Case 1) Let γ_m be an action of the finite abelian group Γ on $\mathbb{Z}^m$ given by an action σ_m on the simplicial basis $\{e_1, \ldots, e_m\}$. Let $S_1, \ldots, S_k \subseteq \{1, \ldots, m\}$ denote the orbits of σ_m. We assume, without loss of generality, that the orbits are listed sequentially among the subscripts of the simplicial basis, so $S_1 = \{e_1 \ldots e_{|S_1|}\}$ etc. For each orbit S_j, let Γ_j denote the stabilizer subgroup, $\Gamma_j = \{g \in \Gamma \,|\, \gamma_g(x) = x \text{ for all } x \in S_j\}$.

Like the original Effros Handelman Shen proof, our proof proceeds by induction on the degree of a relation in $\mathbb{Z}^n$. This is defined as the ordered pair (x, y), where x is the absolute value of the largest coefficient, and y is the number of times it occurs. This set is then ordered lexicographically, which is a well ordering. As in the usual EHS theorem proof, if the coefficients all have the same sign, then the xs must all be 0, and the map $\varphi = 0$. In this case setting $\psi = 0$ and $\varphi_2 = 0$ will suffice, so we may suppose we have nonzero coefficients of both signs. Similarly, if a relation has degree $(0, y)$, then all the coefficients are zero and $\psi = id$, $\varphi_2 = \varphi$ will do. This case forms the basis step for our induction.

Suppose that $p \in \mathbb{Z}^m$ is a relation in our first special form, i.e., p is fixed by γ_m. In this case, for each orbit S_i, the coefficients are constant. Our relation p is of the form

$$\sum_{i\in I} p_i\Bigg(\sum_{l\in S_i} e_l\Bigg).$$

Apply $\phi : \mathbb{Z}^m \to G, e_j \mapsto x_j$ and we get

$$0 = \sum_{i\in I} p_i\Bigg(\sum_{l\in S_i} x_l\Bigg).$$

Let $B \subseteq I$ be $B = \{i \in I \mid p_i \geq 0\}$ and $A \subseteq I$ be $A = I \setminus B = \{i \in I \mid p_i < 0\}$.

Then our relation is

$$\sum_{i\in B} p_i\Big(\sum_{l\in S_i} x_l\Big) = \sum_{j\in A} p_j\Big(\sum_{l\in S_j} x_l\Big). \tag{1}$$

For each $i \in I$, $\sum_{l\in S_i} x_l$ is in G^Γ, the fixed point subgroup. Let $w_j = \sum_{l\in S_j} x_l$ for $j \in I$. We then have $\sum_{i\in B} p_i w_i = \sum_{j\in A} p_j w_j$ in G^Γ. We may assume without loss of generality that the largest coefficient is p_1 and $1 \in B$. We then have $p_1 w_1 \le \sum_{j\in A} p_j w_j$ in G^Γ. It follows that $p_1 w_1 \le p_1(\sum_{j\in A} w_j)$.

Since G^Γ is a dimension group (see appendix) it follows from unperforation that $w_1 \le \sum_{j\in A} w_j$. From the Riesz decomposition property, there exist $y_j \in G^\Gamma$ with $0 \le y_j \le w_j$ for $j \in A$ such that $w_1 = \sum_{j\in A} y_j$.

We have $x_1 \le w_1 = \sum_{j\in A} y_j$, and $x_1 \in G^{\Gamma_1}$, so by using the Riesz decomposition property in G^{Γ_1}, we find $z_j \in G^{\Gamma_1}$ with $0 \le z_j \le w_j$ for $j \in A$ such that

$$x_1 = \sum_{j\in A} z_j. \tag{2}$$

Since $w_j = \sum_{l\in S_j} x_l$, and the x_l are orthogonal over the orbits, we see that $z_j = \sum_{l\in S_j}(z_j \wedge x_l)$, with the $z_j \wedge x_l$ being pairwise orthogonal, and satisfying $z_j \wedge x_l \in G^{\Gamma_1\cap\Gamma_j}$.

It will now be convenient to identify S_j with Γ/Γ_j, in such a way that the first element of S_j corresponds to the identity. We then have $\{x_l \mid l \in S_1\}$ identified with $\{h(x_1) \mid h \in \Gamma/\Gamma_1\}$, etc. Applying elements of Γ to equation 2 above, we have $h(x_1) = \sum_{j\in A} h(z_j)$, where we may consider the h only up to their class in Γ/Γ_1. We have that the $h(z_j)$ are pairwise orthogonal for distinct $h \in \Gamma/\Gamma_1$, as the $h(x_1)$s are.

Let x_{1j} denote the first element in the orbit S_j. Consider the set $V_j = \{v^j_{(h,t)} = (h(z_j)\wedge t(x_{1j})) \mid (h,t) \in (\Gamma/\Gamma_1)\times(\Gamma/\Gamma_j)\}$. By construction, the $v^j_{(h,t)}$ are pairwise orthogonal, and all belong to $G^{\Gamma_1\cap\Gamma_j}$. If $g \in \Gamma$, we have $g(v^j_{(h,t)}) = (g(h(z_j)) \wedge g(t(x_{1j}))) = (([g]_1 h)(z_j)) \wedge (([g]_j t)(x_{1j})) = v^j_{([g]_1 h, [g]_j t)}$, where $[g]_1, [g]_j$ denote the classes of g in Γ/Γ_1 and Γ/Γ_j respectively.

Consider the element $y^j_t = t(x_{1j}) - \sum_{h\in\Gamma/\Gamma_1} v^j_{(h,t)}$. One can easily verify that these are positive, pairwise orthogonal for $t \in \Gamma/\Gamma_j$, and that they form an orbit under the action γ with stabilizer subgroup Γ_j.

Notice that $z_j = \sum_{t\in\Gamma/\Gamma_j} v^j_{(e,t)}$, so $x_1 = \sum_{j\in A}\Big(\sum_{t\in\Gamma/\Gamma_j} v^j_{(e,t)}\Big)$, and for any $g \in \Gamma$ we have $g(x_1) = \sum_{j\in A}\Big(\sum_{t\in\Gamma/\Gamma_j} v^j_{([g]_1,[g]_j t)}\Big) = \sum_{j\in A}\Big(\sum_{t\in\Gamma/\Gamma_j} v^j_{([g]_1,t)}\Big)$, so we may write

$$h(x_1) = \sum_{j\in A}\Bigg(\sum_{t\in\Gamma/\Gamma_j} v^j_{(h,t)}\Bigg) \tag{3}$$

for any $h \in \Gamma/\Gamma_1$. Summing over the first orbit, we thus have

$$\begin{aligned}\Big(\sum_{l\in S_1} x_l\Big) &= \sum_{h\in\Gamma/\Gamma_1}\sum_{j\in A}\Big(\sum_{t\in\Gamma/\Gamma_j} v^j_{(h,t)}\Big)\\ &= \sum_{j\in A}\Big(\sum_{h\in\Gamma/\Gamma_1}\sum_{t\in\Gamma/\Gamma_j} v^j_{(h,t)}\Big).\end{aligned}\tag{4}$$

Now we insert (4) into (1). This gives

$$\begin{aligned}&p_1\Big(\sum_{j\in A}\Big(\sum_{h\in\Gamma/\Gamma_1}\sum_{t\in\Gamma/\Gamma_j} v^j_{(h,t)}\Big)\Big) + \sum_{i\in B\setminus\{1\}} p_i\Big(\sum_{l\in S_i} x_l\Big)\\ &= \sum_{j\in A} p_j\Big(\sum_{l\in S_j} x_l\Big).\end{aligned}\tag{5}$$

We rearrange this to get

$$\begin{aligned}&\sum_{j\in A}(p_1 - p_j)\Big(\sum_{h\in\Gamma/\Gamma_1}\sum_{t\in\Gamma/\Gamma_j} v^j_{(h,t)}\Big) + \sum_{i\in B\setminus\{1\}} p_i\Big(\sum_{l\in S_i} x_l\Big)\\ &= \sum_{j\in A} p_j\Big(\sum_{l\in S_j} x_l - \sum_{h\in\Gamma/\Gamma_1}\sum_{t\in\Gamma/\Gamma_j} v^j_{(h,t)}\Big)\\ &= \sum_{j\in A} p_j\Big(\sum_{t\in\Gamma/\Gamma_j} y^j_t\Big).\end{aligned}\tag{6}$$

Notice that the sets of equations in (3) and

$$t(x_{1j}) = y^j_t + \sum_{h\in\Gamma/\Gamma_1} v^j_{(h,t)}\tag{7}$$

for $t \in \Gamma/\Gamma_j$ are invariant under the action of Γ.

Consider the new set of variables

$$Y = \Big(\cup_{i\in B\setminus\{1\}}\{x_l \mid l \in S_i\}\Big) \cup \Big(\cup_{j\in A} V_j\Big) \cup \Big(\{y^j_t \mid j \in A, t \in \Gamma/\Gamma_j\}\Big).$$

This set is invariant under Γ, so there exists a simplicial group $H \cong \mathbb{Z}^{|Y|}$ and an action $\gamma_{|Y|}$ of Γ on H such that the map sending each simplicial basis element of H to the corresponding element of Y extends to an equivariant positive homomorphism $\varphi_2 : H \to G$. Lifting the equations (3) and (7) to the basis elements of $\mathbb{Z}^m$ and H in the canonical way defines a positive equivariant homomorphism $\psi_1 : \mathbb{Z}^m \to H$ that takes the relation $p \in \mathbb{Z}^m$ to the relation p' in H defined by (6). Since (6) is a relation with orthogonal variables across orbits, constant coefficients across orbits, and lower degree than (1), we may apply our induction hypothesis to H, p', and $\varphi_2 : H \to G$. We get $\mathbb{Z}^t$, $\varphi_3 : \mathbb{Z}^t \to G$ and $\psi_2 : H \to \mathbb{Z}^t$ such that $\varphi_3 \circ \psi_2 = \varphi_2$ and $\psi_2(p') = 0$. Our desired ψ is then $\psi = \psi_2 \circ \psi_1$. □

In the next section, we show that we can prove the orthogonalization property and Case 2 of the decomposition property for the group $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

3. The Case $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ In this section, we give the details of the remaining steps for $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, beginning with the orthogonalization of variables.

Theorem 3.1. *The group* $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ *satisfies the orthogonalization property.*

Proof. (Orthogonalization Property for $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$) Let α and β be two commuting $\mathbb{Z}/2\mathbb{Z}$ actions on a lattice ordered group G, so $\alpha\beta = \beta\alpha$. Then we have an action of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Suppose we have $x, \alpha(x) \in G^+$. Let $r = x \wedge \alpha(x)$. Then let $s = x - x \wedge \alpha(x) = x - r$ and $\alpha(s) = \alpha(x) - x \wedge \alpha(x) = \alpha(x) - r$. So r is fixed by α and s and $\alpha(s)$ are flipped by α and positive. We can show that $s \perp \alpha(s)$. Let $z \leq s$ and $z \leq \alpha(s)$. Then $z + r \leq x$ and $z + r \leq \alpha(x)$. Thus, $z + x \wedge \alpha(x) \leq x \wedge \alpha(x)$, so $z \leq 0$.

Now we incorporate the action β. Consider $r \wedge \beta(r)$, which is fixed by both α and β. Then $r - r \wedge \beta(r)$ and $\beta(r) - r \wedge \beta(r)$ are orthogonal, fixed by α, and flipped by β. Then $s \wedge \beta(s)$ and $\alpha(s) \wedge \alpha\beta(s)$ are orthogonal since $s \perp \alpha(s)$, flipped by α, and fixed by β. Next, $s - s \wedge \beta(s)$, $\beta(s) - s \wedge \beta(s)$, $\alpha(s) - \alpha(s) \wedge \alpha\beta(s)$, and $\alpha\beta(s) - \alpha(s) \wedge \alpha\beta(s)$ are flipped by both α and β. Elementary calculations show that $(s - s \wedge \beta(s)) \perp (\beta(s) - s \wedge \beta(s))$ and $(\alpha(s) - \alpha(s) \wedge \alpha\beta(s)) \perp (\alpha\beta(s) - \alpha(s) \wedge \alpha\beta(s))$. Since $s \perp \alpha(s)$, we get $(s - s \wedge \beta(s)) \perp (\alpha(s) - \alpha(s) \wedge \alpha\beta(s))$. Also, since β is an automorphism, it preserves orthogonality so $(\beta(s) - s \wedge \beta(s)) \perp (\alpha\beta(s) - \alpha(s) \wedge \alpha\beta(s))$.

Summarizing, we now have new orbits $\{r \wedge \beta(r)\}$, $\{r - r \wedge \beta(r), \beta(r) - r \wedge \beta(r)\}$, $\{s \wedge \beta(s), \alpha(s) \wedge \alpha\beta(s)\}$, and $\{s - s \wedge \beta(s), \beta(s) - s \wedge \beta(s), \alpha(s) - \alpha(s) \wedge \alpha\beta(s), \alpha\beta(s) - \alpha(s) \wedge \alpha\beta(s)\}$. In each of these orbits, the elements are pairwise orthogonal. We have $x = s + r = (s - s \wedge \beta(s)) + s \wedge \beta(s) + (r - r \wedge \beta(r)) + (r \wedge \beta(r))$. Applying α and β to this equation, we can express the elements in the orbit of x as positive integer combination of elements of our new orbits. Lifting these equations to the simplicial groups in the canonical way, and taking direct sums for multiple orbits of xs, gives the required new simplicial group and maps. □

Theorem 3.2. *The group* $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ *has the decomposition property with orthogonalized orbits.*

Proof. We begin by reducing the decomposition with orthogonalized variables to special cases. For $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, we use the following four special cases:

$$\alpha_n(p) = p, \beta_n(p) = p$$

$$\alpha_n(p) = p, \beta_n(p) = -p$$

$$\alpha_n(p) = -p, \beta_n(p) = p$$

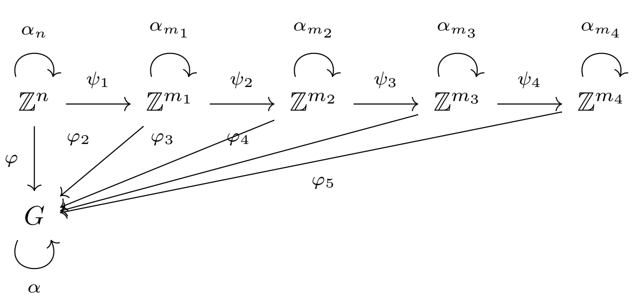
$$\alpha_n(p) = -p, \beta_n(p) = -p.$$

If the problem can be solved in these four special cases, then we can solve the problem in the general case. If $\varphi(p) = 0$, then $\varphi(\alpha_n(p)) = 0, \varphi(\beta_n(p)) = 0$, and $\varphi(\alpha_n\beta_n(p)) = 0$. Thus, we also have $\varphi(p + \alpha_n(p) + \beta_n(p) + \alpha\beta_n(p)) = 0$, $\varphi(p + \alpha_n(p) - \beta_n(p) - \alpha\beta_n(p)) = 0, \varphi(p - \alpha_n(p) + \beta_n(p) - \alpha\beta_n(p)) = 0$, and $\varphi(p - \alpha_n(p) - \beta_n(p) + \alpha\beta_n(p)) = 0$. Conversely, if $\varphi(p + \alpha_n(p) + \beta_n(p) + \alpha\beta_n(p)) = 0, \varphi(p + \alpha_n(p) - \beta_n(p) - \alpha\beta_n(p)) = 0, \varphi(p - \alpha_n(p) + \beta_n(p) - \alpha\beta_n(p)) = 0$, and $\varphi(p - \alpha_n(p) - \beta_n(p) + \alpha\beta_n(p)) = 0$, then $\varphi(4p) = 0$. Since G is torsion free, $\varphi(p) = 0$.

Let $q_1 = p + \alpha_n(p) + \beta_n(p) + \alpha\beta_n(p), q_2 = p + \alpha_n(p) - \beta_n(p) - \alpha\beta_n(p), q_3 = p - \alpha_n(p) + \beta_n(p) - \alpha\beta_n(p)$ and $q_4 = p - \alpha_n(p) - \beta_n(p) + \alpha\beta_n(p)$.

If we can solve the first special case, then we get a new group $\mathbb{Z}^{m_1}$ and two maps $\varphi_2 : \mathbb{Z}^{m_1} \to G$ and $\psi_1 : \mathbb{Z}^n \to \mathbb{Z}^{m_1}$, both equivariant, such that $\varphi_2 \circ \psi_1 = \varphi$ and $\psi_1(q_4) = 0$. If we can solve the second special case, then applying this to φ_2 and $\psi_1(q_3)$, we get a new group $\mathbb{Z}^{m_2}$ and two new maps $\varphi_3 : \mathbb{Z}^{m_2} \to G$ and $\psi_2 : \mathbb{Z}^{m_1} \to \mathbb{Z}^{m_2}$ such that $\varphi_3 \circ \psi_2 = \varphi_2$ and $\psi_2(\psi_1(q_3)) = 0$. If we can solve the third special case, then applying this to φ_3 and $\psi_2(\psi_1(q_2))$, we get a new group $\mathbb{Z}^{m_3}$ and two new maps $\varphi : \mathbb{Z}^{m_3} \to G$ and $\psi_3 : \mathbb{Z}^{m_2} \to \mathbb{Z}^{m_3}$ such that $\varphi_4 \circ \psi_3 = \varphi_3$ and $\psi_3(\psi_2(\psi_1(q_2))) = 0$. If we can solve the fourth special case, then applying this to φ_4 and $\psi_3(\psi_2(\psi_1(q_1)))$, we get a new group $\mathbb{Z}^{m_4}$ and two new maps $\varphi : \mathbb{Z}^{m_4} \to G$ and $\psi_3 : \mathbb{Z}^{m_3} \to \mathbb{Z}^{m_4}$ such that $\varphi_5 \circ \psi_4 = \varphi_4$ and $\psi_4(\psi_3(\psi_2(\psi_1(q_1)))) = 0$.

If we combine all cases, we get the commutative diagram,



Let $\psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$. Then,

$$\begin{aligned}
&\psi(4p)\\
&= \psi(q_1 + q_2 + q_3 + q_4)\\
&= \psi_4[\psi_3[\psi_2[\psi_1(q_1) + \psi_1(q_2) + \psi_1(q_3) + \psi_1(q_4)]]]\\
&= \psi_4[\psi_3[\psi_2[\psi_1(q_1) + \psi_1(q_2) + \psi_1(q_3)]]]\\
&= \psi_4[\psi_3[\psi_2(\psi_1(q_1)) + \psi_2(\psi_1(q_2)) + \psi_2(\psi_1(q_3))]]\\
&= \psi_4[\psi_3[\psi_2(\psi_1(q_1)) + \psi_2(\psi_1(q_2))]]\\
&= \psi_4[\psi_3(\psi_2(\psi_1(q_1))) + \psi_3(\psi_2(\psi_1(q_2)))]\\
&= \psi_4(\psi_3(\psi_2(\psi_1(q_1))))\\
&= 0.
\end{aligned}$$

Since all of the new groups are torsion free, $\psi(p) = 0$. This completes the reduction to four special cases.

Notice the first case is that already done in general in chapter 2, and the other three cases subdivide the case $\Theta(p) = 0$. We may write $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) = \{e, \alpha, \beta, \delta\}$, where e is the identity, $\alpha^2 = \beta^2 = \delta^2 = e$, $\alpha\beta = \beta\alpha$, and $\alpha\beta = \delta$. The three remaining cases all take the form that two of α, β, and δ change the sign of p, while the third necessarily leaves it fixed. Treating the three symmetrically, we need only consider the case $\alpha(p) = \beta(p) = -p$. In this case, we have two types of orbit to consider: $\{x, \alpha(x), \beta(x), \delta(x)\}$ with elements orthogonal, and $\{y, \alpha(y) = \beta(y)\}$ with elements orthogonal. Our relation, after possibly permuting and relabelling, is of the form

$$\sum_{i \in A} p_i(x_i - \alpha(x_i) - \beta(x_i) + \delta(x_i)) + \sum_{j \in B} q_j(y_j - \alpha(y_j)) = 0, \tag{8}$$

with all of the ps and qs strictly positive. As before, we proceed by induction on the degree of the relation. We have two cases to consider: where the largest coefficient is one of the ps, and where it is one of the qs.

Consider first the case the largest coefficient is one of the ps, which we may suppose is p_1. We have

$$p_1 x_1 \leq \sum_{i \in A} p_i(\alpha(x_i)+\beta(x_i)) + \sum_{j \in B} q_j(\alpha(y_j)) \leq p_1\big(\sum_{i \in A}(\alpha(x_i)+\beta(x_i)) + \sum_{j \in B}(\alpha(y_j))\big),$$

so by unperforation, $x_1 \leq \sum_{i \in A}(\alpha(x_i)+\beta(x_i)) + \sum_{j \in B}(\alpha(y_j))$. Since x_1 is orthogonal to $\alpha(x_1)$ and $\beta(x_1)$, we have $x_1 \leq \sum_{i \in A\setminus\{1\}}(\alpha(x_i) + \beta(x_i)) + \sum_{j \in B}(\alpha(y_j))$. By the Riesz decomposition property, there exist positive elements a_i, b_i for $i \in A\setminus\{1\}$ and c_j for $j \in B$ such that $0 \leq a_i \leq \alpha(x_i)$, $0 \leq b_i \leq \beta(x_i)$, $0 \leq c_j \leq \alpha(y_j)$ and

$$x_1 = \sum_{i \in A\setminus\{1\}} (a_i + b_i) + \sum_{j \in B} c_j. \tag{9}$$

Notice that $\{g(a_i) \,|\, g \in \Gamma\}$ and $\{g(b_i) \,|\, g \in \Gamma\}$ form orbits of pairwise orthogonal elements for each $i \in A \backslash \{1\}$. Let $z_i = x_i - \alpha(a_i) - \beta(b_i)$, for $i \in A \backslash \{1\}$. Then the z_i are positive and $\{g(z_i) \,|\, g \in \Gamma\}$ forms an orbit of pairwise orthogonal elements. Let $w_j = y_j - \alpha(c_j) - \beta(c_j)$, for $j \in B$. Then the w_j are positive, and $\{w_j, \alpha(w_j)\}$ forms an orbit of orthogonal elements. Now we insert (9) into (8). We get:

$$\begin{aligned} &p_1\Bigg(\sum_{i \in A\backslash\{1\}} (a_i - \alpha(a_i) - \beta(a_i) + \delta(a_i) + b_i - \alpha(b_i) - \beta(b_i) + \delta(b_i)) \\ &+ \sum_{j \in B}(c_j - \alpha(c_j) - \beta(c_j) + \delta(c_j))\Bigg) \\ &+ \sum_{i \in A\backslash\{1\}} p_i(x_i - \alpha(x_i) - \beta(x_i) + \delta(x_i)) \\ &+ \sum_{j \in B} q_j(y_j - \alpha(y_j)) = 0. \end{aligned} \tag{10}$$

Rearranging this, we get

$$\begin{aligned} &\sum_{i \in A\backslash\{1\}} (p_1 - p_i)(a_i - \alpha(a_i) - \beta(a_i) + \delta(a_i) + b_i - \alpha(b_i) - \beta(b_i) + \delta(b_i)) \\ &+ \sum_{j \in B}(p_1 - q_j)(c_j - \alpha(c_j) - \beta(c_j) + \delta(c_j)) \\ &+ \sum_{i \in A\backslash\{1\}} p_i(z_i - \alpha(z_i) - \beta(z_i) + \delta(z_i)) \\ &+ \sum_{j \in B} q_j(w_j - \alpha(w_j)) = 0. \end{aligned} \tag{11}$$

Notice that this is a relation expressed in terms of variables that are pairwise orthogonal within orbits, that the coefficients across orbits sum to zero, and that its degree is lower than that of the relation in (8). Also, the set of equations giving the old variables in terms of the new is invariant under the action of Γ. The rest of the proof for this case is similar to the proof for fixed relations in Section 2 above.

Now we consider the case where the largest coefficient is one of the qs, which we may assume is q_1. As in the case above, we get $q_1 y_1 \leq \sum_{i \in A} p_i(\alpha(x_i) + \beta(x_i)) + \sum_{j \in B} q_j(\alpha(y_j))$, and using unperforation and the orthogonality of the variables within orbits, we get $y_1 \leq \sum_{i \in A} p_i(\alpha(x_i) + \beta(x_i)) + \sum_{j \in B\backslash\{1\}} q_j(\alpha(y_j))$. Notice that the ys and the terms $(\alpha(x_i) + \beta(x_i))$ all lie in Γ^δ, the fixed point subgroup of δ. Thus there exist positive w_i for $i \in A$ and c_j for $j \in B\backslash\{1\}$ such that $y_1 = \sum_{i \in A} w_i + \sum_{j \in B\backslash\{1\}} c_j$ and the w_is and c_js all lie in Γ^δ. Setting

$a_i = w_i \wedge \alpha(x_i)$ and $b_i = w_i \wedge \beta(x_i)$, we get, as in the case above, positive elements a_i, b_i for $i \in A$ and c_j for $j \in B \setminus \{1\}$ such that $0 \leq a_i \leq \alpha(x_i)$, $0 \leq b_i \leq \beta(x_i)$, $0 \leq c_j \leq \alpha(y_j)$ and

$$y_1 = \sum_{i \in A}(a_i + b_i) + \sum_{j \in B\setminus\{1\}} c_j, \tag{12}$$

and in addition, $\delta(c_j) = c_j$, $\delta(a_i) = b_i$, and $\delta(b_i) = a_i$. Notice that we have $\alpha(b_i) = \beta(a_i)$, so $\{a_i, b_i, \alpha(a_i), \alpha(b_i)\}$ is an orbit with pairwise orthogonal elements. Set $s_i = x_i - \alpha(a_i)$ and $t_j = y_j - \alpha(c_j)$. Then s_i and t_j are both positive.

Now we insert (12) into (8), giving

$$\begin{aligned} &\sum_{i \in A} p_i(x_i - \alpha(x_i) - \beta(x_i) + \delta(x_i)) \\ &+ \sum_{j \in B\setminus\{1\}} q_j(y_j - \alpha(y_j)) \\ &+ q_1(\sum_{i \in A}(a_i + b_i - \alpha(a_i) - \alpha(b_i)) + \sum_{j \in B\setminus\{1\}} (c_j - \alpha(c_j))) = 0. \end{aligned} \tag{13}$$

Rearranging this we get

$$\begin{aligned} &\sum_{i \in A}(q_1 - p_i)(a_i + b_i - \alpha(a_i) - \alpha(b_i)) + \sum_{j \in B\setminus\{1\}} (q_1 - q_j)(c_j - \alpha(c_j)) \\ &+ \sum_{i \in A} p_i(s_i - \alpha(s_i) - \beta(s_i) + \delta(s_i)) \\ &+ \sum_{j \in B\setminus\{1\}} q_j(t_j - \alpha(t_j)). \end{aligned} \tag{14}$$

This is a new relation of the same type, but of lower degree. The rest of the proof is similar to the cases above. □

4. Appendix

In this section, we gather some conventions and basic facts about lattice ordered groups that will be useful. These results are probably well known, but we provide proofs for the reader's convenience. Throughout this paper, all of our groups will be countable. If G is a lattice ordered dimension group and $x, y \in G^+$, we shall write $x \perp y$, and say x is orthogonal to y, if $x \wedge y = 0$.

Lemma 4.1. *Let G be a lattice ordered dimension group, and let c, $a_1, \ldots, a_m$ be positive elements of G. Suppose $a_i \perp a_j$ for $i \neq j$, and $a_i \leq c$ for $1 \leq i \leq m$. Then $(a_1 + \cdots + a_m) \leq c$.*

Proof. Suppose $m = 2$, G, a_1 and a_2 are as above. Then $c = a_2 + (c - a_2)$, where $a_2, (c - a_2) \geq 0$, and $a_1 \leq a_2 + (c - a_2)$. By the Riesz interpolation property, there exist $y_1, y_2 \in G^+$ with $a_1 = y_1 + y_2$, $y_2 \leq a_2$ and $y_1 \leq (c - a_2)$. Since $y_2 \leq a_1$ and $y_2 \leq a_2$, we have $y_2 \leq a_1 \wedge a_2 = 0$, so $y_2 = 0$ and $a_1 \leq c - a_2$. Thus $a_1 + a_2 \leq c$, as required.

Now we proceed inductively. Suppose the statement is true for k, and let $m = k+1$. We have $c = (a_2 + \cdots + a_m) + (c - (a_2 + \cdots + a_m))$, where we may assume that $(c - (a_2 + \cdots + a_m)) \geq 0$. Since $a_1 \leq c$, the Riesz interpolation property gives $y_2, \ldots, y_m, z \in G^+$ such that $a_1 = y_2 + \cdots + y_m + z$, $y_i \leq a_i$ for $2 \leq i \leq m$, and $z \leq (c - (a_2 + \cdots + a_m))$. We then have $y_i \leq a_1 \wedge a_i = 0$ for $2 \leq i \leq m$, so $a_1 \leq z$, and the result follows. □

LEMMA 4.2. *Let G be a lattice ordered dimension group, and let c, $a_1, \ldots, a_m$ be positive elements of G. If we have $a_i \perp a_j$ for $i \neq j$, then $c \wedge (a_1 + \cdots + a_m) = c \wedge a_1 + \cdots + c \wedge a_m$.*

PROOF. Let G, c, and $a_1, \ldots, a_m$ be as above. Suppose $z \in G^+$ and $z \leq c \wedge (a_1 + \cdots + a_m)$. Then $z \leq c$ and $z \leq (a_1 + \cdots + a_m)$. By Riesz interpolation, there exists $y_1, \ldots, y_m \in G^+$ with $z = y_1 + \cdots + y_m$ and $y_i \leq a_i$ for $1 \leq i \leq m$. We have $y_i \leq z \leq c$ for each i as well, so $y_i \leq c \wedge a_i$ for each i. Thus $z \leq (c \wedge a_1) + \cdots + (c \wedge a_m)$. It follows that $c \wedge (a_1 + \cdots + a_m) \leq c \wedge a_1 + \cdots + c \wedge a_m$.

Now suppose that $z \leq (c \wedge a_1) + \cdots + (c \wedge a_m)$. Then, by Riesz interpolation, there exist $y_1, \ldots, y_m \in G^+$ such that $z = y_1 + \cdots + y_m$ and $y_i \leq (c \wedge a_i)$ for $1 \leq i \leq m$. Since $y_i \leq a_i$ for each i, we have $y_i \perp y_j$ for $i \neq j$. Since $y_i \leq c$ for each i, it follows from the previous lemma that $z = y_1 + \cdots + y_m \leq c$. We also have $z \leq a_1 + \cdots + a_m$, so $z \leq c \wedge (a_1 + \cdots + a_m)$. It follows that $(c \wedge a_1) + \cdots + (c \wedge a_m) \leq c \wedge (a_1 + \cdots + a_m)$. □

LEMMA 4.3. *Let G be a lattice ordered dimension group, and let c, $a_1, \ldots, a_m$ be positive elements of G. Suppose $c \perp a_i$ for $1 \leq i \leq m$. Then $c \perp (a_1 + \cdots + a_m)$.*

PROOF. Let G, c, and $a_1, \ldots, a_m$ be as above. From the previous lemma we have $c \wedge (a_1 + \cdots + a_m) = c \wedge a_1 + \cdots + c \wedge a_m = 0$. □

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