C. R. Math. Rep. Acad. Sci. Canada Vol. **46** (3) 2024, pp. 79–104

Optimal Specification Testing for the Fixed Design Spatial Regression Model

Stéphane Bouka

Presented by George A. Elliott, FRSC

Abstract. In this paper, we address the problem of testing the specification of the regression function in a fixed N-dimensional setting when the errors are stationary isotropic mixing random fields. We propose a test statistic that takes into account the proximity between sites, establish its asymptotic normality and prove that it achieves the minimax rate.

Résumé. Dans cet article, nous abordons le problème du test de spécification de la fonction de régression dans un cadre de design fixe de dimension N, lorsque les erreurs sont des champs aléatoires mélangeant isotropiques stationnaires. Nous proposons une statistique de test qui tient compte de la proximité entre les sites, établissons sa normalité asymptotique et prouvons qu'elle atteint la vitesse minimax du test.

1. **Introduction** Spatially dependent data appear in several disciplines such as econometrics, geology, hydrology, epidemiology, environmental sciences, neuroimaging and genomics. In these fields, non-parametric regression models are essential tools for statistical analysis of data. However, it is well known that a poor choice of regression function leads to incorrect and inefficient results for the practical problem under consideration. It is therefore worthwhile using a specification test for this regression function. In this article, we focus on a minimax test for the multivariate discrete regression function for spatially dependent data. Since the seminal paper [9] on the minimax testing, asymptotic and non-asymptotic approaches have been developed mainly for independent data; see, e.g., [1], [7], [11,12], [14,15], [18–20] and [24] for minimax results and [2,3], [22,23] for non-asymptotic approaches. A review on specification tests for models with functional data is given in [13]. For instance, [14] considered a parametric family of regression functions in the null hypothesis, while [11, 12] considered a class of functions instead of a family. However, [11] proposed a test statistic that depends on the parameter of regularity β of the considered function class, whereas [12] proposed a more general test statistic which does not depend of β. The main approach used to evaluate the efficiency of test is based on sequences of local alternatives which ensures the optimality of the sequence ψ_n, which separates the hypotheses H_1 from H_0. In the fixed design case, [11] obtained that

Received by the editors on June 24, 2024; revised July 19, 2024.
AMS Subject Classification: Primary: 62G10, 62G07; secondary: 62M30 , 62C20.
Keywords: Optimal rate of testing, regression model, spatial data.
© Royal Society of Canada 2024.

$\psi_n = n^{-2\beta/(4\beta+1)}$ (with $\beta > 1/4$), and in the non-spatial N-dimensional random design case, [14], [7] and [1] obtained that $\psi_n = n^{-2\beta/(4\beta+N)}$ (with $\beta > N/4$), where n is the sample size. In this paper, we consider the following fixed design spatial regression model where the spatial scalar observations $(Y_{\mathbf{i}}, \mathbf{i} \in \mathcal{I}_{\mathbf{n}})$ are related to the discrete spatial observations $(x_{\mathbf{i}}, \mathbf{i} \in \mathcal{I}_{\mathbf{n}})$ through

$$Y_{\mathbf{i}} = r(x_{\mathbf{i}}) + \epsilon_{\mathbf{i}}, \quad \mathbf{i} \in \mathcal{I}_{\mathbf{n}}, \tag{1.1}$$

where $\mathcal{I}_{\mathbf{n}}$ is the set of sampling sites. Besides, $(\epsilon_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^N}$ is a strictly stationary isotropic zero-mean random field such that $\mathrm{Var}(\epsilon_{\mathbf{i}}) = \sigma^2$ and $\mathrm{Cov}(\epsilon_{\mathbf{i}}, \epsilon_{\mathbf{j}}) = \gamma_0(\|\mathbf{i}-\mathbf{j}\|) \neq 0$ for some $\mathbf{i} \neq \mathbf{j}$, where γ_0 is a known function; it's the main difference between this paper and existing works in the literature in which it is assumed that $\mathrm{Cov}(\epsilon_{\mathbf{i}}, \epsilon_{\mathbf{j}}) = 0$, for all $\mathbf{i} \neq \mathbf{j}$.

We are interested, in the asymptotically minimax framework, with testing for the hypothesis

$$H_0 : r = r_0 \in \sum(\beta, L, M) \quad \text{versus} \quad H_1(\psi_{\mathbf{n}}) : r \in \Lambda_{\mathbf{n}}(\psi_{\mathbf{n}}), \tag{1.2}$$

where r_0 is a given function in $\sum(\beta, L, M)$ and

$$\Lambda_{\mathbf{n}}(\psi_{\mathbf{n}}) = \left\{ r \in \sum(\beta, L, M) : \|r - r_0\|_2 \geq \psi_{\mathbf{n}} \right\},$$

where $\|.\|_2$ denotes the L_2-norm and $\psi_{\mathbf{n}} \searrow 0$ as $\mathbf{n} \to +\infty$ ($\min_{k=1,\cdots,N}\{n_k\} \to +\infty$). More precisely, we extend the work of [11] to the spatial case. Such a test (1.2) turns out to be useful before using a lagged curve model in the context of curve registration ([7]). Indeed, if there exists $\theta \in [0,1]^N$ such that $r(x) = r_0(x - \theta)$ for all $x \in [0,1]^N$ and the function r is one-periodic, then necessarily $r = r_0$. Thus, rejection of the null hypothesis implies the inadequacy of the shifted curve model. In the context given by the model (1.1), consider the following situation: a true image r is affected by a correlated additive noise $\epsilon_{\mathbf{i}}$, which gives $Y_{\mathbf{i}}$ for the observed image (see [8]). If the goal is to detect the true image r, we can consider the test problem (1.2) with $r_0 = 0$. This paper is organized as follow. Notation and the minimax principle are presented in Section 2 whereas the required optimal test is given in Section 3; assumptions and the main results are in Section 4. Section 5 is devoted to some concluding remarks whereas the proofs of the main results are presented in Section 6.

2. Notation and Minimax Principle

We define that $\mathcal{I}_{\mathbf{n}} = \{1, \cdots, n_1\} \times \cdots \times \{1, \cdots, n_N\}$, $\mathbf{n} = (n_1, ..., n_N)^T \in (\mathbb{N}^*)^N$ (where u^T denotes the transposed of a vector u), $x_{\mathbf{i}} = \dfrac{\mathbf{i}}{\mathbf{n}} = \left(\dfrac{i_1}{n_1}, \cdots, \dfrac{i_N}{n_N}\right)^T \in [0,1]^N$ (see [5,8]), $N \in \{1,2,3\}$, r is a function from $[0,1]^N$ to $\mathbb{R}$ belonging to the class

$$\sum(\beta, L, M) = \begin{cases} \{r \in H(\beta, L) : \|r\|_\infty \leq M\} & \text{if } \dfrac{3N-1}{8} \leq \beta \leq 1 \\ \{r \in H(\beta, L) : \|r\|_\infty \leq M, \|\nabla r\|_\infty \leq M\} & \text{if } \beta > 1 \end{cases},$$

where $\beta > 0$, $L > 0$, $M > 0$, ∇r stands for the gradient vector of r, $H(\beta, L)$ is the Hölder class of functions f satisfying $|D^\lambda f(x) - D^\lambda f(y)| \leq L \|x - y\|^{\beta - s}$, $\forall x, y \in [0,1]^N$, $s = \lfloor \beta \rfloor$ stands for the integer part of β, $\forall \lambda = (\lambda_1, \cdots, \lambda_N)^T$ such that $|\lambda| = \sum_{i=1}^N \lambda_i \leq s$, $D^\lambda f(x) = \dfrac{\partial^{|\lambda|} f}{\partial^{\lambda_1} x_1 \cdots \partial^{\lambda_N} x_N}(x)$ and $\|.\|$ stands for the euclidean norm. For $N = 1$, we consider $\beta > \dfrac{1}{4}$ and $\sum(\beta, L, M)$ becomes the class defined in [11].

The minimax principle consists in determining the optimal distance $\psi_{\mathbf{n}}$ between the null hypothesis H_0 and the set of alternatives such that a successful testing is possible. Let us describe more precisely the sense of the optimality point of view that we adopt. Let $\overline{\Delta}_{\mathbf{n}}$ denote a test statistic i.e. an arbitrary function with values 0, 1 which is measurable with respect to the observations $Y_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{I}_{\mathbf{n}}$. We accept H_0 if $\overline{\Delta}_{\mathbf{n}} = 0$ and it's rejected if $\overline{\Delta}_{\mathbf{n}} = 1$. We study the properties of such tests $\overline{\Delta}_{\mathbf{n}}$ in considering the asymptotic behaviour of both errors $R_0(\overline{\Delta}_{\mathbf{n}})$ and $R_1(\overline{\Delta}_{\mathbf{n}}, \psi_{\mathbf{n}})$ defined as follows:

$$R_0(\overline{\Delta}_{\mathbf{n}}) \quad = \quad \mathbb{P}_{r_0}\{\overline{\Delta}_{\mathbf{n}} = 1\}, \quad R_1(\overline{\Delta}_{\mathbf{n}}, \psi_{\mathbf{n}}) = \sup_{r \in \Lambda_{\mathbf{n}}(\psi_{\mathbf{n}})} \mathbb{P}_r\{\overline{\Delta}_{\mathbf{n}} = 0\}.$$

The index r means that the measure P_r is generated by the observations $Y_{\mathbf{i}}$ when their means are $r(x_{\mathbf{i}})$. The sequence of tests $\overline{\Delta}_{\mathbf{n}}$ is of a given asymptotic level $\alpha_1 \in (0,1)$ if

$$\limsup_{\mathbf{n} \to \infty} R_0(\overline{\Delta}_{\mathbf{n}}) \leq \alpha_1. \tag{2.1}$$

The purpose of this paper is to determine the minimal sequence $\psi_{\mathbf{n}}$ which separates H_1 from H_0 and to define a sequence of tests $\overline{\Delta}_{\mathbf{n}}$ of asymptotic level α_1 for which one can find, for any $\alpha_2 \in (0,1)$, a constant $A > 0$ (large enough) such that

$$\limsup_{\mathbf{n} \to \infty} R_1(\overline{\Delta}_{\mathbf{n}}, A\widehat{\mathbf{n}}^{-2\beta/(4\beta+N)}) \leq \alpha_2, \tag{2.2}$$

where $\widehat{\mathbf{n}} = n_1 \times \cdots \times n_N$. The lower bound for the sequence $\psi_{\mathbf{n}}$, which ensures the minimality of $\psi_{\mathbf{n}}$, is implied by proving that for any $\alpha_2 \in [0,1]$, there exists a positive constant a (small enough) such that

$$\liminf_{\mathbf{n} \to \infty} \inf_{\overline{\Delta}_{\mathbf{n}}} R_1(\overline{\Delta}_{\mathbf{n}}, a\widehat{\mathbf{n}}^{-2\beta/(4\beta+N)}) \geq \alpha_2, \tag{2.3}$$

where the infimum is taken over all tests $\overline{\Delta}_{\mathbf{n}}$ of asymptotic level α_1. The inequalities (2.2) and (2.3) imply that $\psi_{\mathbf{n}} = \widehat{\mathbf{n}}^{-2\beta/(4\beta+N)}$ is the minimax rate of testing (borrowing the terminology in [19]) in the problem (1.2).

3. The Optimal Test Procedure We first make a partition of the interval $[0,1]$ into m (with $m := m(\mathbf{n}) \in \mathbb{N}^*$, $m < \min_{i=1,\cdots,N}\{n_i\}$ and $\lim_{\mathbf{n}\to+\infty} m = +\infty$) subintervals $B_1, ..., B_m$, of length $\dfrac{1}{m}$ by taking:

$$B_k = \left[\frac{k-1}{m}, \frac{k}{m}\right[\quad \text{for} \quad k = 1, ..., m-1, \text{ and } \quad B_m = \left[\frac{m-1}{m}, 1\right].$$

We put, for any $\mathbf{k} = (k_1, \cdots, k_N)^T \in \mathcal{K} = \{1, \cdots, m\}^N$, $A_{\mathbf{k}} = B_{k_1} \times \cdots \times B_{k_N}$, $\mathcal{J}_{\mathbf{k}} = \{\mathbf{i} \in \mathcal{I}_{\mathbf{n}} : x_{\mathbf{i}} \in A_{\mathbf{k}}\}$, and $h_{\mathbf{ij}} := ||\mathbf{i} - \mathbf{j}|| = \left(\sum_{l=1}^N (i_l - j_l)^2\right)^{1/2}$ for some $(\mathbf{i}, \mathbf{j}) \in (\mathcal{J}_{\mathbf{k}})^2$. Then, we introduce the test statistic $T_{\mathbf{n}}$ defined as

$$T_{\mathbf{n}} = \frac{1}{m^N} \sum_{\mathbf{k}\in\mathcal{K}} \frac{m^{2N}}{\widehat{\mathbf{n}}^2 \widehat{\sigma}^2 \sqrt{1 + V_{\mathbf{k}}}} \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathcal{J}_{\mathbf{k}} \\ \mathbf{i}\neq\mathbf{j}}} \left[(Y_{\mathbf{i}} - r_0(x_{\mathbf{i}}))(Y_{\mathbf{j}} - r_0(x_{\mathbf{j}})) - \widehat{\gamma}_0(h_{\mathbf{ij}})\right],$$

where

$$V_{\mathbf{k}} = \frac{m^{2N}}{\widehat{\mathbf{n}}^2} \sum_{\substack{\mathbf{i},\mathbf{j},\mathbf{q},\boldsymbol{\ell}\in\mathcal{J}_{\mathbf{k}} \\ \mathbf{i}\neq\mathbf{j}\neq\mathbf{q}\neq\boldsymbol{\ell}}} \left[\varphi(h_{\mathbf{ij}})\varphi(h_{\mathbf{q}\boldsymbol{\ell}}) + \varphi(h_{\mathbf{i}\boldsymbol{\ell}})\varphi(h_{\mathbf{qj}})\right] = O(1),$$

$\widehat{\sigma}^2$ and $\widehat{\gamma}_0$ are the estimators of σ^2 and γ_0 (defined in Assumption 1) given by:

$$\widehat{\sigma}^2 = \frac{1}{2\left((n_1 - p)\prod_{k=2}^N n_k\right)} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}^p} (Y_{\mathbf{i}} - Y_{\mathbf{i}-pe_1})^2, \quad \widehat{\gamma}_0(h_{\mathbf{ij}}) = \widehat{\sigma}^2\varphi(h_{\mathbf{ij}}),$$

where $\mathcal{I}_{\mathbf{n}}^p = \{\mathbf{i} = (i_1, ..., i_N)^T \in (\mathbb{N}^*)^N : p + 1 \leq i_1 \leq n_1, 1 \leq i_k \leq n_k, k = 2, ..., N\}$ with $p = \left\lfloor (\log \widehat{\mathbf{n}})^{1/N} \right\rfloor$, $e_1 = (1, 0, \cdots, 0)^T \in \mathbb{N}^N$. Under Assumptions 1–3, it is shown in [5, Theorem 2] that for all $\tau > 0$,

$$\sup_{r\in\sum(\beta,L,M)} \mathbb{P}_r(|\widehat{\sigma}^2 - \sigma^2| > \tau) = O\left(\max\left(\frac{\log \widehat{\mathbf{n}}}{\widehat{\mathbf{n}}^{4/N}}, \frac{\log \widehat{\mathbf{n}}}{\widehat{\mathbf{n}}}\right)\right). \tag{3.1}$$

Then, our test is

$$\overline{\Delta}_{\mathbf{n}} = \mathbf{1}_{\{T_{\mathbf{n}} > c_0\rho_{\mathbf{n}}\}} \tag{3.2}$$

where $c_0 = \Phi^{-1}(1 - \alpha_1)$ is the $(1 - \alpha_1)$-quantile of the standard normal distribution $\mathcal{N}(0, 1)$ and $\rho_{\mathbf{n}} = \sqrt{\dfrac{m^N}{\widehat{\mathbf{n}}^2}}$.

4. Assumptions and Main Results

4.1. Assumptions Assume the following:

Assumption 1. For $(\mathbf{i}_1, \cdots, \mathbf{i}_{m^N})^T \in \mathcal{J}_{\mathbf{k}_1} \times \cdots \times \mathcal{J}_{\mathbf{k}_{m^N}}$, $(\epsilon_{\mathbf{i}_1}, \cdots, \epsilon_{\mathbf{i}_{m^N}})^T$ is a random vector such that:

$$\gamma_0(\|\mathbf{i}-\boldsymbol{\ell}\|) = \begin{cases} \sigma^2\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|) & \text{if} \quad (\mathbf{i},\boldsymbol{\ell}) \in \mathcal{J}_{\mathbf{k}}^2 \ , \\ 0 \quad \text{if} \quad (\mathbf{i},\boldsymbol{\ell}) \in \mathcal{J}_{\mathbf{k}} \times \mathcal{J}_{\mathbf{s}} & \text{with} \quad \mathbf{k} \neq \mathbf{s}, \end{cases} \tag{4.1}$$

where $\varphi(t) = \dfrac{1}{2^{\mu-1}\Gamma(\mu)}\left(\dfrac{t}{b}\right)^{\mu} K_\mu\left(\dfrac{t}{b}\right)$ with $b>0$, Γ being the Gamma function and K_μ the modified Bessel function of the third kind of order $\mu \in \{1/2, 3/2, 5/2\}$ or $\mu \to +\infty$ with $b \to 0$ in such a way that $2\mu^{1/2}b$ remains constant; and for all $(\mathbf{i},\ \mathbf{j},\ \mathbf{q},\ \boldsymbol{\ell}) \in (\mathcal{J}_{\mathbf{k}})^4$ such that $\mathbf{i} \neq \mathbf{j} \neq \mathbf{q} \neq \boldsymbol{\ell}$,

$$\mathrm{Cov}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}}, \epsilon_{\mathbf{q}}\epsilon_{\boldsymbol{\ell}}) = \mathrm{Cov}(\epsilon_{\mathbf{i}}, \epsilon_{\mathbf{q}})\mathrm{Cov}(\epsilon_{\mathbf{j}}, \epsilon_{\boldsymbol{\ell}}) + \mathrm{Cov}(\epsilon_{\mathbf{i}}, \epsilon_{\boldsymbol{\ell}})\mathrm{Cov}(\epsilon_{\mathbf{j}}, \epsilon_{\mathbf{q}}). \tag{4.2}$$

Assumption 2. $\sup_{\mathbf{u}\in\mathcal{J}_{\mathbf{n}}}|\epsilon_{\mathbf{u}}| \leq M_1 < +\infty$.

Assumption 3. The field $(\epsilon_{\mathbf{i}})_{\mathbf{i}\in\mathcal{J}_{\mathbf{k}}}$ satisfies the mixing condition defined by

$$\alpha(\mathcal{B}(E_1), \mathcal{B}(E_2)) = \sup_{B\in\mathcal{B}(E_1), C\in\mathcal{B}(E_2)} |\mathbb{P}(B\cap C) - \mathbb{P}(B)\mathbb{P}(C)|, \tag{4.3}$$

where $\mathcal{B}(E_1)$ (resp. $\mathcal{B}(E_2)$) stands for the Borel σ-field generated by $\{\epsilon_{\mathbf{i}}, \mathbf{i} \in E_1\}$ (resp. $\{\epsilon_{\mathbf{i}}, \mathbf{i} \in E_2\}$), and E_1 and E_2 are two subsets of $\mathcal{J}_{\mathbf{k}}$. Moreover, there exists a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\phi(t) \searrow 0$ as $t \to +\infty$, such that for all $E_1, E_2 \subset \mathcal{J}_{\mathbf{k}}$ with finite cardinals:

$$\alpha(\mathcal{B}(E_1), \mathcal{B}(E_2)) \leq \widehat{f}(card(E_1), card(E_2))\phi(dist(E_1, E_2)), \tag{4.4}$$

where $\widehat{f} : \mathbb{N}^2 \longrightarrow \mathbb{R}^+$ is a symmetric positive non-decreasing function, $dist(E_1, E_2)$ stands for the Euclidean distance between E_1 and E_2, $Card(E)$ denotes the cardinality of E, $\widehat{f}$ is such that

$$\widehat{f}(i,j) \leq c\min(i,j), \quad i,j \in \mathbb{N}, \tag{4.5}$$

for some constant $c > 0$, and ϕ satisfies a polynomial mixing condition:

$$\phi(t) = O(t^{-\xi}), \quad t \in \mathbb{R}, \ \text{and} \ \xi > 4(4\beta + N). \tag{4.6}$$

Assumption 4. for $\mathbf{i}_i \in \mathcal{J}_{\mathbf{k}}$, $(\epsilon_{\mathbf{i}_1}, \cdots, \epsilon_{\mathbf{i}_{\hat{\mathbf{n}}/m^N}}) \sim \mathcal{N}(\mathbf{0}, \widetilde{\Lambda})$, where $\widetilde{\Lambda} = (\widetilde{\Lambda}_{ij})_{ij}$ with $\widetilde{\Lambda}_{ij} = \sigma^2\varphi(\|\mathbf{i}_i - \mathbf{i}_j\|)$, $\mathbf{i}_i$, $\mathbf{i}_j \in \mathcal{J}_{\mathbf{k}}$.

Remark 1. The mixing condition given in (4.3) is defined as in [25]. Condition (4.5), weaker than strong mixing condition (when $\widehat{f} \neq 1$), has been used for finite dimensional variables in [4]. It is satisfied by many stochastic processes, such as the time series. Condition (4.2) in Assumption 1 is also assumed in [10, Result 3, p. 287] and is satisfied when $(\epsilon_{\mathbf{i}}, \epsilon_{\mathbf{j}}, \epsilon_{\mathbf{q}}, \epsilon_{\boldsymbol{\ell}})$ is a Gaussian random vector with covariance defined as in (4.1). Also, if $\mathbf{i} = \mathbf{j}$ and $\mathbf{q} = \boldsymbol{\ell}$, condition (4.2) gives $\mathrm{Cov}(\epsilon_{\mathbf{i}}^2, \epsilon_{\boldsymbol{\ell}}^2) = 2\sigma^4 \varphi(\|\mathbf{i} - \boldsymbol{\ell}\|)^2$, for $\mathbf{i} \neq \boldsymbol{\ell}$, and this result is well-known when $(\epsilon_{\mathbf{i}}, \epsilon_{\boldsymbol{\ell}})$ is a Gaussian random vector [16, Lemma 4, p. 1552]. The function φ is an exponentially decreasing function such that

$$\varphi(t) = \begin{cases} \exp\left(-\dfrac{t}{b}\right) & \text{if } \ \mu = 1/2 \\ \left(1 + \dfrac{t}{b}\right) \exp\left(-\dfrac{t}{b}\right) & \text{if } \ \mu = 3/2 \\ \left(1 + \dfrac{t}{b} + \dfrac{1}{3}\left(\dfrac{t}{b}\right)^2\right) \exp\left(-\dfrac{t}{b}\right) & \text{if } \ \mu = 5/2 \\ \exp\left(-\left(\dfrac{t}{2b\sqrt{\mu}}\right)^2\right) & \text{if } 2b\sqrt{\mu} \to \text{constant as } \mu \to +\infty \text{ and } b \to 0 \end{cases}.$$

For verifying all these assumptions, it is sufficient to consider bounded stationary Gaussian random subsamples with a sufficiently large polynomial decay (4.6) of correlation and covariance model belonging to the Matern's spatial covariance model class. Besides, as pointed out in [16, p. 1540], the α-mixing condition is suitable if one needs more delicate results, such as for instance a central limit theorem. In this paper, it is used to establish the asymptotic normality of our test statistic under our null hypothesis H_0 and for establishing a large deviation inequality needed to obtain the upper bound of the second-type error under the alternative hypothesis $H_1(A\psi_{\mathbf{n}})$. Assumption 4 is a classical assumption in the regression case. It is needed to prove the lower bound of the second-type error under the alternative $H_1(a\psi_{\mathbf{n}})$. However Assumption 4 on the Gaussian distribution may be generalized to elliptic processes by setting $\epsilon_{\mathbf{i}} \stackrel{D}{=} W_{\mathbf{i}} \epsilon_{\mathbf{i}}^\star$ where $W_{\mathbf{i}}$ is a strictly positive real random variable and $\epsilon_{\mathbf{i}}^\star$ is a Gaussian random field with mean zero and covariance function $\gamma_0^\star(||\mathbf{i} - \mathbf{j}||) = \mathrm{Cov}(\epsilon_{\mathbf{i}}^\star, \epsilon_{\mathbf{j}}^\star)$, $W_{\mathbf{i}}$ and $\epsilon_{\mathbf{i}}^\star$ being independent of each other (see [26]). Assumption 2 is used to simplify the proofs and may be relaxed by assuming that

$$\sup_{\mathbf{u} \in \mathcal{I}_{\mathbf{n}}} |\epsilon_{\mathbf{u}}| \leq D(\mathbf{n}),$$

where $D(\mathbf{n}) = \log(\widehat{\mathbf{n}})$ with $\mathbf{n}$ large enough.

4.2. Results The following theorem gives the asymptotic normality of the test statistic under the null hypothesis.

Theorem 4.1. *Under Assumptions* 1–3*, with* $m = \widehat{\mathbf{n}}^{2/(4\beta+N)}$*,* $\beta \geq (3N-1)/8$ *and* $N \in \{1, 2, 3\}$*, we have, under* H_0*,*

$$\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}}\left(\widehat{\sigma}^2 T_{\mathbf{n}}\right) \xrightarrow{D} \mathcal{N}(0, \sigma^4),$$

where $\xrightarrow{D}$ *stands for the convergence in distribution.*

The following theorem shows that our test has an asymptotic level equals to α_1, and gives a upper bound for the second-type error under the alternative.

Theorem 4.2. *Assume that Assumptions* 1–3 *are satisfied. Then, under* H_0*, Relation* (2.1) *holds. In addition, for any* $\alpha_2 \in (0, 1)$*, there exists a constant* $A > 0$ *such that, for* $\psi_{\mathbf{n}}^2 = \rho_{\mathbf{n}} = \dfrac{m^{N/2}}{\widehat{\mathbf{n}}}$*,* $m = \widehat{\mathbf{n}}^{2/(4\beta+N)}$*,* $\beta \geq (3N-1)/8$ *and* $N \in \{1, 2, 3\}$*, Relation* (2.2) *holds.*

The following result shows that $\psi_{\mathbf{n}} = \widehat{\mathbf{n}}^{-2\beta/(4\beta+N)}$ is the minimax rate of testing.

Theorem 4.3. *Suppose that Assumptions* 1–4 *hold and* r_0 *is such that* $r_0 \in \sum(\beta, L^{'}, M^{'})$*, where* $L^{'} < L$ *and* $M^{'} < M$*. Then there exists* $a > 0$ *such that Relation* (2.3) *holds.*

5**. Concluding Remarks** In this work, we are interested by the problem of testing for specification of the regression function in a N-dimensional fixed design setting when the errors are stationary mixing isotropic random fields. The originality of the proposed method comes from the fact that these errors are spatially correlated. In this general setting, the difficulty encountered is to establish the lower bound of the second-type error which ensures the optimality of the sequence $\psi_{\mathbf{n}}$ which separates the hypotheses H_1 from H_0. For handling this difficulty, we assume that the sub-sampled residuals distribution is Gaussian (see Assumption 4). So, we obtain that $\psi_{\mathbf{n}} = \widehat{\mathbf{n}}^{-2\beta/(4\beta+N)}$ is the minimax rate of our testing in the context of multivariate spatial data, where the condition $\beta \geq \dfrac{3N-1}{8}$ is needed to establish the asymptotic normality, under the null hypothesis H_0, of the proposed test statistic and $N \in \{1, 2, 3\}$ is the encountered temporal or spatial dimension in practical application. Besides, we distinguish two cases when the data are independent:

- $N = 1$ (see [11]);
- $N \geq 1$ (see [14] and [1]).

In the first case with a fixed design, [11] obtained that $\psi_n = n^{-2\beta/(4\beta+1)}$ (with $\beta > 1/4$); and in the second case with a non-spatial N-dimensional random design, [14] and [1] obtained that $\psi_n = n^{-2\beta/(4\beta+N)}$ (with $\beta > N/4$). So, the proposed methodology can be seen as an alternative to [11], [14] and [1] when the available data are spatially dependent. The anisotropic case can be realized assuming what follows: $\epsilon_{\boldsymbol{\ell}}$ is constructed from a shared dependence on a Gaussian random field $X(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in A_{\mathbf{k}}$ defined as follows: $\epsilon_{\boldsymbol{\ell}} = \int_{A_{\mathbf{k}}} K\left(\frac{\boldsymbol{\ell}}{\mathbf{n}} - \boldsymbol{\lambda}\right) X(\boldsymbol{\lambda}) d\boldsymbol{\lambda}$ and $\epsilon_{\boldsymbol{\mu}} = \int_{A_{\mathbf{s}}} K\left(\frac{\boldsymbol{\mu}}{\mathbf{n}} - \boldsymbol{\lambda}\right) X(\boldsymbol{\lambda}) d\boldsymbol{\lambda}$, where K is some smoothing kernel; so $\epsilon_{\boldsymbol{\ell}}$ and $\epsilon_{\boldsymbol{\mu}}$ are dependent when $(\boldsymbol{\ell}, \boldsymbol{\mu}) \in \mathcal{J}_{\mathbf{k}} \times \mathcal{J}_{\mathbf{s}}$ with $\mathbf{k} = \mathbf{s}$, but independent when $\mathbf{k} \neq \mathbf{s}$. A good choice of K might permit one to resolve this problem. However, some other questions related to this problem of testing remain open for spatially dependent observations; this is for example true in the adaptive case as in [12].

6. Proofs of main results

6.1. Preliminary Lemmas In this section, letters c and c_i, $i = 1, 2, \cdots$ will be used to denote constants whose values are unimportant. We need the following results for proving Theorems 4.1–4.3.

Set

$$K_{\mathbf{n}} = \frac{1}{m^N} \sum_{\mathbf{k} \in \mathcal{K}} \widetilde{S}_{\mathbf{k},\mathbf{n}} \quad \text{and} \quad \widetilde{S}_{\mathbf{k},\mathbf{n}} = \sum_{\mathbf{i} \in \mathcal{J}_{\mathbf{k}}} \Gamma_{\mathbf{i},\mathbf{k}},$$

$$\text{with} \quad \Gamma_{\mathbf{i},\mathbf{k}} = \frac{m^{2N}}{\widehat{\mathbf{n}}^2 \sqrt{1 + V_{\mathbf{k}}}} \sum_{\substack{\mathbf{j} \in \mathcal{J}_{\mathbf{k}} \\ \mathbf{i} \neq \mathbf{j}}} \left[\epsilon_{\mathbf{i}} \epsilon_{\mathbf{j}} - \gamma_0(h_{\mathbf{ij}})\right].$$

We will use the spatial block decomposition used by [27]. Let us fix two integers $p_1, q_1 \geq 1$ that are smaller than $\frac{n_k}{m}$, $k = 1, \cdots, N$. Let us make v_k packets of $p_1 + q_1$ in $\dfrac{n_k}{m}$ (where v_k is an integer) such that $\dfrac{n_k}{m} = v_k(p_1 + q_1)$. The random variables $\Gamma_{\mathbf{i},\mathbf{k}}$ are now set into large blocks and small blocks for $\mathbf{l} = (l_1, \cdots, l_N) \in \mathcal{T} = \{0, \cdots, v_1 - 1\} \times \cdots \times \{0, \cdots, v_N - 1\}$. Let

$$\begin{aligned}
U(1, \mathbf{n}, \mathbf{l}, \mathbf{k}) &= \sum_{\substack{i_k = l_k(p_1+q_1)+1 \\ k=1,\ldots,N}}^{l_k(p_1+q_1)+p_1} \Gamma_{\mathbf{i},\mathbf{k}} \\
U(2, \mathbf{n}, \mathbf{l}, \mathbf{k}) &= \sum_{\substack{i_k = l_k(p_1+q_1)+1 \\ k=1,\ldots,N-1}}^{l_k(p_1+q_1)+p_1} \sum_{i_N = l_N(p_1+q_1)+p_1+1}^{(l_N+1)(p_1+q_1)} \Gamma_{\mathbf{i},\mathbf{k}} \\
U(3, \mathbf{n}, \mathbf{l}, \mathbf{k}) &= \sum_{\substack{i_k = l_k(p_1+q_1)+1 \\ k=1,\ldots,N-2}}^{l_k(p_1+q_1)+p_1} \sum_{i_{N-1} = l_{N-1}(p_1+q_1)+p_1+1}^{(l_{N-1}+1)(p_1+q_1)} \sum_{i_N = l_N(p_1+q_1)+1}^{l_N(p_1+q_1)+p_1} \Gamma_{\mathbf{i},\mathbf{k}} \\
U(4, \mathbf{n}, \mathbf{l}, \mathbf{k}) &= \sum_{\substack{i_k = l_k(p_1+q_1)+1 \\ k=1,\ldots,N-2}}^{l_k(p_1+q_1)+p_1} \sum_{i_{N-1} = l_{N-1}(p_1+q_1)+p_1+1}^{(l_{N-1}+1)(p_1+q_1)} \sum_{i_N = l_N(p_1+q_1)+1}^{(l_N+1)(p_1+q_1)} \Gamma_{\mathbf{i},\mathbf{k}}
\end{aligned}$$

and so on. We have

$$U(2^{N-1}+1,\mathbf{n},\mathbf{l},\mathbf{k}) \quad = \quad \sum_{\substack{i_k=l_k(p_1+q_1)+p_1+1\\ k=1,\dots,N-1}}^{(l_k+1)(p_1+q_1)} \; \sum_{i_N=l_N(p_1+q_1)+1}^{l_N(p_1+q_1)+p_1} \Gamma_{\mathbf{i},\mathbf{k}}.$$

Then

$$U(2^{N},\mathbf{n},\mathbf{l},\mathbf{k}) \quad = \quad \sum_{\substack{i_k=l_k(p_1+q_1)+p_1+1\\ k=1,\dots,N}}^{(l_k+1)(p_1+q_1)} \Gamma_{\mathbf{i},\mathbf{k}}.$$

For each integer $i=1,\dots,2^N$, we take

$$T(\mathbf{k},\mathbf{n},i)=\sum_{\mathbf{j}\in\mathcal{T}} U(i,\mathbf{n},\mathbf{j},\mathbf{k}).$$

Then, we derive the decomposition $\widetilde{S}_{\mathbf{k},\mathbf{n}}=\sum_{i=1}^{2^N} T(\mathbf{k},\mathbf{n},i)$.

Proposition 6.1. *Under Assumption* 1, *we have:*

(i) $\dfrac{m^{2N}}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{J}_\mathbf{k}}\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{i}\neq\mathbf{j}}} Cov(\epsilon_\mathbf{i}^2,\epsilon_\mathbf{j}^2)=O\left(\dfrac{m^N}{\widehat{\mathbf{n}}}\right)$,

(ii) $\dfrac{m^{2N}}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{J}_\mathbf{k}}\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{i}\neq\mathbf{j}}} \left[Cov(\epsilon_\mathbf{i},\epsilon_\mathbf{j})\right]^2=O\left(\dfrac{m^N}{\widehat{\mathbf{n}}}\right)$.

Proof. (i) From Assumption 1, we have:

$$\begin{aligned}
\frac{m^{2N}}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{J}_\mathbf{k}}\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{i}\neq\mathbf{j}}}\operatorname{Cov}\left(\epsilon_\mathbf{i}^2,\epsilon_\mathbf{j}^2\right) &= \frac{2m^{2N}\sigma^2}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{J}_\mathbf{k}}\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{i}\neq\mathbf{j}}}\varphi^2\left(\|\mathbf{i}-\mathbf{j}\|\right)\\
&\le \frac{2m^{2N}\sigma^2}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{J}_\mathbf{k}}\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \|\mathbf{j}\|=t>0}}\varphi^2\left(\|\mathbf{j}\|\right)\\
&\le \frac{2m^{N}\sigma^2}{\widehat{\mathbf{n}}}\sum_{t=1}^{+\infty}t^{N-1}\varphi^2(t)\\
&\le \frac{2m^{N}\sigma^2}{\widehat{\mathbf{n}}}\sum_{t=1}^{+\infty}t^{N-1}\varphi(t)=O\left(\frac{m^N}{\widehat{\mathbf{n}}}\right).
\end{aligned}$$

Similarly, we obtain (ii). □

LEMMA 6.1. *Under the conditions of Theorem* 4.1, *we have*

$$\frac{\widehat{\mathbf{n}}^2}{m^N}\, Var(K_{\mathbf{n}}) = \frac{\widehat{\mathbf{n}}^2}{m^{2N}}\, Var(\widetilde{S}_{\mathbf{k_1},\mathbf{n}}) \longrightarrow \sigma^4.$$

PROOF. Since

$$\mathrm{Var}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}}) = [\mathrm{Cov}(\epsilon_{\mathbf{i}}, \epsilon_{\mathbf{j}})]^2 + \sigma^4 = \sigma^4\varphi^2(\|\mathbf{i}-\mathbf{j}\|) + \sigma^4,$$

from Assumption 1 and Proposition 6.1 (ii), we have

$$\begin{aligned}
\frac{\widehat{\mathbf{n}}^2}{m^N}\mathrm{Var}(K_{\mathbf{n}}) &= \frac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathrm{Var}(\widetilde{S}_{\mathbf{k_1},\mathbf{n}}) \\
&= \frac{m^{2N}}{\widehat{\mathbf{n}}^2(1+V_{\mathbf{k_1}})} \sum_{\substack{\mathbf{i},\mathbf{i'}\in\mathcal{J}_{\mathbf{k_1}} \\ \mathbf{i}\neq\mathbf{i'}}} \mathrm{Var}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{i'}}) \\
&\quad + \frac{m^{2N}}{\widehat{\mathbf{n}}^2(1+V_{\mathbf{k_1}})} \sum_{\substack{\mathbf{i},\mathbf{i'},\mathbf{j},\mathbf{j'}\in\mathcal{J}_{\mathbf{k_1}} \\ \mathbf{i}\neq\mathbf{i'}\neq\mathbf{j}\neq\mathbf{j'}}} \mathrm{Cov}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{i'}}, \epsilon_{\mathbf{j}}\epsilon_{\mathbf{j'}}) \\
&= \frac{m^{2N}\sigma^4}{\widehat{\mathbf{n}}^2(1+V_{\mathbf{k_1}})} \sum_{\substack{\mathbf{i},\mathbf{i'}\in\mathcal{J}_{\mathbf{k_1}} \\ \|\mathbf{i}-\mathbf{i'}\|>0}} \varphi^2(\|\mathbf{i}-\mathbf{i'}\|) + \sigma^4 - \frac{m^N\sigma^4}{\widehat{\mathbf{n}}(1+V_{\mathbf{k_1}})} \\
&\longrightarrow \sigma^4.
\end{aligned}$$

□

LEMMA 6.2. *Under Assumptions* 1–3, *there exist* p_1 *and* q_1 *such that we have:*
(i) For all $t \in \mathbb{R}$,
$\mathbb{E}\left[\exp\left(it\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}}T(\mathbf{k},\mathbf{n},1)\right)\right] - \prod_{\mathbf{l}\in\mathcal{I}}\mathbb{E}\left[\exp\left(it\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}}U(1,\mathbf{n},\mathbf{l},\mathbf{k})\right)\right] \to 0$, *with* $i^2 = -1$.
(ii) $\frac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathbb{E}\left(\sum_{i=2}^{2^N} T(\mathbf{k},\mathbf{n},i)\right)^2 \longrightarrow 0.$
(iii) $\frac{\widehat{\mathbf{n}}^2}{m^{2N}} \sum_{\mathbf{l}\in\mathcal{I}} \mathbb{E}\left[U(1,\mathbf{n},\mathbf{l},\mathbf{k})^2\right] \longrightarrow \sigma^4.$
(iv) $\frac{\widehat{\mathbf{n}}^2}{m^{3N}} \sum_{\mathbf{k}\in\mathcal{K}} \sum_{\mathbf{l}\in\mathcal{I}} \mathbb{E}\left[U(1,\mathbf{n},\mathbf{l},\mathbf{k})^2 \mathbf{1}_{\left\{|U(1,\mathbf{n},\mathbf{l},\mathbf{k})|>\tau\sqrt{\frac{m^{3N}}{\widehat{\mathbf{n}}^2}}\right\}}\right] \longrightarrow 0,\ \forall \tau > 0.$

PROOF. We use the decomposition into spatial blocks, where integers $p_1 = p_1(\mathbf{n})$ and $q_1 = q_1(\mathbf{n})$ are defined by $p_1 = \lfloor m^{1/4} \rfloor$ and $q_1 = o\left(m^{1/8}\right)$. We have then $q_1 p_1^{-1} \leq cm^{-1/8} \longrightarrow 0$. Thus $q_1 < p_1$ asymptotically.

Proof of (i). Let us enumerate the random variables $U(1,\mathbf{n},\mathbf{l},\mathbf{k})$, $\mathbf{l} \in \mathcal{T}$ and denote them as $\widetilde{U}_1,...,\widetilde{U}_{\widehat{v}}$. Note that $Card\ \mathcal{T} := \widehat{v} = \widehat{\mathbf{n}}(p_1+q_1)^{-N} \leq \widehat{\mathbf{n}}p_1^{-N}$. Let $I(1,\mathbf{n},\mathbf{l}) = \{\mathbf{i} \in \mathcal{J}_\mathbf{k} : l_\ell(p_1+q_1)+1 \leq i_\ell \leq l_\ell(p_1+q_1)+p_1\}$. Distinct sets of sites $I(1,\mathbf{n},\mathbf{l})$ are separated by a distance of at least q_1 and each set contains p_1^N sites. $I(1,\mathbf{n},\mathbf{l})$ is the set of sites involving $U(1,\mathbf{n},\mathbf{l},\mathbf{k})$. [27, Lemma 3.1] shows that:

$$
\begin{aligned}
Q &= \left|\mathbb{E}\left[\exp\left(it\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}}T(\mathbf{n},1)\right)\right] - \prod_{\mathbf{l}\in\mathcal{T}}\mathbb{E}\left[\exp\left(it\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}}U(1,\mathbf{n},\mathbf{l})\right)\right]\right| \\
&\leq \sum_{k=1}^{\widehat{v}-1}\sum_{j=k+1}^{\widehat{v}}\left|\mathbb{E}(\exp(it\widetilde{U}_k)-1)(\exp(it\widetilde{U}_j)-1)\prod_{s=j+1}^{\widehat{v}}\exp(it\widetilde{U}_s)\right. \\
&\qquad \left. -\mathbb{E}(\exp(it\widetilde{U}_k)-1)\mathbb{E}(\exp(it\widetilde{U}_j)-1)\prod_{s=j+1}^{\widehat{v}}\exp(it\widetilde{U}_s)\right|.
\end{aligned}
$$

Let $\widetilde{I}_j$ denote the set of sites involving $\widetilde{U}_j$. Under Assumption 3, an application of [27, Lemma 2.1] gives

$$
\begin{aligned}
Q_{kj} &= \left|\mathbb{E}\left[(\exp(it\widetilde{U}_k)-1)(\exp(it\widetilde{U}_j)-1)\prod_{s=j+1}^{\widehat{v}}\exp(it\widetilde{U}_s)\right]\right. \\
&\qquad \left. -\mathbb{E}(\exp(it\widetilde{U}_k)-1)\mathbb{E}\left[(\exp(it\widetilde{U}_j)-1)\prod_{s=j+1}^{\widehat{v}}\exp(it\widetilde{U}_s)\right]\right| \\
&\leq c\phi(dist(\widetilde{I}_j,\widetilde{I}_k))p_1^N.
\end{aligned}
$$

Since $\xi > 4(4\beta+N)$ and $m = \widehat{\mathbf{n}}^{\frac{2}{4\beta+N}}$ with $\beta \geq (3N-1)/8$, it follows that

$$
\begin{aligned}
Q &\leq cp_1^N\sum_{k=1}^{\widehat{v}-1}\sum_{j=k+1}^{\widehat{v}}\phi(dist(\widetilde{I}_j,\widetilde{I}_k)) \leq cp_1^N\widehat{v}\sum_{j=2}^{\widehat{v}}\phi(dist(\widetilde{I}_1,\widetilde{I}_k)) \\
&\leq cp_1^N\widehat{v}\sum_{i=1}^{\infty}\sum_{k:iq_1\leq dist(\widetilde{I}_1,\widetilde{I}_k)<(i+1)q_1}\phi(dist(\widetilde{I}_1,\widetilde{I}_k)) \\
&\leq c\widehat{\mathbf{n}}\sum_{i=1}^{\infty}i^{N-1}\phi(iq_1) \leq c_1\widehat{\mathbf{n}}^{-[\xi-4(4\beta+N)]/[4(4\beta+N)]} \longrightarrow 0.
\end{aligned}
$$

Proof of (ii). To prove (ii), it is enough to show that $\dfrac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathbb{E}[T(\mathbf{k},\mathbf{n},i)]^2 \longrightarrow 0$ for each $2 \leq i \leq 2^N$. Without loss of generality, consider $\mathbb{E}[T(\mathbf{k},\mathbf{n},2)]^2$. Enumerate the random variables $U(2,\mathbf{n},\mathbf{l},\mathbf{k})$, $\mathbf{l}\in\mathcal{T}$ and denote them by $\widehat{U}_1,...,\widehat{U}_{\widehat{v}}$. Now,

$$\mathbb{E}[T(\mathbf{k},\mathbf{n},2)]^2=\sum_{j=1}^{\widehat{v}}\mathrm{Var}(\widehat{U}_i)+2\sum_{i=1}^{\widehat{v}}\sum_{\substack{j=1\\ i>j}}^{\widehat{v}}\mathrm{Cov}(\widehat{U}_i,\widehat{U}_j)=A1+A2.$$

Since

$$\mathrm{Var}(\widehat{U}_i)=\mathbb{E}\left[\left(\sum_{\substack{i_k=1\\ k=1,\dots,N-1}}^{p_1}\sum_{i_N=1}^{q_1}\Gamma_{\mathbf{i},\mathbf{k}}\right)^2\right]-\left(\mathbb{E}\left(\sum_{\substack{i_k=1\\ k=1,\dots,N-1}}^{p_1}\sum_{i_N=1}^{q_1}\Gamma_{\mathbf{i},\mathbf{k}}\right)\right)^2,$$

it follows that

$$\begin{aligned}\mathrm{Var}(\widehat{U}_i)\quad\leq\quad&\sum_{\substack{i_k=1\\ k=1,\dots,N-1}}^{p_1}\sum_{i_N=1}^{q_1}\mathbb{E}(\Gamma^2_{\mathbf{i},\mathbf{k}})\\ &+\sum_{\substack{i_k=1\\ k=1,\dots,N-1}}^{p_1}\sum_{i_N=1}^{q_1}\sum_{\substack{l_k=1\\ k=1,\dots,N-1}}^{p_1}\sum_{\substack{l_N=1\\ i_k\neq l_k\ for\ 1\leq k\leq N}}^{q_1}\mathbb{E}(\Gamma_{\mathbf{i},\mathbf{k}}\Gamma_{\mathbf{l},\mathbf{k}}).\end{aligned}$$

However, from Assumption 1, we have $\mathbb{E}(\Gamma^2_{\mathbf{i},\mathbf{k}})=O\left(\frac{m^{3N}}{\widehat{\mathbf{n}}^3}\right)$, then

$$\sum_{\substack{i_k=1\\ k=1,\dots,N-1}}^{p_1}\sum_{i_N=1}^{q_1}\mathbb{E}(\Gamma^2_{\mathbf{i},\mathbf{k}})\leq c\frac{m^{3N}}{\widehat{\mathbf{n}}^3}p_1^{N-1}q_1.$$

Since, from Assumption 1 with $\mu=1/2$ (the reasoning remains the same for the others values of μ), we have $\mathrm{Cov}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}},\epsilon_{\mathbf{l}}\epsilon_{\mathbf{j}})\leq 2\sigma^4\varphi(\|\mathbf{i}-\mathbf{l}\|)$ and

$$\mathrm{Cov}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}},\epsilon_{\mathbf{l}}\epsilon_{\mathbf{j}'})=\sigma^4\varphi(\|\mathbf{i}-\mathbf{l}\|)\varphi(\|\mathbf{j}-\mathbf{j}'\|)+\sigma^4\varphi(\|\mathbf{i}-\mathbf{j}'\|)\varphi(\|\mathbf{j}-\mathbf{l}\|),$$

$$\begin{aligned}\mathbb{E}(\Gamma_{\mathbf{i},\mathbf{k}}\Gamma_{\mathbf{l},\mathbf{k}})\quad&=\quad\frac{m^{4N}}{\widehat{\mathbf{n}}^4(1+V_{\mathbf{k}})}\left[\sum_{\substack{\mathbf{j}\in\mathcal{I}_{\mathbf{k}}\\ \mathbf{i}\neq\mathbf{l}\neq\mathbf{j}}}\mathrm{Cov}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}},\epsilon_{\mathbf{l}}\epsilon_{\mathbf{j}})+\sum_{\substack{\mathbf{j},\mathbf{j}'\in\mathcal{I}_{\mathbf{k}}\\ \mathbf{i}\neq\mathbf{l}\neq\mathbf{j}\neq\mathbf{j}'}}\mathrm{Cov}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}},\epsilon_{\mathbf{l}}\epsilon_{\mathbf{j}'})\right]\\ &\leq\quad\frac{c_1m^{3N}\sigma^4}{\widehat{\mathbf{n}}^3}\varphi(\|\mathbf{i}-\mathbf{l}\|)+\frac{\sigma^4m^{4N}}{\widehat{\mathbf{n}}^4}\left(\sum_{\substack{\mathbf{j}\in\mathcal{I}_{\mathbf{k}}\\ \|\mathbf{j}\|=t>0}}\varphi(t)\right)^2\\ &=\quad O\left(\frac{m^{3N}}{\widehat{\mathbf{n}}^3}\varphi(\|\mathbf{i}-\mathbf{l}\|)\right)+O\left(\frac{m^{4N}}{\widehat{\mathbf{n}}^4}\right).\end{aligned}$$

If $\varphi(\|\mathbf{i}-\mathbf{l}\|) > \frac{m^N}{\widehat{\mathbf{n}}}$, then $\mathbb{E}(\Gamma_{\mathbf{i},\mathbf{k}}\Gamma_{\mathbf{l},\mathbf{k}}) = O\left(\frac{m^{3N}}{\widehat{\mathbf{n}}^3}\varphi(\|\mathbf{i}-\mathbf{l}\|)\right)$ and thus

$$\begin{aligned}
&\sum_{\substack{i_k=1\\ k=1,\ldots,N-1}}^{p_1}\sum_{i_N=1}^{q_1}\sum_{\substack{l_k=1\\ k=1,\ldots,N-1}}^{p_1}\sum_{\substack{l_N=1\\ i_k\neq l_k\ for\ 1\le k\le N}}^{q_1}|\mathbb{E}(\Gamma_{\mathbf{i},\mathbf{k}}\Gamma_{\mathbf{l},\mathbf{k}})|\\
&\le \frac{2c_1 m^{3N}p_1^{N-1}q_1}{\widehat{\mathbf{n}}^3}\sum_{t=1}^{\infty}t^{N-1}\varphi(t) \le \frac{c_2 m^{3N}p_1^{N-1}q_1}{\widehat{\mathbf{n}}^3}.
\end{aligned}$$

Therefore, for $\mathbf{n}$ large enough, we have

$$\begin{aligned}
\frac{\widehat{\mathbf{n}}^2}{m^{2N}}|A1| &\le \frac{\widehat{\mathbf{n}}^2}{m^{2N}}\left(c\frac{m^{3N}}{\widehat{\mathbf{n}}^3}p_1^{N-1}q_1\widehat{v} + c_2\frac{m^{3N}p_1^{N-1}q_1\widehat{v}}{\widehat{\mathbf{n}}^3}\right)\\
&\le \frac{c_3 m^N}{\widehat{\mathbf{n}}}p_1^{N-1}q_1\frac{\widehat{\mathbf{n}}}{m^N}(p_1+q_1)^{-N} \le c_4\left(\frac{q_1}{p_1}\right) \le \frac{c_5}{m^{1/8}} \longrightarrow 0.
\end{aligned}$$

If $0 \le \varphi(\|\mathbf{i}-\mathbf{l}\|) \le \frac{m^N}{\widehat{\mathbf{n}}}$, then $\mathbb{E}(\Gamma_{\mathbf{i},\mathbf{k}}\Gamma_{\mathbf{l},\mathbf{k}}) = O\left(\frac{m^{4N}}{\widehat{\mathbf{n}}^4}\right)$ and since $\beta \ge \frac{3N-1}{8}$ thus

$$\begin{aligned}
\frac{\widehat{\mathbf{n}}^2}{m^{2N}}|A1| &\le \frac{\widehat{\mathbf{n}}^2}{m^{2N}}\left(c\frac{m^{3N}}{\widehat{\mathbf{n}}^3}p_1^{N-1}q_1\widehat{v} + c_2\frac{m^{4N}p_1^{2(N-1)}q_1^2\widehat{v}}{\widehat{\mathbf{n}}^4}\right)\\
&\le \frac{c_3 m^N}{\widehat{\mathbf{n}}}p_1^{N-1}q_1\frac{\widehat{\mathbf{n}}}{m^N}(p_1+q_1)^{-N}\left[1+\frac{m^N p_1^{N-1}q_1}{\widehat{\mathbf{n}}}\right]\\
&\le c_4\left(\frac{q_1}{p_1}\right)\left[1+\frac{m^N p_1^{N-1}q_1}{\widehat{\mathbf{n}}}\right]\\
&\le \frac{c_5}{m^{1/8}} + c_5\widehat{\mathbf{n}}^{-(8\beta-3N+1)/[2(4\beta+N)]} \longrightarrow 0.
\end{aligned}$$

Let $I(2,\mathbf{n},\mathbf{l}) = \{\mathbf{j} : l_k(p_1+q_1+1) \le j_k \le l_k(p_1+q_1)+p_1, 1 \le k \le N-1,$ $l_N(p_1+q_1)+p_1+1 \le j_N \le (l_N+1)(p_1+q_1)\}$. Then $U(2,\mathbf{n},\mathbf{l},\mathbf{k})$ is the sum of $\Gamma_{\mathbf{i},\mathbf{k}}$ with sites in $I(2,\mathbf{n},\mathbf{l})$. Since $p_1 > q_1$, if $\mathbf{l}$ and $\mathbf{i}$ belong to two distinct sets $I(2,\mathbf{n},\mathbf{l})$ and $I(2,\mathbf{n},\mathbf{i})$, then $l_k \neq i_k$ for some $1 \le k \le N$ and $||\mathbf{l}-\mathbf{i}|| > q_1$. Since $\Gamma_{\mathbf{i},\mathbf{k}} = O\left(\frac{m^N}{\widehat{\mathbf{n}}}\right)$, then from Assumptions 2, 3 and [27, Lemma 2.1] with $b_1 = b_2 = 4$, $b_3 = 2$, we obtain

$$\begin{aligned}
\frac{\widehat{\mathbf{n}}^2}{m^{2N}}|A2| &\le c\frac{\widehat{\mathbf{n}}^2}{m^{2N}}\sum_{\substack{i_k=1\\ k=1,\ldots,N}}^{n_k/m}\sum_{\substack{l_k=1\\ k=1,\ldots,N\\ ||\mathbf{i}-\mathbf{l}||>q_1}}^{n_k/m}|cov(\Gamma_{\mathbf{i},\mathbf{k}},\Gamma_{\mathbf{l},\mathbf{k}})|\\
&\le c_1\sum_{t=q_1+1}^{\infty}t^{N-1}\{\phi(t)\}^{1/2} \longrightarrow 0. \qquad (6.1)
\end{aligned}$$

Thus $\dfrac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathbb{E}[T(\mathbf{k},\mathbf{n},2)]^2 \longrightarrow 0$.

Proof of (iii). We have:

$$\widetilde{S}_{\mathbf{k},\mathbf{n}} = \sum_{i=1}^{2^N} T(\mathbf{k},\mathbf{n},i) = T(\mathbf{k},\mathbf{n},1) + \sum_{i=2}^{2^N} T(\mathbf{k},\mathbf{n},i) = S_{\mathbf{n}}^{(1)} + S_{\mathbf{n}}^{(2)}\,;$$

then $S_{\mathbf{n}}^{(1)}$ is the sum of random variables $\Gamma_{\mathbf{i},\mathbf{k}}$ in large blocks and $S_{\mathbf{n}}^{(2)}$ is the sum of random variables in small blocks. Applying Lemma 6.1, we have $\dfrac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathbb{E}[\widetilde{S}^2_{\mathbf{k},\mathbf{n}}] \longrightarrow \sigma^4$. This combined with Lemma 6.2 (ii) $\left(i.e. \dfrac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathbb{E}\left[\left|S_{\mathbf{n}}^{(2)}\right|^2\right] \longrightarrow 0\right)$, implies that $\dfrac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathbb{E}\left[\left|S_{\mathbf{n}}^{(1)}\right|^2\right] \longrightarrow \sigma^4$. Now

$$\begin{aligned}\frac{\widehat{\mathbf{n}}^2}{m^{2N}}\mathbb{E}\left[\left|S_{\mathbf{n}}^{(1)}\right|^2\right] &= \frac{\widehat{\mathbf{n}}^2}{m^{2N}}\sum_{\mathbf{l}\in\mathcal{T}}\mathbb{E}[U(1,\mathbf{n},\mathbf{l},\mathbf{k})^2] \\ &+ \frac{\widehat{\mathbf{n}}^2}{m^{2N}}\sum_{\substack{\mathbf{l},\mathbf{j}\in\mathcal{T}\\ \mathbf{l}\neq\mathbf{j}}}\operatorname{Cov}[U(1,\mathbf{n},\mathbf{l},\mathbf{k}),U(1,\mathbf{n},\mathbf{j},\mathbf{k})].\end{aligned}$$

However, from Inequality (6.1), we obtain

$$\frac{\widehat{\mathbf{n}}^2}{m^{2N}}\sum_{\substack{\mathbf{l},\mathbf{j}\in\mathcal{T}\\ \mathbf{l}\neq\mathbf{j}}}\operatorname{Cov}[U(1,\mathbf{n},\mathbf{l},\mathbf{k}),U(1,\mathbf{n},\mathbf{j},\mathbf{k})] \quad\leq\quad c_1\sum_{t=q_1+1}^{\infty} t^{N-1}\left\{\phi(t)\right\}^{1/2} \longrightarrow 0.$$

Thus $\dfrac{\widehat{\mathbf{n}}^2}{m^{2N}}\sum_{\mathbf{l}\in\mathcal{T}}\mathbb{E}[U(1,\mathbf{n},\mathbf{l},\mathbf{k})^2] \longrightarrow \sigma^4$.

Proof of (iv). From Assumption 2, we have $|\Gamma_{\mathbf{i},\mathbf{k}}| \leq \dfrac{2M_1^2 m^N}{\widehat{\mathbf{n}}}$; since $p_1 = \left\lfloor m^{1/4}\right\rfloor$, it follows that:

$$\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{3N}}}|U(1,\mathbf{n},\mathbf{l},\mathbf{k})| \leq \frac{M_1^2 p_1^N}{m^{N/2}} \leq \frac{M_1^2}{m^{N/4}} \longrightarrow 0.$$

Thus for all $\mathbf{l}$, $\mathbf{k}$ and for $\mathbf{n}$ large enough, we have

$$\mathbb{P}\left(|U(1,\mathbf{n},\mathbf{l},\mathbf{k})| > \tau\sqrt{\frac{m^{3N}}{\widehat{\mathbf{n}}^2}}\right) = 0$$

and

$$\frac{\widehat{\mathbf{n}}^2}{m^{3N}} \sum_{\mathbf{k}\in\mathcal{K}} \sum_{\mathbf{l}\in\mathcal{J}} \mathbb{E}\left(U(1,\mathbf{n},\mathbf{l},\mathbf{k})^2 \mathbf{1}_{\left\{|U(1,\mathbf{n},\mathbf{l},\mathbf{k})|>\tau\sqrt{\frac{m^{3N}}{\widehat{\mathbf{n}}^2}}\right\}} \right) = 0.$$

□

We need the following results to prove Lemma 6.3. The following proposition adapts the inequality of [21, Proposition 2.16].

Proposition 6.2. *Denote by L_m the linear subspaces generated by the indicator functions $\phi_{\mathbf{k},m} = \sqrt{m^N}\mathbf{1}_{A_{\mathbf{k}}}$, where the $A_{\mathbf{k}}$ are disjoint $\frac{\widehat{\mathbf{n}}}{m^N}$ hypercubes of $[0,1]^N$ whose volume is $\frac{1}{m^N}$. Then,*

$$\|Pr_{L_m} r\|_2 \geq C_1\|r\|_2 - C_2 m^{-\beta N},$$

where $Pr_{L_m} r$ is the projection of r on L_m, and C_1 and C_2 are positive constant depending only on β and L.

Proof. The only term in our proposition which differs from that of Proposition 2.16 in [21] (with $p = 2$) is $C_2 m^{-\beta N}$. We only deal with the appearance of this term. Since $r \in H(\beta, L)$, with $\lfloor\beta\rfloor = s$ and $\forall x, y, \in [0,1]^N$, $\quad \forall\lambda$ s.t. $|\lambda| = s$, we have

$$|D^\lambda r(x) - D^\lambda r(y)| \leq L\|x-y\|^{\beta-s}. \tag{6.2}$$

Let us show that $\|r - P_{s,k}\|_2 \leq C_2 m^{-\beta N}$, where $P_{s,k}$ is the piecewise polynomial approximation of r with degree less than s; this latter coincides with the Taylor polynomial of degree s for r in $x \in A_{\mathbf{k}}$ around x_0, i.e.,

$$p_{s,k}(x) = r(x_0) + \sum_{\lambda\in\Lambda} \frac{1}{\lambda!} D^\lambda r(x_0)(x-x_0)^\lambda, \forall x \in A_{\mathbf{k}},$$

where $\lambda! = \lambda_1! \dots \lambda_N!$ and $(x-x_0)^\lambda = (x_1 - x_{0,1})^{\lambda_1} \dots (x_N - x_{0,N})^{\lambda_N}$.

Then, take the difference between r and $p_{s,k}$ (the Taylor Lagrange expansions with integral remainder) and apply (6.2) : $\forall x \in A_k$,

$$\begin{aligned}
|p_{s,k}(x) - r(x)| &= \Bigg| \sum_{\lambda:|\lambda|=s} \frac{s}{\lambda!}(x-x_0)^\lambda \int_0^1 (D^\lambda r(x_0 + t(x-x_0)) \\
&\quad -D^\lambda r(x_0))(1-t)^{s-1} dt \Big| \\
&\leq \sum_{\lambda:|\lambda|=s} \left| \frac{s}{\lambda!}(x-x_0)^\lambda \right| L\|x-x_0\|^{\beta-s} \\
&\leq \sum_{\lambda:|\lambda|=s} \frac{s}{\lambda!} L\|x-x_0\|^s \|x-x_0\|^{\beta-s} \leq s^{N+1} L\|x-x_0\|^\beta \\
&\leq C_2 m^{-\beta N},
\end{aligned}$$

where we use the following simple inequality : $|(x-x_0)^\lambda| < \|x-x_0\|^s$. $\square$

We need the following result to find the upper bound of the second-type error under the alternative.

LEMMA 6.3. *For a sufficiently large positive constant A, under conditions of Theorem 4.2, we have $\mathbb{P}_r(G) \longrightarrow 0$ as $\mathbf{n} \longrightarrow \infty$ uniformly over $\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})$, where for any $\alpha_2 \in (0,1)$, $C = \Phi^{-1}(1-\alpha_1) - \Phi^{-1}(\alpha_2)$ and*

$$G = \left\{ (K_{\mathbf{n}2} + K_{\mathbf{n}3}) \leq C\rho_{\mathbf{n}} \frac{\widehat{\sigma}^2}{\sigma^2} \right\},$$

where

$$\begin{aligned} K_{\mathbf{n}2} &= \frac{2}{m^N} \sum_{\mathbf{k}\in\mathcal{K}} \frac{m^{2N}}{\widehat{\mathbf{n}}^2\sigma^2\sqrt{1+V_{\mathbf{k}}}} \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathcal{I}_{\mathbf{k}} \\ \mathbf{i}\neq\mathbf{j}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\epsilon_{\mathbf{i}}, \\ K_{\mathbf{n}3} &= \frac{1}{m^N} \sum_{\mathbf{k}\in\mathcal{K}} \frac{m^{2N}}{\widehat{\mathbf{n}}^2\sigma^2\sqrt{1+V_{\mathbf{k}}}} \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathcal{I}_{\mathbf{k}} \\ \mathbf{i}\neq\mathbf{j}}} (r(x_{\mathbf{i}}) - r_0(x_{\mathbf{i}}))(r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}})). \end{aligned}$$

PROOF. The proof is very similar to that of Lemma 1 in [11]. However, we deal here with multivariate functions so we will give some details. The proof is then done in two steps:

(a) Firstly, we obtain a lower bound for $K_{\mathbf{n}3}$.
(b) Secondly, we use a Large Deviation Inequality after (a) is established.

(a) The lower bound for $K_{\mathbf{n}3}$ is obtained from Proposition 6.2 and arguing as in [11], that is: $K_{\mathbf{n}3} \geq \widetilde{C}^2 A^2 \psi_{\mathbf{n}}^2$, where $\widetilde{C}$ is a positive constant depending on β, L, M, σ^2.

(b) We need the following results to establish a Large Deviation Inequality.

LEMMA 6.4. *(See [6]) Suppose $E_1, ..., E_{w_1}$ are sets containing u sites each with $dist(E_i, E_j) \geq Q$ for all $i \neq j$ where $1 \leq i \leq w_1$ and $1 \leq j \leq w_1$. Suppose $Z_1, ..., Z_{w_1}$ is a sequence of real-valued r.v.'s measurable with respect to $\mathcal{B}(E_1), ..., \mathcal{B}(E_{w_1})$ respectively, and Z_i takes values in $[c1, c2]$. Then, there exists a sequence of independent r.v.'s $Z_1^*, ..., Z_{w_1}^*$ independent of $Z_1, ..., Z_{w_1}$ such that Z_i^* has the same distribution as Z_i and satisfies: $\sum_{i=1}^{w_1} \mathbb{E}(|Z_i - Z_i^*|) \leq 2w_1(c2-c1)\widehat{f}((w_1-1)u, u)\phi(Q)$.*

PROPOSITION 6.3. *Let $\Omega_{\mathbf{i},\mathbf{k}} = \dfrac{m^N}{\widehat{\mathbf{n}}^2} \sum_{\substack{\mathbf{j}\in\mathcal{I}_{\mathbf{k}} \\ \mathbf{j}\neq\mathbf{i}}} \dfrac{(r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))}{\sqrt{(1+V_{\mathbf{k}})K_{\mathbf{n}3}}} \epsilon_{\mathbf{i}}$. Then:*

$$\sum_{\mathbf{k}\in\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{k}}} \mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}^2) + \sum_{\mathbf{i}\neq\mathbf{l}} |\mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}})| = O\left(\frac{1}{\widehat{\mathbf{n}}}\right).$$

Proof of Proposition 6.3. Since $K_{\mathbf{n}3} \geq \widetilde{C}^2 A^2 \psi_{\mathbf{n}}^2$, $\psi_{\mathbf{n}}^2 = \rho_{\mathbf{n}} = \dfrac{m^{N/2}}{\widehat{\mathbf{n}}}$, then if $\beta > 1$,

$$\begin{aligned}
m^N \mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}^2) &= \frac{m^{2N}\sigma^2}{\widehat{\mathbf{n}}^4} \frac{\sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{1+V_{\mathbf{k}}} \left(\sum_{\substack{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}\\ \mathbf{j}\neq\mathbf{i}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\right)^2}{K_{\mathbf{n}3}} \\
&\leq \frac{cm^{2N}\sigma^2}{\widehat{\mathbf{n}}^4} \frac{\sum_{\mathbf{k}\in\mathcal{K}} \left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\right)^2}{\frac{m^N}{\widehat{\mathbf{n}}^2} \sum_{\mathbf{k}\in\mathcal{K}} \left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\right)^2 (1+o(1))} \\
&\quad + \frac{cm^{3N-2}}{\widehat{\mathbf{n}}^4 \psi_{\mathbf{n}}^2} \\
&\leq c_1 \frac{m^N}{\widehat{\mathbf{n}}^2} + \frac{cm^{(5N-4)/2}}{\widehat{\mathbf{n}}^3} \leq c_1 \frac{m^N}{\widehat{\mathbf{n}}^2} \left(1 + c_2 \frac{m^{(3N-4)/2}}{\widehat{\mathbf{n}}}\right).
\end{aligned}$$

Since $m = \widehat{\mathbf{n}}^{2/(4\beta+N)}$, with $N \in \{1,2,3\}$, then $\dfrac{m^{(3N-4)/2}}{\widehat{\mathbf{n}}} \leq \dfrac{1}{\widehat{\mathbf{n}}^{(4-N)/(4\beta+N)}}$. Thus

$$\sum_{\mathbf{k}\in\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{J}_{\mathbf{k}}} \mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}^2) \leq \frac{c_3}{\widehat{\mathbf{n}}}.$$

Similarly, if $1 \geq \beta \geq \dfrac{3N-1}{8}$, we obtain $\sum_{\mathbf{k}\in\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{J}_{\mathbf{k}}} \mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}^2) \leq \dfrac{c_4}{\widehat{\mathbf{n}}}$.

Besides, in the same way, if $\beta > 1$ for $\mathbf{i} \neq \mathbf{l}$, we have

$$\begin{aligned}
&|\mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}})| \\
&= \left| \left(\frac{m^N}{\widehat{\mathbf{n}}^2} \sum_{\substack{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}\\ \mathbf{j}\neq\mathbf{i}}} \frac{(r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))}{\sqrt{K_{\mathbf{n}3}}} \right) \left(\frac{m^N}{\widehat{\mathbf{n}}^2} \sum_{\substack{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}\\ \mathbf{j}\neq\mathbf{l}}} \frac{(r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))}{\sqrt{K_{\mathbf{n}3}}} \right) \frac{\mathbb{E}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{l}})}{1+V_{\mathbf{k}}} \right| \\
&\leq c_3 \left(\frac{m^N}{\widehat{\mathbf{n}}^2} \sqrt{\frac{\left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\right)^2}{\frac{m^N}{\widehat{\mathbf{n}}^2} \sum_{\mathbf{k}\in\mathcal{K}} \left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\right)^2 (1+o(1))}} + c\frac{m^{N-1}}{\widehat{\mathbf{n}}^2 \psi_{\mathbf{n}}} \right)^2 \\
&\times |\mathbb{E}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{l}})| \\
&\leq c_4 \left(\psi_{\mathbf{n}}^4 \frac{\left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\right)^2}{\sum_{\mathbf{k}\in\mathcal{K}} \left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}} (r(x_{\mathbf{j}}) - r_0(x_{\mathbf{j}}))\right)^2 (1+o(1))} + \frac{\psi_{\mathbf{n}}^6}{m^2} \right) \varphi(\|\mathbf{i}-\mathbf{l}\|).
\end{aligned}$$

Since $\sum_{\mathbf{i}\neq\mathbf{l}}|\mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}})| = \sum_{\mathbf{k}\in\mathcal{K}}\sum_{\substack{\mathbf{i},\mathbf{l}\in\mathcal{J}_{\mathbf{k}}\\ \mathbf{i}\neq\mathbf{l}}}|\mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}})|$ and $\psi_{\mathbf{n}}^4 = \dfrac{m^N}{\widehat{\mathbf{n}}^2}$, then

$$\begin{aligned}
&\sum_{\mathbf{k}\in\mathcal{K}}\sum_{\substack{\mathbf{i},\mathbf{l}\in\mathcal{J}_{\mathbf{k}}\\ \mathbf{i}\neq\mathbf{l}}}|\mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}})| \\
&\leq \frac{c_5}{\widehat{\mathbf{n}}}\left(\frac{\sum_{\mathbf{k}\in\mathcal{K}}\left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}}(r(x_{\mathbf{j}})-r_0(x_{\mathbf{j}}))\right)^2}{\sum_{\mathbf{k}\in\mathcal{K}}\left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}}(r(x_{\mathbf{j}})-r_0(x_{\mathbf{j}}))\right)^2(1+o(1))}+\frac{\psi_{\mathbf{n}}^2}{m^2}\right)\sum_{\mathbf{k}\in\mathcal{K}}\sum_{\substack{\mathbf{l}\in\mathcal{J}_{\mathbf{k}}\\ \|\mathbf{i}-\mathbf{l}\|=t>0}}\varphi(t) \\
&\leq \frac{2c_5}{\widehat{\mathbf{n}}}\sum_{t=1}^{\infty}t^{N-1}\varphi(t)\leq\frac{c_6}{\widehat{\mathbf{n}}}.
\end{aligned}$$

Thus if $\beta > 1$, then $\sum_{\mathbf{i}\neq\mathbf{l}}|\mathbb{E}\left(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}}\right)| = O\left(\dfrac{1}{\widehat{\mathbf{n}}}\right)$.

Similarly, if $\dfrac{3N-1}{8}\leq\beta\leq 1$, we have

$$\begin{aligned}
&\sum_{\mathbf{k}\in\mathcal{K}}\sum_{\substack{\mathbf{i},\mathbf{l}\in\mathcal{J}_{\mathbf{k}}\\ \mathbf{i}\neq\mathbf{l}}}|\mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}})| \\
&\leq \frac{c_4}{\widehat{\mathbf{n}}}\left(\frac{\sum_{\mathbf{k}\in\mathcal{K}}\left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}}(r(x_{\mathbf{j}})-r_0(x_{\mathbf{j}}))\right)^2}{\sum_{\mathbf{k}\in\mathcal{K}}\left(\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{k}}}(r(x_{\mathbf{j}})-r_0(x_{\mathbf{j}}))\right)^2(1+o(1))}+\frac{\psi_{\mathbf{n}}^2}{m^{2\beta}}\right)\sum_{\mathbf{k}\in\mathcal{K}}\sum_{\substack{\mathbf{l}\in\mathcal{J}_{\mathbf{k}}\\ \|\mathbf{i}-\mathbf{l}\|=t>0}}\varphi(t) \\
&\leq \frac{2c_4}{\widehat{\mathbf{n}}}\sum_{t=1}^{\infty}t^{N-1}\varphi(t)\leq\frac{c_5}{\widehat{\mathbf{n}}}.
\end{aligned}$$

Thus for all $\beta\geq\dfrac{3N-1}{8}$, we obtain $\sum_{\mathbf{i}\neq\mathbf{l}}|\mathbb{E}(\Omega_{\mathbf{i},\mathbf{k}}\Omega_{\mathbf{l},\mathbf{k}})| = O\left(\dfrac{1}{\widehat{\mathbf{n}}}\right)$. Therefore for $\mathbf{n}$ large enough, we obtain the result.

LEMMA 6.5. *(A Large Deviation Inequality) For any $r\in\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})$, where $\psi_{\mathbf{n}}^2 = \rho_{\mathbf{n}} = \dfrac{m^{N/2}}{\widehat{\mathbf{n}}} = \widehat{\mathbf{n}}^{-4\beta/(4\beta+N)}$ and under Assumptions 1–3, we have:*

$$\begin{aligned}
\mathbb{P}_r\left(F_1\cap F_2^c\right) \quad \leq \quad & c_7\frac{p_1^N D(\mathbf{n})m^N\phi(p_1)}{\widehat{\mathbf{n}}\psi_{\mathbf{n}}^2\left(\widetilde{C}A-\dfrac{C}{\widetilde{C}A}\right)} \\
&+ c_8\exp\left(-\frac{c_5\psi_{\mathbf{n}}^2\left(\widetilde{C}A-\dfrac{C}{\widetilde{C}A}\right)^2}{\dfrac{c_3}{\widehat{\mathbf{n}}}+c_6\left(\widetilde{C}A-\dfrac{C}{\widetilde{C}A}\right)\dfrac{p_1^N D(\mathbf{n})m^N}{\widehat{\mathbf{n}}^2}}\right),
\end{aligned}$$

where $F_2 = \left\{\sup_{\mathbf{i}\in\mathcal{I}_\mathbf{n}}|\epsilon_\mathbf{i}| > D(\mathbf{n})\right\}$, $D(\mathbf{n}) = \log\widehat{\mathbf{n}}$, $p_1 = \left\lfloor\left(\dfrac{\widehat{\mathbf{n}}}{m^N(\log\widehat{\mathbf{n}})^2}\right)^{1/(2N)}\right\rfloor$ *and*

$F_1 = \left\{\left|\dfrac{K_{\mathbf{n}2}}{\sqrt{K_{\mathbf{n}3}}}\right| \geq \widetilde{C}A\psi_\mathbf{n} - \dfrac{C\rho_\mathbf{n}\widehat{\sigma}^2/\sigma^2}{\psi_\mathbf{n}\widetilde{C}A}\right\}$.

Proof of Lemma 6.5. We give an upper bound for $\mathbb{P}_r\left(\left|\frac{K_{\mathbf{n}2}}{\sqrt{K_{\mathbf{n}3}}}\right| > \tau\right), \quad \forall\tau > 0$. We recall that $\Omega_{\mathbf{i},\mathbf{k}} = \dfrac{m^N}{\widehat{\mathbf{n}}^2\sqrt{1+V_\mathbf{k}}}\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{j}\neq\mathbf{i}}}\dfrac{(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))}{\sqrt{K_{\mathbf{n}3}}}\epsilon_\mathbf{i}$. We consider $\Omega_{\mathbf{i},\mathbf{k}}$ instead of $\Gamma_{\mathbf{i},\mathbf{k}}$ in the block decomposition defined in Section 5 and we choose $q_1 = p_1$, where $p_1 = p_1(\mathbf{n})$ is defined in Lemma 6.5. So, $\dfrac{K_{\mathbf{n}2}}{\sqrt{K_{\mathbf{n}3}}} = \dfrac{2}{\sigma^2}\sum_{i=1}^{2^N} T(\mathbf{k},\mathbf{n},i)$. For the sake of simplicity, we will only consider the case $i = 1$. We enumerate the $\widehat{v}$ terms $U(1,\mathbf{n},\mathbf{j},\mathbf{k}), \mathbf{j}\in\mathcal{T}$ of the sum $T(\mathbf{k},\mathbf{n},1)$ that we call $W_1, ..., W_{\widehat{v}}$. Observe that the set of sites associated with each of these random variables contains p_1^N sites, that two distinct sets of sites are separated by a distance of at least p_1, and finally, that for all $j = 1, ..., \widehat{v}$, there exists an integer $u(j)$ such that $W_j = U(1,\mathbf{n},,u(j),\mathbf{k})$ and if $\beta > 1$, since $K_{\mathbf{n}3} \geq \widetilde{C}^2A^2\psi_\mathbf{n}^2$, $\psi_\mathbf{n}^2 = \rho_\mathbf{n} = \dfrac{m^{N/2}}{\widehat{\mathbf{n}}}$, r and r_0 belong to $\sum(\beta, L, M)$, we have

$$\left|\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{j}\neq\mathbf{i}}}\frac{(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))}{\sqrt{K_{\mathbf{n}3}}}\right| \leq \left|\sum_{\mathbf{j}\in\mathcal{J}_\mathbf{k}}\frac{(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))}{\sqrt{K_{\mathbf{n}3}}}\right| + \frac{|r(x_\mathbf{i}) - r_0(x_\mathbf{i})|}{\sqrt{K_{\mathbf{n}3}}}$$

$$\leq \left(1 + \frac{m^N}{\widehat{\mathbf{n}}}\right)\left|\sum_{\mathbf{j}\in\mathcal{J}_\mathbf{k}}\frac{(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))}{\sqrt{K_{\mathbf{n}3}}}\right| + \frac{2M\left(\sum_{l=1}^N(x_{\mathbf{i},l} - x_{\mathbf{j},l})^2\right)^{1/2}}{\sqrt{K_{\mathbf{n}3}}}$$

and then

$$\begin{aligned}\left|\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{j}\neq\mathbf{i}}}\frac{(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))}{\sqrt{K_{\mathbf{n}3}}}\right| &\leq 2\sqrt{\frac{\left(\sum_{\mathbf{j}\in\mathcal{J}_\mathbf{k}}(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))\right)^2}{\frac{m^N}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{k}\in\mathcal{K}}\left(\sum_{\mathbf{j}\in\mathcal{J}_\mathbf{k}}(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))\right)^2(1+o(1))}} \\ &\quad + \frac{2MN^{1/2}m^{-1}}{\widetilde{C}A\psi_\mathbf{n}} \\ &\leq \frac{c}{\psi_\mathbf{n}}(1 + m^{-1}) \leq \frac{c}{\psi_\mathbf{n}}.\end{aligned}$$

Similarly, if $\beta \leq 1$, we obtain

$$\left|\sum_{\substack{\mathbf{j}\in\mathcal{J}_\mathbf{k}\\ \mathbf{j}\neq\mathbf{i}}}\frac{(r(x_\mathbf{j}) - r_0(x_\mathbf{j}))}{\sqrt{K_{\mathbf{n}3}}}\right| \leq \frac{c_1}{\psi_\mathbf{n}}(1 + m^{-\beta}) \leq \frac{c_1}{\psi_\mathbf{n}}.$$

Therefore

$$|\Omega_{\mathbf{i},\mathbf{k}}| \leq c\frac{|\epsilon_{\mathbf{i}}|m^N}{\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}} \leq c\frac{D(\mathbf{n})m^N}{\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}};$$

it follows

$$|W_i| \leq cp_1^N\frac{D(\mathbf{n})m^N}{\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}}.$$

From Lemma 6.4, there exists a sequence of independent random variables $W_1^*,\cdots,W_{\widehat{v}}^*$ such that for all $j=1,...,\widehat{v}$, W_j^* has the same distribution as W_j, and from Assumption 3, we have

$$\begin{aligned}
\mathbb{E}|W_j - W_j^*| &\leq 2\left(2cp_1^N\frac{D(\mathbf{n})m^N}{\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}}\right)\widehat{f}((\widehat{t}-1)p_1^N, p_1^N)\phi(p_1) \\
&\leq c_1p_1^{2N}\frac{D(\mathbf{n})m^N}{\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}}\phi(p_1).
\end{aligned}$$

Applying the Markov inequality, one has:

$$\mathbb{P}_r\left(\sum_{j=1}^{\widehat{v}}|W_j - W_j^*| > \tau\right) \leq \frac{\widehat{v}\mathbb{E}|W_j - W_j^*|}{\tau} \leq c_2\widehat{v}p_1^{2N}\frac{D(\mathbf{n})m^N}{\tau\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}}\phi(p_1).$$

Since $\widehat{\mathbf{n}} = 2^N\widehat{v}p_1^N$ then

$$\mathbb{P}_r\left(\sum_{j=1}^{\widehat{v}}|W_j - W_j^*| > \tau\right) \leq c_3p_1^N\frac{D(\mathbf{n})m^N}{\tau\widehat{\mathbf{n}}\psi_{\mathbf{n}}}\phi(p_1).$$

The Bernstein inequality leads to

$$\mathbb{P}_r\left(\left|\sum_{j=1}^{\widehat{v}}W_j^*\right| > \tau\right) \leq 2\exp\left(-\frac{\tau^2}{2\left(\sum_{j=1}^{\widehat{v}}\mathbb{E}[(W_j^*)^2] + \frac{\tau c_4p_1^N D(\mathbf{n})m^N}{3\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}}\right)}\right).$$

From Proposition 6.3,

$$\mathbb{P}_r\left(\left|\sum_{j=1}^{\widehat{v}}W_j^*\right| > \tau\right) \leq 2\exp\left(-\frac{\tau^2}{\frac{c_5}{\widehat{\mathbf{n}}} + \frac{\tau c_6p_1^N D(\mathbf{n})m^N}{3\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}}}\right).$$

Thus

$$\begin{aligned}
\mathbb{P}_r\left(\left|\frac{K_{\mathbf{n}2}}{\sqrt{K_{\mathbf{n}3}}}\right| > \tau\right) &\leq 2^N\mathbb{P}_r\left(\frac{2}{\sigma^2}\left|T(\mathbf{k},\mathbf{n},1)\right| > \frac{\tau}{2^N}\right) \\
&\leq c_8\exp\left(-\frac{\tau^2}{\frac{c_5}{\widehat{\mathbf{n}}} + \frac{\tau c_6p_1^N D(\mathbf{n})m^N}{\widehat{\mathbf{n}}^2\psi_{\mathbf{n}}}}\right) + c_7p_1^N\frac{D(\mathbf{n})m^N}{\tau\widehat{\mathbf{n}}\psi_{\mathbf{n}}}\phi(p_1).
\end{aligned}$$

□

6.2. Proof of Theorem 4.1

Proof of Theorem 4.1. Under H_0, we have:

$$\widehat{\sigma}^2 T_{\mathbf{n}} \quad = \quad K_{\mathbf{n}} + \frac{1}{m^N} \sum_{\mathbf{k} \in \mathcal{K}} \frac{m^{2N}}{\widehat{\mathbf{n}}^2 \sqrt{1+V_{\mathbf{k}}}} \sum_{\substack{\mathbf{i},\mathbf{j} \in \mathcal{J}_{\mathbf{k}} \\ \mathbf{i} \neq \mathbf{j}}} (\gamma_0(h_{\mathbf{ij}}) - \widehat{\gamma}_0(h_{\mathbf{ij}})).$$

Since $V_{\mathbf{k}} = O(1)$ and $N = 1, 2, 3$, it follows from Assumptions 1–3 and by Relation (3.1) that:

$$\begin{aligned}
&\mathbb{P}_{r_0}\left(\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}}\left|\frac{1}{m^N} \sum_{\mathbf{k} \in \mathcal{K}} \frac{m^{2N}}{\widehat{\mathbf{n}}^2 \sqrt{1+V_{\mathbf{k}}}} \sum_{\substack{\mathbf{i},\mathbf{j} \in \mathcal{J}_{\mathbf{k}} \\ \mathbf{i} \neq \mathbf{j}}} (\gamma_0(h_{\mathbf{ij}}) - \widehat{\gamma}_0(h_{\mathbf{ij}}))\right| > \tau\right) \\
&\leq \mathbb{P}_{r_0}\left(\frac{m^{N/2}}{\widehat{\mathbf{n}}} \sum_{\mathbf{k} \in \mathcal{K}} \sum_{\mathbf{i} \in \mathcal{J}_{\mathbf{k}}} \sum_{\substack{\mathbf{j} \in \mathcal{J}_{\mathbf{k}} \\ \|\mathbf{j}\| = t > 0}} \varphi(t) |\sigma^2 - \widehat{\sigma}^2| > c_1 \tau\right) \\
&\leq \mathbb{P}_{r_0}\left(m^{N/2} \sum_{t=1}^{\infty} t^{N-1} \varphi(t) |\sigma^2 - \widehat{\sigma}^2| > c_1 \tau\right) \\
&\leq \sup_{r \in \sum(\beta, L, M)} \mathbb{P}_r\left(|\sigma^2 - \widehat{\sigma}^2| > \frac{c_1 \tau}{m^{N/2} \sum_{t=1}^{\infty} t^{N-1} \varphi(t)}\right) \\
&= O\left(\max\left(\frac{m^N \log \widehat{\mathbf{n}}}{\widehat{\mathbf{n}}^{4/N}}, \frac{m^N \log \widehat{\mathbf{n}}}{\widehat{\mathbf{n}}}\right)\right) \longrightarrow 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}} K_{\mathbf{n}} = \sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}} \left(\frac{1}{m^N} \sum_{\mathbf{k} \in \mathcal{K}} \widetilde{S}_{\mathbf{k},\mathbf{n}}\right) \\
&= \frac{1}{\sqrt{m^N}} \sum_{\mathbf{k} \in \mathcal{K}} \left(\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}} T(\mathbf{k}, \mathbf{n}, 1)\right) + \frac{1}{\sqrt{m^N}} \sum_{\mathbf{k} \in \mathcal{K}} \left(\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}} \sum_{i=2}^{2^N} T(\mathbf{k}, \mathbf{n}, i)\right).
\end{aligned}$$

Since the random variables $T(\mathbf{k}, \mathbf{n}, i)$, $\mathbf{k} \in \mathcal{K}$ are independent (see Assumption 1), from Markov inequality and Lemma 6.2 (ii), we obtain

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{m^N}} \sum_{\mathbf{k} \in \mathcal{K}} \left(\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}} \sum_{i=2}^{2^N} T(\mathbf{k}, \mathbf{n}, i)\right) > \tau\right) \\
&\leq \frac{\mathrm{Var}\left(\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{2N}}} \sum_{i=2}^{2^N} T(\mathbf{k_1}, \mathbf{n}, i)\right)}{\tau^2} \longrightarrow 0.
\end{aligned}$$

Moreover since, from Lemma 6.2 (i) and Assumption 1, the random variables $U(1,\mathbf{n},\mathbf{l},\mathbf{k})$, $\mathbf{l}\in\mathcal{T}$, $\mathbf{k}\in\mathcal{K}$, are asymptotically independent, we then deduce from Lemma 6.2 $(iii),(iv)$, that $\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^{3N}}}\sum_{\mathbf{k}\in\mathcal{K}}T(\mathbf{k},\mathbf{n},1)\xrightarrow{D}\mathcal{N}(0,\sigma^4)$. Thus, from Slutsky's Theorem, we obtain that $\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}}K_{\mathbf{n}}\xrightarrow{D}\mathcal{N}(0,\sigma^4)$. So, from Slutsky's Theorem, we conclude that $\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}}\left(\widehat{\sigma}^2T_{\mathbf{n}}\right)\xrightarrow{D}\mathcal{N}(0,\sigma^4)$. □

6.3. Proof of Theorem 4.2.

PROOF OF THEOREM 4.2. Let r be the true regression function in the model (1.1). We rewrite $\frac{\widehat{\sigma}^2}{\sigma^2}T_{\mathbf{n}}$ in the following way: $\frac{\widehat{\sigma}^2}{\sigma^2}T_{\mathbf{n}}=K_{\mathbf{n}1}+K_{\mathbf{n}2}+K_{\mathbf{n}3}$ with $K_{\mathbf{n}2}$, $K_{\mathbf{n}3}$ defined in Lemma 6.3 and

$$K_{\mathbf{n}1} = \frac{1}{m^N}\sum_{\mathbf{k}\in\mathcal{K}}\frac{m^{2N}}{\widehat{\mathbf{n}}^2\sigma^2\sqrt{1+V_{\mathbf{k}}}}\sum_{\substack{\mathbf{i},\mathbf{j}\in\mathcal{J}_{\mathbf{k}},\\ \mathbf{i}\neq\mathbf{j}}}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}}-\widehat{\gamma}_0(h_{\mathbf{ij}})).$$

Proof of Relation (2.1): We show that under H_0, our test (3.2) is of asymptotic level α_1. We have

$$T_{\mathbf{n}}=\frac{1}{m^N}\sum_{\mathbf{k}\in\mathcal{K}}\frac{m^{2N}}{\widehat{\mathbf{n}}^2\widehat{\sigma}^2\sqrt{1+V_{\mathbf{k}}}}\sum_{\substack{\mathbf{i},\mathbf{j}\in\mathcal{J}_{\mathbf{k}},\\ \mathbf{i}\neq\mathbf{j}}}(\epsilon_{\mathbf{i}}\epsilon_{\mathbf{j}}-\widehat{\gamma}_0(h_{\mathbf{ij}})).$$

Relation (3.1), Theorem 4.1 and Slutsky's Theorem imply

$$\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}}T_{\mathbf{n}}\xrightarrow{D}\mathcal{N}(0,1).$$

Thus, the first-type error can be bounded,

$$\mathbb{P}_{r_0}(\overline{\Delta}_{\mathbf{n}}=1)=\mathbb{P}\left(T_{\mathbf{n}}>c_0\rho_{\mathbf{n}}\right)=\mathbb{P}\left(\sqrt{\frac{\widehat{\mathbf{n}}^2}{m^N}}T_{\mathbf{n}}>c_0\right)=\alpha_1+o(1),$$

since c_0 is the $1-\alpha_1$ quantile of a standard Gaussian.

Proof of Relation (2.2). Denoting G^c the complement of G, where G is defined in Lemma 6.3, and for any r in $\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})$, we have

$$\begin{aligned}
&\sup_{r\in\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})}\mathbb{P}_r(\overline{\Delta}_{\mathbf{n}}=0)=\sup_{r\in\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})}\mathbb{P}_r\left(K_{\mathbf{n}1}+K_{\mathbf{n}2}+K_{\mathbf{n}3}\le c_0\rho_{\mathbf{n}}\frac{\widehat{\sigma}^2}{\sigma^2}\right)\\
&=\sup_{r\in\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})}\mathbb{P}_r(\{\overline{\Delta}_{\mathbf{n}}=0\}\cap G^c)+\sup_{r\in\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})}\mathbb{P}_r(\{\overline{\Delta}_{\mathbf{n}}=0\}\cap G)\\
&\le\sup_{r\in\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})}\mathbb{P}_r(K_{\mathbf{n}1}\le(c_0-C)\rho_{\mathbf{n}}\frac{\widehat{\sigma}^2}{\sigma^2})+\sup_{r\in\Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})}\mathbb{P}_r(G).
\end{aligned}$$

Since $K_{\mathbf{n}1} = \widehat{\sigma}^2 T_{\mathbf{n}}/\sigma^2$ under $\mathbb{P}_{r_0}$, applying Theorem 4.1 and Relation (3.1), we obtain, for any $r \in \Lambda_{\mathbf{n}}(A\psi_{\mathbf{n}})$, that

$$\lim_{\mathbf{n}\longrightarrow\infty} \mathbb{P}_r\left(K_{\mathbf{n}1} \leq (c_0 - C)\rho_{\mathbf{n}}\frac{\widehat{\sigma}^2}{\sigma^2}\right) = \alpha_2.$$

Applying Lemma 6.3, we obtain A defined in Lemma 6.3 such that Relation (2.2) is satisfied. □

6.4. Proof of Theorem 4.3

PROOF OF RELATION (2.3). First, we construct a regression function $r_\eta \in \sum(\beta, L, M)$ and a probability measure π for which we will prove $\pi\left(r_\eta \in \Lambda(a\psi_{\mathbf{n}})\right) = 1$. Let $H(\beta, 1)$ be a Hölder class of smoothness parameter β and of constant 1, let f be a function in $H(\beta, 1)$ whose the support is $[0,1]^N$ and verifies $\int_{[0,1]^N} f(x)dx = 0$ and $\int_{[0,1]^N} f^2(x)dx = \mu^2$. Put $\kappa_{\mathbf{n}}^2 = \frac{1+\psi_{\mathbf{n}}^{1/(4\beta)}}{\mu^2}(a\psi_{\mathbf{n}})^2$, $h = \left(\frac{\kappa_{\mathbf{n}}}{2^{\beta-s-1}(L-L')}\right)^{1/\beta}$ with $s = \lfloor\beta\rfloor$; $m^N = \frac{1}{h^N}$ (where we assume, without loss of generality, that m is an integer), and let $(\eta_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}}$ be a sequence of binary i.i.d. random variables with values in $\{-1, 1\}$. We consider the probability measure $\pi = \prod_{\mathbf{k}\in\mathcal{K}} \pi_{\mathbf{k}}$ such that $\pi_{\mathbf{k}}(\eta_{\mathbf{k}} = 1) = \pi_{\mathbf{k}}(\eta_{\mathbf{k}} = -1) = 1/2$. We define

$$r_\eta(x) = r_0(x) + \sum_{\mathbf{k}\in\mathcal{K}} \eta_{\mathbf{k}}\kappa_{\mathbf{n}} h^{N/2} f_{\mathbf{k},h}(x)$$

where $f_{\mathbf{k},h}(x) = f\left((x - z_{\mathbf{k}})/h\right)/h^{N/2}$ if $x \in A_{\mathbf{k}}$ and $f_{\mathbf{k},h}(x) = 0$ if $x \notin A_{\mathbf{k}}$, and $z_{\mathbf{k}} = (z_{k_1}, ..., z_{k_N})$ is the lower end set left of the hypercube of $A_{\mathbf{k}}$. Now, since the proof that $r_\eta \in \sum(\beta, L, M)$ is very similar to the one in [11], it is then omitted. Finally, by construction of f, we have,

$$\|r_\eta - r_0\|_2^2 = h^N m^N \mu^2 \kappa_{\mathbf{n}}^2 = \left(1 + \psi_{\mathbf{n}}^{1/(4\beta)}\right) a^2 \left(\psi_{\mathbf{n}}\right)^2 \geq \left(a\psi_{\mathbf{n}}\right)^2$$

and therefore $\pi\left(r_\eta \in \Lambda\left(a\psi_{\mathbf{n}}\right)\right) = 1$.

Second, we prove the lower bound. One define $\mathbb{P}_{\mathbf{n},\pi}$ ($\mathbb{P}_{\mathbf{n},0}$ respectively) as the joint probability measure of $(Y_{\mathbf{i}_1}, \cdots, Y_{\mathbf{i}_{\widehat{\mathbf{n}}}})^T$ when r_η (r_0 respectively) is the true regression function. Notice that a lower bound for the second type error for any test of a given asymptotic level α_1 is obtained by a lower bound of $\gamma_{\mathbf{n}} \geq \gamma_{\mathbf{n},\pi} = 1 - \frac{1}{2}var\left(\mathbb{P}_{\mathbf{n},0}, \mathbb{P}_{\mathbf{n},\pi}\right)$ (see [19], section 4.1), where $var\left(A, B\right)$ stands for the total variation distance. Now, it is a question to get an asymptotic upper bound for $var\left(\mathbb{P}_{\mathbf{n},0}, \mathbb{P}_{\mathbf{n},\pi}\right)$. From the Cauchy-Bunyakovskii inequality, we obtain

$$var\left(\mathbb{P}_{\mathbf{n},0}, \mathbb{P}_{\mathbf{n},\pi}\right) \leq \frac{1}{2}\left(\int\left(\frac{d\mathbb{P}_{\mathbf{n},\pi}}{d\mathbb{P}_{\mathbf{n},0}}\right)^2 d\mathbb{P}_{\mathbf{n},0} - 1\right)^{1/2}.$$

Putting $S_{\mathbf{n}}^{\mathbf{i}} = \kappa_{\mathbf{n}} f\left(\frac{X_{\mathbf{i}}-z_{\mathbf{k}}}{h}\right)$, with $X_{\mathbf{i}} = (X_{i_1}, ..., X_{i_N})^T \in [0,1]^N$, under Assumption 4, $(\epsilon_{\mathbf{i}_i}, \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}}) \sim \mathcal{N}(\mathbf{0}, \widetilde{\Lambda})$, where $\widetilde{\Lambda} = (\widetilde{\Lambda}_{ij})_{ij}$ with $\widetilde{\Lambda}_{ij} = \sigma^2 \varphi(\|\mathbf{i}_i - \mathbf{i}_j\|)$, $\mathbf{i}_i$, $\mathbf{i}_j \in \mathcal{J}_{\mathbf{k}}$. Then, the joint density $\widetilde{g_{\mathbf{k}}}$ of vector $(\epsilon_{\mathbf{i}_i}, \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}})$ exists and we have

$$
\begin{aligned}
\frac{d\mathbb{P}_{\mathbf{n},\pi}}{d\mathbb{P}_{\mathbf{n},0}}(y_{\mathbf{i}_1}, \cdots, y_{\mathbf{i}_{\hat{\mathbf{n}}}}) &= \prod_{\mathbf{k}\in\mathcal{K}} \int_{\{-1,1\}} \frac{\widetilde{g_{\mathbf{k}}}((y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}) + \eta_{\mathbf{k}} S_{\mathbf{n}}^{\mathbf{i}_i})_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}})}{\widetilde{g_{\mathbf{k}}}((y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}))_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}})} d\pi_{\mathbf{k}}(\eta_{\mathbf{k}}) \\
&= \prod_{\mathbf{k}\in\mathcal{K}} \frac{1}{2} \left(\frac{\widetilde{g_{\mathbf{k}}}((y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}) + S_{\mathbf{n}}^{\mathbf{i}_i})_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}})}{\widetilde{g_{\mathbf{k}}}((y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}))_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}})} + \frac{\widetilde{g_{\mathbf{k}}}((y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}) - S_{\mathbf{n}}^{\mathbf{i}_i})_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}})}{\widetilde{g_{\mathbf{k}}}((y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}))_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}})} \right). \\
&= \prod_{\mathbf{k}\in\mathcal{K}} \exp\left(-\frac{1}{2} (S_{\mathbf{n}}^{\mathbf{i}_i}, \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}})^T \widetilde{\Lambda}^{-1} (S_{\mathbf{n}}^{\mathbf{i}_i}, \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}}) \right) \\
&\qquad \times \cosh\left((y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}), \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}})^T \widetilde{\Lambda}^{-1} (S_{\mathbf{n}}^{\mathbf{i}_i}, \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}}) \right).
\end{aligned}
$$

Let $\delta_{\max}(\widetilde{\Lambda})$ (resp. $\delta_{\min}(\widetilde{\Lambda}^{-1})$) be the largest (resp. the smallest) eigenvalue of the matrix $\widetilde{\Lambda}$ (resp. $\widetilde{\Lambda}^{-1}$). Since, from the Geršgorin's theorem ([17]), it can be proved that $0 < \delta_{\max}(\widetilde{\Lambda}) \le \max_i \sum_j |\widetilde{\Lambda}_{ij}| = \sigma^2 \max_i \sum_j \varphi(\|\mathbf{i}_i - \mathbf{i}_j\|)$, then from Assumption 1, we have

$$
\delta_{\max}(\widetilde{\Lambda}^{-1}) > \delta_{\min}(\widetilde{\Lambda}^{-1}) = \frac{1}{\delta_{\max}(\widetilde{\Lambda})} \ge \frac{1}{\sigma^2 \max_i \sum_j \varphi(\|\mathbf{i}_i - \mathbf{i}_j\|)} = c_1 > 0,
$$

where c_1 is a positive constant. So $\delta_{\min}(\widetilde{\Lambda}) = \dfrac{1}{\delta_{\max}(\widetilde{\Lambda}^{-1})} > 0$, then for $\mathbf{n}$ large enough, we have $\delta_{\min}(\widetilde{\Lambda}) \ge \dfrac{1}{m^{N/8}}$ and from Assumption 2, we have

$$
\begin{aligned}
C_{\mathbf{k}} : &= \left| (y_{\mathbf{i}_i} - r_0(x_{\mathbf{i}_i}), \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}})^T \widetilde{\Lambda}^{-1} (S_{\mathbf{n}}^{\mathbf{i}_i}, \mathbf{i}_i \in \mathcal{J}_{\mathbf{k}}) \right| \\
&\le \frac{1}{\delta_{\min}(\widetilde{\Lambda})} \left(\sum_{\mathbf{i}\in\mathcal{J}_{\mathbf{k}}} (y_{\mathbf{i}} - r_0(x_{\mathbf{i}}))^2 \right)^{1/2} \left(\sum_{\mathbf{i}\in\mathcal{J}_{\mathbf{k}}} \left(S_{\mathbf{n}}^{\mathbf{i}}\right)^2 \right)^{1/2} \\
&\le \frac{c_2}{m^{N/8}} \left[\frac{1}{m^N} \sum_{\mathbf{i}\in\mathcal{J}_{\mathbf{k}}} (f_{\mathbf{k},h}(x_{\mathbf{i}}))^2 \right]^{1/2} \longrightarrow 0,
\end{aligned}
$$

where c_2 is some positive constant. By dominated convergence, it follows that

$$
\begin{aligned}
\sum_{\mathbf{k}\in\mathcal{K}} \log\left[\int (\cosh(C_{\mathbf{k}}))^2 \, d\mathbb{P}_{\mathbf{n},0} \right] &\sim \sum_{\mathbf{k}\in\mathcal{K}} \log\left[1 + \int C_{\mathbf{k}}^2 \, d\mathbb{P}_{\mathbf{n},0} \right] \\
&\sim \int \sum_{\mathbf{k}\in\mathcal{K}} C_{\mathbf{k}}^2 \, d\mathbb{P}_{\mathbf{n},0} \longrightarrow 0.
\end{aligned}
$$

Therefore

$$\begin{aligned}
\left|\log\left(\int\left(\frac{d\mathbb{P}_{\mathbf{n},\pi}}{d\mathbb{P}_{\mathbf{n},0}}\right)^2 d\mathbb{P}_{\mathbf{n},0}\right)\right| &= \Bigg| -\sum_{\mathbf{k}\in\mathcal{K}}(S_{\mathbf{n}}^{\mathbf{i}_i},\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}})^T\widetilde{\Lambda}^{-1}(S_{\mathbf{n}}^{\mathbf{i}_i},\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}) \\
&\quad + \sum_{\mathbf{k}\in\mathcal{K}}\log\left[\int(\cosh(C_{\mathbf{k}}))^2\, d\mathbb{P}_{\mathbf{n},0}\right]\Bigg| \\
&\le \frac{1}{\delta_{\min}(\widetilde{\Lambda})}\sum_{\mathbf{k}\in\mathcal{K}}\sum_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}}\left(S_{\mathbf{n}}^{\mathbf{i}_i}\right)^2 \\
&\quad + \left|\sum_{\mathbf{k}\in\mathcal{K}}\log\left[\int(\cosh(C_{\mathbf{k}}))^2\, d\mathbb{P}_{\mathbf{n},0}\right]\right| \\
&\le m^{N/8}\kappa_{\mathbf{n}}^2\sum_{\mathbf{k}\in\mathcal{K}}\sum_{\mathbf{i}_i\in\mathcal{J}_{\mathbf{k}}}\left[f\left(\frac{X_{\mathbf{i}_i}-z_{\mathbf{k}}}{h}\right)\right]^2 \\
&\quad + \left|\sum_{\mathbf{k}\in\mathcal{K}}\log\left[\int(\cosh(C_{\mathbf{k}}))^2\, d\mathbb{P}_{\mathbf{n},0}\right]\right| \\
&\longrightarrow 0.
\end{aligned}$$

So, we obtain the lower bound of $\gamma_{\mathbf{n},\pi}$ and $\gamma_{\mathbf{n}}$. $\square$

References

1. J. Aston, F. Autin, G. Claeskens, J.-M. Freyermuth, and C. Pouet, Minimax optimal procedures for testing the structure of multidimensional functions, Appl. Comput. Harmon. Anal. **46**(2019), 288–311.
2. Y. Baraud, S. Huet, and B. Laurent, Adaptive tests of linear hypotheses by model selection, Ann. Statist. **31**(2003), 225–251.
3. Y. Baraud, S. Huet, and B. Laurent, Testing convex hypotheses on the mean of a Gaussian vector. Application to testing qualitative hypotheses on a regression function, Ann. Statist. **33**(2005), 214–257.
4. G. Biau and B. Cadre, Nonparametric Spatial Prediction, Stat. Inference Stoch. Process. **7**(2004), 327–349.
5. S. Bouka, Estimation of the trend function and auto-covariance for spatial models, C.R. Acad. Sci. Paris, Ser. I **357**(2019), 907–911.
6. M. Carbon, L. T. Tran, and B. Wu, Kernel density estimation for random fields: the L_1 theory, J. Nonparametr. Stat. **6**(1997), 157–170.
7. L. Commiges and A.S. Dalalyan, Minimax testing of a composite null hypothesis defined via a quadratic functional in the model of regression, Electron. J. Statist. **7**(2013), 146–190.
8. M. El Machkouri and R. Stoica, Asymptotic normality of kernel estimates in a regression model for random fields, J. Nonparametr. Stat. **22**(2010), 955–971,
9. M.S. Ermakov, Minimax detection of a signal in Gaussian white noise, Teor. Veroyatmost. i Primenen. **35**(1990), 704–715.
10. M. Francisco-Fernandez and J.D. Opsomer, Smoothing parameter selection methods for nonparametric regression with spatially correlated errors, The Canadian Journal of Statistics **33**(2005), 279–295.
11. G. Gayraud and Ch. Pouet, Minimax testing composite null hypotheses in the discrete regression scheme, Math. Methods Statist. **4**(2001), 375–394.

12. G. Gayraud and Ch. Pouet, Adaptive minimax testing in the discrete regression scheme, PTRF, pp. 28 (2005).
13. W. González-Manteiga, A review on specification tests for models with functional data, Spanish Journal of Statistics **66**(2019), 1—6.
14. E. Guerre and P. Lavergne, Optimal minimax rates for nonparametric specification testing in regression models, Econom. Theory **18**(2002), 1139–1171.
15. K. Hitomi, M. Iwasawa, and Y. Nishiyama, Optimal minimax rates of specification testing with data-driven bandwidth, Econometric Rev. **42**(2023), 487—512.
16. S. Hörmann and P. Kokoszka, Consistency of the mean and the principal components of spatially distributed functional data, Bernoulli **19**(2013), 1535–1558.
17. R. A. Horn and C. A. Johnson, Matrix analysis, Cambridge University Press, 1985.
18. J. L. Horowitz and V. G. Spokoiny, An adaptive, rate-optimal test of a parametric mean-regression model against a nonparametric alternative, Econometrica **69**(2001), 599–631.
19. Yu. I. Ingster, Asymptotically minimax hypothesis testing for nonparametric alternatives, I-III, Math. Methods Statist. **2**(1993), 85–114; 171–189; 249–268 .
20. Yu. I. Ingster and T. Sapatinas, Minimax godness-of-fit testing in multivariate nonparametric regression, Math. Methods Statist. **18**(2009), 241–269.
21. Yu. I. Ingster and I. A. Suslina, Nonparametric goodness-of-fit testing under Gaussian models, Lecture Notes in Statistics, Springer-Verlag, New York, 2003.
22. B. Laurent, J. M. Loubes, and C. Marteau, Testing inverse problems: a direct or an indirect problem?, J. Statist. Plann. Inference **141**(2011), 1849–1861.
23. B. Laurent, J. M. Loubes, and C. Marteau, Non asymptotic minimax rates of testing in signal detection with heterogeneous variances, Electron. J. Stat. **6**(2012), 91–122.
24. C. Pouet, An asymptotically optimal test for a parametric set of regression functions against a non-parametric alternative, J. Statist. Plann. Inference **98**(2001), 177–189.
25. M. Rosenblatt, A central limit theorem and a strong mixing condition, Proc. Natl. Acad. Sci. USA **42**(1956), 43–47.
26. A. Shirvani, O. Khademnoe, and M. Hosseini-Nasab, Hypothesis testing for points of impact in functional linear regression, Computational and Applied Mathematics (2024). https://doi.org/10.1007/s40314-024-02723-5
27. L. T. Tran, Kernel density estimation on random fields, J. Multivariate Anal. **34**(1990), 37–53.

Laboratoire de Probabilités, Statistique et Informatique,r Unité de Recherche en Mathématiques et Informatique, Université des Sciences et Techniques de Masuku, BP 813 Franceville, GABON
e-mail: stephane.bouka@univ-masuku.org