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Representation of a Zero Trace Matrix as a Commutator

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Abstract. A new solution to the problem of representing a zero-trace matrix as a commutator of a pair of matrices is presented. For diagonalizable matrices, the solution first consists of a Toeplitz matrix H with 1 on the superdiagonal, and a second matrix with cumulative sums of eigenvalues on the subdiagonal. Defective matrices use the Jordan Normal Form to add cumulative sums of the ones and zeros in the Jordan superdiagonal to the diagonal of the second matrix. We show that every matrix is a polynomial in H and its transpose.

Résumé. Une nouvelle solution au problème de la représentation d'une matrice sans trace comme commutateur d'une paire de matrices est présentée. Pour les matrices diagonalisables, la solution consiste d'abord en une matrice de Toeplitz H avec 1 sur le superdiagonale, et une deuxième matrice avec des sommes cumulées de valeurs propres sur la sous-diagonale. Les matrices défectueuses utilisent la forme normale de Jordan pour ajouter les sommes cumulèes des uns et des zéros de la superdiagonale de Jordan à la diagonale de la deuxième matrice. Nous montrons que toute matrice est un polynôme dans H et sa transposée.

1. Introduction Denote by $\mathcal{M}_n(\mathbb{C})$ or simply $\mathcal{M}_n$ the space of $n \times n$ matrices over the complex numbers $\mathbb{C}$. $\mathcal{M}_n$ is a n^2 dimensional vector space with inner product the sum of the coefficients of the Hadamard product. The $n^2 - 1$ dimensional subspace of matrices with trace zero form a hyperplane through the origin. It is the orthogonal complement to the one dimensional subspace spanned by the identity matrix. Since the diagonalizable matrices in $\mathcal{M}_n$ are known to be a dense subset, they are also dense in the trace zero subspace.

2. Representing a Zero Trace Matrix as a Commutator The commutator $M = [A, B] = AB - BA$ of matrices A and $B \in \mathcal{M}_n$ has trace equal to zero. The converse, that every matrix of trace zero is a commutator, was proven by Shoda [1] over fields of characteristic zero.

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Albert and Muckenhoupt [2] used companion matrices to prove this converse over any field. Gaines [3] showed that if one first finds a unitary matrix U, such that U^*MU has a zero diagonal, then one can choose A to be Hermitian. Working over algebraically closed fields, Gibson [4] showed that one could prescribe eigenvalues for A and B so that $[A, B] = M$, the given zero trace matrix. We provide an overlooked simple solution.

Example 1. Let L be a diagonal trace zero 4×4 matrix, with diagonal λ_1, λ_2, λ_3, λ_4. Then $L = [H_4, X_4] = Ad_H(X_4)$[1], using the fact that $\lambda_4 = -\lambda_1 - \lambda_2 - \lambda_3$, for

$$(1) \qquad H_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 + \lambda_3 & 0 \end{bmatrix}$$

The Jordan Normal Form (2) is used to extend this idea to all of $\mathcal{M}_n$. [2]

$$(2) \qquad \mathbf{J}_n = \begin{bmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 & 0 \\ 0 & J_{n_2}(\lambda_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & J_{n_k}(\lambda_k) \end{bmatrix}$$

$$\text{where } J_{n_i} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 \cdots & 0 & 0 & \\ \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 0 \cdots & 0 & \lambda_i \end{bmatrix}$$

We introduce a class of matrices

$$(3) \qquad X_n^\diamond = \begin{bmatrix} y_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_1 & y_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x_2 & y_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-2} & y_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & x_{n-1} & y_n \end{bmatrix}.$$

Theorem 2. *Let C_ϕ be a trace zero matrix in $\mathcal{M}_\mathbf{n}$ with JNF $C_\phi = S\mathbf{J}S^{-1}$, where S is a nonsingular matrix in $\mathcal{M}_\mathbf{n}$. Denote the diagonal of $\mathbf{J}$ by $\{\lambda_1, \cdots, \lambda_n\}$ and the super-diagonal of $\mathbf{J}$ by $\{\delta_1, \cdots, \delta_{n-1}\}$ where $\delta_j \in \{0, 1\}$. H_n is the Toeplitz matrix with 1's on the superdiagonal and 0 elsewhere. The coefficients of $X_n^\diamond$ are given by the 1-parameter family $x_k = \sum_{j=1}^k \lambda_j, \quad 1 \le k \le n-1$, and $y_1 = z,\ y_j = z + \sum_{i=1}^{j-1} \delta_i \quad 2 \le j \le n, z \in \mathbb{C}$. C_ϕ is equal to the commutator*

$$C_\phi = \left[SH_nS^{-1}, SX_n^\diamond S^{-1}\right] = S\,[H_n, X_n^\diamond]\,S^{-1}.$$

[1] $Ad_H(X)$ is the adjoint map of Lie Algebras.

[2] When the subset of defective matrices in $\mathcal{M}_n$ is excluded, the JNF collapses to the diagonal matrix of eigenvalues, since every Jordan block is 1×1.

Proof. A solution for $\{x_1, \cdots, x_{n-1}\}$ is determined by solving the equations from the main diagonal of $[H_n, X_n^\diamond] = \mathbf{J}$. These equations have the form

$$\begin{array}{rrrcrrcr} x_1 & +0 & +0 & \cdots & +0 & +0 & = & \lambda_1 \\ -x_1 & +x_2 & +0 & \cdots & +0 & +0 & = & \lambda_2 \\ 0 & -x_2 & +x_3 & \cdots & +0 & +0 & = & \lambda_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & = & \vdots \\ 0 & +0 & +0 & \cdots & -x_{n-2} & +x_{n-1} & = & \lambda_{n-1} \\ 0 & +0 & +0 & \cdots & +0 & -x_{n-1} & = & \lambda_n \end{array}$$

The sum of all of these rows gives the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = \sum_{j=1}^{n} \lambda_j = \mathrm{trace}(C_\phi).$$

This system is consistent if and only if the trace is zero, so $\lambda_n = -\sum_{j=1}^{n-1} \lambda_j$. There is an immediate unique solution $x_k = \sum_{j=1}^{k} \lambda_j, \quad 1 \le k \le n-1$.

A solution for $\{y_1, \cdots, y_n\}$ is given by the equations from the super-diagonal of $[H_n, X_n^\diamond] = \mathbf{J}$. These equations have the form:

$$\begin{array}{rrrrcrrcr} -y_1 & +y_2 & +0 & +0 & \cdots & +0 & +0 & = & \delta_1 \\ 0 & -y_2 & +y_3 & +0 & \cdots & +0 & +0 & = & \delta_2 \\ 0 & +0 & -y_3 & +y_4 & \cdots & +0 & +0 & = & \delta_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & = & \vdots \\ 0 & +0 & +0 & +0 & \cdots & -y_{n-1} & +y_n & = & \delta_{n-1} \end{array}$$

This system of $n-1$ equations in n unknowns has a 1-parameter family of solutions:

$$y_1 = z, \; y_j = z + \sum_{i=1}^{j-1} \delta_i \quad 2 \le j \le n. \tag{4}$$

$X_n^\diamond$ is determined; $[H_n, X_n^\diamond] = \mathbf{J}$; $C_\phi = \left[SH_nS^{-1}, SX_n^\diamond S^{-1}\right]$ so C_ϕ is a commutator. □

Example 3. H is an upper triangular nilpotent matrix with zero eigenvalues and trace zero. We can represent H as a commutator: $H = [H, X^\heartsuit]$, where

$$X_n^\heartsuit = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n-1 \end{bmatrix}.$$

Any matrix M in $\mathfrak{M}_n$ is the sum of a commuting pair $M = D + N, DN = ND$, where D is diagonalizable and N nilpotent [[5],p. 139]. For the JNF of $M = SJS^{-1}$, $J = J_D + J_U$ where J_D is the diagonal of J and J_U its nilpotent, zero trace super-diagonal. Not all $\delta_j = 1$ in J_U. The diagonal of $X_n^{\heartsuit}$ for $J_U = [H, X_n^{\heartsuit}]$ is a monotonically increasing sequence (see (4)) of natural numbers with repetitions, e.g., $\{0, 1, 2, 2, 2, 3, 4, 4, 5\}$. Then, one could describe any matrix $M = SJ_DS^{-1} + S[H, X_n^{\heartsuit}]S^{-1}$.

Example 4. The Pauli spin matrices are hermitian and have trace zero.

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Their common eigenvalue matrix Λ_P equals a commutator $\Lambda_P = [H, X]$

$$X = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\Lambda_P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Their eigenvector matrices are unitary, e.g., $E_j^{-1} = \bar{E_j}^T$,[3]

$$E_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad E_2 = \begin{bmatrix} \frac{\sqrt{2}\,\mathrm{i}}{2} & \frac{\sqrt{2}\,\mathrm{i}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $P_j = E_j \Lambda_P \bar{E}_j^T = E_j[H, X]\bar{E}_j^T$, for $j = 1, 2, 3$.

3. Toeplitz Matrices

Ye [6] has shown that every matrix is a product of Toeplitz matrices. Ye notes that every Toeplitz matrix is spanned by I and super (and sub-) diagonals of $1's$ which he calls the natural basis. Observe that these are just powers of H and its transpose, e.g., H^2 is the Toeplitz matrix with $1's$ on the second super diagonal. Let $H = H_4$ from (1). We can write the 4×4 elementary matrices $E_{4,1} = (H^T)^3$, $E_{3,1} = H(H^T)^3$, and $E_{4,2} = (H^T)^3 H$.

THEOREM 5. *Every matrix is a polynomial in* $\{H, H^T\}$.

PROOF. For $A = [a_{i,j}] \in \mathfrak{M}_n$, let $H = H_n$. The elementary matrix $E_{i,j} = H^{n-i}(H^T)^{n-1}H^{j-1}$, and $A = \sum \sum a_{i,j} E_{i,j}$ is a polynomial in $\{H, H^T\}$. □

This representation is not unique, since $E_{i,j} = (H^T)^{i-1}\, H^{n-1}\, (H^T)^{n-1}\, H^{j-1}$.

[3]The transpose of E is denoted by E^T. The complex conjugate is denoted by $\bar{E}$.

4. Conclusions We have presented an alternative approach to representing a zero trace matrix as a commutator. Since diagonalizable matrices are dense in $\mathcal{M}_n$, this representation often does not require the JNF. Example 3 shows that the nilpotent portion of any matrix has a representation as the commutator of H and a diagonal natural number matrix. Theorem 5 demonstrates that every matrix can be written as a polynomial in Toeplitz matrices.

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