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CATASTROPHE THEORIES

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Presented by J. Aczél, F.R.S.C.

0. The discussion about catastrophe theory seems not to be closed (see the book reviews [1],[2]). In fact the theory hitherto presented suffers from two conditions: (I) the condition of C^{∞} for the mappings involved, (II) the use of equivalence relations for describing the "catastrophic character" of a state. Especially the last does not meet the situation in economics. Take for instance the <u>principle</u> <u>of</u> <u>bounded</u> <u>growth</u>:

Consider for simplicity a system of states where a state is determined by one real parameter x. A move $x \rightsquigarrow x'$ from a state x to a state x' is called "tolerable" if

(t) $x' \geq x$ for $x < 0$ and $x' \geq 0$ for $x > 0$.

It is easily seen that there is no equivalence relation $\sim$ in the system of real numbers for which (t) would be equivalent to $x \sim x'$.

The following report of some papers of mine ([3],[4],[5]) will contribute to a clarification. The theory there developed -- I have called it also "catastrophe theory" although I don't insist on that name; "conflict theory" would be as good a name -- is not based on the conditions (I) and (II) and has its final setting in lattice theory. The main idea is the introduction of a mapping T by which one can decide whether a move $x \rightsquigarrow x'$ is "tolerable" or not. Specializations yield the known theories, in particular, if T is derived from an equivalence relation in the system of states.

1. The dynamical model (X,A,T).

Let X be a set of elements x,y, ...(the "states") which are subject to certain transitions

$$x \rightsquigarrow x' \ , \ x,x' \in X.$$

We don't need to know more about the mechanism of these transitions than the practical effect: We are given the

mapping of attainability $\qquad A : X \times \mathbb{R}^+ \rightarrow P(X)$

($\mathbb{R}^+$ is the set of all positive real numbers, P(X) is the set of all subsets of X), furthermore the

mapping of tolerability $\qquad T : X \rightarrow P(X)$

with the following meaning:

$$x' \in A(x,t)$$

means that a move $x \rightsquigarrow x'$ can happen within a time interval from t_0 to t_0+t of length t (independent of t_0).

$$x' \in T(x)$$

means that a move $x \rightsquigarrow x'$ is considered to be "tolerable".

Definition. An element $y \in X$ is called stable or non-catastrophic (in the system (X,A,T) if "there is no immediate danger of an intolerable move", or in terms of A and T, if

(s) $$\bigvee_{t>0} A(y,t) \subset T(y).$$

In applications one certainly is interested in an estimate of t, the "reprieve", but here we shall replace (s) by a "static" condition (i.e. one not containing the time parameter explicitly). This is possible if we ask some rather natural properties for A, namely: For all $x,x',x'' \in X$ and $t,t' \in T$ we have

(A1) $\qquad x \in A(x,t)$,

(A2) $\qquad x' \in A(x,t) \wedge x'' \in A(x',t') \Longrightarrow x'' \in A(x, t+t')$, (A3) $A(x,t) = \bigcup_{0<\tau<t} A(x,\tau)$.

From (A1),(A2) and (A3) it follows that X is a topological space with the sets $A(x,t)$, $t > 0$, as a base of open neighborhoods of x so that (s) is equivalent to "There is an open set G with $y \in G \subset T(y)$ ", or , by introducing the open kernel operator k of that topology in X ($k(Y)$ is the largest open subset of X contained in Y), is equivalent to

(s') $y \in k(T(y))$.

This is the "static" condition for the stability of an element $y \in X$. We see that if y is stable then $y \in T(y)$. It makes sense to demand

(T1) $\bigwedge_{x \in X} x \in T(x)$

generally; the meaning of (T1) is the <u>rule of the conservative</u> , " Rest is tolerable", and it is an essential part of the "conflict" character of the theory. Occasionally we shall add a second condition on T :

(T2) $\bigwedge_{x,x',x'' \in X} x' \in T(x) \wedge x'' \in T(x') \Rightarrow x'' \in T(x)$,

which I call the <u>syllogism of liberalism or tolerance</u>:

If $x \rightsquigarrow x'$ and $x' \rightsquigarrow x''$ are tolerable moves so is $x \rightsquigarrow x''$.

<u>2</u>. <u>Lattice theoretical models</u> [3].

<u>2.1</u>. It is natural to extend the definition (s') for stability to subsets Y of X. For this purpose we extend T to subsets Y of X by introducing the

<u>mapping</u> $h: P(X) \rightarrow P(X)$

with the definition $h(Y) := \bigcup\{T(x) : x \in Y\}$ for $Y \in P(X)$. Then (s') reads $y \in k(h\{y\})$ and so we get the extended

<u>Definition</u>: $Y \in P(X)$ is <u>stable with respect to</u> k,h , if

(s") $\qquad Y \subset k(h(Y)).$

With the assumptions (A) and (T), condition (s") refers to a structure $(P(X),k,h)$ where

1. $(P(X),\subset)$ is a complete lattice;
2. k is a kernel operator in $P(X)$ (i.e. $k: P(X) \to P(X)$, $k \leq id$ (that means $\bigwedge_{Y \in P(X)} k(Y) \subset Y$), k is isotone ($Y \subset Y' \Rightarrow k(Y) \subset k(Y')$) and k is idempotent (i.e. $k \circ k = k$); additionally k is topological what means that k is also $\cap$ -distributive;
3. h is a hull operator in $P(X)$, i.e. $h: P(X) \to P(X)$, $h \geq id$, h is isotone and idempotent.

Under the conditions 2. and 3. we call $(P(X),k,h)$ a topological $\mathcal{H}$ -structure on X with (s") as definition of stability (with no reference to some origin of k and h).

Remarks. 1. $h \geq id$ comes from (T1); as k and h act in opposite directions in $P(X)$ we have a "conflict situation". - 2. The idempotency of h is a consequence of (T2), and has the effect to reduce the stability of a point to a common continuity condition: Introducing the

mapping $\qquad h^+ : X \to H$

defined by $h^+(x) := h(\{x\})$, $x \in X$, and $H := \{h(\{x\}) : x \in X\}$, and the

topology ω in H

defined by " $G \subset H$ is open (in H) if for all $x,y \in X$

$$h(\{x\}) \in G \wedge y \in h(\{x\}) \Rightarrow h(\{y\}) \in G \text{ "},$$

we have the

Theorem [4] . In the topological $\mathcal{H}$ -structure $(P(X),k,h)$ $y \in X$ is stable iff h^+ is continuous at y with respect to ω in H. -

3. (T2) is in particular satisfied in the case where T is derived from an equivalence relation $\sim$ in X, i.e. if T(x) is the equivalence class of x with respect to $\sim$. In THOMs theory [6] for instance, X is a system of germs of C^∞-mappings of $\mathbb{R}^n$ into $\mathbb{R}^m$ considered under some equivalence relation defined by diffeomorphisms.

2.2. The general lattice theoretical model.

Definition. Any triple (L,k,h), where L is a complete lattice $(L,\leqslant)$, $k: L \to L$ is a kernel operator in L and $h: L \to L$ satisfies $h \geqslant id$ and is isotone, is called a (general) $\mathcal{H}$-structure; $x \in L$ is called $\mathcal{H}$-stable (with respect to k,h) if

$$(\mathcal{H}) \qquad x \leq k(h(x)).$$

Remark. The weakening in the assumptions on k and h as compared with those for topological $\mathcal{H}$-structures conforms to a minimum of requirements for theorems and constructions.

As to k we have to refer to a

LEMMA. If k is a kernel operator in the complete lattice L then the set F_k of all fixed elements f of k (i.e. with k(f) = f) is upward complete in L, i.e. L-sup L' exists in F_k for each $L' \subset F_k$. Inversely, every upward complete subset F in L is the F_k of a uniquely determined kernel operator k in L.

Two propositions ([3], [5]).

I. For each $\mathcal{H}$-structure (L,k,h) the set of all $\mathcal{H}$-stable elements is upward complete in L, and inversely, every upward complete subset of L is the set of all $\mathcal{H}$-stable elements of some $\mathcal{H}$-structure (L,k,h).

II. $y \in L$ is $\mathcal{H}$-stable with respect to k,h iff the interval

$[y,h(y)] := \{x : x \in L \wedge y \leq x \leq h(y)\}$ contains a fixed element of k.

3. Stabilizers.

Definition. Given a complete lattice L, h as in definition 2.2. and a subset $S \subset L$; a kernel operator k in L is called a stabilizer of S if each element of S is $\mathcal{H}$-stable with respect to k,h.

For instance, k = id is a stabilizer of L, so also for any $S \subset L$. Every kernel operator k', larger than a stabilizer k of S, $k' \geq k$, is also a stabilizer of S.

Construction of all stabilizers [5].

This is done by constructing the fixed elements of a stabilizer: We choose in each interval $[s,h(s)]$, $s \in S$, a non-void subset Q_s, form $F := \bigcup\{Q_s : s \in S\}$ and finally F^+, the upward completion of F in L. Then the kernel operator k with $F_k = F^+$ is a stabilizer of S, and evidently we get all stabilizers of S this way.

Remark. Examples (for instance on the lattice $([0,1],\leq)$ of real numbers [5])show that there may be no smallest stabilizer and that there may be stabilizers which are incomparable.

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ON THE GENERAL SOLUTION OF A FUNCTIONAL EQUATION WHICH OCCURS IN THE THEORY OF SINGULAR INTEGRAL EQUATIONS

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Abstract. The paper determines the most general solution f of the following functional equation

$$f(x+y)\{f(x)+f(y)-1\} = f(x)f(y) \quad ,$$

where f is a mapping of a subset of reals into the reals.

1. In the theory of certain kinds of singular integral equations the following functional equation is of importance [1]:

$$h(x+y)\{h(x)+h(y)\} - h(x)h(y) = 1 \ .$$

by substituting $h(x) = 2f(x)-1$ we get

$$(1) \qquad f(x+y)\{f(x)+f(y)-1\} = f(x)f(y) \ ,$$

where the unknown function f is defined on the reals. In the theory of singular integral equations only the fact plays a role, that the characteristic function of certain intervals satisfies the equation (1).

In the present paper we investigate the situation where the unknown function maps a given set $S \subset R$ into the reals. In this case the equation (1) has the following meaning: If x,y and $x+y$ belong to S, then (1) should be satisfied.

The following theorem holds:

Theorem 1. The most general solution $f : S \to R$ $(S \subset R)$, is the following

$$(2) \qquad f(x) = \begin{cases} 0 & \text{if } x \in S_0 \\ 1 & \text{if } x \in S_1 \\ 1/(1-g(x)) & \text{if } x \in S_2 \end{cases},$$

where S_0, S_1, S_2 are disjoint half-groupoids with $S = S_0 \cup S_1 \cup S_2$, such that the following conditions are fulfilled

$$(3a) \qquad S \cap (S_0 + S_2) \subset S_0$$

$$(3b) \qquad S \cap (S_1 + S_2) \subset S_1$$

and g is an arbitrary solution of the Cauchy functional equation

$$(4) \qquad g(x+y) = g(x)g(y)$$

for which $g(x) \neq 0,1$ $(x \in S_2)$.

Remarks. a) We call M a half-groupoid in S if $x,y \in M$ and $x+y \in S$ imply $x+y \in M$.

b) The sign + in (3a) and (3b) is used in the sense $T+U = \{t+u : t \in T, u \in U\}$.

c) Some of the sets S_i $(i=0,1,2)$ may be empty. In this case one of the conditions (3a) or (3b) (or both) is obviously satisfied.

d) If $0 \in S$, then $0 \in S_0 \cup S_1$ and the conditions (3a) and (3b) (3b) can be fulfilled only if $S_2 = \emptyset$. On the other hand, if S_2 is an arbitrary half-groupoid which does not contain 0, then we can always find a solution g of (4) for which $g(x) \neq 0,1$ $(x \in S_2)$.

Proof. Let f be a solution of (1) and define

$$S_0 = \{x:\ x\epsilon S, f(x)=0\} .$$

We see at once from (1) that S_0 is a half-groupoid. The complementary set $S_0^c=S\backslash S_0$ of S_0 is also a half-groupoid. Let us define

$$g(x) = 1 - (1/f(x)) \quad \text{if} \quad x\epsilon S_0^c .$$

g maps S_0^c into $R\backslash\{1\}$. Now we have

$$f(x) = 1/(1-g(x)) \qquad x\epsilon S_0^c .$$

Substituting this into (1), we see immediately that g satisfies (4) for every x,y and $x+y$ of S_0^c.

Let us consider now the following subsets of S_0^c:

$$S_1 = \{x:\ x\epsilon S, f(x)=1\} = \{x:\ x\epsilon S_0^c, g(x)=0\} ,$$

$$S_2 = S_0^c\backslash S_1 = \{x:\ x\epsilon S_0^c, g(x)\neq 0\} .$$

It is obvious, that both sets S_1 and S_2 are half-groupoids. The functional equation (1) implies the relation (3a) and from (4) follows (3b). The sets S_i $(i=0,1,2)$ are disjoint, this follows from their definitions.

Every function of the form (2) clearly satisfies the equation (1). So the theorem is proved.

From the remark d) we get immediately the following statement:

Corollary. If the domain of f contains the origin, then the most general solution of (1) is the characteristic function of a half-groupoid contained in S.

This corollary can be obtained also directly from (1). For $x=0$, equation (1) yields

$$f(y)\{f(y)-1\} = 0,$$

which implies the statement.

This corollary shows that the equation (1) has solutions similar to those of the functional equation

$$f(\tfrac{x+y}{2})f(xy) = f(x)f(y)$$

which occurs in the theory of waves and the general solutions of which were recently found by W. Benz [2].

2. The following problem was suggested by J. Aczél: Given an arbitrary set $S\subset R\setminus\{0\}$, is it always possible to cut it into three disjoint non-empty half-groupoids so that conditions (3a) and (3b) are fulfilled?

A partial answer is given in the following theorem:

Theorem 2. Let S be a set contained in $R\setminus\{0\}$ and let $V(S)\subset(R,Q,+,\cdot)$ be the subspace generated by S. If $\dim V(S)>2$, then it is possible to find three disjoint non-empty half-groupoids S_i $(i=0,1,2)$ for which the conditions (3a) and (3b) are fulfilled.

Proof. Let x_0,x_1,x_2 be three rational linearly independent numbers in S, and let us define $h_0=(x_0-x_1)/2$, $h_1=(x_0+x_2)/2$, $h_2=x_2$. We consider the following sets

$$H_1 = \{x\colon x=\sum_j c_j h_j,\ c_j\in Q,\ h_j\in H\setminus\{h_0\}\},\ \text{(here } \sum_j \text{ denotes a finite sum)}$$

$$H_0^+ = \{x\colon x=ch_0,\ c>0,\ c\in Q\},\ H_0^- = \{x\colon x=ch_0, c<0,\ c\in Q\}.$$

Then $R=H_1\cup(H_1+H_0^+)\cup(H_1+H_0^-)$. In this way we define $S_0=S\cap(H_1+H_0^+)$, $S_1=S\cap(H_1+H_0^-)$, $S_2=S\cap H_1$. Now we can verify at once, that $x_i\in S_i$ $(i=0,1,2)$, $S=S_0\cup S_1\cup S_2$, S_i are non-empty disjoint half-groupoids for which (3a) and (3b) holds.

Remark. In a more general way we can state that the answer to the above question is certainly affirmative if a maximal hyperplane H_1 exists for which $S\cap H_1\neq\emptyset$ and which divides all other elements of S into two disjoint parts.

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INFORMATION FUNCTIONS ON OPEN DOMAINS

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Abstract. The n-dimensional fundamental equation of information of degree α is considered in the interior of its usual domain. Its general solution is presented when the weight of α is not equal to one.

Introduction. The operations addition, subtraction, multiplication and division on the Euclidean space R^n are performed co-ordinatewise. For $x = (x_1,x_2,\dots,x_n)$ with $x_i > 0$ and $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)$ we define

$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n} .$$

We denote the open interval $]0,1[^n$ by I and abbreviate $(a,a,\dots,a)$ simply as a. We call a function $f : I \to R$ an n-place information function of degree α if it satisfies

$$(1) \qquad f(x) + (1-x)^\alpha f(\tfrac{y}{1-x}) = f(y) + (1-y)^\alpha f(\tfrac{x}{1-y}) ,$$

which is supposed to hold for all (x,y) in the open domain

$$D^o = \{(x,y)\,|\,x,y\in I \text{ with } x+y\in I\} .$$

This equation is motivated, discussed and well documented by J. Aczél and Z. Daróczy in [2]. However in such treatments it is supposed to hold not only on D^o, but at some boundary points of D^o as well and thus allowing us the convenience of boundary substitutions. In view of some applications of 2-place information functions, J. Aczél has raised the problem of determining the general solutions of (1) on D^o alone. The case $n = 1$ and $\alpha = 0$ is solved in [3] and the case $n = 1$ and $\alpha = 1$ is solved in [5]. The purpose of this note is to find the general solution of (1) on D^o for all n and all α with $\sum\alpha_i \neq 1$.

<u>Theorem</u>. Under the assumption that $\sum\alpha_i \neq 1$, the general solution of the n-dimensional fundamental equation (1) on D^o is given by

$$(2)\qquad \begin{cases} f(x) = ax^\alpha + b(1-x)^\alpha - b & \text{if } \alpha \neq 0 \\ f(x) = a + L(1-x) & \text{if } \alpha = 0 \end{cases}$$

on I, where a, b are constants and L is a solution of the Cauchy equation

$$(3)\qquad L(xy) = L(x) + L(y) \quad \text{for all } x,y \in I.$$

<u>Proof of the theorem</u>. We consider the function $\Delta : D^o \to R$ defined by

$$(4)\qquad \Delta(x,y) = f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) - f(x+y) \quad ,$$

and claim that it satisfies, for all $x,y,z,t \in I$ with $x+y+z \in I$,

$$(5)\qquad \Delta(x,y) = \Delta(y,x)$$

$$(6)\qquad \Delta(x,y) + \Delta(x+y,z) = \Delta(x,y+z) + \Delta(y,z)$$

$$(7)\qquad \Delta(tx,ty) = t^\alpha \Delta(x,y) \quad .$$

In fact, (5) follows directly from (1). In order to get (6) and (7) we consider the following calculations using (1)

$$\Delta(x,y)+\Delta(x+y,z)$$

$$= f(x)+(1-x)^{\alpha}f(\frac{y}{1-x})-f(x+y)+f(x+y)+(1-x-y)^{\alpha}f(\frac{z}{1-x-y})-f(x+y+z)$$

$$= f(x)+(1-x)^{\alpha}[f(\frac{y}{1-x})+(1-\frac{y}{1-x})^{\alpha}f(\frac{z/(1-x)}{1-y/(1-x)})]-f(x+y+z)$$

$$= f(x)+(1-x)^{\alpha}[\Delta(\frac{y}{1-x},\frac{z}{1-x})+f(\frac{y+z}{1-x})]-f(x+y+z)$$

$$= \Delta(x,y+z)+(1-x)^{\alpha}\Delta(\frac{y}{1-x},\frac{z}{1-x}) \quad .$$

By (5), the symmetry of Δ, the left side of the above equation is symmetric in x and y, while its right side is symmetric in y and z. Hence both sides are symmetric in x and y as well as in y and z, and are therefore fully symmetric in x, y and z. The full symmetry of the left side yields (6), and comparison of the right side with that of (6) reveals $\Delta(y,z) = (1-x)^{\alpha}\Delta(y/(1-x),z/(1-x))$, which is (7) after renaming the variables.

We can extend Δ to $\bar{\Delta} :]0,\infty[^n \times]0,\infty[^n \to R$ by

$$\bar{\Delta}(x,y) = s^{\alpha}\Delta(\frac{x}{s},\frac{y}{s}) \quad \text{where} \quad (\frac{x}{s},\frac{y}{s}) \in D^{o} \quad .$$

Because of (7), $\bar{\Delta}$ is unambiguously defined. It is not difficult to see that $\bar{\Delta}$ satisfies (5) to (7) for all $x, y, z, t \in]0,\infty[^n$.

We now proceed to determine $\bar{\Delta}$. By (7) we obtain the diagonal of $\bar{\Delta}$,

(8) $\bar{\Delta}(x,x) = cx^{\alpha}$ where $c = \bar{\Delta}(1,1)$ is a constant.

Put $z = x + y$ into (6) and then use (5) to (7) repeatedly we get

$$\bar{\Delta}(x,y)+\bar{\Delta}(x+y,x+y) = \bar{\Delta}(x,x+2y)+\bar{\Delta}(y,x+y)$$

$$= [\bar{\Delta}(x,x+2y)+\bar{\Delta}(x,2y)]-\bar{\Delta}(x,2y)+[\bar{\Delta}(x,y)+\bar{\Delta}(x+y,y)]-\bar{\Delta}(x,y)$$

$$= [\bar{\Delta}(x,x)+\bar{\Delta}(2x,2y)]-\bar{\Delta}(x,2y)+[\bar{\Delta}(x,2y)+\bar{\Delta}(y,y)]-\bar{\Delta}(x,y)$$

$$= \bar{\Delta}(x,x)+\bar{\Delta}(y,y)+(2^{\alpha}-1)\bar{\Delta}(x,y) \quad .$$

Under the assumption that $\sum\alpha_i \neq 1$, the above leads to

$$\bar{\Delta}(x,y) = \frac{1}{2-2^{\alpha}}(\bar{\Delta}(x,x)+\bar{\Delta}(y,y)-\bar{\Delta}(x+y,x+y))$$

which is a representation of $\bar{\Delta}$ through its diagonal. This, along with (8), gives the explicit form of $\bar{\Delta}$.

Since $\bar{\Delta}$ is an extension of Δ, we get in particular

(9) $$\Delta(x,y) = ax^{\alpha} + ay^{\alpha} - a(x+y)^{\alpha} \quad \text{on } D^{o}$$

where $a = c/(2-2^{\alpha})$ is a constant. With this, (4) is reduced to

(10) $$\phi(x) + (1-x)^{\alpha}\phi(\frac{y}{1-x}) - \phi(x+y) = 0$$

on D^{o}, where

(11) $$\phi(x) := f(x) - ax^{\alpha} \quad .$$

With the replacement $y = (1-x)z$, we rewrite (10) as

(12) $$\phi(x) + (1-x)^{\alpha}\phi(z) = \phi(x+z-xz)$$

for all $x, z \in I$. There are two cases.

In case $\alpha \neq 0$ we use the symmetry of (12) on the right to get $\phi(x) + (1-x)^{\alpha}\phi(z) = \phi(z) + (1-z)^{\alpha}\phi(x)$ and by fixing in it $z = z_0$ with $(1-z_0)^{\alpha} \neq 1$ we get explicitly $\phi(x) = b(1-x)^{\alpha} - b$ where

b is a constant. This and (11) give f as asserted in (2).

In case $\alpha = 0$ equation (12) is reduced to $\phi(x) + \phi(z) = \phi(x+z-xz)$ and is equivalent to the Cauchy equation (3) where $L(x) := \phi(1-x)$. Thus by (11), $f(x) = ax^{\alpha} + \phi(x) = a + L(1-x)$ as asserted in (2).

The verification that functions defined by (2) indeed satisfy (1) is straightforward. □

Remark. The general solution of the Cauchy equation (2) is given by $L(x) = \Sigma \ell_i(x_i)$, where each $\ell_i :]0,1[\to R$ is a solution of the equation $\ell(uv) = \ell(u) + \ell(v)$. Relevant materials can be found in [1, 4].

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A MIXED THEORY OF INFORMATION. VII.
INSET INFORMATION FUNCTIONS OF ALL DEGREES

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Abstract. In the mixed theory of information (cf. [4,5] the results of which we will not use here), measures of information, e.g. the "inset" entropies, may depend explicitly upon events (messages, outcomes of an experiment, market situations, etc.), not only upon their probabilities. The characterization of (semi-)symmetric recursive inset entropies leads to inset information functions of degree α (IIF_α). All *measurable* IIF_1's have been determined in [4], all *symmetric* IIF_α's ($\alpha\neq 0,1$) in [5,1]. Here we determine all IIF_α's for *all* real α *without any additional supposition*, in particular also all IIF_0's.

1. An *inset information function of degree* α (IIF_α) is a solution of the equation

$$(1)\begin{cases} f(E_1\cup E_2,E_3;x) + (1-x)^\alpha f(E_1,E_2;\frac{y}{1-x}) = \\ f(E_1\cup E_3,E_2;y) + (1-y)^\alpha f(E_1,E_3;\frac{x}{1-y}) \\ \text{for all } (x,y)\in D;\ E_1,E_3,E_3\in\mathbb{B}\quad (E_i\cap E_j=\emptyset \text{ for } i\neq j)\ . \end{cases}$$

Here $\mathbb{B}$ is a ring of sets. This means that $E\in\mathbb{B}$ and $F\in\mathbb{B}$ imply $E\cup F\in\mathbb{B}$, $E\setminus F\in\mathbb{B}$, thus also $E\cap F\in\mathbb{B}$ and $\emptyset\in\mathbb{B}$, cf. [7]. By definition, $D = \{(x,y)\,|\,x,y\in[0,1[,x+y\leq 1\}$. The IIF_α is symmetric, if

$$(2)\quad f(E_1,E_2;x) = f(E_2,E_1;1-x)\quad (E_1,E_2\in\mathbb{B},\ E_1\cap E_2=\emptyset,\ x\in[0,1])\ .$$

We will determine here all IIF_α's (symmetric or not). - In the classical, purely probabilistic case, f is independent of its first two variables and we get

$$(3)\quad f(x) + (1-x)^\alpha f(\tfrac{y}{1-x}) = f(y) + 1-y)^\alpha f(\tfrac{x}{1-y}) \text{ on } D.$$

We use the result ([1,2]) that the validity of the generalization

$$(4)\quad f_1(x) + (1-x)^\alpha f_2(\tfrac{y}{1-x}) = f_3(y) + (1-y)^\alpha f_4(\tfrac{x}{1-y}) \quad \text{on } D,$$

of equation (3) <u>implies</u>

$$(5)\quad \begin{cases} f_2(t) = cf(t)+\kappa(\alpha)d(t)+A_2t^\alpha+B_2, \\ f_4(t) = cf(t)-\kappa(\alpha)d(t)+A_4t^\alpha+B_4 \quad (t\in[0,1]) \text{ for } \alpha\neq 0 \end{cases},$$

$$(6)\quad f_2(t) = \begin{cases} A_2 & (t=0) \\ B_2 & (t\in]0,1[) \\ C_2 & (t=1) \end{cases}, \qquad f_4(t) = \begin{cases} A_4 & (t=0) \\ B_4 & (t\in]0,1[) \\ C_4 & (t=1) \end{cases} \text{ for } \alpha=0,$$

where $\kappa(2)=1$, $\kappa(\alpha)=0$ for $\alpha\neq 2$, $c,A_2,B_2,C_2,A_4,B_4,C_4$ are constants, d a derivation $[d(x+y)=d(x)+d(y),\ d(xy)=xd(y)+yd(x)]$ and f is a solution of (3) which satisfies also $f(x)=f(1-x)$ $(x\in[0,1])$.

From [4,secs.3.5 and 6.2] we know that under these circumstances (up to a multiplicative constant, which can be submerged in c)

$$(7)\quad f(x) = x^\alpha + (1-x)^\alpha - 1 \quad (x\in[0,1];\ 0^\alpha := 0) \text{ if } \alpha\neq 0,1$$

$$(8)\quad \begin{cases} f(x) = L(x) + L(1-x) \quad \text{if } \alpha=1, \text{ where} \\ L(xy) = xL(y) + yL(x) \text{ and } L(x)\not\equiv 0 \quad (x,y\in[0,1]) \end{cases}.$$

<u>2</u>. Fixing E_1,E_2 and E_3 temporarily in (1), we see that it goes over into (4) with the notations

$$(9)\quad f_2(t) = f(E_1,E_2;t) \quad f_4(t) = f(E_1,E_3;t) \quad (t\in[0,1]), \text{ etc.}$$

Comparing (9) and (6) and letting E_1,E_2,E_3 vary again, we get

$$(10)\quad f(E_1,E_2;t) = \begin{cases} A(E_1,E_2) & \text{for } t=0 \\ B(E_1,E_2) & \text{for } t\in\,]0,1[\qquad \text{if } \alpha=0 \\ C(E_1,E_2) & \text{for } t=1 \end{cases} .$$

If $\alpha\neq 0$, comparison of (9) and (5) gives, with variable E_1, E_2, E_3, that $\kappa(\alpha)d(t)=0$, c depends only upon E_1 and

$$(11)\quad f(E_1,E_2;t) = c(E_1)f(t) + A(E_1,E_2)t^\alpha + B(E_1,E_2) \quad .$$

We substitute (11) into (1) with $y=0$ (note that $f(0)=0$ in (7) and (8) because $0^\alpha = L(0)=0$):

$$(12)\begin{cases} c(E_1\cup E_2)f(x)+A(E_1\cup E_2,E_3)x^\alpha+B(E_1\cup E_2,E_3)+B(E_1,E_2)(1-x)^\alpha \\ = B(E_1\cup E_3,E_2)+c(E_1)f(x)+A(E_1,E_3)x^\alpha+B(E_1,E_3) \end{cases}$$

($E_j\in\mathbb{B}, E_i\cap E_j=\emptyset$ for $i\neq j=1,2,3$). Now the $f(x)$'s [cf. (7), (8)], appearing in (12), are linearly independent of x^α and 1, but $(1-x)^\alpha$ is not linearly independent of these 3 functions. Taking this into consideration and comparing coefficients of x^α and also those of $f(x)$, we get

$$(13)\quad A(E_1\cup E_2,E_3)-B(E_1,E_2) = A(E_1,E_3) \quad \text{for all } \alpha\neq 0$$

$$(14)\begin{cases} c(E_1\cup E_2)+B(E_1,E_2) = c(E_1) & \text{for } \alpha\neq 0,1, \\ c(E_1\cup E_2) = c(E_1) & \text{for } \alpha=1 \ . \end{cases}$$

We put into (13) first $E_3=\emptyset$, $b(E):=A(E,\emptyset)$, then $E_1=\emptyset$, $a(E):=A(\emptyset,E)-b(\emptyset)$, and get for $\alpha\neq 0$

$$(15)\quad B(E_1,E_2) = b(E_1\cup E_2)-b(E_1),\ A(E_2,E_3) = b(E_2)+a(E_3) \ .$$

Finally, we put $E_1=\emptyset$ into (14) and obtain

$$(16)\quad c(E_2) = \gamma-b(E_2) \text{ if } \alpha\neq 0,1 \quad \text{and} \quad c(E_2) = \gamma \text{ if } \alpha=1 \ .$$

As to (10), substitution into (1) gives, for $x,y\in]0,1[$ or $x=0$, $y\in]0,1[$ or $y=1-x\in]0,1[$,

$$(17)\quad B(E_1\cup E_2,E_3)+B(E_1,E_2) = B(E_1\cup E_3,E_2)+B(E_1,E_3)\ ,$$

$$(18)\quad A(E_1\cup E_2,E_3)+B(E_1,E_2) = B(E_1\cup E_3,E_2)+A(E_1,E_3)\ ,$$

$$(19)\quad B(E_1\cup E_2,E_3)+C(E_1,E_2) = B(E_1\cup E_3,E_2)+C(E_1,E_3)\ ,$$

respectively. In [6] it has been proved that the general symmetric $[B(E_1,E_2)=B(E_2,E_1)]$ solution of (17) is of the form $B(E_1,E_2)=g(E_1)+g(E_2)-g(E_1\cup E_2)$. Putting $E_1=\emptyset$ into (17), we see that $B(E,F)-B(\emptyset,F)$ is symmetric and satisfies (17) too, so

$$(20)\quad B(E_1,E_2) = g(E_1)+h(E_2)-g(E_1\cup E_2)\ ,$$

$[h(E):=g(E)+B(\emptyset,E)]$. Subtraction of (17) from (18) or (19) gives $(A-B)(E_1\cup E_2,E_3)=(A-B)(E_1,E_3)$ and $(C-B)(E_1,E_2)=(C-B)(E_1,E_3)$. So

$$(21)\quad (A-B)(E_2,E_3) = \phi(E_3) \quad\text{and}\quad (C-B)(E_1,E_2) = \psi(E_1)\ .$$

3. We summarize (7), (8), (11), (15), (16) and (10), (20), (21) [we are writing $g=\gamma-b$, $h=\gamma+a$ in (22), $g=-b$, $h=a$ in (23) and $e=h+\phi$, $k=g+\psi$ in (24)]:

Theorem. The general solution of (1) is given by

$$(22)\quad f(E_1,E_2;t) = g(E_1)(1-t)^\alpha+h(E_2)t^\alpha-g(E_1\cup E_2)\ \text{for}\ \alpha\neq 0,1\ ,$$

$$(23)\quad f(E_1,E_2;t) = \gamma L(x)+\gamma L(1-x)+g(E_1)(1-t)+h(E_2)t-g(E_1\cup E_2)\ \text{for}\ \alpha=1$$

$$(24)\quad f(E_1,E_2;t) = \begin{cases} g(E_1)+e(E_2)-g(E_1\cup E_2) & (t=0) \\ g(E_1)+h(E_2)-g(E_1\cup E_2) & (t\in]0,1[) \\ k(E_1)+h(E_2)-g(E_1\cup E_2) & (t=1) \end{cases}\quad \text{for}\ \alpha=0\ ,$$

$(E_1,E_2\in\mathbb{B}, E_1\cap E_2=\emptyset,\ t\in[0,1])$ where γ is an arbitrary constant, g,h,k,e are arbitrary real valued functions on $\mathbb{B}$ and $L\neq 0$ is an arbitrary solution of the second equation in (8). - Both (1) and (2) hold iff $g=h$ and $e=k$ in (22), (23) and (24).

Indeed, substitution shows that, under these circumstances, (22), (23) and (24) always satisfy (1) [or (1) and (2)].

Note: While $f(x)$ in (7) is clearly independent of the E's even in the situation described at the beginning of section 2, it looks as if the $f(x)$ in (8) could depend upon E_1 (but not upon E_2 as comparison of (9) and (5) shows), because the general solution of the second equation in (8) depends on the choice of values on a Hamel basis, which may depend upon E_1. But (13) [and (14) for $\alpha\neq 0,1$] and thus (15) would still hold and, putting (15) back into (12) for $\alpha=1$, we would get $c(E_1\cup E_2)f_{E_1\cup E_2}(x) = c(E_1)f_{E_1}(x)$ or, for $E_1=\emptyset$, $E_2=E$, $c(E)f_E(x) = \gamma f(x)$ with one fixed function f of the form (8), which is exactly what we have above [cf. (23) and the second equations in (14) and (16)].

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A MAXIMUM PRINCIPLE IN IR^3

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In this paper we extend the results of Weinberger[4] and Sather[3], so that the coefficients in the given equation are allowed to depend on t, and x. Through this paper the influence of Douglis [1] is apparent. See [2].

Assume the equation

$$(+) \quad Lv=k(t).(v_{rr}+(1/r^2).v_{pp})+v_{tt}+(k(t)/r).v_r$$
$$+b_1.v_r+a.v_t+(b_2+m).v=f,$$

where $k(t)<o$ for $t<o$, and all the coefficients of (+) are continuously differentiable functions of x,t, in D, such that

$$(1) \quad x=(x_1,x_2),x_1=r.\cos p,x_2=r.\sin p,$$

$$(2) \quad D=\{(x,t):T_o\leq t\leq T<o,\ x_1^2+x_2^2<(\int_t^T(-k(s))^{1/2}.ds)^2\},$$

$f\in C^o(\bar{D})$, with initial conditions

$$(++) \quad v(r,p,T_o)=o, v_t(r,p,t=T_o)=g(r,p),$$

$g\in C^o(\bar{D})$. Let S_o be the portion of the boundary of D, which lies in the plane $t=T_o$, and S_4 be the remainder of the boundary D, which is a characteristic conoid with respect to the above-mentioned equation.

A MAXIMUM PRINCIPLE FOR L. *Assume that* $v(r,p,t)\in C^2(\bar{D})$ *satisfies the differential inequality* $Lv=f\leq o$, *with initial conditions* (++), *such that* $g\leq o$, *and* $D\subset IR^3$ *is defined above. If the following conditions hold in* D

$$(A): d/dr(w.(k'/2(-k)^{1/2}+(b_1-B)))-d/dr(d^+w).(-k)^{1/2}\geq o,$$
$$B=\int_o^r(b_1)_r.dr,\ d^+=\partial/\partial t+(-k)^{1/2}.\partial/\partial r,\ a=k^{-1}.\int_o^r b_1.dr-(-k)^{-1/2}$$
$$.(b_1-B),\ (b_2)_p-m+\int_o^r(b_1)_p/2k.dr\geq o,\ w=r^{1/2}.(-k)^{-1/4}$$

$.\exp(\int_o^r (1/2k).b_1.dr)$, *then* $v \leq o$ *in* $D \subset IR^3$.

PROOF. Let S_4^e be the part of S_4, where $r_o > e$, such that

$$(3) \qquad r_o = \int_{T_o}^{T} (-k(s))^{1/2}.ds \ , \ e = \int_{T^o}^{T} (-k(s))^{1/2}.ds, \qquad T_o \leq t \leq T^o \leq T < o \ .$$

The direct characteristic conoid S_4 as well as the truncated one S_4^e are generated by the bicharacteristics of space time.

Assume the operators

$$(4) \qquad R_e v = \int_e^{r_o} dr. \oint v(t,r,p)dp \quad \text{on} \quad S_4^e,$$

and define

$$(5) \qquad Rv = \lim_{e \to o} R_e v,$$

when the limit exists.

By applying (4-5) and integrating by parts we find the following *fundamental integral identity*

$$\begin{aligned}(6) \qquad RwLv = & -\oint dp.\int_o^{r_o} v(P).(d/dr\ (d^+w).dr)(-k(s))^{1/2} \\ & -(-k(s))^{1/2}.\oint wd^+v|_Q.dp - (-k(s))^{1/2}.\oint(v-v(P))d^+w|_Q dp \\ & -\oint dp.\int_o^{r_o}(v-v(P)).(d/dr(w(k'/2(-k)^{1/2}+(b_1-B)))).dr \\ & +\oint w(k'/2(-k)^{1/2}+(b_1-B)).(v-v(P))|_Q\ dp \\ & +v.\oint dp.\int_o^{r_o}((d/dr\ d^+w).(-k)^{1/2}+m.w-(w.b_2)_p).dr,\end{aligned}$$

where $Q=(r_o,p,T_o), P=P(o,o,T(<o)), s=t$.

Let P be the point with the smallest t-coordinate T for which v vanishes. The identity (6) implies

$$(7) \qquad o \geq -\oint w.v_t|_Q\ dp,$$

which is a condradiction. **Q.E.D.**

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THE CAYLEY ALGEBRA AND LINEAR TRANSFORMATIONS OF $\mathbb{R}^8$

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Presented by G. de B. Robinson, F.R.S.C.

Abstract: By a classical result every linear transformation of $\mathbb{R}^4$ can be written as a sum of four transformations of the form

$$\varphi(x) = a\,x\,b,$$

where the product is taken in the algebra of quaternions. In this paper it is shown that a similar result holds for linear transformations of $\mathbb{R}^8$, if the algebra of quaternions is replaced by the Cayley algebra. The result is applied to construct a canonical isomorphism from the Clifford algebra of a 6-dimensional real vector space with a negative definite inner product to the algebra of linear transformations of $\mathbb{R}^8$.

1. The Cayley algebra.

Let $\mathbb{R}^8$ be an 8-dimensional Euclidean space. Choose a complex structure $\mathcal{J}$ in $\mathbb{R}^8$ (that is, an isometry $\mathcal{J}$ which satisfies the relation $\mathcal{J}^2 = -\iota$) and use it to make $\mathbb{R}^8$ into a 4-dimensional complex vector space $\mathbb{C}^4$. Introduce a Hermitian inner product in $\mathbb{C}^4$ by setting

$$(x,y)_H = (x,y) + i(\mathcal{J}x,y)$$

Next, fix a unit vector e in $\mathbb{C}^4$. Then its orthogonal complement is a 3-dimensional complex subspace $\mathbb{C}^3$ of $\mathbb{C}^4$. Let Δ denote the (complex) normed determinant function in $\mathbb{C}^3$. Then, given $a \in \mathbb{C}^3$ and $b \in \mathbb{C}^3$, there is a unique vector $a \times b \in \mathbb{C}^3$ such that the identity

$$\Delta(x, a, b) = (x, a \times b)_H$$

holds. It is called the <u>complex cross product</u> in $\mathbb{C}^3$ and has the following properties:

$$\begin{aligned} (\lambda a)\times b &= \bar\lambda\, a\times b \\ a\times\lambda b &= \bar\lambda\, a\times b \qquad \lambda\in\mathbb{C} \\ a\times b &= -\,b\times a \\ (a\times b)\times c &= (a,c)_H\, b-(b,c)_H\, a \end{aligned}$$

Next, define a multiplication in $\mathbb{R}^8$ as follows:

$$\begin{aligned} p\cdot q &= -(p,q)_H\, e + p\times q \qquad p,q\in\mathbb{C}^3 \\ (\lambda e)q = \lambda q, &\quad e(\lambda q)=\bar\lambda q \qquad \lambda,\mu\in\mathbb{C} \\ (\lambda e)(\mu e) &= (\lambda\mu)e \end{aligned}$$

It makes $\mathbb{R}^8$ into a <u>real</u> (non-associative) 8-dimensional algebra, called the <u>Cayley algebra</u>. The conjugate of an element $x\in\mathbb{R}^8$ is defined by

$$\bar x = -\xi e + y \quad \text{where} \quad x=\xi e+y\,, \quad \xi\in\mathbb{R},\ y\in\mathbb{R}^7.$$

It satisfies

$$\bar x\cdot x = x\cdot\bar x = (x,x)\,e.$$

This relation shows that every non-zero element x has a (left and right) inverse and so the Cayley algebra is a division algebra. The (real) linear map $x\to\bar x$ is called the <u>conjugation</u> and will be denoted by T.

<u>2. The map Φ</u>. Fix two vectors a and b and consider the linear transformation $\Phi_{a,b}$ of $\mathbb{R}^8$ given by

$$\Phi_{a,b}(x) = (ax)\,\bar b\,, \qquad x\in\mathbb{R}^8.$$

In particular,

$$\Phi_{a,e}(x) = ax\,, \quad \Phi_{e,b}(x) = x\,\bar b$$

<u>Theorem</u>. Let $\Phi:\mathbb{R}^8\otimes\mathbb{R}^8\to\mathbb{R}^8$ be the linear map defined by

$$\Phi(a\otimes b) = \Phi_{a,b}.$$

Then Φ is a linear isomorphism.

To outline the proof of the theorem observe first that $\dim \mathbb{R}^8 \otimes \mathbb{R}^8 = 64 = \dim L(\mathbb{R}^8)$ and so it is sufficient to show that Φ is surjective. This will be done by constructing certain projection maps P_b^a and Q_b^a which are contained in the image of Φ and which generate the space $L(\mathbb{R}^8)$.

It will be convenient to introduce the maps

$$\Phi^+(a \otimes b) = \frac{1}{2}\, \Phi(a \otimes b + ia \otimes ib)$$

and

$$\Phi^-(a \otimes b) = \frac{1}{2}\, \Phi(a \otimes b - ia \otimes ib)$$

Then, clearly, $\operatorname{Im} \Phi^+ \subset \operatorname{Im} \Phi$ and $\operatorname{Im} \Phi^- \subset \operatorname{Im} \Phi$.

3. The decomposition of $L(\mathbb{R}^{2n})$. A linear transformation φ of a complex vector space $\mathbb{C}^n$ will be called antilinear, if

$$\varphi(ix) = -i\varphi(x) \qquad x \in \mathbb{C}^n$$

The antilinear transformations of $\mathbb{C}^n$ form a n-dimensional complex vector space denoted by $AL(\mathbb{C}^n)$. On the other hand, we have the space $L(\mathbb{C}^n)$ of complex linear transformations and the space $L(\mathbb{R}^{2n})$ of linear transformations of $\mathbb{R}^{2n}$ (the underlying real vector space). Now let $\varphi \in L(\mathbb{R}^{2n})$ and set

$$\varphi_1 = \frac{1}{2}(\varphi - J\varphi J) \quad \text{and} \quad \varphi_2 = \frac{1}{2}(\varphi + J\varphi J).$$

Then

$$\varphi_1 + \varphi_2 = \varphi$$

This yields a direct decomposition

$$L(\mathbb{R}^{2n}) = L(\mathbb{C}^n) \oplus AL(\mathbb{C}^n)$$

4. The transformations P, Q and S. Fix a and b in $\mathbb{C}^4$ and consider the transformations

$$P_b^a(x) = (x,a)_H\, b, \quad Q_b^a(x) = (a,x)_H\, b.$$

Then P^c_b is complex linear while Q^a_b is antilinear. Moreover we have the relations

$$P^{\lambda a}_b = \bar{\lambda}\, P^a_b \quad \text{and} \quad Q^{\lambda a}_b = \lambda\, Q^a_b, \qquad \lambda \in \mathbb{C}$$

The transformations P^e_e and Q^e_e will be simply denoted by P and Q. Observe that if $p \in \mathbb{C}^3$, $q \in \mathbb{C}^3$ then the maps P^p_q and Q^p_q restrict to linear transformations of $\mathbb{C}^3$.

Next, fix $r \in \mathbb{C}^3$ and set

$$S_r(x) = r \times \pi(x) \qquad x \in \mathbb{C}^4$$

where $\pi : \mathbb{C}^4 \to \mathbb{C}^3$ is the complex projection. Then S_r is antilinear. The triple identity for the complex cross-product yields the formula

$$S_{p\times q} = \frac{1}{2}\left(Q^p_q - Q^q_p\right) \qquad p, q \in \mathbb{C}^3.$$

Proposition I. Let $p, q, r \in \mathbb{C}^3$. Then the following relations hold:

$$P^q_p = \underline{\Phi}^+(p \otimes q) - (p,q)_H\, I$$

$$Q^q_p = \underline{\Phi}^-(p \otimes q - q \otimes p) + \frac{1}{2} S_{p\times q}$$

$$P^e_r = \underline{\Phi}^+(e \otimes r)$$

$$Q^e_r = \underline{\Phi}^-(r \otimes e)$$

Corollary. $P^r_e \in \operatorname{Im} \Phi$ and $Q^e_r \in \operatorname{Im} \Phi$.

5. The co-cross product. To show that the transformations S_r, P^e_r and Q^r_e are in the image of Φ let $\mathbb{R}_6$ denote the underlying real vector space of $\mathbb{C}^3$. Then the cross-product determines a dual map

$$\Delta : \mathbb{R}^6 \to \mathbb{R}^6 \otimes \mathbb{R}^6$$

called the co-cross product. It can be written in the form

$$\vartheta(r) = \sum_\nu e_\nu \otimes (r \times e_\nu) + \sum_\nu \tilde{e}_\nu \otimes (r \times \tilde{e}_\nu),$$

where e_1, e_2, e_3 is an orthonormal basis of $\mathbb{C}^3$ and $\tilde{e}_\nu = ie_\nu$.

Proposition II. Let $r \in \mathbb{C}^3$ and set $\theta_r = \frac{1}{2}\Phi(\vartheta(r))$. Then

$$S_r = \theta_r + 2\underline{\Phi}^-(e \otimes r)$$

$$P_r^e = \theta_r + \underline{\Phi}^-(e \otimes r)$$

$$Q_e^r = S_r - \underline{\Phi}^+(r \otimes e).$$

Corollary. The maps S_r, P_r^e and Q_e^r are contained in $\operatorname{Im}\Phi$.

6. The projections P and Q and the conjugative T

Proposition III. Set

$$z_1 = \sum_\nu e_\nu \otimes e_\nu \quad \text{and} \quad z_2 = \sum_\nu e_\nu \otimes \tilde{e}_\nu$$

where e_1, e_2, e_3 is an orthonormal basis of $\mathbb{C}^3$ and $\tilde{e}_\nu = ie_\nu$.

Then

$$P = \underline{\Phi}^+(e \otimes e) \qquad iP = \underline{\Phi}^+(j \otimes e), \ (j = ie)$$

$$Q = \frac{1}{3}\left[2\underline{\Phi}^-(e \otimes e) - \underline{\Phi}^+(z_1)\right]$$

$$iQ = \frac{1}{3}\left[2\underline{\Phi}^-(j \otimes e) + \underline{\Phi}^+(z_2)\right]$$

$$T = -\frac{1}{3}\left[\underline{\Phi}^-(e \otimes e) + \underline{\Phi}^+(z_1)\right]$$

$$iT = \frac{1}{3}\left[\underline{\Phi}^+(j \otimes e) + \underline{\Phi}^+(z_2)\right].$$

Corollary: The maps P, Q, T, iP, iQ, S,T are contained in Im Φ.

Propositions I, II and III show that for all $a, b \in \mathbb{C}^4$

$$P^a_b \in \mathrm{Im}\,\Phi \quad \text{and} \quad Q^a_b \in \mathrm{Im}\,\Phi.$$

7. Proof of the theorem: Consider first a complex linear transformation ϕ of $\mathbb{C}^4$. Then ϕ is a complex linear combination of 16 transformations P^a_b. Since $P^a_{ib} = iP^a_b$ it follows that ϕ can be written as a real linear combination of 32 transformations P^a_b. It follows that $L(\mathbb{C}^4) \subset \mathrm{Im}\,\Phi$. The same argument shows that $AL(\mathbb{C}^4) \subset \mathrm{Im}\,\Phi$. Thus the map Φ is surjective and hence a linear isomorphism.

Corollary. Every linear transformation of $\mathbb{R}^8$ can be written as a sum of eight transformations of the form $(ax)b$.

8. Application to the Clifford algebra C_6^-. Let C_6^- denote the Clifford algebra over a 6-dimensional real vector space with a negative definite inner product. With the help of structure theorems for Clifford algebras it can be shown that C_6^- is isomorphic to the algebra of linear transformations of $\mathbb{R}^8$. We shall now use the theorem in sec. 2 to construct an explicit isomorphism $C_6^- \xrightarrow{\cong} L(\mathbb{R}^8)$. Regard $\mathbb{R}^6$ as the underlying vector space of $\mathbb{C}^3$ (cf. sec. 1) and define a negative definite inner product in $\mathbb{R}^6$ by setting $(p,q)^- = -(p,q)$. Consider the linear map $\psi : \mathbb{R}^6 \to L(\mathbb{R}^8)$ given by $\psi_p(x) = p \cdot x$. Then $\psi_p^2 = (p,p)^- \cdot 1$ and so ψ extends to a homorphism $\psi : C_6^- \to L(\mathbb{R}^8)$. Then the theorem in sec. 2 implies that ψ is surjective and hence an isomorphism.

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LOCALLY COMPACT DESARGUESIAN HJELMSLEV PLANES OF LEVEL n

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Presented by H.S.M. Coxeter, F.R.S.C.

We correct some statements made in [4] and characterize the locally compact T_2 desarguesian planes of level n as the ones which are connected or 0-dimensional.

0. Introduction

Throughout this discussion, H is a topological Hjelmslev plane. For definitions, elementary results and notation pertaining to such a plane we refer the reader to [3] and [4]. Details of the proofs in this announcement will appear elsewhere.

1. Corrections to [4]

We use the numbering from [4] to correct some statements made in that announcement, and to add some assumptions concerning connectedness to other results.

The assumption of local compactness is dropped from 1.1 which becomes

1.1 Theorem. Every T_2 H-plane is $\sim$-connected or totally $\sim$-disconnected.

Result 1.2 should read as follows.

1.2 Theorem. Let ℓ be a line of a locally compact T_2 H-plane.

(A) ℓ is locally connected or for each point

P : If W is a compact neighbourhood of $P \in \ell$ then the connected component of P in W lies in the neighbour class of P.

(B) The following statements are equivalent.

(1) ℓ is locally connected.

(2) ℓ is locally arcwise connected.

(3) ℓ is arcwise connected.

If any of the above conditions hold, then ℓ is connected.

Remark. For ordinary planes connectedness implies local connectedness. For H-planes 1.2(B) shows the converse is obtained.

Because of this correction we require the assumption of local connectedness in several results from [4]. Also the possibility of 0-dimensional planes must be included in the claims for other results. We list these changes now, and underline the additional assumptions or claims.

(a) 1.4 Theorem. ... or a locally compact locally connected translation ... .

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(b) The first claim of 2.8 should be

(1) $A(P)$ is a connected or <u>0-dimensional</u> ... .

(c) 2.10 <u>Theorem</u>. Let H be a locally compact H-plane.

(1) If H is of height n, then H is connected <u>or</u> <u>0-dimensional</u>.

(2) If H is a <u>locally</u> <u>connected</u> translation ... .

(d) The assumption that H is <u>locally</u> <u>connected</u> must be added to 2.11, 2.12 and 2.13.

2. <u>Locally Compact</u> T_2 <u>Hjelmslev Rings</u>

All rings R are associative with unit. R^+ is the additive structure of R. For the definition of a Hjelmslev ring (H-ring) we refer the reader to [5] 5.1.

A ring R is a topological ring if the maps $(a,b) \mapsto a-b$ and $(a,b) \mapsto ab$ are continuous. In addition R is a Gelfand ring if the group of units, $U(R)$, is an open topological group.

2.1 <u>Proposition</u>. A locally compact T_2 H-ring is a Gelfand ring.

2.2 <u>Theorem</u>. Let H be a locally compact T_2 H-ring so that the interior of the radical is void. Then, one of the following statements holds:

(1) H is connected.

(2) H is 0-dimensional.

(3) $H^+ = \mathbb{R}^n \oplus C$ (direct sum as topological groups) where $\mathbb{R}^n$ is the n-dimensional real vector group and C is a 0-dimensional subgroup of H^+.

We wish to characterize the first two cases in 2.2.

A H-ring is an E-ring if its radical J is nilpotent. For equivalent definitions of an E-ring consult [1] and [5].

2.3 Theorem. Let H be a locally compact T_2 H-ring whose radical has a void interior. Then, H is an E-ring if and only if H is connected or 0-dimensional. Moreover, H is connected <=> H is locally connected.

The above result is rather lengthy and depends on many deep results from the theory of topological rings.

3. Locally Compact T_2 PH-planes of Level n

If H is a H-ring, then $\mathcal{H}(H)$ is the desarguesian PH-ring defined via homogeneous coordinates (see [5], §6). From [3] 8.1 we have that $\mathcal{H}(H)$ is a topological PH-plane <=> H is a Gelfand H-ring.

Then 2.1 gives

3.1 Proposition. If H is a locally compact T_2 H-ring, then $\mathcal{H}(H)$ is a locally compact T_2 PH-plane.

For the definition of a PH-plane of level n we refer the reader to [2]. From [1], [2] we have the result that $H(H)$ is of level n <=> H is an E-ring. Combining this fact with 2.3 we obtain

3.2 Theorem. Let $H(H)$ be a locally compact T_2 desarguesian PH-plane so that $H(H)/\sim$ is not discrete. Then, $H(H)$ is of level n <=> $H(H)$ is connected or 0-dimensional.

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SOME RESULTS ON GENERALIZED LINE GRAPHS

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Presented by N. S. Mendelsohn, F.R.S.C.

1. One property of a line graph is that the least eigenvalue, $\lambda(G)$, of its 0-1 adjacency matrix is bounded from below by -2 [5]. The cocktail party graph, CP(n), the regular graph on $2n$ vertices of degree $2n-2$, has the same property. A generalized line graph [5], $L(H;a_1,\dots,a_n)$, is constructed from a graph H with n vertices $v_1,\dots,v_n$ and nonnegative integers $a_1,\dots,a_n$; it consists of disjoint copies of $L(H)$ and $CP(a_i)$, $i = 1,\dots,n$, with additional edges joining a vertex in $L(H)$ with a vertex in $CP(a_i)$ if the vertex in $L(H)$ corresponds to an edge in H that has v_i as an end point. This graph also has its least eigenvalue bounded below by -2. Cameron, et. al. [2] have used root systems to construct all graphs whose least eigenvalue is bounded from below by -2. These root systems are vectors in R^n of length $\sqrt{2}$ whose pairwise inner products are 0 or ± 1. If the matrix B has such vectors for its rows with inner products of 0 or 1, then $BB^T = 2I + A$ where A is the adjacency matrix of a graph whose eigenvalues are not less than -2. Conversely, given the graph the matrix B can be formed, and the minimal vectors in the integral span of the rows of B will form a root system. The root systems yielding graphs have been classified; they are A_n (corresponding to line graphs of bipartite graphs), D_n (corresponding to generalized line graphs), E_6, E_7, and E_8. E_6 is contained in E_7, which in turn is contained in E_8; this gives the following theorem:

Theorem 1.1 (Cameron, Goethals, Seidel, Shult) [2]. If G is a connected graph whose least eigenvalue is at least -2 then G is a generalized line graph or G can be constructed from the root system E_8.

The next result has been noted by both F. Bussemaker and B. McKay.

Theorem 1.2. If a graph is constructed from vectors in E_8, then it contains a subgraph that can be constructed from vectors in E_6.

The generalized line graph was introduced as an extension of the spectral concept of a line graph. The results presented below show that the definition is a good one since it also generalizes topologocal properties of line graphs.

2. In [6] J. Krausz characterized line graphs by their clique structure in the following way:

Theorem 2.1. A graph is a line graph iff its edges can be partitioned into cliques such that each vertex is in at most two cliques, and two cliques have at most one common vertex.

A generalized cocktail party graph (GCP) is a graph obtained by the deletion of independent edges from the complete graph, K_n. A vertex of degree $n-1$ is called a-type while the others are called b-type.

Theorem 2.2. G is a generalized line graph iff its edges can be partitioned into GCPs such that each vertex is in at most two GCPs, two GCPs have at most one common vertex, and if two GCPs have a common vertex, then it is of a-type in both of them.

A cover is a partition of the edges of G into GCPs. It is called proper if it satisfies the three conditions in the conclusion of Theorem 2.2.

Theorem 2.3. If G is a connected generalized line graph with more than 6 vertices, then there exists one and only one proper cover of G.

3. In [7] H. Whitney showed that, except for the one pair of graphs K_3 and $K_{1,3}$, if two connected line graphs are isomorphic, then, except for isolated vertices, the root graphs are isomorphic too. By allowing multi-graphs as root graphs, this result can be extended to generalized line graphs.

Theorem 3.1. Except for the pairs in Figure 1, if two connected generalized line graphs are isomorphic then the root graphs are also isomorphic.

4. In [3] the multiplicity of the eigenvalue -2 of $L(H)$ is given by the cycle structure of H, and a matroid construction yields all the eigenvectors. The same is true for generalized line graphs.

Theorem 4.1 [3]. If H is a connected graph with vertex set $V(H)$ and edge set E(H), then the multiplicity of the eigenvalue -2 of $L(H)$ is $|E(H)| - |V(H)|$ if H is not bipartite and $|E(H)| - |V(H)| + 1$ otherwise.

Theorem 4.2. Let $G = L(H;a_1,\ldots,a_n)$, and let H have n vertices, m edges, and not all $a_i = 0$. Then the multiplicity of -2 is $\sum a_i + m - n$.

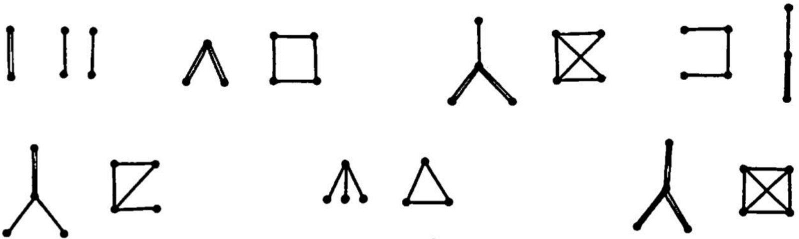

Figure 1

5. In [1] Beineke characterized line graphs by showing that there are exactly nine minimal nonline graphs. For generalized line graphs there are exactly 31 minimal nongeneralized line graphs (MNGLG); they are displayed in Figure 2. From Theorem 1.1 it follows that a MNGLG with $\lambda(G) > -2$ must be generated by vectors from the root system E_8. Since $A + 2I$ is non-singular, G must have 6, 7, or 8 vertices which correspond to vectors that generate E_6, E_7, or E_8. Now by Theorem 1.2 any such graph must contain a graph on six vertices that generates E_6. The 20 such graphs are the first 20 displayed in Figure 2. There are no MNGLGs with $\lambda(G) = -2$. Finally consider the case those MNGLGs with $\lambda(G) < -2$.

Lemma 5.1. Let G be a MNGLG with $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ as eigenvalues and $\lambda_n < -2$. Then $\lambda_{n-1} > -2$.

Corollary 5.2. If G is a MNGLG, and G_1 is a proper induced subgraph then $\lambda(G_1) > -2$ and the multiplicity of the eigenvalue -2 is at most 1.

The only graphs with six or fewer vertices that satisfy the conclusion of Corollary 5.2 are those in Figure 2.

Theorem 5.3. A graph G is a generalized line graph if and only if it does not contain any of the 31 graphs in Figure 2 as an induced subgraph.

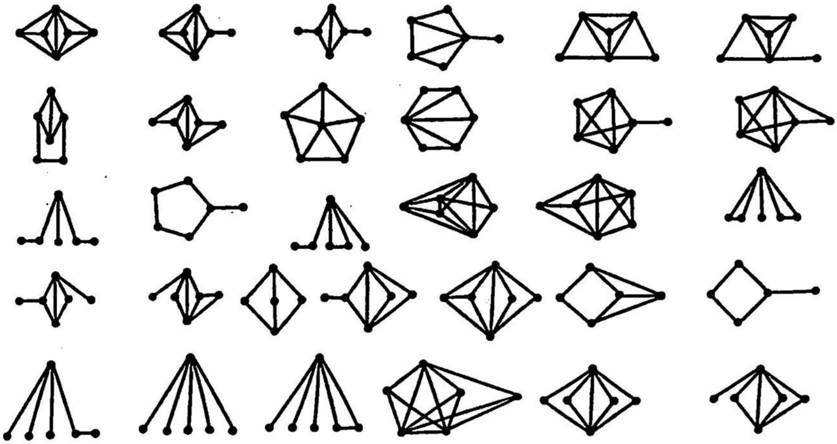

Figure 2

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