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Bounded Circles on a Complex Hyperbolic Space are Expressed by Trajectories on Geodesic Spheres

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Abstract. We take a bounded circle on a complex hyperbolic space. We show that if it has complex torsion either ±1 or 0 then it is expressed by a geodesic on some geodesic sphere, and show that if it has complex torsion τ with $0 < |\tau| < 1$ then it is uniquely expressed by a non-geodesic trajectory on a geodesic sphere up to congruency.

Résumé. Nous prenous un cercle borné en l'espace hyperbolique complexe. Nous montrons que il est exprimé par une géodésique sur une sphère géodésique si sa torsion complexe est 0 ou ±1, et montrons que il est uniquement exprimé par une trajectoire sur une sphère géodésique qui n'est pas une géodésique si sa torsion complexe est $0 < |\tau| < 1$.

1. **Introduction** In our previous paper [5], we pose the question whether every circle on a complex projective space $\mathbb{C}P^n$ can be seen as a geodesic on some geodesic sphere. This question comes from the elementary result that every circle on a Euclidean 3-space $\mathbb{R}^3$ can be seen as a geodesic on a standard sphere of suitable radius and such an expression is unique up to congruency on $\mathbb{R}^3$. Our answer to the question on circles on $\mathbb{C}P^n$ is negative. Circles on $\mathbb{C}P^n$ are classified by their geodesic curvatures and complex torsions. Complex torsion of a circle of positive geodesic curvature measures the angle between the velocity vector and the complex line formed by the acceleration vector. When a circle on $\mathbb{C}P^n$ is a geodesic or has complex torsion either ±1 or 0, it can be seen as a geodesic on some geodesic sphere. But when it has complex torsion τ with $0 < |\tau| < 1$, it cannot be seen as geodesics. If we consider trajectories for Sasakian magnetic fields, which are natural generalizations of geodesics from the dynamical theoretic point of view, it can be seen as a trajectory on some geodesic sphere.

There are many studies how geodesics on submanifolds can be seen in their ambient spaces. For example, Sakamoto [11] studied isometric immersions which

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map each geodesic to a curve contained in a totally geodesic 2-dimensional submanifold, and Maeda [7] gave a characterization of geodesic spheres in a complex hyperbolic space $\mathbb{C}H^n$ by the property that geodesics can be seen as circles in the ambient space. Our study gives another view on curve theoretic study on submanifolds.

In this paper, corresponding to [5], we study circles on a complex hyperbolic space $\mathbb{C}H^n$. Of course, since there are circles whose images are not bounded sets, it is clear that not all circles can be seen as geodesics on geodesic spheres. But if we restrict ourselves to circles whose images are bounded sets, our question is still meaningful. Moreover, for circles on $\mathbb{C}P^n$, we have their multiple expressions up to a congruency relation. We are hence also interested in uniqueness of expressions of bounded circles by trajectories. We show that for bounded circles of complex torsion ±1 we have many expressions but in other case their expressions are unique up to a congruency relation.

2. Circles on a Complex Hyperbolic Space

A smooth curve γ parameterized by its arc-length on a Riemannian manifold $\widetilde{M}$ is said to be a *circle* if it satisfies the equations

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = k_\gamma Y_\gamma, \\ \widetilde{\nabla}_{\dot\gamma} Y_\gamma = -k_\gamma \dot\gamma. \end{cases} \tag{2.1}$$

with a nonnegative constant k_γ and a field Y_γ of unit tangent vectors along γ ([10]). Here, $\widetilde{\nabla}$ denotes the Riemannian connection on $\widetilde{M}$. We call k_γ and $\{\dot\gamma, Y_\gamma\}$ the *geodesic curvature* and Frenet frame of γ, respectively. We note that the equations (2.1) is equivalent to the equation

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -{k_\gamma}^2\dot\gamma. \tag{2.2}$$

For a circle γ of positive geodesic curvatureon a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c which satisfies (2.1), by using the complex structure J on $\mathbb{C}H^n$, we set $\tau_\gamma = \langle \dot\gamma, JY_\gamma\rangle$, and call it its *complex torsion.* Since we have

$$\frac{d}{dt}\tau_\gamma = \langle k_\gamma Y_\gamma, JY_\gamma\rangle + \langle \dot\gamma, -k_\gamma J\dot\gamma\rangle = 0,$$

we find that τ_γ is constant along γ. When $\tau_\gamma = 0$, it is said to be *totally real.* We can classify circles on $\mathbb{C}H^n(c)$ by their geodesic curvatures and complex torsions. We say two smooth curves γ_1, γ_2 on a Riemannian manifold $\widetilde{M}$ which are parameterized by their arc-length to be *congruent* to each other (in strong sense) if there is an isometry $\tilde\varphi$ of $\widetilde{M}$ satisfying $\tilde\varphi \circ \gamma_1(t) = \gamma_2(t)$ for all t. Since $\mathbb{C}H^n(c)$ is a symmetric space of rank one and its isometries are ±-holomorphic, we find the following.

LEMMA 1 ([3, 8]). *Two circles* γ_1, γ_2 *on* $\mathbb{C}H^n(c)$ *are congruent to each other if and only if their geodesic curvatures and complex torsions satisfy either* $k_{\gamma_1} = k_{\gamma_2} = 0$ *or* $k_{\gamma_1} = k_{\gamma_2} > 0$ *and* $|\tau_{\gamma_1}| = |\tau_{\gamma_2}|$.

By this lemma, we find that the moduli space $\mathcal{C}(\mathbb{C}H^n)$, the set of all congruence classes, of circles on $\mathbb{C}H^n(c)$ is set-theoretically identified with the set $[0,\infty) \times [0,1]/\sim$, where we define $(k_1,\tau_1) \sim (k_2,\tau_2)$ for $(k_1,\tau_1),(k_2,\tau_2) \in [0,\infty) \times [0,1]$ if and only if either $k_1 = k_2 = 0$ or $k_1 = k_2 > 0$ and $\tau_1 = \tau_2$.

We here recall some basic properties of circles on $\mathbb{C}H^n(c)$. We define a function $\nu : [0,\infty) \to \mathbb{R}$ by

$$\nu(k) = \begin{cases} 0, & \text{if } k < \sqrt{|c|}/2, \\ (4k^2+c)^{3/2}/(3\sqrt{3}|c|k), & \text{if } \sqrt{|c|}/2 \le k \le \sqrt{|c|}, \\ 1, & \text{if } k > \sqrt{|c|}. \end{cases}$$

A circle γ on $\mathbb{C}H^n(c)$ has the following properties (for more, see [3]):

(1) It is an orbit of a one-parameter family of isometries of $\mathbb{C}H^n(c)$;
(2) If $k_\gamma > \sqrt{|c|}$ or $\tau_\gamma < \nu(k_\gamma)$, it is bounded, that is, the set $\gamma\big((-\infty,\infty)\big)$ is a bounded set;
(3) When $\tau_\gamma = \pm 1$, it lies on a totally geodesic $\mathbb{C}H^1$;
(4) When $k_\gamma > \sqrt{|c|}$ and $\tau_\gamma = \pm 1$, it is closed of length $\pi/\sqrt{{k_\gamma}^2 + c}$;
(5) When $\tau_\gamma = 0$, it lies on a totally geodesic $\mathbb{R}H^2$;
(6) When $k_\gamma > \sqrt{|c|}/2$ and $\tau_\gamma = 0$, it is closed of length $2\pi/\sqrt{4{k_\gamma}^2 + c}$.

3. **Expressions by Geodesics on Geodesic Spheres** Generally, for a smooth curve σ on a real hypersurface M in $\mathbb{C}H^n(c)$, we call the curve $\iota \circ \sigma$ on $\mathbb{C}H^n(c)$ defined by use of an isometric immersion $\iota : M \to \mathbb{C}H^n(c)$ the *extrinsic shape* of σ. For a curve γ on $\mathbb{C}H^n(c)$, if there is a real hypersurface M and a curve σ on M satisfying $\gamma(t) = \iota \circ \sigma(t)$ for all t, we say that γ is expressed by σ and call (M,σ) an expression of γ. In this section, we study expressions of circles by geodesics on geodesic spheres. If we have two expressions (M_1,σ_1) and (M_2,σ_2) of a circle γ on $\mathbb{C}H^n(c)$, we say they are *congruent* to each other if there is an isometry $\tilde{\varphi}$ of $\mathbb{C}H^n(c)$ which either preserves γ or reverses γ, that is, either $\tilde{\varphi} \circ \gamma(t) = \gamma(t)$ for all t or $\tilde{\varphi} \circ \gamma(t) = \gamma(-t)$ for all t. Our main result in this section is the following.

THEOREM 1. *Let* γ *be a bounded circle on* $\mathbb{C}H^n(c)$.

(1) *When* $\tau_\gamma = \pm 1$ *or when* $\tau_\gamma = 0$, *it is expressed by a geodesic on some geodesic sphere in* $\mathbb{C}H^n(c)$. *Such an expression is unique up to congruency.*
(2) *If* $0 < |\tau_\gamma| < 1$, *it cannot be expressed by any geodesics on any geodesic spheres in* $\mathbb{C}H^n(c)$.

We take a geodesic sphere $M = G(r)$ of radius r in $\mathbb{C}H^n(c)$. It admits an almost contact metric structure $(\xi, \eta, \phi, \langle\ ,\ \rangle)$ induced by the complex structure J on $\mathbb{C}H^n(c)$. Let $\mathcal{N}$ be the inward unit normal of this geodesic sphere. With the induced metric $\langle\ ,\ \rangle$ on M, the characteristic vector field ξ is defined by $\xi = -J\mathcal{N}$, the 1-form η by $\eta(v) = \langle v, \xi\rangle$, and the structure tensor field ϕ which is a $(1,1)$-tensor field on M by $\phi(v) = Jv - \langle v, \xi\rangle\mathcal{N}$. The shape operator A_M of M with respect to $\mathcal{N}$ satisfies $A_M\xi = \delta_M\xi$ and $A_Mv = \lambda_Mv$ for each tangent vector $v \in TM$ orthogonal to ξ. Here, the principal curvatures are $\delta_M = \sqrt{|c|}\coth\bigl(\sqrt{|c|}\,r\bigr)$ and $\lambda_M = \bigl(\sqrt{|c|}/2\bigr)\coth\bigl(\sqrt{|c|}\,r/2\bigr)$ (see [9], for example). In particular, A_M and ϕ are simultaneously diagonalizable, that is, $A_M\phi = \phi A_M$.

For a geodesic σ on a geodesic sphere M in $\mathbb{C}H^n(c)$, we set $\rho_\sigma = \langle\dot\sigma, \xi\rangle$ and call it its *structure torsion*. By the Weingarten formula which states that $\widetilde\nabla_X\mathcal{N} = -A_MX$ for a vector field X tangent to M, we have $\nabla_X\xi = \phi A_MX$. Since A_M is symmetric and ϕ is anti-symmetric, we have

$$\frac{d}{dt}\rho_\sigma = \langle\dot\sigma, \phi A_M\dot\sigma\rangle = -\langle A_M\phi\dot\sigma, \dot\sigma\rangle,$$

hence we have

$$\frac{d}{dt}\rho_\sigma = \frac{1}{2}\langle\dot\sigma, (\phi A_M - A_M\phi)\dot\sigma\rangle = 0.$$

This means that the structure torsion of a geodesic σ is constant along σ. We can classify geodesics on a geodesic sphere by their structure torsions.

LEMMA 2 ([4]). *Two geodesics σ_1, σ_2 on a geodesic sphere in $\mathbb{C}H^n(c)$ are congruent to each other if and only if their structure torsions satisfy $|\rho_{\sigma_1}| = |\rho_{\sigma_2}|$.*

We study the condition that the extrinsic shape of a geodesic to be a circle. For the sake of simplicity, we denote $\iota \circ \sigma$ also by σ.

LEMMA 3. *Let σ be a geodesic on a geodesic sphere M in $\mathbb{C}H^n(c)$. Its extrinsic shape is a circle if and only if either $\rho_\sigma = \pm 1$ or $\rho_\sigma = 0$. When $\rho_\sigma = \pm 1$, the geodesic curvature of the extrinsic shape is δ_M and the complex torsion is ∓ 1. When $\rho_\sigma = 0$, they are λ_M and 0.*

PROOF. By the Gauss formula which states that $\widetilde\nabla_XY = \nabla_XY + \langle A_MX, Y\rangle\mathcal{N}$ for arbitrary vector fields X, Y tangent to M and by the Weingarten formula, we have

$$\begin{aligned}
\widetilde\nabla_{\dot\sigma}\dot\sigma &= \langle A_M\dot\sigma, \dot\sigma\rangle\mathcal{N} = \bigl\{\lambda_M + (\delta_M - \lambda_M)\rho_\sigma^2\bigr\}\mathcal{N},\\
\widetilde\nabla_{\dot\sigma}\mathcal{N} &= -A_M\dot\sigma = -\lambda_M\bigl(\dot\sigma - \rho_\sigma\xi\bigr) - \delta_M\rho_\sigma\xi\\
&= -\bigl\{\lambda_M + (\delta_M - \lambda_M)\rho_\sigma^2\bigr\}\dot\sigma + (\delta_M - \lambda_M)\rho_\sigma\bigl(\rho_\sigma\dot\sigma - \xi\bigr).
\end{aligned}$$

Since $\delta_M > \lambda_M > 0$, we find that the extrinsic shape of σ is a circle if and only if either $\rho_\sigma = \pm 1$ or $\rho_\sigma = 0$. When $\rho_\sigma = \pm 1$, we find that the geodesic curvature of the extrinsic shape is δ_M and its complex torsion is ∓ 1, because its Frenet frame is $\{\dot\sigma = \pm\xi, \mathcal{N}\}$. When $\rho_\sigma = 0$, they are λ_M and 0, because its Frenet frame is $\{\dot\sigma, \mathcal{N}\}$. □

As a consequence of Lemmas 2 and 3, we have the following

Proposition 1. *Let γ be a circle on $\mathbb{C}H^n(c)$. If we have two expressions of γ by geodesics on geodesic spheres, they are congruent to each other.*

Proof. Let (M_1, σ_1) and (M_2, σ_2) be two expressions of γ by geodesics on geodesic spheres. By Lemma 3, the complex torsion τ_γ of γ is either ± 1 or 0. Corresponding to these cases, the radii of these geodesic spheres satisfy either $\coth\bigl(\sqrt{|c|}\,r\bigr) = k_\gamma/\sqrt{|c|}$ or $\coth\bigl(\sqrt{|c|}\,r/2\bigr) = 2k_\gamma/\sqrt{|c|}$. This means that the radii of these geodesic spheres are determined by γ. Hence they are isometric to each other and there is an isometry $\tilde{\varphi}$ of $\mathbb{C}H^n(c)$ satisfying $\tilde{\varphi}(M_1) = M_2$. The curve $\tilde{\varphi} \circ \sigma_1$ is a geodesic on M_2. Since $\tilde{\varphi}$ is $\pm$-holomorphic, that is, it satisfies $d\tilde{\varphi} \circ J = \pm J \circ d\tilde{\varphi}$, and since $d\tilde{\varphi}(\mathcal{N}_{M_1}) = \mathcal{N}_{M_2}$, we have

$$\begin{aligned}\rho_{\tilde{\varphi}\circ\sigma_1} &= \langle d\tilde{\varphi} \circ \dot{\sigma}_1, -J\mathcal{N}_{M_2}\rangle = \pm\langle d\tilde{\varphi} \circ \dot{\sigma}_1, -d\tilde{\varphi}(J\mathcal{N}_{M_1})\rangle \\ &= \pm\langle \dot{\sigma}_1, -J\mathcal{N}_{M_1}\rangle = \pm\rho_{\sigma_1}.\end{aligned}$$

Therefore, we obtain $|\rho_{\tilde{\varphi}\circ\sigma_1}| = |\rho_{\sigma_2}|$ and find that $\tilde{\varphi} \circ \sigma_1$ and σ_2 are congruent geodesics on M_2 by Lemma 2. We therefore have an isometry ψ of M_2 with $\psi \circ (\tilde{\varphi} \circ \sigma_1)(t) = \sigma_2(t)$ for all t. It is well known that isometries on a geodesic sphere in $\mathbb{C}H^n$ are equivariant. This means that for the isometry ψ of M_2 there is an isometry $\tilde{\psi}$ of $\mathbb{C}H^n(c)$ with $\tilde{\psi}\big|_{M_2} = \psi$. Considering the isometry $\Phi = \tilde{\psi} \circ \tilde{\varphi}$ of $\mathbb{C}H^n(c)$, we find that it satisfies $\Phi(M_1) = M_2$ and preserves γ. □

We are now in a position to prove Theorem 1. The second assertion is a direct consequence of Lemma 3. To show the other assertions, we need the result on congruency of circles. For each circle of complex torsion either ± 1 or 0, the existence of its expressions is guaranteed by Lemmas 1 and 3. Uniqueness of its expression is a consequence of Proposition 1. This completes the proof of Theorem 1.

4. Expressions by Trajectories on Geodesic Spheres

Since we cannot express all circles on $\mathbb{C}H^n(c)$ by geodesics on geodesic spheres, we extend the family of curves on geodesic spheres. Though circles are simplest curves next to geodesics from the viewpoint of Frenet-Serret formula, they are determined by their initial velocity and acceleration vectors. We therefore take a family of curves which are determined only by their initial velocity vectors. On a geodesic sphere M, we have a natural 2-form $\mathbb{F}_\phi$ associated with the almost contact metric structure. It is defined by $\mathbb{F}_\phi(v, w) = \langle v, \phi w\rangle$ for all tangent vectors $v, w \in T_pM$ at an arbitrary point $p \in M$. Since the complex structure on $\mathbb{C}H^n(c)$ is parallel, we find that this 2-form is closed. We call a constant multiple $\mathbb{F}_\kappa = \kappa\mathbb{F}_\phi$ ($\kappa \in \mathbb{R}$) a *Sasakian magnetic field.* Generally, a closed 2-form on a Riemannian manifold is said to be a magnetic field because it can be regarded as a generalization of static magnetic fields on a Euclidean 3-space (see [12]). A smooth curve σ on a geodesic sphere M parameterized by its arclength is said to be a *trajectory* for $\mathbb{F}_\kappa$ if it satisfies the differential equation $\nabla_{\dot\sigma}\dot\sigma = \kappa\phi\dot\sigma$. For the canonical magnetic field $\mathbb{F}_0$, its trajectories are geodesics. Thus, trajectories for Sasakian magnetic fields are generalizations of geodesics which are closely related with the almost contact metric structure of the underlying geodesic sphere. Moreover, the equation of circles on $\mathbb{C}H^n$ of complex torsion ± 1 is given as $\widetilde\nabla_{\dot\gamma}\dot\gamma = \mp k_\gamma J\dot\gamma$, hence such circles can be interpreted as trajectories for Kähler magnetic fields ([1]). Therefore, the authors consider that to study expressions by trajectories for Sasakian magnetic fields is natural.

For a trajectory σ for $\mathbb{F}_\kappa$ on a geodesic sphere M, we define its structure torsion by $\rho_\sigma = \langle\dot\sigma, \xi\rangle$. By the same computation as for geodesics, we find that it is constant along σ. We can classify trajectories on a geodesic sphere by their strengths of magnetic fields and complex torsions.

LEMMA 4 ([2]). *Two trajectories σ_1, σ_2 for Sasakian magnetic fields $\mathbb{F}_{\kappa_1}$ and $\mathbb{F}_{\kappa_2}$ on a geodesic sphere M in $\mathbb{C}H^n(c)$ are congruent to each other if and only if they satisfy one of the following conditions*:

(i) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| = 1$,
(ii) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| < 1$, $|\kappa_1| = |\kappa_2|$ *and* $\kappa_1\rho_{\sigma_1} = \kappa_2\rho_{\sigma_2}$.

Remark 1. Every trajectory σ with $\rho_\sigma = \pm 1$ is a geodesic, and does not depend on Sasakian magnetic fields.

This lemma guarantees the condition on two expressions to be congruent.

LEMMA 5. *Let (M_1, σ_1) and (M_2, σ_2) be two expressions of a bounded circle γ on $\mathbb{C}H^n(c)$ by trajectories for Sasakian magnetic fields $\mathbb{F}_{\kappa_1}$ and $\mathbb{F}_{\kappa_2}$ on geodesic spheres. They are congruent to each other if and only if the base geodesic spheres have the same radii and one of the following condition holds*:

(i) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| = 1$,
(ii) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| < 1$ *and* $\kappa_1\rho_{\sigma_1} = \kappa_2\rho_{\sigma_2}$.

PROOF. Through isometric embeddings, we regard geodesic spheres M_1, M_2 as subsets of $\mathbb{C}H^n(c)$. Therefore, we have $\sigma_1(t) = \sigma_2(t) = \gamma(t)$ for all t.

Suppose that these two expressions are congruent to each other. We then have an isometry $\tilde\varphi$ of $\mathbb{C}H^n(c)$ with $\tilde\varphi(M_1) = M_2$ satisfying either $\tilde\varphi\circ\sigma_1(t) = \sigma_2(t)$ for all t or $\tilde\varphi\circ\sigma_1(t) = \sigma_2(-t)$ for all t. In particular, the radii of M_1 and M_2 coincide with each other. We set

$$\epsilon = \begin{cases} 1, & \text{when } \tilde\varphi\circ\sigma_1(t) = \sigma_2(t) \text{ holds,} \\ -1, & \text{when } \tilde\varphi\circ\sigma_1(t) = \sigma_2(-t) \text{ holds.} \end{cases}$$

We then have $(d\tilde\varphi\circ\dot\sigma_1)(t) = \epsilon\dot\sigma_2(\epsilon t)$. Since we have $d\tilde\varphi(\mathcal{N}_{M_1}) = \mathcal{N}_{M_2}$, and since $\tilde\varphi$ is $\pm$-holomorphic, we find

$$\begin{aligned} \rho_{\sigma_2} &= \langle\dot\sigma_2, \xi_{M_2}\rangle = \epsilon\langle d\tilde\varphi(\dot\sigma_1), -Jd\tilde\varphi(\mathcal{N}_{M_1})\rangle = \pm\epsilon\langle d\tilde\varphi(\dot\sigma_1), -d\tilde\varphi J(\mathcal{N}_{M_1})\rangle \\ &= \pm\epsilon\langle d\tilde\varphi(\dot\sigma_1), d\tilde\varphi(\xi_{M_1})\rangle = \pm\epsilon\langle\dot\sigma_1, \xi_{M_1}\rangle = \pm\epsilon\rho_{\sigma_1}. \end{aligned}$$

In particular, we have $|\rho_{\sigma_1}| = |\rho_{\sigma_2}|$. Also we have

$$\begin{aligned}\kappa_2\phi\dot{\sigma}_2 &= \nabla_{\dot{\sigma}_2}\dot{\sigma}_2 = \nabla_{d\tilde{\varphi}\circ\dot{\sigma}_1}(d\tilde{\varphi}\circ\dot{\sigma}_1) = d\tilde{\varphi}\big(\nabla_{\dot{\sigma}_1}\dot{\sigma}_1\big) = d\tilde{\varphi}(\kappa_1\phi\dot{\sigma}_1) \\ &= \kappa_1 d\tilde{\varphi}(J\dot{\sigma}_1 - \rho_{\sigma_1}\mathcal{N}_{M_1}) = \pm\kappa_1 J(d\tilde{\varphi}\circ\dot{\sigma}_1) - \kappa_1\rho_{\sigma_1}\mathcal{N}_{M_2} \\ &= \pm\epsilon\kappa_1 J\dot{\sigma}_2 \mp \epsilon\kappa_1\rho_{\sigma_2}\mathcal{N}_{M_2} = \pm\epsilon\kappa_1\phi\dot{\sigma}_2.\end{aligned}$$

When $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| = 1$, because the $\phi\dot{\sigma}_2$ is the null vector, the above tells nothing. When $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| < 1$, we have $\kappa_2 = \pm\epsilon\kappa_1$. Hence, we find $|\kappa_1| = |\kappa_2|$ and $\kappa_2\rho_{\sigma_2} = \kappa_1\rho_{\sigma_2}$. Thus, if two expressions (M_1, σ_1) and (M_2, σ_2) of γ are congruent to each other, then one of the conditions in the assertion holds.

On the other hand, we suppose that (M_1, σ_1) and (M_2, σ_2) satisfy the conditions in the assertion. We have an isometry $\tilde{\varphi}$ of $\mathbb{C}H^n(c)$ with $\tilde{\varphi}(M_1) = M_2$, because M_1, M_2 are of the same radius. By the same computation as above, we find $\rho_{\tilde{\varphi}\circ\sigma_1} = \pm\epsilon\rho_{\sigma_1}$ and $\nabla_{d\tilde{\varphi}\circ\dot{\sigma}_1}(d\tilde{\varphi}\circ\dot{\sigma}_1) = \pm\epsilon\kappa\phi(d\varphi\circ\dot{\sigma}_1)$ because $\tilde{\varphi}$ is $\pm$-holomorphic. Thus, $\tilde{\varphi}\circ\sigma_1$ and σ_2 are trajectories on M_2 which satisfy one of the conditions in Lemma 4. We therefore find an isometry ψ of M_2 satisfying $\psi\circ(\tilde{\varphi}\circ\sigma_1)(t) = \sigma_2(t)$. It is known that there is an isometry $\tilde{\psi}$ of $\mathbb{C}H^n(c)$ satisfying $\tilde{\psi}\big|_{M_2} = \psi$. We hence find that the isometry $\tilde{\psi}\circ\tilde{\varphi}$ of $\mathbb{C}H^n$ satisfies $(\tilde{\psi}\circ\tilde{\varphi})(M_1) = M_2$ and $(\tilde{\psi}\circ\tilde{\varphi})\circ\sigma_1(t) = \sigma_2(t)$ for all t. $\square$

We are now in a position to study expressions of circles by trajectories on geodesic spheres. If we take a trajectory σ for $\mathbb{F}_\kappa$ on a geodesic sphere M, its extrinsic shape satisfies

$$\begin{aligned}\widetilde{\nabla}_{\dot{\sigma}}\widetilde{\nabla}_{\dot{\sigma}}\dot{\sigma} &= \widetilde{\nabla}_{\dot{\sigma}}\big\{\kappa J\dot{\sigma} + (\langle A_M\dot{\sigma}, \dot{\sigma}\rangle - \kappa\rho_\sigma)\mathcal{N}_M\big\} \\ &= -\Big\{\kappa^2(1-\rho_\sigma^2) + \big\{\lambda_M + (\delta_M - \lambda_M)\rho_\sigma^2\big\}^2\Big\}\dot{\sigma} \\ &\quad + \big\{\lambda_M - \kappa\rho_\sigma + (\delta_M - \lambda_M)\rho_\sigma^2\big\}\big\{\kappa + (\delta_M - \lambda_M)\rho_\sigma\big\}\big(\rho_\sigma\dot{\sigma} - \xi\big).\end{aligned}$$

Therefore, we obtain the condition on trajectories on a geodesic sphere to be circles in $\mathbb{C}H^n$.

LEMMA 6. *Let σ be a trajectory for $\mathbb{F}_\kappa$ on a geodesic sphere M in $\mathbb{C}H^n(c)$. Its extrinsic shape is a circle on $\mathbb{C}H^n(c)$ if and only if one of the following condition holds*:

(i) $\rho_\sigma = \pm 1$,
(ii) $\lambda_M - \kappa\rho_\sigma + (\delta_M - \lambda_M)\rho_\sigma^2 = 0$,
(iii) $\kappa + (\delta_M - \lambda_M)\rho_\sigma = 0$.

Corresponding to these cases, the geodesic curvature k_σ and the complex torsion τ_σ of the extrinsic shape of σ are as follows:

(i) $k_\sigma = |\delta_M|$, $\tau_\sigma = \mp 1$,
(ii) $k_\sigma = |\kappa|$, $\tau_\sigma = -\mathrm{sgn}(\kappa)\,1$,

(iii) $k_\sigma = \sqrt{\kappa^2 - 2\lambda_M \kappa \rho_\sigma + \lambda_M^2}$, $\tau_\sigma = (2\kappa\rho_\sigma^2 - \kappa - \lambda_M \rho_\sigma)/k_\sigma$.

Here, we denote by $\mathrm{sgn}(\kappa)$ *the signature of* κ*, and ignore complex torsions in cases that the extrinsic shape is a geodesic.*

By use of this lemma, we can show expressions of all bounded circles on $\mathbb{C}H^n(c)$ by trajectories on geodesic spheres. We view the result in [6] from another angle.

THEOREM 2. *Let* γ *be a bounded circle of positive geodesic curvature* k_γ *and of complex torsion* τ_γ *on* $\mathbb{C}H^n(c)$.

(1) *When* $\tau_\gamma = \pm 1$*, there are infinitely many its expressions by non-geodesic trajectories on geodesic spheres up to a congruency relation.*
(2) *When* $0 < |\tau_\gamma| < 1$*, there exists its unique expression by a non-geodesic trajectory on some geodesic sphere up to a congruency relation.*
(3) *When* $\tau_\gamma = 0$*, it has no expressions by non-geodesic trajectories on geodesic spheres.*

PROOF. (1) We study the second case in Lemma 6. Since $\lambda_M > 0$, we see $\rho_\sigma \neq 0$. The function $\kappa(\rho) = \lambda_M/\rho + (\delta_M - \lambda_M)\rho$ on the interval $(0,1)$ is monotone decreasing and takes values in the interval (δ_M, ∞). If we vary the radius of geodesic spheres, the principal curvature δ_M vary in the interval $(\sqrt{|c|}, \infty)$. As circles of complex torsion ± 1 are bounded if and only if their geodesic curvatures are greater than $\sqrt{|c|}$, we get the first assertion.

(2), (3) We study the third case in Lemma 6. Since $\kappa = -(\delta_M - \lambda_M)\rho_\sigma = -|c|\rho_\sigma/(4\lambda_M)$, we obtain

$$k_\sigma = \sqrt{\lambda_M^2 + \frac{|c|\rho_\sigma^2}{2} + \frac{c^2\rho_\sigma^2}{16\lambda_M^2}}, \qquad \tau_\sigma = \frac{\rho_\sigma(|c| - 2|c|\rho_\sigma^2 - 4\lambda_M^2)}{4k_\sigma\lambda_M}. \tag{4.1}$$

Since we consider non-geodesic trajectories, we have $0 < |\rho_\sigma| < 1$. By the first equality, we have $\lambda_M \leq k_\sigma \leq \lambda_M + |c|/(4\lambda_M) = \delta_M$. By using two equlities in (4.1), we have ${\tau_\sigma}^2 = g_M(k_\sigma)$ with the continuous function

$$g_M(x) = \frac{(x^2 - \lambda_M^2)(32\lambda_M^2 x^2 + 4c\lambda_M^2 - c^2)^2}{|c|(8\lambda_M^2 - c)^3 x^2}$$

defined in the interval $\big(\lambda_M, \delta_M\big)$. This function is monotone increasing and $\lim_{x\downarrow\lambda_M} g_M(x) = 0$, $\lim_{x\uparrow\delta_M} g_M(x) = 1$. We set a subset

$$\mathit{\Gamma}_M = \big\{[k,\tau] \in \mathcal{BC}(\mathbb{C}H^n) \bigm| \tau^2 = g_M(k),\ \lambda_M < k < \delta_M\big\}$$

of the moduli space $\mathcal{BC}(\mathbb{C}H^n)$ of all bunded circles on $\mathbb{C}H^n(c)$ which is expressed as

$$\big\{(k,\tau) \in \big(\sqrt{c}/2, \infty\big) \times [0,1] \bigm| k > \sqrt{c} \text{ or } \tau < \nu(k)\big\}.$$

Since $g_M(x)$ is continuous with respect to λ_M, and since λ_M takes all values in the interval $\bigl(|c|/4,\infty\bigr)$ when we vary radii of geodesic spheres, we find that the union $\bigcup \mathit{\Gamma}_M$ covers $\mathcal{BC}(\mathbb{C}H^n)$ (see Fig. 1). Thus, we obtain the assertions on existence by Lemma 1.

For each k with $k>\sqrt{|c|}/2$, we set a function $h_k(y)$ on the interval $(\sqrt{|c|}/2,\infty)$ by

$$h_k(y)=\frac{(k^2-y)(32k^2y+4cy-c^2)^2}{|c|(8y-c)^3k^2}.$$

We note $g_M(k)=h_k(\lambda_M^2)$. Therefore, the function h_k shows the behavior of complex torsions of extrinsic shapes of trajectories having geodesic curvature k with respect to radii of geodesic spheres. Since we have

$$\frac{dh_k}{dy}=-\frac{(4y+c)(32k^2y+4cy-c^2)}{|c|(8y-c)^2k^2},$$

under the condition $k>\sqrt{|c|}/2$, we find that the function $h_k(y)$ is monotone decreasing. This means that $\mathit{\Gamma}_M\cap\mathit{\Gamma}_{M'}=\varnothing$ if $\lambda_M\neq\lambda_{M'}$ (see Fig. 1). Thus, we get the assertion on uniqueness with the aid of Lemma 5. $\square$

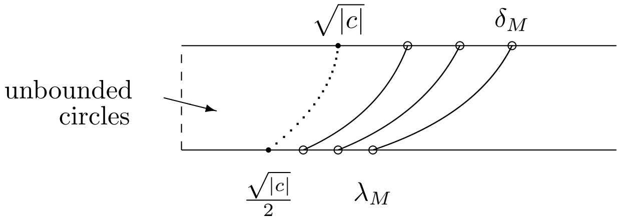

Fig. 1: $\mathit{\Gamma}_M$ on $\mathcal{BC}(\mathbb{C}H^n)$

Remark 2. Our proof of Theorem 2 shows that we have infinitely many expressions of bounded circles with complex torsion ±1 by non-geodesic trajectories on geodesic spheres. If we fix the radius r of the underlying geodesic sphere, each circle γ of geodesic curvature $k_\gamma>\sqrt{|c|}\coth\bigl(\sqrt{|c|}\,r\bigr)$ and of complex torsion ±1 is uniquely expressed by a non-geodesic trajectory on some geodesic sphere of radius r.

COROLLARY 1. *Two expressions of a bounded circle* γ *on* $\mathbb{C}H^n$ *with* $|\tau_\gamma|<1$ *are congruent to each other.*

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