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Weakly Purely Infinite C*-algebras with Topological Dimension Zero are Purely Infinite

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Abstract. We show that a C*-algebra with topological dimension zero is purely infinite if it is weakly purely infinite (a question of Kirchberg and Rørdam). We give an application of this result.

Résumé. On démontre qu'une C*-algèbre de dimension topologique égale à zéro est purement infinie si elle est faiblement purement infinie (une question de Kirchberg et de Rørdam). On donne une application de ce résultat.

1. Introduction The notion of topological dimension zero of a C*-algebra was introduced in [4], and it is a non-Hausdorff version of total disconnectedness of the primitive spectrum. A C*-algebra A has topological dimension zero if $Prim(A)$ has a basis for its topology consisting of compact open sets (see [4, Remark 2.5(vi)]). It is known that real rank zero implies the ideal property (see [8]), and the ideal property implies topological dimension zero.

In [6], Kirchberg and Rørdam introduced and studied three different notions of infiniteness for a C*-algebra (namely, strong pure infiniteness, pure infiniteness, and weak pure infiniteness), and they asked whether these three notions are equivalent (see [6, Question 9.5]).

Kirchberg and Rørdam in [6, Proposition 4.18 and Corollary 6.9] (resp., Pasnicu and Rørdam in [8, Proposition 2.14]) showed that for C*-algebras with real rank zero (or just with the ideal property), weak pure infiniteness, pure infiniteness, and strong pure infiniteness coincide. It was also shown in [9, Theorem 3.15] that for C*-algebras with topological dimension zero, pure infiniteness and strong pure infiniteness are equivalent. Here we show that these two properties are equivalent to weak pure infiniteness, for C*-algebras with topological dimension zero.

In this paper, by an ideal, we always mean a closed two-sided ideal, unless otherwise specified. For an ideal I (resp., C*-subalgebra B) in a C*-algebra A, we write $I \trianglelefteq A$ (resp., $B \leq A$). Also, for a hereditary (resp., and full) C*-subalgebra B in a C*-algebra A, we write $B \leq_h A$ (resp., $B \leq_{h,f} A$). For a subset S of a C*-algebra A, $\overline{ASA}$, or just $\overline{AaA}$, when $S = \{a\}$, is the ideal generated by S. An element $a \in A$ is called *full* if $A = \overline{AaA}$ (see [1, page 91]).

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Let $Prim(A)$ denote the set of primitive ideals in a C*-algebra A. Then $Prim(A)$ is a topological space with the Jacobson (hull-kernel) topology [1]. Throughout this paper, the symbol $\otimes$ will mean the minimal tensor product of C*-algebras. Given $a, b \in A^+$, we say that a is Cuntz subequivalent to b (and write $a \precsim b$), if there is a sequence $\{x_k\}_{k=1}^{\infty} \subseteq A$ such that $x_k^* b x_k \to a$, in norm. We say that a and b are *Cuntz equivalent* (and write $a \sim_{cu} b$), if $a \precsim b$ and $b \precsim a$ [5]. A positive element a in a C*-algebra A is called *infinite* if there exists a non-zero positive element b in A such that $a \oplus b \precsim a \oplus 0$ in $M_2(A)$. If a is non-zero and if $a \oplus a \precsim a \oplus 0$ in $M_2(A)$, then a is said to be *properly infinite* (see [5, Definition 3.2]). A C*-algebra A is said to be *purely infinite* if there are no characters on A, and for every pair of positive elements a, b in A, if $a \in \overline{AbA}$, then $a \precsim b$ (see [5, Definition 4.1]). A C*-algebra A will be said to have property *pi-n* if the n-fold direct sum $a \oplus a \oplus ... \oplus a = a \otimes 1_n$ is properly infinite in $M_n(A)$ for every non-zero positive element a in A. If A is pi-n for some n, then we shall call A *weakly purely infinite* (see [6, Definition 4.3]). We refer the reader to [6, Definition 5.1] for the definition of *strongly purely infinite* C*-algebras. An element $a \in A^+$ is *strictly full* if $(a-\varepsilon)_+$ is full for some (and so for all sufficiently small) $\varepsilon > 0$ [7, page 46].

2. Main Results

Theorem 2.1. *Let A be a C*-algebra with topological dimension zero. The following statements are equivalent:*

(1) *A is purely infinite.*
(2) *A is weakly purely infinite.*

Proof. Since every purely infinite C*-algebra is weakly purely infinite (see [6, Definition 4.3]), it is enough to prove that (2) implies (1).

Let A be a weakly purely infinite C*-algebra with topological dimension zero. If we show that every non-zero hereditary C*-subalgebra in any quotient of A contains an infinite projection, then [8, Proposition 2.11($(iv) \Rightarrow (ii)$)] (where separability is not necessary) implies that A is purely infinite.

Suppose that Q is a quotient of A and $0 \neq H \leq_h Q$.

According to [6, Proposition 4.5 (ii) and (iii)], weakly purely infinite passes to hereditary C*-subalgebras and quotients. Thus H is weakly purely infinite, and hence is pi-n, for an $n \in \mathbb{N}$ (see [6, Definition 4.3]). Since H has no finite dimensional representations, Glimm's lemma (see [5, Proposition 4.10]) implies that there exist mutually orthogonal non-zero, positive, elements $r_1, r_2, ..., r_n$ in H all equivalent to r_1. Set $J := \overline{Hr_1H}$. Then, since (H and so also) J is pi-n (by [6, Proposition 4.5(ii)]), $r_1 \otimes 1_n$ is properly infinite in $M_n(J)$, and we have:

$$
\begin{aligned}
r_1 \otimes 1_n &\sim r_1 \oplus r_2 \oplus ... \oplus r_n \\
&\sim \sum_{j=1}^{n} r_j,
\end{aligned}
$$

(see [5, Lemma 2.8]).

On the other hand, it was shown in [9, Theorem 3.20((1) $\Rightarrow$ (3))] that topological dimension zero is equivalent to residual (IC) (i.e., every non-zero ideal of every quotient of A contains a non-zero C*-subalgebra with compact primitive spectrum). Moreover, residual (IC) passes to hereditary C*-subalgebras and quotients (see [9, Remark 3.22]). Thus, H has residual (IC), and hence J contains a non-zero C*-subalgebra with compact primitive spectrum, say D. Since $Prim(D)$ is compact, [9, Lemma 2.1] implies that D has a strictly full element d, i.e., there is $\varepsilon > 0$ such that $(d-\varepsilon)_+$ is full in D. Thus the C*-subalgebra

$$L_1 = \overline{(D \otimes 1_n)(d \otimes 1_n)(D \otimes 1_n)}$$

is equal to

$$L_2 = \overline{(D \otimes 1_n)((d-\varepsilon)_+ \otimes 1_n)(D \otimes 1_n)}.$$

Since

$$\overline{M_n(J)L_1 M_n(J)} = \overline{M_n(J)L_2 M_n(J)},$$

and $(d \otimes 1_n)$ and $((d-\varepsilon)_+ \otimes 1_n)$ are properly infinite in $M_n(J)$ (because J is pi-n), we have:

$$(d \otimes 1_n) \sim_{cu} ((d-\varepsilon)_+ \otimes 1_n)$$

in $M_n(J)$, by [5, Proposition 3.5(ii)]. But

$$((d-\varepsilon)_+ \otimes 1_n) = ((d \otimes 1_n) - \varepsilon)_+.$$

Hence

$$(d \otimes 1_n) \sim_{cu} ((d \otimes 1_n) - \varepsilon)_+$$

in $M_n(J)$. Now [2, Lemma 7.2] implies that there is a full and properly infinite projection p in $M_n(J)$.

Thus we have:

$$\overline{M_n(J)pM_n(J)} = \overline{M_n(J)(r_1 \otimes 1_n)M_n(J)}.$$

Again, according to [5, Proposition 3.5(ii)], we have:

$$p \sim_{cu} (r_1 \otimes 1_n),$$

and hence

$$p \sim_{cu} \sum_{j=1}^{n} r_j,$$

in $M_n(J)$, and so in $M_n(H)$. Set $x := \sum_{j=1}^{n} r_j$. Since $p \precsim x$, there is a z in $M_n(H)$ such that $p = z^*xz$ (see the observations after [5, Proposition 2.6]). Hence

$$p \sim x^{\frac{1}{2}} zz^* x^{\frac{1}{2}}.$$

Set $q := x^{\frac{1}{2}}zz^*x^{\frac{1}{2}}$. Then

$$q \in \overline{xM_n(H)x},$$

and q is a projection (see [10, Exercise 3.4.3(ii)]). But

$$x \in H \leq_h M_n(H),$$

and $\overline{xM_n(H)x}$ is the smallest hereditary C*-subalgebra of $M_n(H)$ containing x. Therefore, q belongs to H, and it is properly infinite, because $p \sim q$. □

The conditions (1)−(3) in the following corollary are equivalent, by Theorem 2.1 above, and [9, Theorem 3.15(i)]. Moreover, according to [11, Corollary 3.2], and [6, Proposition 5.11(iii) and Theorem 8.6], a separable, nuclear C*-algebra is strongly purely infinite if and only if it is $\mathcal{O}_\infty$-stable.

COROLLARY 2.2. *Let A be a C*-algebra with topological dimension zero. The following statements are equivalent:*

(1) *A is strongly purely infinite.*
(2) *A is purely infinite.*
(3) *A is weakly purely infinite.*

If A is separable and nuclear, then the conditions (1)-(3) *above are equivalent to:*

(4) *A is $\mathcal{O}_\infty$-stable.*

THEOREM 2.3. *Suppose that A is a Noetherian C*-algebra. The following statements are equivalent:*

(1) *A is (strongly, or weakly) purely infinite.*
(2) *Every non-zero hereditary C*-subalgebra of A contains a full, properly infinite, projection.*

If A has real rank zero, then conditions (1) *and* (2) *are equivalent to:*

(3) *Every non-zero projection of A is properly infinite.*

PROOF. Let A be a Noetherian C*-algebra.

First, we show (1) ⇔ (2):

Note first that, since every Noetherian C*-algebra A has topological dimension zero, the three properties strong pure infiniteness, pure infiniteness and weak pure infiniteness are equivalent, by Corollary 2.2. Now, let A be purely infinite and let $0 \neq H \leq_h A$. Since H is Noetherian, $Prim(H)$ is compact, and hence H has a strictly full element x. Thus there is $\varepsilon > 0$ such that

$$\overline{HxH} \cong \overline{H(x-\varepsilon)_+H}.$$

Since H is purely infinite, by [5, Definition 4.1, or Proposition 3.5(ii)], we have:

$$x \sim_{cu} (x-\varepsilon)_+,$$

and so [2, Lemma 7.2] implies that H has a full, properly infinite, projection. For the converse, suppose that (2) holds, and a is a non-zero positive element of A, and set $A_a := \overline{aAa}$. By hypothesis, A_a has a full, properly infinite, projection p. Moreover, a is strictly positive in A_a. Thus $p \sim_{cu} a$ (see [5, Proposition 3.5(ii)]), and therefore a is properly infinite. Hence A is purely infinite. (A is pi-1 and so weakly purely infinite.)

Now let us show that if A has real rank zero, then (1) and (3) are equivalent. First, suppose that (3) holds. If $0 \neq a \in A^+$, with $A_a := \overline{aAa}$, by [3, Theorem 2.6], A_a has an approximate unit of projections, say $\{p_i\}_{i \in I}$. We may consider this approximate unit as increasing and sequential, because A_a is σ-unital. Hence we have:

$$A_a := \overline{\bigcup_{i=1}^{\infty} \overline{A_a p_i A_a}}.$$

But (A and so) A_a is Noetherian. Thus there is an $n \in \mathbb{N}$ such that $A_a = \overline{A_a p_n A_a}$. By hypothesis, p_n is properly infinite. Now, since $p_n \sim_{cu} a$, the element a is also properly infinite. Hence A is purely infinite (and non-zero). The converse holds, by [5, Theorem 4.16]. □

It is perhaps worth mentioning that Noetherian (and Artinian) C*-algebras do not necessarily have real rank zero or the ideal property (see [9, Example 3.28 (i) and (ii)]).

References

1. B. Blackadar, *Operator algebras. Theory of C*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, **122**, Springer-Verlag, Berlin, 2006.
2. J. Bosa, J. Gabe, A. Sims, and S. White, The nuclear dimension of $\mathcal{O}_\infty$-stable C*-algebras, *Adv. Math.* **401** (2022), 108250.
3. L.G. Brown and G.K. Pedersen, C*-algebras of real rank zero, *J. Funct. Anal.* **99**(1) (1991), 131–149.
4. L.G. Brown and G.K. Pedersen, Limits and C*-algebras of low rank or dimension, *J. Operator Theory* **61**(2) (2009), 381–417.
5. E. Kirchberg and M. Rørdam, Non-simple purely infinite C*-algebras, *Amer. J. Math.* **122**(3) (2000), 637–666.
6. E. Kirchberg and M. Rørdam, Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$, *Adv. Math.* **167**(2) (2002), 195–264.
7. E. Ortega, F. Perera, and M. Rørdam, The corona factorization property, stability, and the Cuntz semigroup of a C*-algebra, *Int. Math. Res. Not.* **2012**(1) (2012), 34−66.
8. C. Pasnicu and M. Rørdam, Purely infinite C*-algebras of real rank zero, *J. Reine Angew. Math.* **613** (2007), 51–73.
9. C. Pasnicu and M. Rouzbehani, Topological dimension zero and some related properties, submitted.
10. K.R. Strung, *An Introduction to C*-Algebras and the Classification Program*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser/Springer, Cham, 2021.
11. A.S. Toms and W. Winter, Strongly self-absorbing C*-algebras, *Trans. Amer. Math. Soc.* **359**(8) (2007), 3999–4029.

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