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Maximal Regularity for the Neumann-Stokes Problem in $H^{r/2,r}$ Spaces

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Abstract. We provide a maximal regularity theorem for the linear Stokes equation with a non-homogeneous divergence condition in a bounded domain $\Omega \subseteq \mathbb{R}^3$ and with the Neumann boundary conditions. We prove the existence and uniqueness of solutions such that the velocity belongs to the space $H^{(s+1)/2,s+1}((0,T)\times\Omega)$, where $s \in [1,1.5)\cup(1.5,2)$.

Résumé. Nous fournissons un théorème de régularité maximale pour l'équation linéaire de Stokes avec une condition de divergence non homogène dans un domaine borné $\Omega \subseteq \mathbb{R}^3$ et avec les conditions aux limites de Neumann. On prouve l'existence et l'unicité de solutions telles que la vitesse appartient à l'espace $H^{(s+1)/2,s+1}((0,T)\times\Omega)$, où $s \in [1,1.5)\cup(1.5,2)$.

1. Introduction In this paper, we provide a maximal regularity type theorem for the linear Stokes system with the Neumann boundary conditions and nonhomogeneous divergence. Such maximal regularity theorems in different functional spaces have been studied early on by Grubb and Solonnikov [7]. Still they have attracted much interest recently [11,12,14] due to their important role in the study of free boundary problems [1–6, 8–10, 13–15].

Our main result, Theorem 2.1, is a variant of maximal regularity for the nonhomogeneous Stokes equation with the non-homogeneous divergence, stated similarly to [11], but in different functional spaces. In [11], Mucha and Zajaczkowski proved the existence for initial data in an anisotropic Sobolev space L^r, where $r > 3$; here we provide existence and uniqueness for the solution in a low regularity L^2-based Hilbert spaces (see (2.6)). We believe that our proof with one simple corrector z is new (the proof in [11] uses two correctors). The feature of the approach in [11] and Theorem 2.1 is the representation (2.10) for g_t. An important new idea is the presence of the fourth term on the left side of (2.12), i.e., we provide the control of the pressure trace $\|p\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}}$. The control of this quantity is obtained by a space-time trace inequality (see Lemma 3.1).

The main difficulty in applying the theorem from [7] is that the boundary term (which is the sixth term on the right-hand side of (2.12)) has a higher time regularity than the Grubb-Solonnikov interior pressure terms (i.e., the second

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and third terms on the left side of (2.12)). To overcome this, we prove an inequality satisfied for a boundary pressure term, which allows us to absorb the high time regularity pressure term on the right. Another important ingredient in the proof of the main theorem is a trace inequality for functions which are Sobolev in the time variable and square-integrable on the boundary (see Lemma 3.1 and (3.1) below). The trace regularity is used essentially in the proof of the existence for the nonhomogeneous Stokes equation.

2. The Main Result We consider the linear Stokes system

$$\begin{aligned} &u_t - \Delta u + \nabla p = f \qquad \text{in } (0,T)\times\Omega \\ &\operatorname{div} u = g \qquad \text{in } (0,T)\times\Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with a C^∞ boundary $\partial\Omega$; the lower regularity assumption can also be treated using partition of unity and straightening of the boundary. The system is supplemented with two boundary conditions

$$\frac{\partial u}{\partial N} - pN = h_1 \qquad \text{on } \ (0,T)\times\Gamma_1 \tag{2.2}$$

$$u = h_2 \qquad \text{on } (0,T)\times\Gamma_2, \tag{2.3}$$

where Γ_1 and Γ_2 are the two disjoint components of the boundary with $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and N is the unit normal vector to the boundary Γ_1. We impose the initial condition

$$u(0,\cdot) = u_0 \qquad \text{in } \ \Omega \tag{2.4}$$

and the compatibility condition

$$u_0|_{\Gamma_2} = h_2|_{t=0}. \tag{2.5}$$

Denote

$$H^{r,s}((0,T)\times\Omega) = H^r((0,T),L^2(\Omega)) \cap L^2((0,T),H^s(\Omega)), \tag{2.6}$$

with the norm

$$\|f\|^2_{H^{r,s}((0,T)\times\Omega)} = \|f\|^2_{H^r((0,T),L^2(\Omega))} + \|f\|^2_{L^2((0,T),H^s(\Omega))}. \tag{2.7}$$

We shall use the notation $H^{r,s}_\Gamma$, involving functions on $(0,T)\times\Gamma$, for the analogous space corresponding to the boundary set Γ.

The following theorem, which states the maximal regularity theorem for the linear Stokes equation with a nonhomogeneous divergence (cf. [7, 11, 16]), is our main result.

THEOREM 2.1. *Let $s \in [1,3/2) \cup (3/2,2)$. Then, assuming*

$$u_0 \in H^s(\Omega) \tag{2.8}$$

and

$$(f,g,h_1,h_2) \in (H^{s/2-1/2,s-1}, H^{s/2,s}, H_{\Gamma_1}^{s/2-1/4,s-1/2}, H_{\Gamma_2}^{s/2+1/4,s+1/2}), \tag{2.9}$$

with the structural condition

$$g_t = \tilde{g} + \operatorname{div} b, \tag{2.10}$$

where

$$\tilde{g}, b \in H^{s/2-1/2,s-1}, \tag{2.11}$$

there exists a unique solution (u,p) to the system (2.1)–(2.5) *satisfying*

$$\begin{aligned}
&\|u\|_{H^{s/2+1/2,s+1}} + \|p\|_{H^{s/2-1/2,s}} + \|\nabla p\|_{H^{s/2-1/2,s-1}} + \|p\|_{H_{\Gamma_1}^{s/2-1/4,s-1/2}} \\
&\quad\lesssim \|u_0\|_{H^s} + \|f\|_{H^{s/2-1/2,s-1}} + \|g\|_{H^{s/2,s}} + \|\tilde{g}\|_{H^{s/2-1/2,s-1}} + \|b\|_{H^{s/2-1/2,s-1}} \\
&\qquad + \|h_1\|_{H_{\Gamma_1}^{s/2-1/4,s-1/2}} + \|h_2\|_{H_{\Gamma_2}^{s/2+1/4,s+1/2}},
\end{aligned} \tag{2.12}$$

where the implicit constant depends on Ω and T.

The proof of the theorem is given in the next section.

3. Proof of Theorem 2.1

First we recall two auxiliary lemmas from [9] which are needed in the proof of the main result. The first lemma provides an estimate for the trace in a space-time norm.

LEMMA 3.1 ([9]). *Let $s > 1/2$ and $\theta \geq 0$. If*

$$u \in L^2((0,T), H^s(\Omega)) \cap H^{2\theta s/(2s-1)}((0,T), L^2(\Omega)),$$

then $u \in H^\theta((0,T), L^2(\Gamma))$, and for all $\epsilon_0 \in (0,1]$ we have the inequality

$$\|u\|_{H^\theta((0,T),L^2(\Gamma))} \lesssim \epsilon_0 \|u\|_{H^{2\theta s/(2s-1)}((0,T),L^2(\Omega))} + C_{\epsilon_0} \|u\|_{L^2((0,T),H^s(\Omega))}, \tag{3.1}$$

where the implicit constant depends on T and $C_{\epsilon_0} > 0$ is a constant depending on ϵ_0.

The second lemma provides a space-time interpolation inequality needed in several instances in the proof of the main theorem.

LEMMA 3.2 ([9]). *Let $\alpha, \beta > 0$. If $u \in H^\alpha((0,T), L^2(\Omega)) \cap L^2((0,T), H^\beta(\Omega))$, then $u \in H^\theta((0,T), H^\lambda(\Omega))$ and for all $\epsilon_0 \in (0,1]$, we have the inequality*

$$\|u\|_{H^\theta((0,T),H^\lambda(\Omega))} \lesssim \epsilon_0 \|u\|_{H^\alpha((0,T),L^2(\Omega))} + C_{\epsilon_0} \|u\|_{L^2((0,T),H^\beta(\Omega))}, \tag{3.2}$$

for all $\theta \in (0,\alpha)$ and $\lambda \in (0,\beta)$ such that

$$\frac{\theta}{\alpha} + \frac{\lambda}{\beta} \le 1, \tag{3.3}$$

where the implicit constant depends on T and $C_{\epsilon_0} > 0$ is a constant depending on ϵ_0.

In the previous two propositions, the implicit constant does not depend on T if $T \ge 1$, but it may grow when $T \to 0$. In (2.12), as well as in the rest of the paper, we do not indicate the domains $(0,T) \times \Omega$ or Ω in the norms as they are understood from the context. However, we always use a complete notation for norms involving the boundary traces on Γ_1 and Γ_2.

PROOF OF THEOREM 2.1. As in [11], let z be the solution of

$$\begin{aligned} &\Delta z = g \qquad \text{in } (0,T)\times\Omega \\ &\frac{\partial z}{\partial N} = 0 \qquad \text{on } (0,T)\times\Gamma_1 \\ &z = 0 \qquad \text{on } (0,T)\times\Gamma_2. \end{aligned} \tag{3.4}$$

Note that the time derivative z_t satisfies

$$\begin{aligned} &\Delta z_t = \tilde{g} + \operatorname{div} b \qquad \text{in } (0,T)\times\Omega \\ &\frac{\partial z_t}{\partial N} = 0 \qquad \text{on } (0,T)\times\Gamma_1 \\ &z_t = 0 \qquad \text{on } (0,T)\times\Gamma_2, \end{aligned} \tag{3.5}$$

by (2.10). The difference $\tilde{u} = u - \nabla z$ satisfies

$$\tilde{u}_t - \Delta\tilde{u} + \nabla p = f - \nabla z_t + \Delta\nabla z \qquad \text{in } (0,T)\times\Omega \tag{3.6}$$

$$\operatorname{div}\tilde{u} = 0 \qquad \text{in } (0,T)\times\Omega, \tag{3.7}$$

subject to the mixed boundary conditions

$$\begin{aligned} &\frac{\partial\tilde{u}}{\partial N} - pN = h_1 - \frac{\partial\nabla z}{\partial N} \qquad \text{on } (0,T)\times\Gamma_1 \\ &\tilde{u} = -\nabla z + h_2 \qquad \text{on } (0,T)\times\Gamma_2 \end{aligned} \tag{3.8}$$

and the initial condition

$$\tilde{u}(0,\cdot) = u_0 - \nabla z(0) \qquad \text{in } \Omega. \tag{3.9}$$

Denote the right-hand sides of (3.6), $(3.8)_1$, $(3.8)_2$, and (3.9) by $\tilde{f}$, $\tilde{h}_1$, $\tilde{h}_2$, and $\tilde{u}_0$, respectively. Note that based on the estimates in the proof of [7, Theorem 7.5], Theorem 2.1 holds for the special case $g = \tilde{g} = b = 0$ since $s \geq 1$, but without the fourth term on the left side of (2.12). Applying it to the system (3.6)–(3.9), we obtain

$$\begin{aligned} &\|\tilde{u}\|_{H^{s/2+1/2,s+1}} + \|p\|_{H^{s/2-1/2,s}} + \|\nabla p\|_{H^{s/2-1/2,s-1}} \\ &\quad \lesssim \|\tilde{u}_0\|_{H^s} + \|\tilde{f}\|_{H^{s/2-1/2,s-1}} + \|\tilde{h}_1\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}} + \|\tilde{h}_2\|_{H^{s/2+1/4,s+1/2}_{\Gamma_2}}, \end{aligned} \tag{3.10}$$

whence

$$\begin{aligned} &\|\tilde{u}\|_{H^{s/2+1/2,s+1}} + \|p\|_{H^{s/2-1/2,s}} + \|\nabla p\|_{H^{s/2-1/2,s-1}} \\ &\quad \lesssim \|u_0\|_{H^s} + \|z(0)\|_{H^{s+1}} + \|f\|_{H^{s/2-1/2,s-1}} + \|\nabla z_t\|_{H^{s/2-1/2,s-1}} \\ &\qquad + \|\Delta \nabla z\|_{H^{s/2-1/2,s-1}} + \|h_1\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}} + \|h_2\|_{H^{s/2+1/4,s+1/2}_{\Gamma_2}} \\ &\qquad + \left\| \frac{\partial \nabla z}{\partial N} \right\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}} + \|\nabla z\|_{H^{s/2+1/4,s+1/2}_{\Gamma_2}}. \end{aligned} \tag{3.11}$$

We now need to show that each of the terms on the right-hand side of (3.11) is dominated by the right-hand side of (2.12). The first, third, sixth, and seventh terms in (3.11) already appear in (2.12), so we only need to estimate the remaining ones. For the second term, we have

$$\|z(0)\|_{H^{s+1}} \lesssim \|g(0)\|_{H^{s-1}} = \|\operatorname{div} u_0\|_{H^{s-1}} \lesssim \|u_0\|_{H^s}. \tag{3.12}$$

For the fourth term in (3.11), we write

$$\begin{aligned} \|\nabla z_t\|_{H^{s/2-1/2,s-1}} &\lesssim \|\nabla z_t\|_{L^2 H^{s-1}} + \|\nabla z_t\|_{H^{s/2-1/2} L^2} \\ &\lesssim \|\tilde{g}\|_{L^2 H^{s-2}} + \|b\|_{L^2 H^{s-1}} + \|\tilde{g}\|_{H^{s/2-1/2} H^{-1}} + \|b\|_{H^{s/2-1/2} L^2} \\ &\lesssim \|\tilde{g}\|_{H^{s/2-1/2,s-1}} + \|b\|_{H^{s/2-1/2,s-1}}, \end{aligned} \tag{3.13}$$

where we used the elliptic regularity for (3.5) in the second inequality. For the fifth term in (3.11), we have

$$\|\Delta \nabla z\|_{H^{s/2-1/2,s-1}} \lesssim \|\nabla g\|_{H^{s/2-1/2,s-1}} \lesssim \|g\|_{L^2 H^s} + \|g\|_{H^{s/2-1/2} H^1} \lesssim \|g\|_{H^{s/2,s}}, \tag{3.14}$$

where we used Lemma 3.2 in the last step. For the eighth term in (3.11), we write

$$\left\|\frac{\partial\nabla z}{\partial N}\right\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}} \lesssim \left\|\frac{\partial\nabla z}{\partial N}\right\|_{L^2H^{s-1/2}(\Gamma_1)} + \left\|\frac{\partial\nabla z}{\partial N}\right\|_{H^{s/2-1/4}L^2(\Gamma_1)}, \tag{3.15}$$

where for simplicity of notation, we write Γ_1 instead of $(0,T)\times\Gamma_1$ as the time interval is understood. For the first term on the right side of (3.15), we use the Sobolev trace inequality and obtain

$$\left\|\frac{\partial\nabla z}{\partial N}\right\|_{L^2H^{s-1/2}(\Gamma_1)} \lesssim \|D^2 z\|_{L^2H^s} \lesssim \|g\|_{L^2H^s}, \tag{3.16}$$

which is dominated by the third term in (2.12). For the second term on the right side of (3.15), we use the space-time trace inequality in Lemma 3.1 to write

$$\left\|\frac{\partial\nabla z}{\partial N}\right\|_{H^{s/2-1/4}L^2(\Gamma_1)} \lesssim \|D^2 z\|_{H^{s/2}L^2} + \|D^2 z\|_{L^2H^s} \lesssim \|g\|_{H^{s/2,s}}. \tag{3.17}$$

Regarding the ninth term in (3.11), we have

$$\|\nabla z\|_{H^{s/2+1/4,s+1/2}_{\Gamma_2}} \lesssim \|\nabla z\|_{L^2H^{s+1/2}(\Gamma_2)} + \|\nabla z\|_{H^{s/2+1/4}L^2(\Gamma_2)}. \tag{3.18}$$

For the first term on the right side of (3.18), we use the trace inequality to write

$$\|\nabla z\|_{L^2H^{s+1/2}(\Gamma_2)} \lesssim \|\nabla z\|_{L^2H^{s+1}} \lesssim \|z\|_{L^2H^{s+2}} \lesssim \|g\|_{L^2H^s}. \tag{3.19}$$

For the second term on the right side of (3.18), we appeal to Lemma 3.1 to get

$$\begin{aligned} \|\nabla z\|_{H^{s/2+1/4}L^2(\Gamma_2)} &\lesssim \|\nabla z\|_{H^{s/2+1/2}L^2} + \|\nabla z\|_{L^2H^{s+1}} \\ &\lesssim \|z\|_{H^{s/2+1/2}H^1} + \|z\|_{L^2H^{s+2}} \\ &\lesssim \|z_t\|_{H^{s/2-1/2}H^1} + \|z\|_{L^2H^{s+2}} \\ &\lesssim \|\tilde{g}\|_{H^{s/2}L^2} + \|b\|_{H^{s/2-1/2}L^2} + \|g\|_{L^2H^s}. \end{aligned} \tag{3.20}$$

Therefore, all the terms on the far right side of (3.20) are bounded by the right-hand side of (2.12). This concludes estimates for all the terms in (3.11), and (2.12) is proven, except for the bound of the fourth term on the left, i.e., the term $\|p\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}}$. Since, by (2.2),

$$\|p\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}} \lesssim \left\|\frac{\partial u}{\partial N}\right\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}} + \|h_1\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}}, \tag{3.21}$$

we only need to prove

$$\left\| \frac{\partial u}{\partial N} \right\|_{H^{s/2-1/4,s-1/2}_{\Gamma_1}} \lesssim \|u\|_{H^{s/2+1/2,s+1}}. \tag{3.22}$$

To bound the space part of the norm on the left side of (3.22), we use the classical trace inequality and write

$$\left\| \frac{\partial u}{\partial N} \right\|_{L^2 H^{s-1/2}(\Gamma_1)} \lesssim \|\nabla u\|_{L^2 H^s} \lesssim \|u\|_{H^{s/2+1/2,s+1}}. \tag{3.23}$$

To bound the time component of the norm on the left side of (3.22), we instead use Lemmas 3.1 and 3.2, which leads to

$$\left\| \frac{\partial u}{\partial N} \right\|_{H^{s/2-1/4} L^2(\Gamma_1)} \lesssim \|\nabla u\|_{H^{s/2} L^2} + \|\nabla u\|_{L^2 H^s} \lesssim \|u\|_{H^{s/2+1/2,s+1}}, \tag{3.24}$$

completing the proof of (3.22) and thus also of (2.12). □

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