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Generalized Tracially Approximated C*-algebras

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Abstract. In this paper, we introduce some classes of generalized tracial approximation C*-algebras. Consider the class of unital C*-algebras which are tracially $\mathcal{Z}$-absorbing (or have tracial nuclear dimension at most n, or have the property SP, or are m-almost divisible). Then A is tracially $\mathcal{Z}$-absorbing (respectively, has tracial nuclear dimension at most n, has the property SP, is weakly (n, m)-almost divisible) for any simple unital C*-algebra A in the corresponding class of generalized tracial approximation C*-algebras. As an application, let A be an infinite-dimensional unital simple C*-algebra, and let B be a centrally large subalgebra of A. If B is tracially $\mathcal{Z}$-absorbing, then A is tracially $\mathcal{Z}$-absorbing. This result was obtained by Archey, Buck, and Phillips in [2].

Résumé. On introduit la notion d'approximation traciale généralisée d'une C*-algèbre par des C*-algèbres dans une class donnée. Cette notion généralise la notion de Lin d'approximation triviale simple, et aussi la notion d'Archey et de Phillips de centralement grande sousalgèbre, deux notions qui se sont démontrées très importantes.

1. **Introduction** The Elliott program for the classification of amenable C*-algebras might be said to have begun with the K-theoretical classification of AF algebras in [6]. A major next step was the classification of simple AH algebras without dimension growth (in the real rank zero case see [8], and in the general case see [9]). This led eventually to the classification of simple separable amenable C*-algebras with finite nuclear dimension in the UCT class (see [25], [33], [11], [19], [20], [41], [10], [17], and [18]).

A crucial intermediate step was Lin's axiomatization of Elliott-Gong's decomposition theorem for simple AH algebras of real rank zero (classified by Elliott-Gong in [8]) and Gong's decomposition theorem ([16]) for simple AH algebras (classified by Elliott-Gong-Li in [9]). For this purpose, Lin introduced the concepts of TAF and TAI ([28] and [29]). (A weaker version of the property TAF had been introduced by Popa in [36].) Instead of assuming inductive limit structure, Lin started with a certain abstract (tracial) approximation property. Elliott and Niu in [12] considered this notion of tracial approximation by other

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classes of unital C*-algebras than the finite-dimensional ones for TAF and the interval algebras for TAI. In [12], Elliott and Niu, and in [7], Elliott, Fan, and Fang showed that certain properties of C*-algebras in a given class Ω are inherited by a simple unital C*-algebra in the class TAΩ.

Large and centrally large subalgebras were introduced in [35] and [3] by Phillips and Archey as abstractions of Putnam's orbit breaking subalgebra of the crossed product algebra $\mathrm{C}^*(X, \mathbb{Z}, \sigma)$ of the Cantor set by a minimal homeomorphism in [37].

In [3], Archey and Phillips showed that if B is centrally large in A and B has stable rank one, then so also does A. In [2], Archey, Buck, and Phillips proved that if A is a simple infinite-dimensional stably finite unital C*-algebra and $B \subseteq A$ is a centrally large subalgebra, then A is tracially $\mathcal{Z}$-absorbing in the sense of [22] if, and only if, B is tracially $\mathcal{Z}$-absorbing.

Inspired by centrally large subalgebras and tracial approximation C*-algebras, we introduce a class of generalized tracial approximation C*-algebras. The notion generalizes both Archey and Phillips's centrally large subalgebras and Lin's notion of tracial approximation.

Let Ω be a class of unital C*-algebras. We define as follows the class of C*-algebras which can be weakly tracially approximated by C*-algebras in Ω, and denote this class by WTAΩ.

DEFINITION 1.1. *A simple unital* C*-*algebra A will be said to belong to the class* WTAΩ *if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any non-zero element $a \geq 0$, there exist a projection $p \in A$, an element $g \in A$ with $0 \leq g \leq 1$, and a unital* C*-*subalgebra B of A with $g \in B, 1_B = p$, and $B \in \Omega$, such that*

(1) $(p-g)x \in_\varepsilon B,\ x(p-g) \in_\varepsilon B$, *for all $x \in F$,*
(2) $\|(p-g)x - x(p-g)\| < \varepsilon$, *for all $x \in F$,*
(3) $1-(p-g) \precsim a$ *(see Section 2), and*
(4) $\|(p-g)a(p-g)\| \geq \|a\| - \varepsilon$.

It follows from the definitions and by the proof of Theorem 4.1 of [12] that if A is a simple unital C*-algebra and $A \in \mathrm{TA}\Omega$ (Definition 2.5), then $A \in \mathrm{WTA}\Omega$. Furthermore, if $\Omega = \{B\}$, and $B \subseteq A$ is a centrally large subalgebra of A (Definition 2.6), then $A \in \mathrm{WTA}\Omega$.

In Theorem 3.9 of [31], Niu shows that if (X, σ, Γ) is a dynamical system (X compact metrizable and Γ countable amenable) with the (URP), then the crossed product C*-algebra can be weakly tracially approximated in a non-unital sense by (not necessarily unital) homogeneous C*-algebras with dimension ratio almost dominated by the mean dimension of (X, σ, Γ). Let $\Omega = \{C^\dagger : C \cong \bigoplus_{s=1}^{N} \mathrm{M}_{K_s}(\mathrm{C}_0(Z_s)),\ N \in \mathbb{Z}_{\geq 0}\}$, where each Z_s is a locally compact Hausdorff space, and $C^\dagger$ is the unitization of C. By the proof of Theorem 3.9 of [31], one can show that Niu's crossed product C*-algebra belongs to the class WTAΩ.

The Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by Connes in [4]. It was adapted by Herman and Ocneanu for UHF-algebras in [21]. In [27] Kishimoto, in [23] and [24] Izumi, and in [39] Rørdam considered the Rokhlin property in a much more general C*-algebra context. More recently, Osaka and Phillips studied actions of a finite group and of the group $\mathbb{Z}$ of integers on certain simple C*-algebras with a modified Rokhlin property in [32] and [34].

In [32], Osaka and Phillips showed the following result (a part of Lemma 2.5 of [32]): Let A be a stably finite simple unital C^*-algebra with real rank zero such that the order on projections in matrix algebras over A is determined by traces (strict comparison of projections). Let $\alpha \in \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then for every finite set $F \subseteq \mathrm{C}^*(\mathbb{Z}, A, \alpha)$, every $\varepsilon > 0$, every $N \in \mathbb{N}$, and every non-zero positive element $z \in A$, there exist a projection $p \in A$, a unital subalgebra $D \subseteq p\mathrm{C}^*(\mathbb{Z}, A, \alpha)p$ with $1_D = p$, and a projection $q \in D$ such that

(1) $qx \in_\varepsilon D,\ xq \in_\varepsilon D$, for all $x \in F$,
(2) $1 - q \precsim z$, and
(3) $N\langle 1 - q\rangle \leq \langle q\rangle$ (see Section 2).

Inspired by this result, in this paper, we introduce another class of generalized tracial approximation C*-algebras. Let Ω be a class of unital C*-algebras. We define as follows the class of C*-algebras which can be tracially almost approximated by C*-algebras in Ω, and denote this class by TAAΩ.

DEFINITION 1.2. *A simple unital* C*-*algebra A will be said to belong to the class* TAAΩ *if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, any non-zero element $a \geq 0$, there exist a projection $p \in A$, a* C*-*subalgebra B of A with $1_B = p$ and $B \in \Omega$, and a projection $q \in B$ such that*

(1) $qx \in_\varepsilon B,\ xq \in_\varepsilon B$, for all $x \in F$,
(2) $1 - q \precsim a$ (see Section 2), and
(3) $\|qaq\| \geq \|a\| - \varepsilon$.

In this paper, we shall prove the following five results:

Let Ω be a class of unital C*-algebras with the property SP. Then A has the property SP for any simple unital C*-algebra $A \in$ WTAΩ. (Theorem 3.1.)

Let Ω be a class of unital C*-algebras which are tracially $\mathcal{Z}$-absorbing (Definition 2.3). Then A is tracially $\mathcal{Z}$-absorbing for any simple unital C*-algebra $A \in$ WTAΩ. (Theorem 3.4.)

Let Ω be a class of unital C*-algebras with tracial nuclear dimension at most n (Definition 2.4). Then A has tracial nuclear dimension at most n for any simple unital C*-algebra $A \in$ WTAΩ. (Theorem 3.7.)

Let Ω be a class of unital C*-algebras which are m-almost divisible (Definition 2.7). Let $A \in$ WTAΩ be a simple unital stably finite C*-algebra such that for any $n \in \mathbb{N}$ the C*-algebra $\mathrm{M}_n(A)$ belongs to the class WTAΩ. Then A is weakly $(2, m)$-almost divisible (Definition 2.8). (Theorem 3.10.)

Let Ω be a class of unital C*-algebras which are m-almost divisible. Let $A \in$ TAAΩ be a simple unital stably finite C*-algebra such that for any $n \in \mathbb{N}$ and any unital hereditary C*-subalgebra D of $\mathrm{M}_n(A)$, D belongs to the class TAAΩ. Then A is weakly $(2, m)$-almost divisible. (Theorem 3.11.)

As applications, the following known results follow from these results.

Let A be a simple unital C*-algebra, and let B be a centrally large subalgebra of A. If B is tracially $\mathcal{Z}$-absorbing, then A is tracially $\mathcal{Z}$-absorbing. This result was obtained by Archey, Buck, and Phillips in [2].

Let Ω be a class of unital C*-algebras which are tracially $\mathcal{Z}$-absorbing. Then A is tracially $\mathcal{Z}$-absorbing for any simple unital C*-algebra $A \in \mathrm{TA}\Omega$. This result was obtained by Elliott, Fan, and Fang in [7].

Let A be a simple unital C*-algebra, and let B be a centrally large subalgebra of A. If B has tracial nuclear dimension at most n, then A has tracial nuclear dimension at most n. This result was obtained by Zhao, Fang, and Fan in [43]

Let Ω be a class of unital C*-algebras which have tracial nuclear dimension at most n. Then A has tracial nuclear dimension at most n for any simple unital C*-algebra $A \in \mathrm{TA}\Omega$. This result was obtained by Fan and Yang in [13].

2. **Preliminaries and Definitions**

Recall that a C*-algebra A has the property SP if every non-zero hereditary C*-subalgebra of A contains a non-zero projection.

Let A be a C*-algebra, and let $\mathrm{M}_n(A)$ denote the C*-algebra of $n\times n$ matrices with entries elements of A. Let $\mathrm{M}_\infty(A)$ denote the algebraic inductive limit of the sequence $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is the canonical embedding as the upper left-hand corner block. Let $\mathrm{M}_\infty(A)_+$ (respectively, $\mathrm{M}_n(A)_+$) denote the positive elements of $\mathrm{M}_\infty(A)$ (respectively, $\mathrm{M}_n(A)$). Given $a, b \in \mathrm{M}_\infty(A)_+$, one says that a is Cuntz subequivalent to b (written $a \precsim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

One says that a and b are Cuntz equivalent (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. We shall write $\langle a\rangle$ for the Cuntz equivalence class of a.

The object $\mathrm{Cu}(A) := (A \otimes K)_+/\sim$ will be called the Cuntz semigroup of A. (See [5].) Observe that any $a, b \in \mathrm{M}_\infty(A)_+$ are Cuntz equivalent to orthogonal elements $a', b' \in \mathrm{M}_\infty(A)_+$ (i.e., $a'b' = 0$), and so $\mathrm{Cu}(A)$ becomes an ordered semigroup when equipped with the addition operation

$$\langle a\rangle + \langle b\rangle = \langle a + b\rangle$$

whenever $ab = 0$, and the order relation

$$\langle a\rangle \le \langle b\rangle \Leftrightarrow a \precsim b.$$

Given a in $\mathrm{M}_\infty(A)_+$ and $\varepsilon > 0$, we denote by $(a-\varepsilon)_+$ the element of $\mathrm{C}^*(a)$ corresponding (via the functional calculus) to the function $f(t) = \max(0, t - \varepsilon)$, $t \in \sigma(a)$. By the functional calculus, it follows in a straightforward manner that $((a - \varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+$.

Let $0 < \varepsilon < 1$ be two positive numbers. Define

$$f_\varepsilon(t) = \begin{cases} 1 & if\ t \ge \varepsilon, \\ (2t-\varepsilon)/\varepsilon & if\ \varepsilon/2 < t \le \varepsilon, \\ 0 & if\ 0 \le t \le \varepsilon/2. \end{cases}$$

The following facts are well known.

THEOREM 2.1. *([1], [22], [35], [40].) Let A be a* C*-*algebra.*

(1) *Let $a,\ b \in A_+$ and $\varepsilon > 0$ be such that $\|a-b\| < \varepsilon$. Then there is a contraction d in A with $(a-\varepsilon)_+ = dbd^*$.*

(2) *Let $a,\ p$ be positive elements in $\mathrm{M}_\infty(A)$ with p a projection. If $p \precsim a$, then there is b in $\mathrm{M}_\infty(A)_+$ such that $bp = 0$ and $b + p \sim a$.*

(3) *Let a be a positive element of A not Cuntz equivalent to a projection. Let $\delta > 0$, and let $f \in C_0(0,1]$ be a non-negative function with $f = 0$ on $(\delta, 1)$, $f > 0$ on $(0,\delta)$, and $\|f\| = 1$. Then $f(a) \neq 0$ and $(a-\delta)_+ + f(a) \precsim a$.*

(4) *Let $a, b \in A$ satisfy $0 \le a \le b$. Let $\varepsilon \ge 0$. Then $(a-\varepsilon)_+ \precsim (b-\varepsilon)_+$ (Lemma 1.7 of [35]).*

Winter and Zacharias introduced the notion of nuclear dimension for C*-algebras in [42].

DEFINITION 2.2. *([42].) Let A be a* C*-*algebra, $m \in \mathbb{N}$. A completely positive contraction $\varphi : F \to A$ is m-decomposable (where F is a finite dimensional* C*-*algebra), if there is a decomposition $F = F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)}$ such that the restriction $\varphi^{(i)}$ of φ to $F^{(i)}$ has order zero (which means preserves orthogonality, i.e., $\psi(e)\psi(f) = 0$ for all $e, f \in \mathrm{M}_n$ with $ef = 0$), for each $i \in \{0, \cdots, m\}$, and we say φ is m-decomposable with respect to the decomposition $F = F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)}$. A has nuclear dimension m, written $\dim_{\mathrm{nuc}}(A) = m$, if m is the least integer such that the following condition holds: For any finite subset $G \subseteq A$ and $\varepsilon > 0$, there is a finite-dimensional completely positive approximation (F, φ, ψ) for G to within ε (i.e., F is finite-dimensional, $\psi : A \to F$ and $\varphi : F \to A$ are completely positive, and $\|\varphi\psi(b) - b\| < \varepsilon$ for any $b \in G$) such that ψ is a contraction, and φ is m-decomposable with completely positive contraction order zero components $\varphi^{(i)}$. If no such m exists, we write $\dim_{\mathrm{nuc}}(A) = \infty$.*

Hirshberg and Orovitz introduced the notion of tracial $\mathcal{Z}$-absorption in [22].

DEFINITION 2.3. *([22].) We say a unital* C*-*algebra A is tracially $\mathcal{Z}$-absorbing if $A \neq \mathbb{C}$, and for any finite set $F \subseteq A$, $\varepsilon > 0$, non-zero positive element $a \in A$, and $n \in \mathbb{N}$, there is a completely positive order zero contraction $\psi : \mathrm{M}_n \to A$, where order zero means preserving orthogonality, i.e., $\psi(e)\psi(f) = 0$ for all $e,\ f \in \mathrm{M}_n$ with $ef = 0$, such that the following properties hold:*

(1) *$1 - \psi(1) \precsim a$, and*

(2) *for any normalized element $x \in \mathrm{M}_n$ (i.e., with $\|x\| = 1$) and any $y \in F$ we have $\|\psi(x)y - y\psi(x)\| < \varepsilon$.*

Note that this property implies that either $A = 0$ or $\dim(\mathrm{A}) = \infty$.

Inspired by Hirshberg and Orovitz's tracial $\mathcal{Z}$-absorption in [22], Fu introduced a notion of tracial nuclear dimension in his doctoral dissertation [14] (see also [15]), and he showed that finite tracial nuclear dimension implies tracial $\mathcal{Z}$-absorption for a separable, exact, simple unital C*-algebra with non-empty tracial state space.

DEFINITION 2.4. *([14].) A unital* C**-algebra A is said to have tracial nuclear dimension at most m, written* $\mathrm{Trdim}_{\mathrm{nuc}}(A) \leq m$, *if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any non-zero positive element a of A, there exist a* C**-subalgebra D of A with* $\dim_{\mathrm{nuc}}(D) \leq m$ *(Definition* 2.2*), a contractive completely positive linear map $\varphi : A \to A$ and a contractive completely positive linear map $\psi : A \to D$ such that*

(1) $\varphi(1) \precsim a$, *and*
(2) $\|x - \varphi(x) - \psi(x)\| < \varepsilon$, *for any $x \in F$.*

Let Ω be a class of unital C*-algebras. Then the class of simple separable C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by $\mathrm{TA}\Omega$, is defined as follows.

DEFINITION 2.5. *([12].) A simple unital* C**-algebra A is said to belong to the class* $\mathrm{TA}\Omega$ *if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any non-zero element $a \geq 0$, there are a projection $p \in A$, and a* C**-subalgebra B of A with $1_B = p$ and $B \in \Omega$, such that*

(1) $\|xp - px\| < \varepsilon$, *for all $x \in F$,*
(2) $pxp \in_{\varepsilon} B$, *for all $x \in F$, and*
(3) $1 - p \precsim a$.

Remark: If Ω is a class of unital C*-algebras, by the proof of Theorem 4.1 of [12], if A is a simple unital C*-algebra and $A \in \mathrm{TA}\Omega$ (Definition 2.5), then $A \in \mathrm{WTA}\Omega$ (Definition 1.1). If Ω is a class of unital C*-algebras then the class $\mathrm{TA}\Omega$ is contained in the class $\mathrm{TAA}\Omega$ of Definition 1.2. (In particular one has Corollaries 3.2, 3.5, and 3.8, below.)

Centrally large and stably centrally large subalgebras were introduced in [3] by Archey and Phillips.

DEFINITION 2.6. *([3].) Let A be an infinite-dimensional simple unital* C**-algebra. A unital* C**-subalgebra $B \subseteq A$ is said to be centrally large in A if for every $m \in \mathbb{N}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that the following conditions hold.*

(1) $0 \leq g \leq 1$.
(2) *For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.*
(3) *For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.*
(4) $g \precsim_B y$ *and* $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.
(6) *For $j = 1, 2, \ldots, m$ we have $\|ga_j - a_jg\| < \varepsilon$.*

Recall from Section 1 that if a simple unital C*-algebra A has a centrally large C*-subalgebra B, then A belongs to the class WTAΩ with $\Omega = \{B\}$. (In particular one has Corollaries 3.3, 3.6, and 3.9, below.)

The property of m-almost divisibility was introduced by Robert and Tikuisis in [38].

Definition 2.7. *([38].) Let $m \in \mathbb{N}$. We say that A is m-almost divisible if for each $a \in \mathrm{M}_\infty(A)_+$, $k \in \mathbb{N}$, and $\varepsilon > 0$, there exists $b \in \mathrm{M}_\infty(A)_+$ such that $k\langle b\rangle \leq \langle a\rangle$ and $\langle (a-\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b\rangle$.*

Definition 2.8. *Let $n,\ m \in \mathbb{N}$. We shall say that A is weakly (n, m)-almost divisible if for each $a \in \mathrm{M}_\infty(A)_+$, $k \in \mathbb{N}$, and $\varepsilon > 0$, there exists $b \in \mathrm{M}_\infty(A)_+$ such that $k\langle b\rangle \leq n\langle a\rangle$ and $\langle (a-\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b\rangle$.*

Note that if A has the property of either Definition 2.7 or Definition 2.8 then so also does any matrix algebra over A.

The following two theorems are Lemma 1.7 and Lemma 1.8 of [2].

Theorem 2.10 will be used in the proof of Theorem 3.4.

Theorem 2.9. *([2].) For every $\varepsilon > 0$ there is $\delta > 0$ such that the following statement holds. Let A be a C*-algebra, $B \subseteq A$ a C*-subalgebra, n a non-zero integer, $\varphi_0 : \mathrm{M}_n \to A$ a completely positive contractive order zero map, and $x \in B$ such that*

(1) $0 \leq x \leq 1$,

(2) with $(e_{j,k}), j, k = 1, 2, \cdots, n$ the standard system of matrix units for M_n, we have $\|\varphi_0(e_{j,k})x - x\varphi_0(e_{j,k})\| < \varepsilon$ for $j, k = 1, 2, \cdots, n$, and

(3) $\varphi_0(e_{j,k})x \in_\varepsilon B$.

Then there is a completely positive contractive order zero map $\varphi : \mathrm{M}_n \to B$ such that for all $z \in \mathrm{M}_n$, with $\|z\| \leq 1$, we have $\|\varphi_0(z)x - \varphi(z)\| < \varepsilon$.

Theorem 2.10. *([2].) For every $\varepsilon > 0$ and non-zero positive integer n, there is $\delta > 0$ such that the following statement holds. Whenever A, B, $\varphi_0 : \mathrm{M}_n \to A$, and $x \in B$ satisfy the conditions of Theorem 2.9, and in addition A is unital and B contains the unit of A, there exists a completely positive contractive order zero map $\varphi : \mathrm{M}_n \to A$ such that*

(1) $\|\varphi_0(z)x - \varphi(z)\| < \varepsilon$, for all $z \in \mathrm{M}_n$ with $\|z\| \leq 1$, and

(2) $1 - \varphi(1) \precsim (1-x) \oplus (1 - \varphi_0(1))$.

3. The Main Results

Theorem 3.1. *Let Ω be a class of unital C*-algebras which have the property SP. Then A has the property SP for any simple unital C*-algebra $A \in$ WTAΩ.*

Proof. Let B be a non-zero hereditary C*-subalgebra of A. We must show that B contains a non-zero projection. Choose a positive element a of B of norm one.

Given $\varepsilon > 0$, with f_ε as above, there exists $\delta_2 > 0$ satisfying Lemma 2.5.11 (2) of [30].

With $F = \{a\}$, and any $\varepsilon' > 0$, since $A \in \mathrm{WTA}\Omega$, there exist a projection $p \in A$, an element $g \in A$ with $0 \le g \le 1$, and a C*-subalgebra D of A with $g \in D$ and $1_D = p$, such that D has the property SP and

(1) $(p-g)a \in_{\varepsilon'} D,\ a(p-g) \in_{\varepsilon'} D$,

(2) $\|(p-g)a - a(p-g)\| < \varepsilon'$, and

(3) $\|(p-g)a(p-g)\| \ge 1 - \varepsilon'$.

By (1) and (2), for sufficiently small ε' (see Lemma 2.5.11 (2) of [30]), there exists an element of norm at most one $b \in D_+$ such that $\|a^{1/2}(p-g)^2a^{1/2} - b\| < \delta_2$ and $\|(p-g)a(p-g) - b\| < \delta_2$.

Since $\|(p-g)a(p-g) - b\| < \delta_2$ and (by (3)) $\|(p-g)a(p-g)\| \ge 1-\varepsilon'$, one has $(b-\varepsilon)_+ \ne 0$ (otherwise, $1 - \varepsilon' < \delta_2 + \varepsilon$). Since D has the property SP, then there exists a non-zero projection $q \in \overline{(b-\varepsilon)_+ D (b-\varepsilon)_+}$.

Since $f_\varepsilon(b)(b-\varepsilon)_+ = (b-\varepsilon)_+$, we have $f_\varepsilon(b)q = q$.

Since $\|a^{1/2}(p-g)^2a^{1/2} - b\| < \delta_2$, by the choice of δ_2,

$$\|f_\varepsilon(a^{1/2}(p-g)^2a^{1/2}) - f_\varepsilon(b)\| < \varepsilon.$$

Hence,

$$\begin{aligned}&\|f_\varepsilon(a^{1/2}(p-g)^2a^{1/2})qf_\varepsilon(a^{1/2}(p-g)^2a^{1/2}) - q\|\\ &= \|f_\varepsilon(a^{1/2}(p-g)^2a^{1/2})qf_\varepsilon(a^{1/2}(p-g)^2a^{1/2}) - f_\varepsilon(b)qf_\varepsilon(b)\|\\ &< 3\varepsilon.\end{aligned}$$

It follows by the functional calculus that, when ε is small enough, there exists a non-zero projection e belonging to the hereditary C*-subalgebra of A generated by a, and since $a \in B$ and B is hereditary, $e \in B$. This shows that A has the property SP. □

COROLLARY 3.2. *Let Ω be a class of unital* C**-algebras which have the property* SP*. Then A has the property* SP *for any simple unital* C**-algebra $A \in \mathrm{TA}\Omega$.*

PROOF. As pointed out in Section 1, $\mathrm{TA}\Omega \subseteq \mathrm{WTA}\Omega$. The statement then follows from Theorem 3.1. □

COROLLARY 3.3. *Let A be a non-zero simple unital* C**-algebra, and let B be a centrally large subalgebra of A. If B has the property* SP*, then A has the property* SP*.*

PROOF. See remark following Definition 2.6. □

THEOREM 3.4. *Let Ω be a class of unital* C**-algebras which are tracially $\mathcal{Z}$-absorbing (Definition* 2.3*). Then A is tracially $\mathcal{Z}$-absorbing, if $A \ne \mathbb{C}$, for any simple unital* C**-algebra $A \in \mathrm{WTA}\Omega$.*

PROOF. We must show that for any finite set $F = \{a_1, a_2, \cdots, a_k\} \subseteq A$ (we may assume that $\|a_i\| < 1$ for all $1 \le i \le k$), any $\varepsilon > 0$, any non-zero positive element $b \in A$, and any $n \in \mathbb{N}$, there is an order zero contraction $\psi : \mathrm{M}_n \to A$ such that the following conditions hold:

(1) $1-\psi(1) \precsim b$, and

(2) for any normalized element $z \in \mathrm{M}_n$ and any $y \in F$, we have $\|\psi(z)y - y\psi(z)\| < \varepsilon$.

Since A is either zero or infinite-dimensional (see the remark following Definition 2.3), and is simple, if $A \neq 0$ then by Lemma 2.3 of [35], there exist elements $b',\ b'' \in A$ of norm one such that $b'b'' = 0$, and $b' + b'' \precsim b$. Also there exist elements ${b_1}',\ {b_2}' \in A$ of norm one such that $b_1'b_2' = 0$, $b_1' \sim b_2'$, and $b_1' + {b_2}' \precsim b'$.

Given $\varepsilon > 0$, with $f(t) = t^{1/2} \in C([0,1])$, there exists $\varepsilon' > 0$ satisfying Lemma 2.5.11 (1) of [30]. Given such $\varepsilon' > 0$, for $G = F \cup \{b'',\ (b'')^{1/2}\}$, since $A \in \mathrm{WTA}\Omega$, there exist a projection $p \in A$, an element $g \in A$ with $0 \leq g \leq 1$, and a tracially $\mathcal{Z}$-absorbing C*-subalgebra B of A with $g \in B$ and $1_B = p$ such that

$(1)'$ $(p-g)x \in_{\varepsilon'} B$, $x(p-g) \in_{\varepsilon'} B$, for $x \in G$,

$(2)'$ $\|(p-g)x - x(p-g)\| < \varepsilon'$, for $x \in G$,

$(3)'$ $1-(p-g) \precsim {b_1}' \sim {b_2}'$, and

$(4)'$ $\|(p-g)b''(p-g)\| \geq 1-\varepsilon'$.

By $(2)'$ with sufficiently small ε', by Lemma 2.5.11 (1) of [30], we have

$(5)'$ $\|(p-g)^{1/2}x - x(p-g)^{1/2}\| < \varepsilon$, for $x \in G$, and

$(6)'$ $\|(1-(p-g))^{1/2}x - x(1-(p-g))^{1/2}\| < \varepsilon$, for $x \in G$.

By $(1)'$, with sufficiently small ε', together with $(5)'$, there exist elements $a_1',\ a_2',\ \cdots,\ a_k' \in B$ and a positive element $b''' \in B$ such that

$$\|(p-g)^{1/2}a_i(p-g)^{1/2} - a_i'\| < \varepsilon, \quad \text{for } 1 \leq i \leq k, \text{and}$$

$$\|(p-g)^{1/2}b''(p-g)^{1/2} - b'''\| < \varepsilon.$$

From the first inequality, together with $(5)'$ and $(6)'$, for any $1 \leq i \leq k$, one has

$$\begin{aligned}
&\|a_i - a_i' - (1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\| \\
&\leq \|a_i - (p-g)a_i - (1-(p-g))a_i\| + \|(p-g)a_i - (p-g)^{1/2}a_i(p-g)^{1/2}\| \\
&+\|(1-(p-g))a_i - (1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\| \\
&+\|(p-g)^{1/2}a_i(p-g)^{1/2} - a_i'\| \\
&< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \qquad (\mathbf{3.4.1}).
\end{aligned}$$

From the second inequality, by (1) of Theorem 2.1, one has

$(7)'$ $\quad (b''' - \varepsilon)_+ \precsim (p-g)^{1/2}b''(p-g)^{1/2}$.

By $(4)'$, if $\varepsilon' \leq \varepsilon$, then

$$\|(p-g)^{1/2}b''(p-g)^{1/2}\| \geq \|(p-g)b''(p-g)\| \geq 1-\varepsilon.$$

Hence by the second inequality again,

$$1-\varepsilon \leq \|(p-g)^{1/2}b''(p-g)^{1/2}\| \leq \|b'''\| + \|(p-g)^{1/2}b''(p-g)^{1/2} - b'''\|$$

$$\leq \|b'''\| + \varepsilon.$$

Therefore,

$$\|(b''' - \varepsilon)_+\| \geq \|b'''\| - \varepsilon \geq 1 - 3\varepsilon.$$

So, if $\varepsilon < 1/3$, then $\|(b''' - \varepsilon)_+\| > 0$.

Since $B \in \Omega$, for $H = \{a'_1,\ a'_2,\ \cdots,\ a'_k,\ p-g,\ (p-g)^{1/2},\ (p-g)a'_i\} \subseteq B$, $\varepsilon > 0$, $(b''' - \varepsilon)_+ > 0$, and n, there is an order zero contraction $\psi_0 : \mathrm{M}_n \to B$ with the following properties:

$(1)''$ $p - \psi_0(1) \precsim (b''' - \varepsilon)_+$, and

$(2)''$ for any element $z \in \mathrm{M}_n$ of norm one, and any $x \in H$,, we have $\|\psi_0(z)x - x\psi_0(z)\| < \varepsilon$.

By Theorem 2.10, applied with both the A and B of 2.10 equal to the present B, there exists a completely positive contractive order zero map $\psi : \mathrm{M}_n \to B$ such that

$(1)'''$ $\|\psi(z) - \psi_0(z)(p-g)\| < \varepsilon$, and

$(2)'''$ $p - \psi(1) \precsim (p - (p-g)) \oplus (p - \psi_0(1))$.

We then have

$$\begin{aligned}
&1 - \psi(1) = 1 - p + p - \psi(1) \precsim (1 + g - p) \oplus (p - \psi(1))\\
&\precsim {b_1}' \oplus (p - (p-g)) \oplus (p - \psi_0(1)) \quad (\text{by} \quad (3)' \text{and} \quad (2)''')\\
&\precsim {b_1}' \oplus (1 - (p-g)) \oplus (b''' - \varepsilon)_+ \quad (\text{by} \quad (1)'')\\
&\precsim {b_1}' \oplus {b_2}' \oplus (p-g)^{1/2} b'' (p-g)^{1/2} \quad (\text{by} \quad (3)' \text{and} \quad (7)')\\
&\precsim b' + b'' \precsim b.
\end{aligned}$$

This is (1) above.

For any element $z \in \mathrm{M}_n$ of norm one, any $a'_i \in F$, we have (by $(1)'''$, $(2)''$, the choice of a'_i, $(2)'$ for small enough ε', the choice of a'_i, and $(1)'''$)

$$\begin{aligned}
&\|\psi(z)a'_i - a'_i\psi(z)\|\\
&\leq \|\psi(z)a'_i - \psi_0(z)(p-g)a'_i\| + \|\psi_0(z)(p-g)a'_i - (p-g)a'_i\psi_0(z)\|\\
&+ \|(p-g)a'_i\psi_0(z) - (p-g)(p-g)^{1/2}a_i(p-g)^{1/2}\psi_0(z)\|\\
&+ \|(p-g)(p-g)^{1/2}a_i(p-g)^{1/2}\psi_0(z) - (p-g)^{1/2}a_i(p-g)^{1/2}(p-g)\psi_0(z)\|\\
&+ \|(p-g)^{1/2}a_i(p-g)^{1/2}(p-g)\psi_0(z) - a'_i(p-g)\psi_0(z)\|\\
&+ \|a'_i(p-g)\psi_0(z) - a'_i\psi(z)\| < \varepsilon + \varepsilon + \varepsilon + \varepsilon + \varepsilon + \varepsilon = 6\varepsilon \qquad (\mathbf{3.4.2}).
\end{aligned}$$

We also have (by $(1)'''$, $(6)'$, $(2)''$, $(2)'$ with $\varepsilon' \leq \varepsilon$, $(5)'$, the choice of a'_i, $(2)''$, the choice of a_i', (5)$'$, (1)$'''$, and (6)$'$)

$$\begin{aligned}
&\|\psi(z)(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}-(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\psi(z)\|\\
&\leq \|\psi(z)(1-(p-g))^{1/2}a_i((1-(p-g))^{1/2}\\
&-\psi_0(z)(p-g)(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\|\\
&+\|\psi_0(z)(p-g)(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}-\psi_0(z)(p-g)(1-(p-g))a_i\|\\
&+\|\psi_0(z)(1-(p-g))(p-g)a_i-(1-(p-g))\psi_0(z)(p-g)a_i\|\\
&+\|(1-(p-g))\psi_0(z)(p-g)a_i-(1-(p-g))\psi_0(z)a_i(p-g)\|\\
&+\|(1-(p-g))\psi_0(z)a_i(p-g)-(1-(p-g))\psi_0(z)(p-g)^{1/2}a_i(p-g)^{1/2}\|\\
&+\|(1-(p-g))\psi_0(z)(p-g)^{1/2}a_i(p-g)^{1/2}-(1-(p-g))\psi_0(z)a_i'\|\\
&+\|(1-(p-g))\psi_0(z)a_i'-(1-(p-g))a_i'\psi_0(z)\|\\
&+\|(1-(p-g))a_i'\psi_0(z)-(1-(p-g))(p-g)^{1/2}a_i(p-g)^{1/2}\psi_0(z)\|\\
&+\|(1-(p-g))(p-g)^{1/2}a_i(p-g)^{1/2}\psi_0(z)-(1-(p-g))a_i(p-g)\psi_0(z)\|\\
&+\|(1-(p-g))a_i(p-g)\psi_0(z)-(1-(p-g))a_i\psi(z)\|\\
&+\|(1-(p-g))a_i\psi(z)-(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\psi(z)\|\\
&<\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon=11\varepsilon \qquad (\mathbf{3.4.3}).
\end{aligned}$$

Therefore, for any $a_i \in F$, we have (by (**3.4.1**), (**3.4.1**), (**3.4.2**), and (**3.4.3**))

$$\begin{aligned}
&\|\psi(z)a_i-a_i\psi(z)\|\\
&\leq \|\psi(z)a_i-\psi(z)(a_i'+(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2})\|\\
&+\|\psi(z)(a_i'+(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2})\\
&-(a_i'+(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2})\psi(z)\|\\
&+\|(a_i'+(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2})\psi(z)-a_i\psi(z)\|\\
&\leq 3\varepsilon+3\varepsilon+\|\psi(z)a_i'-a_i'\psi(z)\|\\
&+\|\psi(z)((1-(p-g))^{1/2}a_i((1-(p-g))^{1/2}\\
&-((1-(p-g))^{1/2}a_i((1-(p-g))^{1/2}\psi(z)\|\\
&\leq 6\varepsilon+6\varepsilon+11\varepsilon=23\varepsilon.
\end{aligned}$$

This is (2) above, with 23ε in place of ε. □

The following corollary was obtained by Elliott, Fan, and Fang in [7].

Corollary 3.5. *([7].) Let Ω be a class of unital* C*-*algebras which are tracially $\mathcal{Z}$-absorbing. Then A is tracially $\mathcal{Z}$-absorbing, or else $A=\mathbb{C}$, for any simple unital* C*-*algebra $A \in \mathrm{TA}\Omega$.*

Proof. As pointed out in Section 1, $\mathrm{TA}\Omega \subseteq \mathrm{WTA}\Omega$. The statement then follows from Theorem 3.4. □

The following corollary was obtained by Archey, Buck, and Phillips in [2].

Corollary 3.6. *([2].) Let A be a simple unital* C*-*algebra, and let B be a centrally large subalgebra of A. If B is tracially $\mathcal{Z}$-absorbing, then A is tracially $\mathcal{Z}$-absorbing.*

Proof. See remark following Definition 2.6. □

THEOREM 3.7. *Let Ω be a class of unital nuclear* C*-*algebras with tracial nuclear dimension at most n (Definition 2.4). Then A has tracial nuclear dimension at most n for any simple unital* C*-*algebra* $A \in \mathrm{WTA}\Omega$.

PROOF. Let A be a simple unital C*-algebra in WTAΩ. We must show that for any $\varepsilon > 0$, any finite subset $F = \{a_1, a_2, \cdots, a_n\}$ of A, and any non-zero positive element b of A, there exist a C*-subalgebra D of A with $\dim_{\mathrm{nuc}}(D) \leq m$, a contractive completely positive linear map $\varphi : A \to A$, and a contractive completely positive linear map $\psi : A \to D$ such that

(1) $\varphi(1) \precsim b$, and

(2) $\|x - \varphi(x) - \psi(x)\| < \varepsilon$, for any $x \in F$.

We shall show this with 8ε in place of ε. By Lemma 2.3 of [35], there exist positive elements $b_1,\ b_2 \in A$ of norm one such that $b_1 b_2 = 0$, $b_1 \sim b_2$, and $b_1 + b_2 \precsim b$.

Given $\varepsilon > 0$, with $f(t) = t^{1/2} \in C([0,1])$, there exists $\varepsilon' > 0$ satisfying Lemma 2.5.11 (1) of [30]. Given such $\varepsilon' > 0$, for $G = F \cup \{b_2\}$, since $A \in \mathrm{WTA}\Omega$ there exist a projection $p \in A$, an element $g \in A$ with $0 \leq g \leq 1$, and a C*-subalgebra B of A with $g \in B$, $1_B = p$, and $\mathrm{Trdim}_{nuc}(B) \leq m$ such that

$(1)'$ $(p-g)x \in_{\varepsilon'} B,\ x(p-g) \in_{\varepsilon'} B$, for $x \in G$,

$(2)'$ $\|(p-g)x - x(p-g)\| < \varepsilon'$, for $x \in G$,

$(3)'$ $1 - (p-g) \precsim b_1 \sim b_2$, and

$(4)'$ $\|(p-g)b_2(p-g)\| > 1 - \varepsilon'$.

By $(2)'$ and Lemma 2.5.11 (1) of [30], if ε' is sufficiently small, we have

$(5)'$ $\|(p-g)^{1/2}x - x(p-g)^{1/2}\| < \varepsilon$, for any $x \in G$,

$(6)'$ $\|(1-(p-g))^{1/2}x - x(1-(p-g))^{1/2}\| < \varepsilon$, for any $x \in G$.

By $(1)'$, with sufficiently small ε', together with $(5)'$, there exist elements $a_1',\ a_2',\ \cdots,\ a_n' \in B$ and a positive element $b_2' \in B$ such that

$$\|(p-g)^{1/2}a_i(p-g)^{1/2} - a_i'\| < \varepsilon, \quad \text{for } 1 \leq \mathrm{i} \leq \mathrm{n}, \quad \text{and}$$

$$\|(p-g)^{1/2}b_2(p-g)^{1/2} - b_2'\| < \varepsilon.$$

From the first inequality, together with $(5)'$ and $(6)'$, for any $1 \leq i \leq n$, one has

$$\begin{aligned}
&\|a_i - a_i' - (1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\| \\
&\leq \|a_i - (p-g)a_i - (1-(p-g))a_i\| + \|(p-g)a_i - (p-g)^{1/2}a_i(p-g)^{1/2}\| \\
&+\|(1-(p-g))a_i - (1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\| \\
&+\|(p-g)^{1/2}a_i(p-g)^{1/2} - a_i'\| \\
&< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \qquad (\mathbf{3.7.1}).
\end{aligned}$$

Since $\|(p-g)^{1/2}b_2(p-g)^{1/2} - b_2'\| < \varepsilon$, by (1) of Theorem 2.1, we have

$(7)'$ $(b_2' - 3\varepsilon)_+ \precsim ((p-g)^{1/2}b_2(p-g)^{1/2} - 2\varepsilon)_+$.

By $(4)'$, with $\varepsilon' \leq \varepsilon < 1/5$,

$$\|(p-g)^{1/2}b_2(p-g)^{1/2}\| \geq \|(p-g)b_2(p-g)\| \geq 1 - \varepsilon.$$

Therefore, by the choice of b_2', one has

$$\|(b_2'-3\varepsilon)_+\| \geq \|(p-g)^{1/2}b_2(p-g)^{1/2}\| - 4\varepsilon \geq 1-5\varepsilon.$$

In particular, $(b_2'-3\varepsilon)_+ \neq 0$.

Define a contractive completely positive linear map $\varphi'' : A \to A$ by $\varphi''(a) = (1-(p-g))^{1/2}a(1-(p-g))^{1/2}$. Since B is a nuclear C*-algebra, by Theorem 2.3.13 of [30] there exists a contractive completely positive linear map $\psi'' : A \to B$ such that $\|\psi''(p-g)-(p-g)\| < \varepsilon$ and $\|\psi''(a_i')-a_i'\| < \varepsilon$ for all $1 \leq i \leq n$.

Since $\mathrm{Trdim}_{\mathrm{nuc}}(B) \leq m$, there exist a contractive completely positive linear map $\varphi' : B \to B$ and a contractive completely positive linear map $\psi' : B \to D$ with $\dim_{\mathrm{nuc}}(D) \leq m$ such that

$(1)''$ $\varphi'(p) \precsim (b_2'-3\varepsilon)_+$, and

$(2)''$ $\|(p-g)-\varphi'(p-g)-\psi'(p-g)\| < \varepsilon$, and $\|a_i'-\varphi'(a_i')-\psi'(a_i')\| < \varepsilon$, for all $1 \leq i \leq n$.

Define $\bar{\varphi} : A \to A$ by $\bar{\varphi}(a) = \varphi''(a)+\varphi'(\psi''((p-g)^{1/2}a(p-g)^{1/2}))$, $\varphi = \frac{1}{1+2\varepsilon}\bar{\varphi}$, and $\psi : A \to D$ by $\psi(a) = \psi'(\psi''((p-g)^{1/2}a(p-g)^{1/2}))$. Then

$$\begin{aligned}
&\|\varphi(1)\| = \tfrac{1}{1+2\varepsilon}\|\bar{\varphi}(1)\| \\
&= \tfrac{1}{1+2\varepsilon}\|\varphi''(1)+\varphi'(\psi''((p-g)^{1/2}1(p-g)^{1/2}))\| \\
&= \tfrac{1}{1+2\varepsilon}\|1-(p-g)-\varphi'(\psi''(p-g))\| \\
&= \tfrac{1}{1+2\varepsilon}\|1-(p-g)+\varphi'(p-g)+\psi'(p-g)+\varphi'(\psi''(p-g))-\varphi'(p-g)-\psi'(p-g)\| \\
&\leq \tfrac{1}{1+2\varepsilon}(\|(p-g)-\varphi'(p-g)-\psi'(p-g)\| \\
&+\|\varphi'(\psi''(p-g))-\varphi'(p-g)\|+\|1-\psi'(p-g)\|) \\
&\leq \tfrac{1}{1+2\varepsilon}(1+2\varepsilon) = 1 \quad (\text{by}(2)'' \text{ and definition of } \psi'').
\end{aligned}$$

Therefore, φ is a contractive completely positive linear map. Also ψ is a contractive completely positive linear map.

We have

$$\begin{aligned}
&\varphi(1) = \tfrac{1}{1+2\varepsilon}(\varphi''(1)+\varphi'(\psi''(p-g))) \\
&\sim \varphi''(1)+\varphi'(\psi''(p-g)) \\
&\precsim 1-(p-g)\oplus\varphi'(p) \precsim b_1 \oplus (b_2'-3\varepsilon)_+ \quad (\text{by } (3)' \text{and } (1)'') \\
&\precsim b_1 \oplus ((p-g)^{1/2}b_2(p-g)^{1/2}-2\varepsilon)_+ \quad (\text{by } (7)') \\
&\precsim b_1 \oplus b_2 \sim b_1+b_2 \precsim b,
\end{aligned}$$

and (by (**3.7.1**), $(2)''$, the choice of ψ'', the definition of a_i', and the last two again)

$$\begin{aligned}
&\|a_i - \varphi(a_i) - \psi(a_i)\| \\
&= \|a_i - \tfrac{1}{1+2\varepsilon}\varphi''(a_i) - \tfrac{1}{1+2\varepsilon}\varphi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2})) \\
&-\psi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2}))\| \\
&\leq \|a_i - \tfrac{1}{1+2\varepsilon}(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2} - a_i'\| \\
&+\|a_i' - \tfrac{1}{1+2\varepsilon}\varphi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2})) - \psi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2}))\| \\
&\leq \|a_i - (1-(p-g))^{1/2}a_i(1-(p-g))^{1/2} - a_i'\| \\
&+\|(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2} - \tfrac{1}{1+2\varepsilon}(1-(p-g))^{1/2}a_i(1-(p-g))^{1/2}\| \\
&+\|a_i' - \varphi'(a_i') - \psi'(a_i')\| \\
&+\|\varphi'(a_i') - \varphi'(\psi''(a_i'))\| \\
&+\|\varphi'(\psi''(a_i')) - \varphi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2}))\| \\
&+\|\varphi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2})) - \tfrac{1}{1+2\varepsilon}\varphi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2}))\| \\
&+\|\psi'(a_i') - \psi'(\psi''(a_i'))\| \\
&+\|\psi'(\psi''(a_i')) - \psi'(\psi''((p-g)^{1/2}a_i(p-g)^{1/2}))\| \\
&\leq 3\varepsilon + \tfrac{2\varepsilon}{1+2\varepsilon} + \varepsilon + \varepsilon + \varepsilon + \tfrac{2\varepsilon}{1+2\varepsilon} + \varepsilon + \varepsilon = 8\varepsilon + \tfrac{4\varepsilon}{1+2\varepsilon}.
\end{aligned}$$

Thus we have (1), and (2) with $8\varepsilon + \frac{4\varepsilon}{1+2\varepsilon}$ in place of ε. □

The following corollary was obtained by Fan and Yang in [13].

COROLLARY 3.8. *Let Ω be a class of unital* C*-*algebras such that* $\mathrm{Trdim_{nuc}}(B) \leq m$ *for any* $B \in \Omega$. *Then* $\mathrm{Trdim_{nuc}}(A) \leq m$ *for any simple unital* C*-*algebra* $A \in \mathrm{TA}\Omega$.

PROOF. As pointed out in Section 1, $\mathrm{TA}\Omega \subseteq \mathrm{WTA}\Omega$. The statement then follows from Theorem 3.7. □

The following corollary was obtained by Zhao, Fang, and Fan in [43].

COROLLARY 3.9. *Let A be a simple unital* C*-*algebra, and let B be a nuclear centrally large subalgebra of A. If* $\mathrm{Trdim_{nuc}}(B) \leq m$, *then* $\mathrm{Trdim_{nuc}}(A) \leq m$.

PROOF. See remark following Definition 2.6. □

THEOREM 3.10. *Let Ω be a class of unital* C*-*algebras which are m-almost divisible (Definition* 2.7*). Let $A \in \mathrm{WTA}\Omega$ be a simple unital stably finite* C*-*algebra such that for any $n \in \mathbb{N}$ the* C*-*algebra* $\mathrm{M}_n(A)$ *belongs to the class* $\mathrm{WTA}\Omega$. *Then A is weakly $(2, m)$-almost divisible (Definition* 2.8*).*

PROOF. We must show that there is $b \in \mathrm{M}_\infty(A)_+$ such that $k\langle b\rangle \leq \langle a\rangle + \langle a\rangle$ and $\langle (a-\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b\rangle$ for any given $a \in A_+$, $\varepsilon > 0$, and $k \in \mathbb{N}$. We may assume that $\|a\| = 1$. (We have replaced $\mathrm{M}_n(A)$ containing a given initially by A.)

For any $\delta_1 > 0$, since $A \in \mathrm{WTA}\Omega$, there exist a projection $p \in A$, an element $g \in A$ with $0 \leq g \leq 1$, and a C^*-subalgebra B of A with $g \in B$, $1_B = p$, and $B \in \Omega$ such that

(1) $(p-g)a \in_{\delta_1} B$, and

(2) $\|(p-g)a - a(p-g)\| < \delta_1$.

By (2), with sufficiently small δ_1, by Lemma 2.5.11 (1) of [30], we have

(3) $\|(p-g)^{1/2}a - a(p-g)^{1/2}\| < \varepsilon/3$, and

(4) $\|(1-(p-g))^{1/2}a - a(1-(p-g))^{1/2}\| < \varepsilon/3$.

By (1) and (2), with sufficiently small δ_1, there exists a positive element $a' \in B$ such that

(5) $\|(p-g)^{1/2}a(p-g)^{1/2} - a'\| < \varepsilon/3$.

By (3), (4), and (5),

$$\begin{aligned}
&\|a - a' - (1-(p-g))^{1/2}a(1-(p-g))^{1/2}\| \\
&\leq \|a-(p-g)a-(1-(p-g))a\| + \|(p-g)a-(p-g)^{1/2}a(p-g)^{1/2}\| \\
&+\|(1-(p-g))a-(1-(p-g))^{1/2}a(1-(p-g))^{1/2}\| \\
&+\|(p-g)^{1/2}a(p-g)^{1/2}-a'\| \\
&< \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon. \qquad (\mathbf{3.10.1})
\end{aligned}$$

Since B is m-almost divisible, and $(a'-3\varepsilon)_+ \in B$, there exists $b_1 \in B$ such that $k\langle b_1\rangle \leq \langle (a'-3\varepsilon)_+\rangle$ and $\langle (a'-4\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b_1\rangle$.

Since B is m-almost divisible, and $(a'-2\varepsilon)_+ \in B$, there exists $b' \in B$ such that $k\langle b'\rangle \leq \langle (a'-2\varepsilon)_+\rangle$ and $\langle (a'-3\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b'\rangle$.

Write $a'' = (1-(p-g))^{1/2}a(1-(p-g))^{1/2}$.

We divide the proof into two cases.

Case (1) We assume that $(a'-3\varepsilon)_+$ is Cuntz equivalent to a projection.

(1.1) We assume that $(a'-4\varepsilon)_+$ is Cuntz equivalent to a projection.

(1.1.1) If $\langle (a'-4\varepsilon)_+\rangle$, the class of a projection, is not equal to $(k+1)(m+1)\langle b_1\rangle$, by Theorem 2.1 (2), there exists non-zero $c \in A_+$ such that $\langle (a'-4\varepsilon)_+\rangle + \langle c\rangle \leq (k+1)(m+1)\langle b_1\rangle$.

For any $\delta_2 > 0$, since $A \in \mathrm{WTA}\Omega$, there exist a projection $p' \in A$, an element $g_1 \in A$ with $0 \leq g_1 \leq 1$, and a C^*-subalgebra D of A with $g_1 \in D$, $1_D = p'$, and $D \in \Omega$ such that

$(1)'$ $(p'-g_1)a'' \in_{\delta_2} D$,

$(2)'$ $\|(p'-g_1)a'' - a''(p'-g_1)\| < \delta_2$, and

$(3)'$ $1-(p'-g_1) \precsim c$.

By $(1)'$ and $(2)'$, with sufficiently small δ_2, as above, via the analogues of (4), (5), and (6) for a'', p', and g_1, there exists a positive element $a''' \in D$ such that

$$\|(p'-g_1)^{1/2}a''(p'-g_1)^{1/2} - a'''\| < \varepsilon/3, \quad \text{and}$$

$$\|a'' - a''' - (1-(p'-g_1))^{1/2}a''(1-(p'-g_1))^{1/2}\| < \varepsilon. \qquad (\mathbf{3.10.2})$$

Since D is m-almost divisible, and $(a'''-3\varepsilon)_+ \in D$, there exists $b_2 \in D_+$ such that $k\langle b_2\rangle \leq \langle (a'''-3\varepsilon)_+\rangle$ and $\langle (a'''-4\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b_2\rangle$.

Since $a' \leq a'+a''$, one has $\langle (a'-\varepsilon)_+\rangle \leq \langle (a'+a''-\varepsilon)_+\rangle$ (by Theorem 2.1 (5)). By (**3.10.1**), $\|a-a'-a''\| < \varepsilon$ and hence by Theorem 2.1 (1), one also has $\langle (a'+a''-\varepsilon)_+\rangle \leq \langle a\rangle$. Therefore, $\langle (a'-2\varepsilon)_+\rangle \leq \langle (a'-\varepsilon)_+\rangle \leq \langle a\rangle$.

Similarly, with $x = (1-(p'-g_1))^{1/2}a''(1-(p'-g_1))^{1/2}$, $a''' \le a''' + x$ implies $\langle(a'''-3\varepsilon)_+\rangle \le \langle(a'''+x-3\varepsilon)_+\rangle$ and (**3.10.2**), i.e., $\|a''-a'''-x\| < \varepsilon$, implies $\langle(a'''+x-3\varepsilon)_+\rangle \le \langle a''\rangle \le \langle a\rangle$.

Therefore, we have

$$\begin{aligned} &k\langle b_1 \oplus b_2\rangle = k\langle b_1\rangle + k\langle b_2\rangle \\ &\le \langle(a'-3\varepsilon)_+\rangle + \langle(a'''-3\varepsilon)_+\rangle \\ &\le \langle(a'-2\varepsilon)_+\rangle + \langle(a'''-3\varepsilon)_+\rangle \\ &\le \langle a\rangle + \langle a\rangle. \end{aligned}$$

By (**3.10.1**) and (**3.10.2**),

$$\|a - a' - a''' - (1-(p'-g_1))^{1/2}a''(1-(p'-g_1))^{1/2}\| < 2\varepsilon,$$

and therefore, by Theorem 2.1 (1) and Theorem 2.1 (4) (twice),

$$\begin{aligned} &\langle(a-10\varepsilon)_+\rangle \\ &\le \langle(a'-4\varepsilon)_+\rangle + \langle(a'''-4\varepsilon)_+\rangle + \langle(1-(p'-g_1))^{1/2}a''(1-(p'-g_1))^{1/2}\rangle \\ &\le \langle(a'-4\varepsilon)_+\rangle + \langle(a'''-4\varepsilon)_+\rangle + \langle 1-(p'-g_1)\rangle \\ &\le \langle(a'-4\varepsilon)_+\rangle + \langle(a'''-4\varepsilon)_+\rangle + \langle c\rangle \text{ (by (3)}'\text{)} \\ &\le (k+1)(m+1)\langle b_1\rangle + (k+1)(m+1)\langle b_2\rangle = (k+1)(m+1)\langle b_1 \oplus b_2\rangle. \end{aligned}$$

These are the desired inequalities, with $b_1 \oplus b_2$ in place of b and 10ε in place of ε.

(1.1.2) If $\langle(a'-4\varepsilon)_+\rangle$ is equal to $(k+1)(m+1)\langle b_1\rangle$, so that $(k+1)(m+1)\langle b_1\rangle \le \langle(a'-3\varepsilon)_+\rangle$, then, as $m+1 \ge 2$, we have $k\langle b_1 \oplus b_1\rangle \le \langle(a'-3\varepsilon)_+\rangle$. Also, $\langle(a'-4\varepsilon)_+\rangle + \langle b_1\rangle \le (k+1)(m+1)\langle b_1 \oplus b_1\rangle$.

As in the part **(1.1.1)**, as $A \in \mathrm{WTA}\Omega$ (with b_1 in place of c in the part **(1.1.1)**), there exist a projection $p'' \in A$, an element $g_2 \in A$ with $0 \le g_2 \le 1$, and a C^*-subalgebra D_1 of A with $g_3 \in D_1$, $1_{D_1} = p''$, and $D_1 \in \Omega$, and there exists a positive element $a^{(4)} \in D_1$ such that

$$\|a'' - a^{(4)} - (1-(p''-g_2))^{1/2}a''(1-(p''-g_2))^{1/2}\| < \varepsilon, \qquad \textbf{(3.10.3)}$$

$$\langle(a^{(4)}-3\varepsilon)_+\rangle \le \langle a\rangle, \quad \text{and} \quad \langle 1-(p''-g_2)\rangle \le \langle b_1\rangle.$$

Since D_1 is m-almost divisible, and $(a^{(4)}-3\varepsilon)_+ \in D_1$, there exists $b_3 \in (D_1)_+$ such that $k\langle b_3\rangle \le \langle(a^{(4)}-3\varepsilon)_+\rangle$ and $\langle(a^{(4)}-4\varepsilon)_+\rangle \le (k+1)(m+1)\langle b_3\rangle$.

Then, as in the part **(1.1.1)**,

$$\begin{aligned} &k\langle b_1 \oplus b_1 \oplus b_3\rangle = k\langle b_1 \oplus b_1\rangle + k\langle b_3\rangle \\ &\le \langle(a'-3\varepsilon)_+\rangle + \langle(a^{(4)}-3\varepsilon)_+\rangle \\ &\le \langle(a'-2\varepsilon)_+\rangle + \langle(a^{(4)}-3\varepsilon)_+\rangle \\ &\le \langle a\rangle + \langle a\rangle. \end{aligned}$$

By (**3.10.1**) and (**3.10.3**),

$$\|a - a' - a^{(4)} - (1 - (p'' - g_2))^{1/2} a'' (1 - (p'' - g_2))^{1/2}\| < 2\varepsilon,$$

and therefore, as in the part **(1.1.1)**

$$\begin{aligned}
&\langle (a - 10\varepsilon)_+ \rangle \\
&\le \langle (a' - 4\varepsilon)_+ \rangle + \langle (a^{(4)} - 4\varepsilon)_+ \rangle + \langle (1 - (p'' - g_2))^{1/2} a'' (1 - (p'' - g_2))^{1/2} \rangle \\
&\le \langle (a' - 4\varepsilon)_+ \rangle + \langle (a^{(4)} - 4\varepsilon)_+ \rangle + \langle 1 - (p'' - g_2) \rangle \\
&\le \langle (a' - 4\varepsilon)_+ \rangle + \langle b_1 \rangle + \langle (a^{(4)} - 4\varepsilon)_+ \rangle \\
&\le (k+1)(m+1)\langle b_1 \oplus b_1 \rangle + (k+1)(m+1)\langle b_3 \rangle \\
&= (k+1)(m+1)\langle b_1 \oplus b_1 \oplus b_3 \rangle.
\end{aligned}$$

These are the desired inequalities, with $b_1 \oplus b_1 \oplus b_3$ in place of b and 10ε in place of ε.

(1.2) We assume that $(a' - 4\varepsilon)_+$ is not Cuntz equivalent to a projection. By Theorem 2.1 (3), there is a non-zero positive element d such that $\langle (a' - 5\varepsilon)_+ \rangle + \langle d \rangle \le \langle (a' - 4\varepsilon)_+ \rangle$.

As in the part **(1.1.1)**, as $A \in \mathrm{WTA}\Omega$ (with d in place of c in the part **(1.1.1)**), there exist a projection $p''' \in A$, an element $g_3 \in A$ with $0 \le g_3 \le 1$, and a C^*-subalgebra D_2 of A with $g_2 \in D_2$, $1_{D_2} = p'''$, and $D_2 \in \Omega$, and there exists a positive element $a^{(5)} \in D_2$ such that

$$\|a'' - a^{(5)} - (1 - (p''' - g_3))^{1/2} a'' (1 - (p''' - g_3))^{1/2}\| < \varepsilon, \qquad (\mathbf{3.10.4})$$

$$\langle (a^{(5)} - 3\varepsilon)_+ \rangle \le \langle a \rangle, \quad \text{and} \quad \langle 1 - (p''' - g_3) \rangle \le \langle d \rangle.$$

Since D_2 is m-almost divisible, and $(a^{(5)} - 3\varepsilon)_+ \in D_2$, there exists $b_4 \in (D_2)_+$ such that $k\langle b_4 \rangle \le \langle (a^{(5)} - 3\varepsilon)_+ \rangle$, and $\langle (a^{(5)} - 4\varepsilon)_+ \rangle \le (k+1)(m+1)\langle b_4 \rangle$.

Then, as in the part **(1.1.1)**,

$$\begin{aligned}
&k\langle (b_1 \oplus b_4) \rangle = k\langle b_1 \rangle + k\langle b_4 \rangle \\
&\le \langle (a' - 3\varepsilon)_+ \rangle + \langle (a^{(5)} - 3\varepsilon)_+ \rangle \\
&\le \langle (a' - 2\varepsilon)_+ \rangle + \langle (a^{(5)} - 3\varepsilon)_+ \rangle \\
&\le \langle a \rangle + \langle a \rangle,
\end{aligned}$$

By (**3.10.1**) and (**3.10.4**),

$$\|a - a' - a^{(5)} - (1 - (p''' - g_3))^{1/2} a'' (1 - (p''' - g_3))^{1/2}\| < 2\varepsilon,$$

and therefore, as in the part **(1.1.1)**,

$$\begin{aligned}
&\langle (a-11\varepsilon)_+\rangle \\
&\leq \langle (a'-5\varepsilon)_+\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle + \langle (1-(p'''-g_3))^{1/2}a''(1-(p'''-g_3))^{1/2}\rangle \\
&\leq \langle (a'-5\varepsilon)_+\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle + \langle 1-(p'''-g_3)\rangle \\
&\leq \langle (a'-5\varepsilon)_+\rangle + \langle d\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle \\
&\leq \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle \\
&\leq (k+1)(m+1)\langle b_1\oplus b_4\rangle.
\end{aligned}$$

These are the desired inequalities, with $b_1\oplus b_4$ in place of b and 11ε in place of ε.

Case (2) We assume that $(a'-3\varepsilon)_+$ is not Cuntz equivalent to a projection.

By (3) of Theorem 2.1, there is a non-zero positive element e such that $\langle (a'-4\varepsilon)_+\rangle + \langle e\rangle \leq \langle (a'-3\varepsilon)_+\rangle$.

As in the part **(1.1.1)**, as $A\in \mathrm{WTA}\Omega$ (with e in place of c in the part **(1.1.1)**), there exist a projection $p''''\in A$, an element $g_4\in A$ with $0\leq g_4\leq 1$, and a C^*-subalgebra D_3 of A with $g\in D_3$, $1_{D_3}=p''''$, and $D_3\in\Omega$, and there exists a positive element $a^{(6)}\in D_3$ such that

$$\|a''-a^{(6)}-(1-(p''''-g_4))^{1/2}a''(1-(p''''-g_4))^{1/2}\|<\varepsilon, \qquad \mathbf{(3.10.5)}$$

$$\langle (a^{(6)}-3\varepsilon)_+\rangle \leq \langle a\rangle, \quad \text{and} \quad \langle 1-(p''''-g_4)\rangle \leq \langle e\rangle.$$

Since D_3 is m-almost divisible, and $(a^{(6)}-3\varepsilon)_+\in D_3$, there exists $b_5\in (D_3)_+$ such that $k\langle b_5\rangle \leq \langle (a^{(6)}-3\varepsilon)_+\rangle$, and $\langle (a^{(6)}-4\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b_5\rangle$.

Then, as in the part **(1.1.1)**,

$$\begin{aligned}
&k\langle (b'\oplus b_5)\rangle = k\langle b'\rangle + k\langle b_5\rangle \\
&\leq \langle (a'-2\varepsilon)_+\rangle + \langle (a^{(6)}-3\varepsilon)_+\rangle \\
&\leq \langle a\rangle + \langle a\rangle,
\end{aligned}$$

By (**3.10.1**) and (**3.10.5**),

$$\|a-a'-a^{(6)}-(1-(p''''-g_4))^{1/2}a''(1-(p''''-g_4))^{1/2}\|<2\varepsilon,$$

and therefore, as before,

$$\begin{aligned}
&\langle (a-10\varepsilon)_+\rangle \\
&\leq \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle + \langle (1-(p''''-g_4))^{1/2}a''(1-(p''''-g_4))^{1/2}\rangle \\
&\leq \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle + \langle 1-(p''''-g_4)\rangle \\
&\leq \langle (a'-4\varepsilon)_+\rangle + \langle e\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle \\
&\leq \langle (a'-3\varepsilon)_+\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle \\
&\leq (k+1)(m+1)\langle b'\oplus b_5\rangle.
\end{aligned}$$

These are the desired inequalities, with $b' \oplus b_5$ in place of b and 10ε in place of ε. □

The proof of Theorem 3.11 which follows is similar to that of Theorem 3.10.

Theorem 3.11. *Let Ω be a class of unital* C*-*algebras which are m-almost divisible (Definition* 2.7*). Let $A \in \mathrm{TAA}\Omega$ be a simple unital stably finite* C*-*algebra such that for any $n \in \mathbb{N}$ and any unital hereditary* C*-*subalgebra D of $\mathrm{M}_n(A)$, D belongs to the class* TAAΩ. *Then A is weakly $(2, m)$-almost divisible (Definition* 2.8*).*

Proof. We must show that there is $b \in \mathrm{M}_\infty(A)_+$ such that $k\langle b\rangle \le \langle a\rangle + \langle a\rangle$ and $\langle (a-\varepsilon)_+\rangle \le (k+1)(m+1)\langle b\rangle$, for any given $a \in A_+$, $\varepsilon > 0$, and $k \in \mathbb{N}$. We may assume that $\|a\| = 1$. (We have replaced $\mathrm{M}_n(A)$ containing a given initially by A.)

For any $\delta_2 > 0$, since $A \in \mathrm{TAA}\Omega$, there exist a projection $p \in A$, a C*-subalgebra B of A with $1_B = p$ and $B \in \Omega$, and a projection $q \in B$ such that

(1) $qa \in_{\delta_2/3} B,\ aq \in_{\delta_2/3} B$.

By (1), $a-(1-q)a(1-q) = (qa+aq-qaq) \in_{\delta_2} B$, i.e., there exists $\bar{a} \in B$ such that $\|a-(1-q)a(1-q)-\bar{a}\| < \delta_2$. With sufficiently small δ_2, by the functional calculus, we may assume that there exists a positive element $a' \in B$ such that

$$\|a-(1-q)a(1-q)-a'\| < \varepsilon. \qquad (\mathbf{3.11.1})$$

Since B is m-almost divisible, and $(a'-3\varepsilon)_+ \in B$, there exists $b_1 \in B_+$ such that $k\langle b_1\rangle \le \langle (a'-3\varepsilon)_+\rangle$ and $\langle (a'-4\varepsilon)_+\rangle \le (k+1)(m+1)\langle b_1\rangle$.

Since B is m-almost divisible, and $(a'-2\varepsilon)_+ \in B$, there exists $b' \in B_+$ such that $k\langle b'\rangle \le \langle (a'-2\varepsilon)_+\rangle$ and $\langle (a'-3\varepsilon)_+\rangle \le (k+1)(m+1)\langle b'\rangle$.

We divide the proof into two cases.

Case (1) We assume that $(a'-3\varepsilon)_+$ is Cuntz equivalent to a projection.

(1.1) We assume that $(a'-4\varepsilon)_+$ is Cuntz equivalent to a projection.

(1.1.1) If $\langle (a'-4\varepsilon)_+\rangle$, the class of a projection, is not equal to $(k+1)(m+1)\langle b_1\rangle$, by Theorem 2.1 (2), there exists a non-zero $c \in A_+$ such that $\langle (a'-4\varepsilon)_+\rangle + \langle c\rangle \le (k+1)(m+1)\langle b_1\rangle$.

For any $\delta_2 > 0$, since $(1-q)A(1-q) \in \mathrm{TAA}\Omega$ (recall that the present A was initially $\mathrm{M}_n(A)$), there exist a projection $p' \in (1-q)A(1-q)$, a C*-subalgebra D of $(1-q)A(1-q)$ with $1_D = p'$ and $D \in \Omega$, and a projection $q' \in D$ such that

$(1)'$ $q'(1-q)a(1-q) \in_{\delta_2/3} D,\ (1-q)a(1-q)q' \in_{\delta_2/3} D$, and

$(2)'$ $1-q-q' \precsim c$.

By $(1)'$, as above, there exists a positive element $a''' \in B$ such that

$$\|(1-q)a(1-q)-a'''-(1-q-q')(1-q)a(1-q)(1-q-q')\| < \varepsilon. \quad (\mathbf{3.11.2})$$

By (**3.11.1**) and (**3.11.2**), one has

$$\|a-a'-a'''-(1-q-q')(1-q)a(1-q)(1-q-q')\| < 2\varepsilon. \quad (\mathbf{3.11.3})$$

Since D is m-almost divisible, and $(a''' - 3\varepsilon)_+ \in D$, there exists $b_2 \in D_+$ such that $k\langle b_2\rangle \le \langle (a''' - 3\varepsilon)_+\rangle$ and $\langle (a''' - 4\varepsilon)_+\rangle \le (k+1)(m+1)\langle b_2\rangle$.

Since $a' \le a' + a''' + (1-q-q')(1-q)a(1-q)(1-q-q')$, by Theorem 2.1 (5), $\langle (a'-2\varepsilon)_+\rangle \le \langle (a'+a'''+(1-q-q')(1-q)a(1-q)(1-q-q')-2\varepsilon)_+\rangle$, and by (**3.11.3**), and Theorem 2.1 (1), $\langle (a'-2\varepsilon)_+\rangle \le \langle (a'+a'''+(1-q-q')(1-q)a(1-q)(1-q-q')-2\varepsilon)_+\rangle \le \langle a\rangle$. By a similar argument (cf. proof of case (1.1.1) of Theorem 3.10), $\langle (a'''-3\varepsilon)_+\rangle \le \langle a\rangle$.

Therefore, we have

$$\begin{aligned} k\langle b_1 \oplus b_2\rangle &= k\langle b_1\rangle + k\langle b_2\rangle \\ &\le \langle (a'-3\varepsilon)_+\rangle + \langle (a'''-3\varepsilon)_+\rangle \\ &\le \langle (a'-2\varepsilon)_+\rangle + \langle (a'''-3\varepsilon)_+\rangle \\ &\le \langle a\rangle + \langle a\rangle. \end{aligned}$$

Since $\|a - a' - a''' - (1-q-q')(1-q)a(1-q)(1-q-q')\| < 2\varepsilon$, therefore, by Theorem 2.1 (1) and Theorem 2.5 (4) (twice),

$$\begin{aligned} &\langle (a-10\varepsilon)_+\rangle \\ &\le \langle (a'-4\varepsilon)_+\rangle + \langle (a'''-4\varepsilon)_+\rangle + \langle (1-q-q')(1-q)a(1-q)(1-q-q')\rangle \\ &\le \langle (a'-4\varepsilon)_+\rangle + \langle (a'''-4\varepsilon)_+\rangle + \langle 1-q-q'\rangle \\ &\le \langle (a'-4\varepsilon)_+\rangle + \langle (a'''-4\varepsilon)_+\rangle + \langle c\rangle \quad \text{(by (2)}'\text{)} \\ &\le (k+1)(m+1)\langle b_1\rangle + (k+1)(m+1)\langle b_2\rangle = (k+1)(m+1)\langle b_1 \oplus b_2\rangle. \end{aligned}$$

These are the desired inequalities, with $b_1 \oplus b_2$ in place of b and 10ε in place of ε.

(1.1.2) If $\langle (a'-4\varepsilon)_+\rangle$ is equal to $(k+1)(m+1)\langle b_1\rangle$, so that $(k+1)(m+1)\langle b_1\rangle \le \langle (a'-3\varepsilon)_+\rangle$, then, as $m+1 \ge 2$, we have $k\langle b_1 \oplus b_1\rangle \le \langle (a'-3\varepsilon)_+\rangle$. Also, $\langle (a'-4\varepsilon)_+\rangle + \langle b_1\rangle \le (k+1)(m+1)\langle b_1 \oplus b_1\rangle$.

As in the part **(1.1.1)**, since $(1-q)A(1-q) \in \mathrm{TAA}\Omega$ (with b_1 in place of c as in the part **(1.1.1)**), there exist a projection $p'' \in (1-q)A(1-q)$, a C*-subalgebra D_1 of $(1-q)A(1-q)$ with $1_{D_1} = p''$ and $D_1 \in \Omega$, and a projection $q'' \in D_1$, and there exists a positive element $a^{(4)} \in D_1$ such that

$$\|(1-q)a(1-q) - a^{(4)} - (1-q-q'')(1-q)a(1-q)(1-q-q'')\| < \varepsilon, \qquad (\mathbf{3.11.4})$$

$$\langle (a^{(4)} - 3\varepsilon)_+\rangle \le \langle a\rangle, \quad \text{and} \quad \langle 1-q-q''\rangle \le \langle b_1\rangle.$$

Since D_1 is m-almost divisible, and $(a^{(4)} - 3\varepsilon)_+ \in D_1$, there exists $b_3 \in (D_1)_+$ such that $k\langle b_2\rangle \le \langle (a^{(4)} - 3\varepsilon)_+\rangle$, and $\langle (a^{(4)} - 4\varepsilon)_+\rangle \le (k+1)(m+1)\langle b_3\rangle$.

Then, as in the part **(1.1.1)**,

$$\begin{aligned} & k\langle b_1 \oplus b_1 \oplus b_3\rangle = k\langle b_1 \oplus b_1\rangle + k\langle b_3\rangle \\ & \le \langle (a'-3\varepsilon)_+\rangle + \langle (a^{(4)}-3\varepsilon)_+\rangle \\ & \le \langle (a'-2\varepsilon)_+\rangle + \langle (a^{(4)}-3\varepsilon)_+\rangle \\ & \le \langle a\rangle + \langle a\rangle. \end{aligned}$$

By (**3.11.1**) and (**3.11.4**), one has

$$\|a - a' - a^{(4)} - (1-q-q'')(1-q)a(1-q)(1-q-q'')\| < 2\varepsilon,$$

and therefore, as in the part **(1.1.1)**,

$$\begin{aligned} & \langle (a-10\varepsilon)_+\rangle \\ & \le \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(4)}-4\varepsilon)_+\rangle + \langle (1-q-q'')(1-q)a(1-q)(1-q-q'')\rangle \\ & \le \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(4)}-4\varepsilon)_+\rangle + \langle 1-q-q''\rangle \\ & \le \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(4)}-4\varepsilon)_+\rangle + \langle b_1\rangle \\ & \le (k+1)(m+1)\langle b_1 \oplus b_1\rangle + (k+1)(m+1)\langle b_3\rangle \\ & = (k+1)(m+1)\langle b_1 \oplus b_1 \oplus b_3\rangle. \end{aligned}$$

These are the desired inequalities, with $b_1 \oplus b_1 \oplus b_3$ in place of b and 10ε in place of ε.

(1.2) We assume that $(a'-4\varepsilon)_+$ is not Cuntz equivalent to a projection. By Theorem 2.1 (3), there is a non-zero positive element d such that $\langle (a'-5\varepsilon)_+\rangle + \langle d\rangle \le \langle (a'-4\varepsilon)_+\rangle$.

As in the part **(1.1.1)**, since $(1-q)A(1-q) \in \mathrm{TAA}\Omega$ (with d in place of c as in the part **(1.1.1)**), there exist a projection $p''' \in (1-q)A(1-q)$, a C*-subalgebra D_2 of $(1-q)A(1-q)$ with $1_{D_2} = p'''$ and $D_2 \in \Omega$, and a projection $q''' \in D_2$, and there exists a positive element $a^{(5)} \in D_2$, such that

$$\|(1-q)a(1-q) - a^{(5)} - (1-q-q''')(1-q)a(1-q)(1-q-q''')\| < \varepsilon, \quad \textbf{(3.11.5)}$$

$$\langle (a^{(5)}-3\varepsilon)_+\rangle \le \langle a\rangle, \quad \text{and} \quad \langle 1-q-q'''\rangle \le \langle d\rangle.$$

Since D_2 is m-almost divisible, and $(a^{(5)}-3\varepsilon)_+ \in D_2$, there exists $b_4 \in (D_2)_+$ such that $k\langle b_4\rangle \le \langle (a^{(5)}-3\varepsilon)_+\rangle$ and $\langle (a^{(5)}-4\varepsilon)_+\rangle \le (k+1)(m+1)\langle b_4\rangle$.

As in the part **(1.1.1)**, one has

$$\begin{aligned} & k\langle b_1 \oplus b_4\rangle = k\langle b_1\rangle + k\langle b_4\rangle \\ & \le \langle (a'-3\varepsilon)_+\rangle + \langle (a^{(5)}-3\varepsilon)_+\rangle \\ & \le \langle (a'-2\varepsilon)_+\rangle + \langle (a^{(5)}-3\varepsilon)_+\rangle \\ & \le \langle a\rangle + \langle a\rangle. \end{aligned}$$

By (**3.11.1**) and (**3.11.5**), one has

$$\|a - a' - a''' - (1-q-q''')(1-q)a(1-q)(1-q-q''')\| < 2\varepsilon,$$

and therefore, as in the part **(1.1.1)**,

$$\begin{aligned}
&\langle (a-11\varepsilon)_+\rangle \\
&\le \langle (a'-5\varepsilon)_+\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle + \langle (1-q-q')(1-q)a(1-q)(1-q-q')\rangle \\
&\le \langle (a'-5\varepsilon)_+\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle + \langle 1-q-q'\rangle \\
&\le \langle (a'-5\varepsilon)_+\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle + \langle d\rangle \\
&\le \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(5)}-4\varepsilon)_+\rangle \\
&\le (k+1)(m+1)\langle b_1\rangle + (k+1)(m+1)\langle b_4\rangle = (k+1)(m+1)\langle b_1\oplus b_4\rangle.
\end{aligned}$$

These are the desired inequalities, with $b_1 \oplus b_4$ in place of b and 11ε in place of ε.

Case (2) We assume that $(a'-3\varepsilon)_+$ is not Cuntz equivalent to a projection.

By Theorem 2.1 (3), there is a non-zero positive element e such that $\langle (a'-4\varepsilon)_+\rangle + \langle e\rangle \le \langle (a'-3\varepsilon)_+\rangle$.

As in the part **(1.1.1)**, since $(1-q)A(1-q) \in \mathrm{TAA}\Omega$ (with e in place of c as in the part **(1.1.1)**), there exist a projection $p'''' \in (1-q)A(1-q)$, a C*-subalgebra D_3 of $(1-q)A(1-q)$ with $1_{D_3} = p''''$ and $D_3 \in \Omega$, and a projection $q'''' \in D_3$, and there exists a positive element $a^{(6)} \in D_3$, such that

$$\|(1-q)a(1-q) - a^{(6)} - (1-q-q')(1-q)a(1-q)(1-q-q')\| < \varepsilon, \quad (\mathbf{3.11.6})$$

$$\langle (a^{(6)}-3\varepsilon)_+\rangle \le \langle a\rangle, \quad \text{and} \quad \langle 1-q-q''''\rangle \le \langle e\rangle.$$

Since D_3 is m-almost divisible, and $(a^{(6)}-3\varepsilon)_+ \in D_3$, there exists $b_5 \in (D_3)_+$ such that $k\langle b_5\rangle \le \langle (a^{(6)}-3\varepsilon)_+\rangle$ and $\langle (a^{(6)}-4\varepsilon)_+\rangle \le (k+1)(m+1)\langle b_2\rangle$.

As in the part **(1.1.1)**,

$$\begin{aligned}
&k\langle b_1'\oplus b_5\rangle = k\langle b_1'\rangle + k\langle b_5\rangle \\
&\le \langle (a'-2\varepsilon)_+\rangle + \langle (a^{(6)}-3\varepsilon)_+\rangle \\
&\le \langle a\rangle + \langle a\rangle.
\end{aligned}$$

By (**3.11.1**) and (**3.11.6**), one has

$$\|a - a' - a^{(6)} - (1-q-q'''')(1-q)a(1-q)(1-q-q'''')\| < 2\varepsilon,$$

and therefore, as before,

$$\begin{aligned}
&\langle (a-10\varepsilon)_+\rangle \\
&\leq \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle + \langle (1-q-q'''')(1-q)a(1-q)(1-q-q'''')\rangle \\
&\leq \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle + \langle 1-q-q''''\rangle \\
&\leq \langle (a'-4\varepsilon)_+\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle + \langle e\rangle \\
&\leq \langle (a'-3\varepsilon)_+\rangle + \langle (a^{(6)}-4\varepsilon)_+\rangle \\
&\leq (k+1)(m+1)\langle b_1'\rangle + (k+1)(m+1)\langle b_5\rangle = (k+1)(m+1)\langle b_1'\oplus b_5\rangle.
\end{aligned}$$

These are the desired inequalities, with $b_1'\oplus b_5$ in place of b and 10ε in place of ε. □

References

1. P. Ara, F. Perera, and A. S. Toms, *K-theory for operator algebras. Classification of C^*-algebras*, Aspects of Operator Algebras and Applications. Contemp. Math. **534**, American Mathematival Society, Providence, RI, 2011, pp. 1–71.

2. D. Archey, J. Buck, and N. C. Phillips, *Centrally large subalgebra and tracial $\mathcal{Z}$-absorbing*, Int. Math. Res. Not. IMRN, **6** (2018), 1857–1877.

3. D. Archey and N. C. Phillips, *Permanence of stable rank one for centrally large subalgebras and crossed products by minimal homeomorphisms*, J. Operator Theory, **83** (2020), 353–389.

4. A. Connes, *Outer conjugacy class of automorphisms of factors*, Ann. Sci. École Norm. Sup. (4), **8** (1975), 383–420.

5. K. T. Coward, G. A. Elliott, and C. Ivanescu, *The Cuntz semigroup as an invariant for C^*-algebras*, J. Reine Angew. Math., **623** (2008), 161–193.

6. G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite dimensional algebras*, J. Algebra, **38** (1976), 29–44.

7. G. A. Elliott, Q. Fan, and X. Fang, *Certain properties of tracial approximation C^*-algebras*, C. R. Math. Acad. Sci. Soc. R. Can., **40** (2018), 104–133.

8. G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero, II*, Ann. of Math., **144** (1996), 497–610.

9. G. A. Elliott, G. Gong, and L. Li, *On the classification of simple inductive limit C^*-algebras II: The isomorphism theorem*, Invent. Math., **168** (2007), 249–320.

10. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, *The classification of simple separable KK-contractible C^*-algebras with finite nuclear dimension*, J. Geom. Phys., **158** (2020), Article 103861, 51 pages.

11. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, *On the classification of simple amenable C^*-algebras with finite decomposition rank, II*, arXiv:1507.03437.

12. G. A. Elliott and Z. Niu, *On tracial approximation*, J. Funct. Anal., **254** (2008), 396–440.

13. Q. Fan and J. Yang, *Regularity properties of certain crossed product C^*-algebras*, Indian J. Pure Appl. Math., **51** (2020), 1515–1531.

14. X. Fu, *Tracial nuclear dimension of C^*-algebras*, Ph.D. thesis, East China Normal University, 2018.

15. X. Fu and H. Lin, *Tracial approximation in simple C^*-algebras*, Canad. J. Math., First View, 26 February 2021, pp. 1–63.

16. G. Gong, *On the classification of simple inductive limit C^*-algebras, I. The reduction theorem*, Doc. Math., **7** (2002), 255–461.

17. G. Gong and H. Lin, *On classification of non-unital simple amenable C^*-algebras, II*, J. Geom. Phys., **158** (2020), Article 103865, 102 pages.

18. G. Gong and H. Lin, *On classification of non-unital simple amenable C^*-algebras III: the range and the reduction*, arXiv:2010.01788v4.

19. G. Gong, H. Lin, and Z. Niu, *Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, I: C^*-algebra with generalized tracial rank one*, C. R. Math. Acad. Sci. Soc. R. Can., **42** (2020), 63–450.

20. G. Gong, H. Lin, and Z. Niu, *Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, II: C^*-algebras with rational generalized tracial rank one*, C. R. Math. Acad. Sci. Soc. R. Can., **42** (2020), 451–539.

21. R. H. Herman and A. Ocneanu, *Stability for integer actions on UHF C^*-algebras*, J. Funct. Anal., **59** (1984), 132–144.

22. I. Hirshberg and J. Orovitz, *Tracially $\mathcal{Z}$-absorbing C^*-algebras*, J. Funct. Anal., **265** (2013), 765–785.

23. M. Izumi, *Finite group action on C^*-algebras with the Rokhlin property I*, Duke Math. J., **122** (2004), 233–280.

24. M. Izumi, *Finite group action on C^*-algebras with the Rokhlin property II*, Adv. Math., **184** (2004), 119–160.

25. E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra O_2*, J. Reine Angew. Math., **525** (2000), 17–53.

26. E. Kirchberg and M. Rørdam, *Divisibility properties for C^*-algebras*, Proc. London Math. Soc., **106** (2013), 1330–1370

27. A. Kishimoto, *The Rohlin property for shifts on UHF algebras*, J. Reine Angew. Math., **465** (1995), 183–196.

28. H. Lin, *Tracially AF C^*-algebras*, Trans. Amer. Math. Soc., **353** (2001), 683–722.

29. H. Lin, *The tracial topological rank of C^*-algebras*, Proc. London Math. Soc., **83** (2001), 199–234.

30. H. Lin, *An Introduction to the Classification of Amenable C^*-Algebras*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2001.

31. Z. Niu, *Comparison radius and mean topological dimension: Rokhlin property, comparison of open sets, and subhomogeneous C^*-algebras*, J. Analyse Math., 2022, https://doi.org/10.1007/s11854-022-0205-8.

32. H. Osaka and N. C. Phillips, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergodic Theory Dynam. Systems, **26** (2006), 1579–1621.

33. N. C. Phillips, *A classification theorem for nuclear purely infinite C^*-algebras*, Doc. Math., **5** (2000), 49–114.

34. N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*, Amer. J. Math., **133** (2011), 581–636.

35. N. C. Phillips, *Large subalgebras*, arXiv:1408.5546v2.

36. S. Popa, *On local finite-dimensional approximation of C^*-algebras*, Pacific J. Math., **181** (1997), 141–158.

37. I. F. Putnam, *The* C*-*algebras associated with mimimal homeomorphisms of the Cantor set*, Pacific J. Math., **136** (1989), 1483–1518.

38. L. Robert and A. Tikuisis, *Nuclear dimension and $\mathcal{Z}$-stability of non-simple C^*-algebras*, Trans. Amer. Math. Soc., **369** (2017), 4631–4670.

39. M. Rørdam, *Classification of certain infinite simple C^*-algebras*, J. Funct. Anal., **131** (1995), 415–458.

40. M. Rørdam and W. Winter, *The Jiang-Su algebra revisited*, J. Reine Angew. Math., **642** (2010), 129–155.

41. A. Tikuisis, S. White, and W. Winter, *Quasidiagonality of nuclear C^*-algebras*, Ann. of Math. (2), **185** (2017), 229–284.

42. W. Winter and J. Zacharias, *The nuclear dimension of a C^*-algebra*, Adv. Math., **224** (2010), 461–498.

43. X. Zhao, X. Fang, and Q. Fan, *Inheritance of tracial nuclear dimension for centrally large subalgebras*, J. Math. Anal. Appl., **492** (2020), Article 124491, 12 pages.

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