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D-module Approach to Special Functions and Generating Functions

Kam Hang Cheng, Yik Man Chiang and Avery Ching

Presented by Luc Vinet, FRSC

Abstract. This is a research announcement on a unifying study of generating functions of various sequences of special functions, using Bernstein's theory of holonomic D-modules. Both new and well-known generating functions have been obtained in a systematic and algebraic way. New difference analogues of some special functions are also discovered. This announcement focuses on particular results about Hermite functions, Bessel functions and polynomials, Laguerre polynomials, and Gegenbauer polynomials.

Résumé. Il s'agit d'une annonce de recherche sur une étude unificatrice des fonctions génératrices de diverses séquences de fonctions spéciales, en utilisant la théorie de Bernstein des D-modules holonomes. Des fonctions génératrices nouvelles et bien connues ont été obtenues de manière systématique et algébrique. De nouveaux analogues discrets de certaines fonctions spéciales sont également découverts. Cette annonce se concentre sur des résultats particuliers concernant les fonctions d'Hermite, les fonctions et polynômes de Bessel, les polynômes de Laguerre et les polynômes de Gegenbauer.

1. **Introduction** It is well known that rich information about a sequence of functions is encoded in its generating function. Apart from conventional methods of obtaining generating functions such as by series manipulation, there have been studies of generating functions of sequences of special functions making use of modern algebraic methods since the second half of the 20th century. For example, Weisner [22] obtained new generating functions from known ones using Lie group symmetry. He obtained generating functions of hypergeometric functions, Hermite functions, and Bessel functions [23, 24] this way. On the other hand, Truesdell [19, 20] established what is now known to be the F-equation theory to deal with generating functions of a wide range of special functions. McBride summarized in [15] several methods of obtaining generating functions,

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including those by Weisner and Truesdell. This research announcement is about a systematic study of generating functions using Bernstein's theory of holonomic D-modules, which is about to appear in [4]. It turns out that the D-modules related to all the special functions we have studied here in [4] and in [6] are holonomic, and this enables us to derive new generating functions for these special functions.

It was shown in [5], [7], and [6] that the D-module methodology plays useful roles in dealing with complex function theory and special functions with respect to difference operators. In fact, after re-casting some results concerning classical special functions in terms of the Weyl-algebraic language, we are able to gain insights into the D-modules of some classical special functions and their generating functions, which appear to be their natural defining frameworks. It will become evident that our D-module approach unifies the study of generating functions of sequences of special functions with respect to the differential operator and the forward difference operator, as well as possibly other operators.

Motivated by the product rule of differentiation which gives the equality $\frac{d}{dx}(xf(x)) - x\frac{d}{dx}f(x) = f(x)$, one naturally considers the free $\mathbb{C}$-algebra generated by two symbols ∂ and X, modulo the two-sided ideal generated by the element $\partial X - X\partial - 1$. The resulting $\mathbb{C}$-algebra

$$\mathcal{A} = \mathbb{C}\langle \partial, X\rangle/(\partial X - X\partial - 1)$$

is called the Weyl algebra $\mathcal{A}_1 = \mathcal{A}$, and left modules over this Weyl algebra $\mathcal{A}$ (or over some variant of $\mathcal{A}$) are called D-modules [8]. The space $\mathcal{O}$ of analytic functions is conventionally realized as the D-module $\mathcal{O}_d$ with

$$(\partial f)(x) = f'(x) \quad \text{and} \quad (Xf)(x) = xf(x),$$

so that the axiom $[\partial, X] = \partial X - X\partial = 1$ is satisfied. In order to better understand and to generalize classical generating functions, one gives different realizations of $\mathcal{O}$ by equipping it with different D-module structures other than the conventional one. An immediate example is to realize $\mathcal{O}$ as the D-module $\mathcal{O}_\Delta$ with

$$(\partial f)(x) = (\Delta f)(x) = f(x+1) - f(x) \quad \text{and} \quad (Xf)(x) = xf(x-1),$$

so that $[\partial, X] = 1$ is again satisfied. Here the subscript d in $\mathcal{O}_d$ means that the action on the variable x is the conventional one, where ∂ acts as differentiation; while the subscript Δ means that ∂ acts as a forward difference. This idea can be extended naturally to the space of two-variable analytic functions by replacing the Weyl algebra $\mathcal{A}$ by $\mathcal{A}_2$, which is the free $\mathbb{C}$-algebra $\mathbb{C}\langle \partial_1, \partial_2, X_1, X_2\rangle$ modulo the relations

$$[\partial_i, X_j] = \delta_{i,j}, \quad [\partial_i, \partial_j] = 0, \quad [X_i, X_j] = 0, \quad \text{for } i,\, j = 1,\, 2,$$

so that the space of two-variable analytic functions is realized as various left $\mathcal{A}_2$-modules, which we denote by $\mathcal{O}_{dd}$, $\mathcal{O}_{\Delta d}$, and so on.

We will see below the reason why two variables are needed to study generating functions of special functions. Many classical functions are defined by the differential equations which they satisfy. For instance, for each $\nu \in \mathbb{C}$ and $n \in \mathbb{Z}$, the Bessel function $J_{\nu+n}$ satisfies the Bessel differential equation that corresponds to the element $(X\partial)^2 + X^2 - (\nu+n)^2 \in \mathcal{A}$ (as well as a simple normalization condition). In the philosophy of the upcoming article [4] and in [6], we regard the whole sequence of Bessel functions $\{J_{\nu+n}\}_n$ as a single object. To account for the change of this whole sequence as the "variable" n changes, the first step is to observe that the defining element $(X\partial)^2 + X^2 - (\nu+n)^2 \in \mathcal{A}$ satisfies the "transmutation formulae"[1]

$$\begin{aligned}
&\left[(X\partial)^2 + X^2 - (\nu+n+1)^2\right]\left[\partial - (\nu+n)X^{-1}\right] \\
&\quad = \left[\partial - (\nu+n+2)X^{-1}\right]\left[(X\partial)^2 + X^2 - (\nu+n)^2\right], \\
&\left[(X\partial)^2 + X^2 - (\nu+n-1)^2\right]\left[\partial + (\nu+n)X^{-1}\right] \\
&\quad = \left[\partial + (\nu+n-2)X^{-1}\right]\left[(X\partial)^2 + X^2 - (\nu+n)^2\right].
\end{aligned}$$

As a particular consequence, they imply the classical recurrence formulae

$$-xJ_{\nu+n+1}(x) = xJ'_{\nu+n}(x) - (\nu+n)J_{\nu+n}(x), \tag{1.1}$$

$$xJ_{\nu+n-1}(x) = xJ'_{\nu+n}(x) + (\nu+n)J_{\nu+n}(x). \tag{1.2}$$

Now let $\mathcal{O}^{\mathbb{Z}}$ be the space of bilateral sequences $\{f_n\}_n = \{\ldots, f_{-1}, f_0, f_1, \ldots\}$ of functions which are analytic in an annulus. This becomes a left $\mathcal{A}_2$-module denoted by $\mathcal{O}^{\mathbb{Z}}_d$ when endowed with the structure

$$\begin{aligned}
(\partial_1 f)_n(x) &= f'_n(x), & (\partial_2 f)_n(x) &= (n+1)f_{n+1}(x), \\
(X_1 f)_n(x) &= xf_n(x), & (X_2 f)_n(x) &= f_{n-1}(x)
\end{aligned}$$

for all bilateral sequences $\{f_n\}_n$ and all x in the annulus. In such a set-up, the sequence of Bessel functions $\{J_{\nu+n}\}_n \in \mathcal{O}^{\mathbb{Z}}_d$ becomes a solution of the system of differential equations (1.1) and (1.2) represented by the two elements

$$X_1\partial_1 - (\nu + X_2\partial_2) + X_1/X_2 \quad \text{and} \quad X_1\partial_1 + (\nu + X_2\partial_2) - X_1X_2 \tag{1.3}$$

in $\mathcal{A}_2$ respectively. As in [7], the key now is to consider the *Bessel module*, which is the left $\mathcal{A}_2$-module generated by the above two elements, i.e.,

$$\mathcal{B}_\nu := \frac{\mathcal{A}_2}{\mathcal{A}_2\big(X_1\partial_1 + (\nu + X_2\partial_2) - X_1X_2\big) + \mathcal{A}_2\big(X_1\partial_1 - (\nu + X_2\partial_2) + X_1/X_2\big)}.$$

Note that the Bessel module, as well as all the D-modules we have worked with so far, are holonomic D-modules as in Bernstein's theory.

[1] For a comprehensive coverage of transmutation formulae, one can consult, e.g., [9, 10]. Some of our transmutation formulae appear to be new.

The above recurrence formulae of Bessel functions (1.1) and (1.2) are an example of Truesdell's general F-equation theory [20] in which the main focus is phrased in terms of the existence and uniqueness of his F-equation. Trusedell's viewpoint was based on a mathematical analysis argument, whilst our approach utilizes Bernstein's theory of holonomic D-modules [8], and hence is algebraic.

The two left $\mathcal{A}_2$-modules $\mathcal{O}_d^{\mathbb{Z}}$ and $\mathcal{B}_\nu$ above are related by a left $\mathcal{A}_2$-linear map from $\mathcal{B}_\nu$ to $\mathcal{O}_d^{\mathbb{Z}}$. This map together with a corresponding z-transform allows us to obtain the classical generating function for Bessel functions in the left $\mathcal{A}_2$-module $\mathcal{O}_{dd}$, which denotes the space of two-variable functions in which ∂_1 and ∂_2 act as partial differentiation on each variable. From this we recover the series expression

$$\sum_{n=-\infty}^{\infty} J_{\nu+n}(x)\, t^n \sim t^{-\nu} \exp\left[\frac{x}{2}\big(t-\frac{1}{t}\big)\right]. \tag{1.4}$$

Note that when ν is not an integer, the principal part of the above Laurent series in t is divergent, and is 1-Gevrey to be precise; here the symbol $\sim$ in (1.4) means that the principal part is interpreted as its Borel resummation [18] whenever it diverges. The special case $\nu = 0$ of (1.4) (in which $\sim$ can be replaced by $=$) is a well-known classical result [21], but the authors are unable to verify if (1.4) is already known when ν is not an integer.

Apart from $\mathcal{O}_{dd}$ as mentioned before, the space of two-variable analytic functions can be alternatively realized as another D-module denoted by $\mathcal{O}_{\Delta d}$, in which ∂_1 acts as a forward difference in the first variable x and ∂_2 acts as (partial) differentiation in the second variable t. Using this we have obtained, for arbitrary $\nu \in \mathbb{C}$, the new generating function

$$\sum_{n=-\infty}^{\infty} J^{\Delta}_{\nu+n}(x)\, t^n \sim e^{i\pi\nu}\,\frac{\sin(x-\nu)\pi}{\sin(\pi x)}\, t^{-\nu}\left[\frac{1}{2}\big(t-\frac{1}{t}\big)+1\right]^{x} \tag{1.5}$$

for the sequence $\{J^{\Delta}_{\nu+n}\}_n$; here J^{Δ}_{ν} is the *difference Bessel function of order* ν recently discovered by Bohner and Cuchta in [3], which is defined by the Newton-type series[2]

$$J^{\Delta}_{\nu}(x) := \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{-\nu-2k}}{k!\,\Gamma(\nu+k+1)}(x)_{\nu+2k} = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{-\nu-2k}}{k!\,\Gamma(\nu+k+1)}\frac{\Gamma(x+1)}{\Gamma(x+1-\nu-2k)}.$$

The generating function on the right-hand side of (1.5) can be treated as a difference analogue of the function $t^{-\nu}\exp[\frac{x}{2}(t-\frac{1}{t})]$ in (1.4) in the variable x; note that the periodic factor $e^{i\pi\nu}\frac{\sin(x-\nu)\pi}{\sin(\pi x)}$ can be treated as a difference analogue of a constant function as in the context of differential calculus. As a

[2]This series converges in the half-plane $\Re(x) > -\frac{1}{2}$ [6].

by-product, we have also obtained an integral representation of these difference Bessel functions

$$J_\nu^\Delta(x) = e^{i\pi\nu}\,\frac{\sin(x-\nu)\pi}{\sin(\pi x)}\,\frac{1}{2\pi i}\int_\infty^{(0+)} t^{-\nu-1}\left[\frac{1}{2}\big(t-\frac{1}{t}\big)+1\right]^x dt$$

for $\Re(x) < \Re(\nu)$, where the path of integration is a Hankel-type contour in the complex t-plane on which $|2t+t^2| \geq 1$. This formula is a difference analogue of Schläfli-Sonine's integral for the classical Bessel functions J_ν [21, p. 176].

Indeed, when realized in the D-module $\mathcal{O}_\Delta^{\mathbb{Z}}$ instead of in $\mathcal{O}_d^{\mathbb{Z}}$, the two elements (1.3) yield two new recurrence formulae which are difference analogues of (1.1) and (1.2):

$$x\left(\Delta\mathscr{C}_{\nu+n}^\Delta\right)(x-1) + (\nu+n)\mathscr{C}_{\nu+n}^\Delta(x) - x\mathscr{C}_{\nu+n-1}^\Delta(x-1) = 0, \tag{1.6}$$

$$x\left(\Delta\mathscr{C}_{\nu+n}^\Delta\right)(x-1) - (\nu+n)\mathscr{C}_{\nu+n}^\Delta(x) + x\mathscr{C}_{\nu+n+1}^\Delta(x-1) = 0. \tag{1.7}$$

In a sequence $\{\mathscr{C}_{\nu+n}^\Delta\}_n$ that satisfies these recurrence formulae (1.6) and (1.7), each term $\mathscr{C}_{\nu+n}^\Delta$ is an analytic solution of the difference Bessel equation of order $\nu+n$ [3]. In particular when $\mathscr{C}_{\nu+n}^\Delta = J_{\nu+n}^\Delta$ for all n, (1.6) and (1.7) recover those recurrence formulae obtained in [3].

A similar idea also enables us to derive a new difference analogue of the reverse Bessel polynomials and their generating function, etc.; see [6]. We denote $\widehat{\mathcal{A}}_2 := \mathcal{A}_2[\frac{1}{X_1}, \frac{1}{X_2}]$ and define the *reverse Bessel polynomial module* by

$$\Theta := \frac{\widehat{\mathcal{A}}_2}{\widehat{\mathcal{A}}_2(\partial_1 - 1 + X_1/\partial_2) + \widehat{\mathcal{A}}_2(X_1\partial_1 - 2X_2\partial_2 - 1 - X_1 + \partial_2)}.$$

This time let $\mathcal{O}_\Delta^{\mathbb{N}}$ be the left $\mathcal{A}_2$-module obtained by equipping the space of sequences $\{f_n\}_n = \{f_0, f_1, f_2, \ldots\}$ of analytic functions with the structure

$$\begin{aligned} (\partial_1 f)_n(x) &= f_n(x+1) - f_n(x), & (\partial_2 f)_n(x) &= f_{n+1}(x), \\ (X_1 f)_n(x) &= x f_n(x-1), & (X_2 f)_n(x) &= n f_{n-1}(x). \end{aligned} \tag{1.8}$$

This strategy yields a left $\widehat{\mathcal{A}}_2$-linear map from Θ to $\mathcal{O}_\Delta^{\mathbb{N}}$, which gives rise to the new generating function

$$\sum_{n=0}^\infty \theta_n^\Delta(x)\,\frac{t^n}{n!} = \frac{e^{-i\pi x}}{2i\sin\pi x\Gamma(-x)}\int_{-\infty}^{(0+)} e^\lambda(-\lambda)^{-x-1}\,\frac{\exp\left[\lambda(1-\sqrt{1-2t})\right]}{\sqrt{1-2t}}\,d\lambda$$

converging in $\mathbb{C}\times\{t : |t| < \frac{1}{2}\}$, for the newly discovered *difference reverse Bessel polynomials* [6]

$$\theta_n^\Delta(x) := \sum_{k=0}^n \frac{(n+k)!}{2^k(n-k)!\,k!}\,(x)_{n-k};$$

see also [11].

Further investigation shows that the above viewpoint also applies to other special functions and orthogonal polynomials. Note that the sequences of functions we have defined using elements of the Weyl algebra have the property of *orthogonality* which can be handled algebraically using the idea of residue maps, and the orthogonality of the above Bessel polynomials has been investigated in [14]. It is the purpose of this announcement to present some of those more important generating functions that we have found in [4]. We note that Wilf and Zeilberger [25, 26] also adopted the theory of holonomic D-modules in their studies of identities from (q-)special functions.

The following are some new and old results which can all be obtained in a unified way making use of our consideration of D-modules. This certainly does not include all such results obtainable by this method. A more comprehensive list of such results will be included in the upcoming article [4]. Some of the special functions or polynomials written down do not appear to be found in the current literatures such as [1], [2], [12], [13], [16], and [17].

2. Generating Functions of Hermite Functions

To study the *Poisson generating function* of a sequence, i.e. a generating function of the form $F(x,t) = \sum f_n(x)\frac{t^n}{n!}$, one lets $\mathcal{O}_d^{\mathbb{N}}$ be the space of sequences of functions analytic in a disk, and endows it with the structure of a left $\mathcal{A}_2$-module similar to (1.8), with ∂_1 and X_1 acting as differentiation and multiplication by x instead. Now recalling that for each $\nu \in \mathbb{C}$ and $n \in \mathbb{N}$, the classical Hermite function $H_{\nu+n}$ is a solution of the differential equation represented by the element $\partial^2 - 2X\partial + 2(\nu+n)$; one finds that this element satisfies the "transmutation formulae"

$$\left[\partial^2 - 2X\partial + 2(\nu+n+1)\right](\partial - 2X) = (\partial - 2X)\left[\partial^2 - 2X\partial + 2(\nu+n)\right],$$
$$\left[\partial^2 - 2X\partial + 2(\nu+n-1)\right]\partial = \partial\left[\partial^2 - 2X\partial + 2(\nu+n)\right],$$

which imply the well-known recurrence relations

$$H'_{\nu+n}(x) - 2xH_{\nu+n}(x) = -H_{\nu+n+1}(x), \tag{2.1}$$

$$H'_{\nu+n}(x) = 2(\nu+n)H_{\nu+n-1}(x). \tag{2.2}$$

The sequence of Hermite functions $\{H_{\nu+n}\}_n$ (as one single object) is hence a solution of the holonomic left $\mathcal{A}_2$-module

$$\frac{\mathcal{A}_2}{\mathcal{A}_2(\partial_1 + \partial_2 - 2X_1) + \mathcal{A}_2(\partial_1 - 2X_2 - 2\nu/\partial_2)} \tag{2.3}$$

in $\mathcal{O}_d^{\mathbb{N}}$; (2.3) is called the *Hermite module*. Alternatively, a solution of (2.3) in $\mathcal{O}_{dd}$ is $\exp(2xt - t^2)H_\nu(x-t)$, which is the Poisson generating function of the sequence $\{H_{\nu+n}\}_n$. This is also known as an addition formula:

$$\sum_{n=0}^{\infty} H_{\nu+n}(x)\frac{t^n}{n!} = \exp(2xt - t^2)H_\nu(x-t). \tag{2.4}$$

As a variation, in the case $\nu = 0$, the left $\mathcal{A}_2$-linear map from (2.3) to $\mathcal{O}_\Delta^{\mathbb{N}}$ is represented algebraically by $(2X - \partial)^n$. A polynomial solution of (2.3) in $\mathcal{O}_\Delta^{\mathbb{N}}$ is therefore given by the sequence $\{H_n^\Delta\}_n$ with

$$H_n^\Delta(x) := (2X - \partial)^n \cdot 1.$$

These are called *difference Hermite polynomials.* The first few of them are given by

$$\begin{aligned}
H_0^\Delta(x) &= 1, \\
H_1^\Delta(x) &= 2x, \\
H_2^\Delta(x) &= 4x(x-1) - 2, \\
H_3^\Delta(x) &= 8x(x-1)(x-2) - 12x, \qquad \text{etc.}
\end{aligned}$$

They satisfy the recurrence relations

$$\begin{aligned}
\left(\Delta H_n^\Delta\right)(x) - 2xH_n^\Delta(x-1) &= -H_{n+1}(x), \\
\left(\Delta H_n^\Delta\right)(x) &= 2nH_{n-1}(x).
\end{aligned}$$

The Poisson generating function of $\{H_n^\Delta\}_n$ is a solution of the Hermite module (2.3) in $\mathcal{O}_{\Delta d}$, which turns out to be the function $(1+2t)^x \exp(-t^2)$. In other words, since (2.3) is holonomic, we have the new formula

$$\sum_{n=0}^{\infty} H_n^\Delta(x) \frac{t^n}{n!} = (1+2t)^x \exp(-t^2).$$

Here the function $(1+2t)^x \exp(-t^2)$ is a difference analogue of the function $\exp(2xt - t^2)$ for the variable x.

By computing the z-transform of the classical Hermite polynomials $\{H_n\}_n$ by the same method, we also obtain a generating function for a sequence $\{f_n\}_n$ that satisfies the same recurrence relations (2.1) and (2.2), namely

$$\sum_{n=-\infty}^{\infty} f_n(x) t^n = \frac{1}{t} \exp\left(x^2 - \frac{x}{t} + \frac{1}{4t^2}\right).$$

It turns out that the sequence of entire functions $\{f_n\}_n$ defined by

$$\begin{aligned}
f_n(x) &= 0 \quad \text{for all } n \geq 0, \\
f_{-n-1}(x) &= \frac{i^n H_n(ix)}{2^n n!} e^{x^2} \quad \text{for all } n \geq 0
\end{aligned}$$

satisfies the recurrence relations (2.1) and (2.2), so we obtain

$$\sum_{n=0}^{\infty} f_{-n-1}(x)t^{-n-1} = \frac{1}{t}\exp\left(x^2 - \frac{x}{t} + \frac{1}{4t^2}\right),$$

which recovers the previous generating function (2.4) again, in the case $\nu = 0$. The sequence of difference Hermite polynomials also has an analogous generating function.

3. Generating Functions of Laguerre Polynomials In a similar way, one recalls that for each $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, the generalized Laguerre polynomial $L_n^{(\alpha)}$ is a solution of the differential equation that corresponds to the element $\partial X \partial + (\alpha - X)\partial + n$; one then finds that this element satisfies the transmutation formulae

$$\begin{aligned}
&[\partial X\partial + (\alpha - X)\partial + n - 1]\,(X\partial - n) \\
&\quad = (\partial X - n)\,[\partial X\partial + (\alpha - X)\partial + n]\,, \\
&[\partial X\partial + (\alpha - X)\partial + n + 1]\,(X\partial - X + \alpha + n + 1) \\
&\quad = (\partial X - X + \alpha + n + 1)\,[\partial X\partial + (\alpha - X)\partial + n]\,,
\end{aligned}$$

which imply the recurrence relations

$$\begin{aligned}
x(L_n^{(\alpha)})'(x) &= nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \\
x(L_n^{(\alpha)})'(x) &= (n+1)L_{n+1}^{(\alpha)}(x) - (n+\alpha+1-x)L_n^{(\alpha)}(x).
\end{aligned}$$

See also [10, §8.8]. The sequence of generalized Laguerre polynomials $\{L_n^{(\alpha)}\}_n$ is a solution of the holonomic left $\mathcal{A}_2$-module

$$\frac{\mathcal{A}_2}{\mathcal{A}_2\big((1-X_2)\partial_1 + X_2\big) + \mathcal{A}_2\big(X_1 + (X_2-1)[\partial_2(X_2-1)+\alpha]\big)} \tag{3.1}$$

in $\mathcal{O}_d^{\mathbb{N}}$; (3.1) is called the *generalized Laguerre module.* Alternatively, a solution of (3.1) in $\mathcal{O}_{dd}$ is $(1-t)^{-\alpha-1}\exp(-\frac{xt}{1-t})$, which is the generating function of the sequence $\{L_n^{(\alpha)}\}_n$ since (3.1) is holonomic:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right).$$

As a variation, a polynomial solution of (3.1) in $\mathcal{O}_\Delta^{\mathbb{N}}$ is given by the sequence $\{L_n^{\Delta(\alpha)}\}_n$ with

$$L_n^{\Delta(\alpha)}(x) := \frac{1}{n!}[X\partial + (\alpha + 1 - X)]^n \cdot 1.$$

These are called *difference generalized Laguerre polynomials.* The first few of them are given by

$$\begin{aligned}
L_0^{\Delta(\alpha)}(x) &= 1,\\
L_1^{\Delta(\alpha)}(x) &= -x + (\alpha+1),\\
L_2^{\Delta(\alpha)}(x) &= \frac{1}{2}[x(x-1) - 2(\alpha+2)x + (\alpha+1)(\alpha+2)],\\
L_3^{\Delta(\alpha)}(x) &= \frac{1}{6}[-x(x-1)(x-2) + 3(\alpha+3)x(x-1) - 3(\alpha+2)(\alpha+3)x\\
&\qquad + (\alpha+1)(\alpha+2)(\alpha+3)], \quad \text{etc.}
\end{aligned}$$

They satisfy the recurrence relations

$$\begin{aligned}
x\left(\Delta L_n^{\Delta(\alpha)}\right)(x-1) &= nL_n^{\Delta(\alpha)}(x) - (n+\alpha)L_{n-1}^{\Delta(\alpha)}(x),\\
x\left(\Delta L_n^{\Delta(\alpha)}\right)(x-1) &= (n+1)L_{n+1}^{\Delta(\alpha)}(x) - (n+\alpha+1)L_n^{\Delta(\alpha)}(x) + xL_n^{\Delta(\alpha)}(x-1).
\end{aligned}$$

The generating function of $\{L_n^{\Delta(\alpha)}\}_n$ is a solution of the generalized Laguerre module (3.1) in $\mathcal{O}_{\Delta d}$, and turns out to be $(1-t)^{-\alpha-1}(1-\frac{t}{1-t})^x$. Since (3.1) is holonomic, we deduce:

$$\sum_{n=0}^{\infty} L_n^{\Delta(\alpha)}(x)t^n = (1-t)^{-\alpha-1}\left(1 - \frac{t}{1-t}\right)^x.$$

Again, the generating function $(1-t)^{-\alpha-1}(1-\frac{t}{1-t})^x$ is a difference analogue of the function $(1-t)^{-\alpha-1}\exp(-\frac{xt}{1-t})$ for the variable x.

Interestingly, we have also found that the sequences of Krawtchouk polynomials and Meixner polynomials in the Askey scheme in fact both share the same left $\mathcal{A}_2$-module structure (3.1) with that of the generalized Laguerre polynomials.

4. **Generating Functions of Gegenbauer Polynomials** For each $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, the Gegenbauer polynomial $C_n^{(\lambda)}$ is a solution of the differential equation that corresponds to the element $(1-X^2)\partial^2 - (2\lambda+1)X\partial + n(n+2\lambda)$. One finds that this element satisfies the transmutation formulae

$$\begin{aligned}
&\left[(1-X^2)\partial^2 - (2\lambda+1)X\partial + (n+1)(n+1+2\lambda)\right]\left[(1-X^2)\partial - (n+2\lambda)X\right]\\
&\quad = \left[(1-X^2)\partial - (n+2+2\lambda)X\right]\left[(1-X^2)\partial^2 - (2\lambda+1)X\partial + n(n+2\lambda)\right],\\
&\left[(1-X^2)\partial^2 - (2\lambda+1)X\partial + (n-1)(n-1+2\lambda)\right]\left[(1-X^2)\partial + nX\right]\\
&\quad = \left[(1-X^2)\partial + (n-2)X\right]\left[(1-X^2)\partial^2 - (2\lambda+1)X\partial + n(n+2\lambda)\right],
\end{aligned}$$

which imply the recurrence relations

$$(1-x^2)(C_n^{(\lambda)})'(x) = (n+2\lambda-1)C_{n-1}^{(\lambda)}(x) - nxC_n^{(\lambda)}(x),$$
$$(1-x^2)(C_n^{(\lambda)})'(x) = (n+2\lambda)xC_n^{(\lambda)}(x) - (n+1)C_{n+1}^{(\lambda)}(x).$$

See also [10, §6.5]. The sequence of Gegenbauer polynomials $\{C_n^{(\lambda)}\}_n$ is a solution of the holonomic left $\mathcal{A}_2$-module

(4.1)
$$\frac{\mathcal{A}_2}{\mathcal{A}_2\big((1-X_1^2)\partial_1+(1-X_1X_2)\partial_2-2\lambda X_1\big)+\mathcal{A}_2\big((1-X_1^2)\partial_1+(X_1X_2-X_2^2)\partial_2-2\lambda X_2\big)}$$

in $\mathcal{O}_d^{\mathbb{N}}$; (4.1) is called the *Gegenbauer module*. Alternatively, a solution of (4.1) in $\mathcal{O}_{dd}$ is $(t^2-2xt+1)^{-\lambda}$ if $\lambda \neq 0$ and is $\ln(t^2-2xt+1)$ if $\lambda = 0$, which is the well-known generating function of $\{C_n^{(\lambda)}\}_n$. Since (4.1) is holonomic, it then follows that

$$\sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n = \begin{cases} (t^2-2xt+1)^{-\lambda} & \text{if } \lambda \neq 0 \\ \ln(t^2-2xt+1) & \text{if } \lambda = 0 \end{cases}. \tag{4.2}$$

The sequence of *difference Gegenbauer polynomials* $\{C_n^{\Delta(\lambda)}\}_n$ is a solution of (4.1) in $\mathcal{O}_\Delta^{\mathbb{N}}$. Its generating function is a solution of (4.1) in $\mathcal{O}_{\Delta d}$, and can be obtained from the previous generating function (4.2) by an integral transform called the *Newton transform*; the generating function of $\{C_n^{\Delta(\lambda)}\}_n$ is

$$\sum_{n=0}^{\infty} C_n^{\Delta(\lambda)}(x)t^n = \begin{cases} \dfrac{1}{\Gamma(-x)}\displaystyle\int_0^{+\infty} e^{-s}(t^2+2ts+1)^{-\lambda}s^{-x-1}\,ds & \text{if } \lambda \neq 0 \\ \dfrac{1}{\Gamma(-x)}\displaystyle\int_0^{+\infty} e^{-s}\ln(t^2+2ts+1)s^{-x-1}\,ds & \text{if } \lambda = 0 \end{cases}.$$

5. **Conclusion** In this announcement, we have described a unifying method for obtaining generating functions of sequences of special functions in general. Examples of several sequences of new and old orthogonal functions/polynomials have been presented. The generating functions of these sequences are obtained as solutions of pairs of PDEs, which arise from holonomic left $\mathcal{A}_2$-modules the sequences themselves give. Various methods of solving ODEs and PDEs, such as integrating factors and the method of characteristics, can all be realized algebraically in our Weyl-algebraic setting which will be presented in [4]; see also [6].

Apart from the issue of generating functions, we have also found that the sequences of Krawtchouk and Meixner polynomials in the Askey scheme both share the same left $\mathcal{A}_2$-module structure with that of the generalized Laguerre polynomials. In the D-module context, these sequences can therefore be regarded as equivalent objects. It is also worth noting that the orthogonality of these algebraically defined sequences can be handled using the idea of a residue map. The issue of D-modules and orthogonality in general will be addressed in a future project.

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Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.
e-mail: henry.cheng@family.ust.hk

Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.
e-mail: machiang@ust.hk

Department of Statistics, The University of Warwick, Coventry, CV4 7AL, United Kingdom. (Current address: Department of Mathematics, University of Dundee, Nethergate, Dundee DD1 4HN, Scotland, United Kingdom. *e-mail:* AChing001@dundee.ac.uk)
e-mail: avery.ching@warwick.ac.uk