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ON EQUIVALENCE RELATIONS IN MAPPING THEORY

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Presented by J. Aczel, F.R.S.C.

0. The study of a mapping $f:X \to Y$ with respect to a property p can be embedded in a general scheme: Let F be the set of all mappings $f:X \to Y$ with fixed X and Y then we can identify p with an equivalence relation (= eq. relation) in F described by two disjoint classes, the one consisting of all f with property p, the other with property non-p. So the general question is how to study an eq. relation $\sim$ in F. This can be done for instance by investigating the behaviour of $\sim$ with respect to systems $E' := \{\sim_i : i \in J\}$ of special eq. relations $\sim_i$ in F. So a preliminary of this way of investigation is the study of systems E' of eq. relations in F. The following notes are a contribution to such a study. First we prove a general theorem about certain mappings of subsets of the equivalence lattice E of an arbitrary set S onto subsets of E; an application of this theorem where S is the aforementioned set F will yield a characterization of the set X as a "closure space" in the sense of E.ČECH by means of a system of eq. relations in F with rather natural properties.

I would like to call such an approach to a (here the closure space) structure a "motivation from above" while the name "motivation from below" may stand for the short but trivial way of giving examples.

1. Let S be a set and E be the set of all eq. relations in S (notation: $f\ \eta\ g$, read: "f η-equivalent to g " with $\eta \in E$ and $f,g \in S$). Definitions:

I. If $f,g \in S$ then

$$[f,g] := \{\eta : \eta \in E \wedge f\ \eta\ g\}$$

(we may call the subset $[f,g]$ in E a "glyph" in E); if E_i is a subset of E we write for short $[f,g]_i := [f,g] \cap E_i$.

II. A subset E_i of E is called periglyphic if

(P) $\bigwedge_{Q \subset E_i,\ f \in S} \bigvee_{g \in S} [f,g]_i = Q$,

in other words: The mapping $g \mapsto [f,g]_i$ of S into $P(E_i)$ (= the power set of E_i) is surjective for each $f \in S$.

Example: If S is the set F mentioned above and Y contains at least two elements then the system $E_1 := \{=_x : x \in X\}$ with

$$f =_x g :\Longleftrightarrow f(x) = g(x)$$

is periglyphic.

For, given $Q \subset X$ and $f \in F$, then g defined according to

$$g(x) = f(x) \text{ for } x \in Q \text{ and } g(x) \neq f(x) \text{ for } x \notin Q$$

satisfies (P) .

III. If E_1, E_2 are subsets of E then we say: E_1 is homoglyphic to E_2 if

(H) $\bigwedge_{f,g,f',g' \in S} [f,g]_1 \subset [f',g']_1 \Rightarrow [f,g]_2 \subset [f',g']_2$.

A consequence of (H) is

$$[f,g]_1 = [f',g']_1 \Rightarrow [f,g]_2 = [f',g']_2 \text{ , or}$$

there exists a mapping

(k) $\quad k : [f,g]_1 \mapsto [f,g]_2 \quad (= k([f,g]_1));$

the domain of k is $D_1 := \{[f,g]_1 : (f,g) \in S \times S\} \subset P(E_1)$ and the range is $P(E_2)$. Because of the reflexivity of the η's we have $[f,f]_i = E_i$, $i=1,2$, and so $k(E_1) = E_2$. Furthermore k is isotone $(Q \subset Q' \Rightarrow k(Q) \subset k(Q'))$.

2. Theorem. Assume that E_1, E_2 are subsets of the equivalence lattice E of a set S, and that E_1 is periglyphic and homoglyphic to E_2. Then k, as defined by (k), is a $\cap$-distributive mapping from $P(E_1)$ into $P(E_2)$ with $k(E_1) = E_2$.

Proof. $D_1 = P(E_1)$ is a consequence of (P). Because of the isotony of k we have for any $X,Y \in P(E_1)$

(1) $\quad k(X) \cap k(Y) \supset k(X \cap Y)$.

Using (P) we find representations $X = [f,g]_1$ and $[g,h]_1 = Y$ with some $f,g,h \in S$. For $Z := [f,h]_1$ we get

(2) $\quad Z \subset C(X \dotplus Y)$, or

$CZ \supset X \dotplus Y := (X \cap CY) \cup (Y \cap CX)$.

For let e.g. $\eta \in X \cap CY$ with $\eta \in E_1$ then $f \eta g \wedge \neg(g \eta h)$, and this implies, because η is an eq. relation, that $\neg(f \eta h)$, or $\eta \notin Z$.

Furthermore the transitivity of the η's implies

$$[f,g]_2 \cap [g,h]_2 \subset [f,h]_2 \quad ,$$

or using (k) and (2) and the isotony of k

$$k(X) \cap k(Y) \subset k(Z) \subset k(C(X \dotplus Y)) \quad \text{for all } X,Y \in P(E_1) .$$

So $k(X) \cap k(Y) \subset k(X) \cap k(Y \cup CX) \subset k(C(X \dotplus (Y \cup CX))) = k(X \cap Y)$ what together with (1) yields

$$k(X) \cap k(Y) = k(X \cap Y) ,$$

the $\cap$-distributivity of k .

3. The foregoing theorem has an inverse.

Theorem. Let be $E_1 \subset E$ and $k' : P(E_1) \to P(A)$ with some set A be $\cap$-distributive. Then for each $w \in k'(E_1)$

(3) $\qquad f \sim_w g :\Leftrightarrow w \in k'([f,g]_1)$ for $f,g \in S$

defines an eq. relation $\sim_w$ in S and E_1 is homoglyphic to $E_2 := \{\sim_w : w \in k'(E_1)\}$.

Proof. First we remark that k' is isotone and so $k'(E_1)$ is the largest set produced by k' .

Symmetry of $\sim_w$: evident,-Reflexivity of $\sim_w$: $f \sim_w f$ for each $f \in S$ because of $w \in k'(E_1) = k'([f,f]_1)$. - Transitivity of $\sim_w$: $k'([f,g]_1) \cap k'([g,h]_1) = k'([f,g]_1 \cap [g,h]_1) \subset k'([f,h]_1)$ (because of $[f,g]_1 \cap [g,h]_1 \subset [f,h]_1$) , so by (3) $f \sim_w g \wedge g \sim_w h \Rightarrow f \sim_w h$. - Finally $[f,g]_2 = [f,g] \cap E_2 = \{\sim_w : \quad w \in k'([f,g]_1)\}$. So $[f,g]_1 \subset [f',g']_1 \Rightarrow k'([f,g]_1) \subset k'([f',g']_1) \Rightarrow [f,g]_2 \subset [f',g']_2$, therefore E_1 is homoglyphic to E_2 .

4. Application. (in case that S is the set F of all mappings $f : X \to Y$ (with the cardinality of Y larger than 1)).

Definition. A set X is said to be a closure space (in the sense of E.ČECH [1]) if there is a mapping $\underline{k}: P(X) \to P(X)$ which is finitely $\cap$-distributive (i.e. $\cap$-distributive with $k(X) = X$ (because of $\bigcap \emptyset = X$)) and satisfies $\underline{k}(A) \subset A$ for all $A \in P(X)$. A characterization of closure space is given by the following

Theorem. Let X, Y be sets and the cardinality of Y be larger than 1, F be the set of all mappings $f : X \to Y$ and $E_1 := \{=_x : x \in X\}$ (see example in 1.). Then X is provided with the structure of a closure space if and only if there is a family $E_2 := \{\sim_x : x \in X\}$ of equivalence relations $\sim_x$ in X such that E_1 is homoglyphic to E_2 and

$$\text{(I)} \qquad \bigwedge_{f,g} \in [f,g]_2 \subset [f,g]_1 .$$

Proof. Apply the theorems of 2. and 3. and write $\underline{k}(A)$ instead of $k(\{\sim_x : x \in A\})$ for $A \in P(X)$. [2]

Interpretation. Using for shortness the notations $[f \sim g] := \{\sim_x : x \in X \wedge f \sim_x g\}$ and $[f=g]$ analogously then (H) and (I) respectively are equivalent to the rather natural requirements

$$\bigwedge_{f,g,f',g' \in F} [f=g] \subset [f'=g'] \Rightarrow [f \sim g] \subset [f' \sim g'] \quad \text{and}$$

$$\bigwedge_{f,g \in F} [f \sim g] \subset [f=g] .$$

Remark [2]. Characterization of topological spaces:

X is a topological space if and only if

$$\text{(T)} \qquad \bigwedge_{f,g,f',g' \in F} [f=g] \subset [f'=g'] \Leftrightarrow [f \sim g] \subset [f' \sim g']$$

where the open kernel of $[f=g]$ is equal to $[f \sim g]$, $f,g \in F$.

In case (T) the statement $f \sim_x g$ ($\Leftrightarrow x \in [f \sim g] \subset [f=g]$) has a well known meaning: x is contained in an open set on which f and g are identical, or f and g are contained in the same "mapping germ" with respect to the point x.

5. The exposition in 1., 2., and 3. may be considered as a step into what I would like to call "geometry in

equivalence lattices".

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GEODESIC SPHERES AND LOCALLY SYMMETRIC SPACES

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1. Let M be a Riemannian manifold and m a point in M. (All Riemannian manifolds are assumed to be connected and of class C^{ω}.) If r is a so small positive number that the exponential map $\exp_m$ is defined on a ball $B_m(r)$ of radius r and center m in the tangent space $T_m(r)$ of m at M, then the image $G_m(r)$ of the hypersphere $S_m(r)$ under $\exp_m$ is called a geodesic sphere centered at m and with radius r.

It is interesting to know to what extent the properties of sufficiently small geodesic spheres determine the Riemannian geometry of the ambient space. Some results are given in [3], [5], [7], [8], [11], in particular by considering the volume of the geodesic spheres or balls, the total curvatures of the geodesic spheres and the mean value of functions. The main technique to derive these results is the determination of several power series expansions.

In this note we continue this study. Our main purpose is to find the relations between the geodesic spheres and locally symmetric spaces. (See also [13],[14].) We announce the following results; the details will appear elsewhere [4].

2. THEOREM 1. *A Riemannian manifold* M *is of constant sectional curvature if and only if every sufficiently small geodesic sphere has either parallel Ricci tensor* (dim M $>$ 3) *or parallel second fundamental form.*

THEOREM 2. *A Riemannian manifold of dimension* $>$ 4 *is of constant sectional curvature if and only if every sufficiently small geodesic sphere is conformally flat.*

THEOREM 3. *If every sufficiently small geodesic sphere of a Riemannian manifold* M (dim M $>$ 3) *has constant scalar curvature, then* M *is Einsteinian and super-Einsteinian. In particular if* M *is 4-dimensional,* M *is locally symmetric.*

For the definition of super-Einstein spaces see [2], [11]. This theorem is similar to the result about harmonic spaces where one considers the trace of the second fundamental form instead of the trace of the Ricci tensor. (See also [2], [12].)

3. Let (N,g) be a compact Riemannian manifold with metric g. Let Δ denote the Laplacian operator of N acting on p-forms; $p \leq \dim N$. Then we have the spectrum for each p:

$$\mathrm{Spec}^p N = \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \leq \dots \uparrow \infty\}$$

where each eigenvalue of Δ is repeated as many times as its multiplicity indicates. We have

THEOREM 4. *Let* M *be a Riemannian (respectively, Kähler or quaternionic) manifold. If for all* $m \in M$ *and all sufficiently small* r, $\mathrm{Spec}^p(G_m(r))$, *for some fixed* p, *is the same as that in a real (respectively, complex or quaternionic) projective space, then* M *is locally isometric to that projective space.*

4. On a Riemannian manifold N with Riemannian connection ∇, a symmetric tensor field T of type $(0,2)$ is called a *Codazzi tensor* if for all vector fields X, Y, Z tangent to N, we have $(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z)$. T is called *cyclic-parallel* if we have $(\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) + (\nabla_Z T)(X,Y) = 0$. This condition is equivalent to $(\nabla_X T)(X,X) = 0$ for all X tangent to N. (See [6] for the case that T is the Ricci tensor of N.)

As a generalization of Theorem 1 we have

THEOREM 5. *A Riemannian manifold* M *is of constant sectional curvature if and only if the Ricci tensor* ($\dim M > 3$) *or the second fundamental form of every sufficiently small geodesic sphere is a Codazzi tensor.*

It can be proved that the Ricci tensor and the second fundamental form of every sufficiently small geodesic sphere in Riemannian manifolds of constant sectional curvature, in Kähler manifolds of constant holomorphic sectional curvature, and in quaternionic manifolds of constant quaternionic sectional curvature is cyclic-parallel. On the other hand we have

THEOREM 6. *If the Ricci tensor of every sufficiently small geodesic sphere in a Riemannian manifold* M ($\dim M > 3$) *is cyclic-parallel, then* M *is locally symmetric.*

THEOREM 7. *If the second fundamental form of every sufficiently small geodesic sphere in a Riemannian manifold* M *is cyclic-parallel, then* M *is either a flat Riemannian manifold or locally isometric to a rank one symmetric space.*

THEOREM 8. *A Kähler (respectively, quaternionic) manifold is of constant holomorphic (respectively, quaternionic) sectional curvature if and only if the second fundamental form of every sufficiently small geodesic sphere is cyclic-parallel.*

5. Geodesic spheres are special cases of tubes about submanifolds. The method of power series expansions for tubes is applied in [1], [9], [10] to study similar problems and will also be used in other forthcoming papers.

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Spectral Sequences for the K-theory of Glued Rings

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Presented by P. Ribenboim, F.R.S.C.

The purpose of this research is to analyze the algebraic K-theory of rings obtained by 'glueing'. By 'K-theory' we mean the classical groups K_0, K_1, the negative K-theory of Bass, and the Karoubi-Villamayor groups KV_*. A ring is 'glued' if it arises as a pullback in a cartesian square, or more generally if it is the inverse limit of a functor A from a small category $\mathcal{G}$ to the category of commutative rings.

1. LIMITS OF RINGS

For convenience, we will make the following standing assumption:

(1.1) R is a commutative ring, $\{I(\alpha):\alpha\in\mathcal{G}\}$ is a finite set of ideals indexed in reverse order by the poset $\mathcal{G}$ and A is the functor $A(\alpha) = R/I(\alpha)$.

We let $\alpha_1,\ldots,\alpha_n$ denote the maximal elements of $\mathcal{G}$ and let $\lim A(\mathcal{G})$ denote the inverse limit of the functor A. Thus

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* This work was completed while the first author was a visitor at Queen's University, Kingston, Ontario.

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$$\lim A(\mathcal{G}) = \{(a_i)\in\prod_i R/I(\alpha_i) : a_i \equiv a_j \bmod I(\gamma) \text{ whenever } \alpha_i, \alpha_j > \gamma\}$$

A sufficient condition for $R \to \lim A(\mathcal{G})$ to be surjective is that for each ℓ, $2 \le \ell \le n$, and for each ℓ-tuple $(\alpha, \beta_2, \ldots, \beta_\ell)$ of maximal elements of $\mathcal{G}$

$$(1.2) \quad I(\alpha) + \bigcap_{k=2}^{\ell} I(\beta_k) = \bigcap\{I(\gamma) : \gamma \le \alpha, \gamma \le \beta_k \text{ some } k\}.$$

For fixed ℓ the conditions (1.2) are called $(CRT)_\ell$ due to the connection of these to the Chinese Remainder Theorem [ZS, p. 280].

If, in addition to (1.1), for each $\alpha \in \mathcal{G}$ $R \to \lim A(\{\gamma < \alpha\})$ is onto, the $(CRT)_\ell$ conditions are equivalent to the surjectivity of $R \to \lim A(\mathcal{G})$.

If $(CRT)_2$ holds, then the topological space (scheme) $\operatorname{Spec}(\lim A(\mathcal{G}))$ is the colimit of the topological spaces (schemes) $\operatorname{Spec} A(\alpha)$. Our terminology stems from the geometric interpretation of this: $\operatorname{Spec}(\lim A(\mathcal{G}))$ is obtained by "glueing" the schemes $\operatorname{Spec} A(\alpha)$.

2. K_1-REGULARITY AND SEMINORMALITY

A ring B is K_i-regular if $K_i(B) = K_i(B[t_1, t_2, \ldots])$. It is known that K_1-regularity implies K_0-regularity and seminormality (Pic-regularity) [DW]. Even when the rings $R/I(\alpha)$ are K_2-regular domains, $\lim A(\mathcal{G})$ may not even be K_0-regular. (In some important cases $\lim A(\mathcal{G})$ is K_1-regular - this can be shown by methods similar to those in [DW]). For seminormality, however, we have the following version of the results of [O].

Theorem 2.1: Assume in addition to (1.1) that each $I(\alpha)$ is a radical ideal of R. Then $\lim A(\mathcal{G})$ is seminormal in $B = \prod R/I(\alpha_i)$. In fact if $(CRT)_2$ holds, $\lim A(\mathcal{G})$ is the seminormalization of the image of R in B. If each $R/I(\alpha_i)$ is seminormal and B is contained in the total ring of quotients of $\lim A(\mathcal{G})$, then $\lim A(\mathcal{G})$ is seminormal.

3. THE SPECTRAL SEQUENCE

We now grade $\mathcal{G}$ by a fixed functor $\dim : \mathcal{G} \to \mathbb{N}$ sending distinct $\beta < \alpha$ to distinct integers, and let $\mathcal{G}^n$ denote the subposet of all α with $\dim(\alpha) \leq n$. The prototype of "dim" is $\dim(\alpha)$ = Krull dimension of $A(\alpha)$.

$\mathcal{G}$ is called cellular if (i) for every α there is a $\nu \leq \alpha$ with $\dim(\nu) = 0$ and (ii) for every n the maps $H^p(\mathcal{G}^{n+1};G) \to H^p(\mathcal{G}^n;G)$ are isomorphisms when $p < n$ and injections when $p = n$. Here $H^p(\mathcal{B};G)$ is the cohomology of the category $\mathcal{B}$ with coefficients in the abelian group G (see [Q]).

$\mathcal{G}$ is cellular iff for every $\alpha \in \mathcal{G}$ with $\dim(\alpha) = p \neq 0$, the comma category $\mathcal{G}^{p-1}\downarrow\alpha$ of all $\beta < \alpha$ has $H^q(\mathcal{G}^{p-1}\downarrow\alpha;G) = 0$ for $q \neq p$. Any abstract simplicial complex $\mathcal{G}$, considered as a graded poset (in the obvious way), is cellular. The posets $\mathcal{F}$ used in [DW] are also cellular.

__Theorem 3.1__: Assume, in addition to the standing hypotheses (1.1), that k is a regular ring, R is a k-algebra, $\mathcal{G}$ is cellular, and that for every $\alpha \in \mathcal{G}$ with $\dim \alpha = p$

i) $A(\alpha) = k[x_1,\ldots,x_p]$ as k-algebras

ii) $R \to \lim A(\mathcal{G}^{p-1}\downarrow\alpha)$ is surjective

iii) $\lim (\mathcal{G}^{p-1}\downarrow\alpha)$ is K_1-regular

Then there is a spectral sequence

$$E_2^{pq} = H^p(\mathcal{G};K_{-q}(k)) \Rightarrow KV_{-p-q}(\lim A(\mathcal{G}))$$

Moreover $H^0(\mathcal{G};K_q(k)) = E_2^{0,-q} = E_\infty^{0,-q}$ is a summand of $KV_q(\lim A(\mathcal{G}))$.

The spectral sequence is functorial with respect to base change. Thus $\lim A(\mathcal{G})$ will be K_0-regular in the situation of (3.1). Note that $K_0 = KV_0$, so we can extract the classical K_0 from the spectral sequence (we obtain the negative K-theory of Bass as well).

4. AN EXAMPLE

Let k be a field of characteristic $\neq 2$ and set $A = \{f(x,y)\in k[x,y] : f(t,0)=f(1-t,t-t^2)\} = k[x-x^2,g,h,hx,hxy]$ where $g = (1-2x)y + x(x-x^2)$ and $h = y(y-x+x^2)$. This ring is the inverse limit of a 2-dimensional $\mathcal{G}$ $(\cdot\rightrightarrows\cdot\rightrightarrows\cdot)$ whose geometric realization is $\mathbb{RP}^2$. Moreover, if $k = \mathbb{R}$ the real points of Spec(A) give an embedding of $\mathbb{RP}^2$ in $\mathbb{R}^5$. Although (1.1) is not satisfied, (3.1) still holds. The spectral sequence degenerates to give $KV_q(A) = K_q(k) \oplus \widetilde{KV}_q(A)$ and an extension

(4.1) $\quad 0 \to K_{q+2}(k) \otimes Z/2Z \to \widetilde{KV}_q(A) \to \mathrm{Tor}(K_{q+1}(k), Z/2Z) \to 0.$

In particular, there is an extension

(4.2) $\quad 0 \to K_2(k)/2K_2(k) \to K_0(A) \to Z/2Z \to 0 .$

Comparison with the Atiyah-Hirzebruch spectral sequence for $KO(\mathbb{RP}^2)$ yields the following result: The sequence (4.2) splits if and only if the symbol $\{-1,-1\}$ of $K_2(k)$ is in $2K_2(k)$. In particular (4.2) does not split when $k = \mathbb{R}$, in which case $K_0(A) = Z \oplus Z/4Z$

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THE DENSITIES OF CERTAIN REGULAR STAR-POLYTOPES

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Presented by H.S.M. Coxeter, F.R.S.C.

Abstract

An elementary technique for dissecting simplexes is used to explain why the densities of many regular star-polytopes are binomial coefficients.

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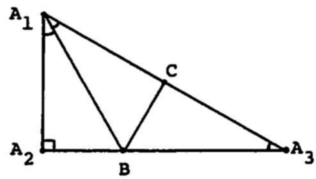
(1) In a right triangle $A_1A_2A_3$ with $\measuredangle A_1 = 2\measuredangle A_3$ and $\measuredangle A_2 = \pi/2$, the bisector A_1B of $\measuredangle A_1$ meets A_2A_3 at an angle $\pi/3$:



Indeed, reflect ΔA_1A_2B in A_1B to obtain ΔA_1CB, and reflect the latter in CB to obtain ΔA_3CB, so that $\measuredangle A_1BA_2 = \pi/3$. This proof holds in the Euclidean plane E^2, in the hyperbolic plane H^2, and on the sphere S^2. We use (1) to derive information about certain regular polytopes in n-space X^n, one of S^n, E^n, or H^n.

As described in [1, pp. 40-44], such polytopes can be constructed using the group G generated by reflections in the walls $H_1, H_2, \ldots, H_{n+1}$ of a simplex $\Lambda \subseteq X^n$. For any such simplex, let π/p_{ij} be the dihedral angle between H_i and H_j, where $p_{ij} \in (1, \infty]$ for $i \neq j$, and each $p_{ii} = 1$. H_i and H_j may be parallel, in which case $p_{ij} = \infty$ and $\pi/p_{ij} = 0$. Many properties of Λ (and G) are hidden in its Coxeter graph [2, pp. 84-86].

Suppose, for example, that $\measuredangle A_1 = \pi/p$ in (1); then $\Delta A_1A_2A_3$ is represented by the graph

(2) 

,

with a node for each side of the triangle. Two nodes are joined by an edge (labelled q) just when the corresponding angle $\pi/q \neq \pi/2$. Thus by (1), $\Delta A_1A_2A_3$ contains three congruent copies

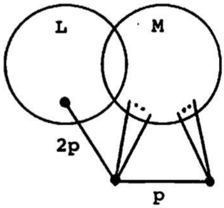
of 

, in which the label q = 3 is omitted and understood.

Similarly, a graph with n + 1 nodes describes the simplex $\Lambda \subseteq X^n$. Suppose that this graph has as a <u>subgraph</u> (2), whose nodes from left to right correspond to H_1, H_2, and H_3. Let $T = H_1 \cap H_2 \cap H_3$; in the 3-space T* completely orthogonal to T let σ be a "sphere" with centre $T \cap T^*$; σ could be a plane in E^n, or a horosphere or equidistant surface in H^n, but it has the intrinsic geometry of S^2, E^2, or H^2. Now H_1, H_2, and H_3 cut σ in a triangle with graph (2), so by (1), the bisector H of the dihedral angle between H_2 and H_3 meets H_1 at an angle $\pi/3$.

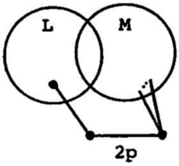
We assume hereafter that for all $j > 3$, H_j makes equal angles with H_2 and H_3. Thus H is perpendicular to H_j, $j > 3$, and Λ has the following graph,

(3)

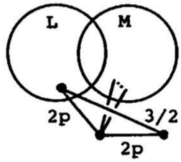


in which H_2 and H_3 make equal angles with those H_j whose nodes lie in the subgraph M, and are perpendicular to those $H_j (j \neq 1)$ whose nodes lie in L. By (1) and the above remarks, we obtain the following

(4) THEOREM. In the simplex Λ of (3), the bisector H of the angle between H_2 and H_3 cuts Λ into two simplexes with graphs

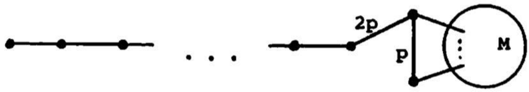


and



In one special case, we may apply the theorem once for $2p = 2 \cdot p$ and r times again, since $3 = 2 \cdot (3/2)$:

(5) COROLLARY. Suppose for the simplex Λ in (3) that L is an unmarked chain of r nodes:



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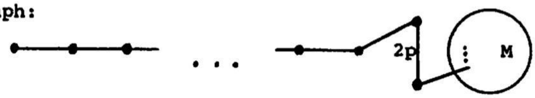
Then Λ contains r+2 (congruent) copies of the simplex Φ with graph:



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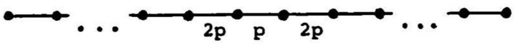
For $0 \le r \le n$, let $K(n+1, r;p)$ denote the graph

in which a chain L of r nodes is linked by a brancn marked "2p" to M, itself the complete graph on n+1-r nodes with each branch marked "p" (when $p \neq 2$). Let $\Lambda(n+1, r;p)$ denote the corresponding simplex; in particular, the simplex $\Lambda(n+1, n;p)$, with graph $K(n+1, n;p)$

,

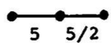
lies in S^n, E^n, or H^n according as $\cos \frac{\pi}{p}$ is <, =, or > n^{-1}; [cf. 2, p. 135, eqn. 7.77].

We now apply (5) first n-r times along the chain L in $\Lambda(n+1, r;p)$, then n-r-1 times within the subgraph $M = K(n+1-r, 0;p)$. $\Lambda(n+1, r;p)$ therefore contains $(n+1)!/(r+1)!$ congruent copies of $\Lambda(n+1, n;p)$, <u>and</u> $(n-r)!$ copies of the simplex $\Phi(n+1, r, p)$,

 ,

whose graph has its (r+1)st edge marked "p". This proves in some generality a remark made in [1, p. 210]:

(6) The simplex $\Phi(n+1, r, p)$ has $\binom{n+1}{r+1}$ times the volume of $\Lambda(n+1, n;p)$.

(7) <u>Examples</u>. (a) Since  has 3 times the area of , the regular star-polyhedron $\{5,\frac{5}{2}\}$ has <u>density</u> 3 (i.e. a general ray from its centre intersects 3 of its pentagonal faces) [cf. 2, pp. 100-113].

(b) The star-polytope $\{3,5,\frac{5}{2}\}$ has density 4; $\{5,\frac{5}{2},5\}$ has density 6 = $\binom{4}{2}$; and in H^4, $\{3,5,\frac{5}{2},5\}$ has density 10 = $\binom{5}{3}$.

(c) The absence of fractional labels in the case  = 6 ×  gives, instead of a star-polytope, a regular compound honeycomb in hyperbolic space H^3, i.e. 3{3,3,6} [6{6,3,6}] 3{6,3,3} , in which the vertices of six{6,3,6}'s, taken 3 at a time, are the vertices of a single {3,3,6}. This "3" indicates that of the six  's, three contain the vertex of  used in constructing {6,3,6} [cf. 2, pp. 47-50].

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A NOTE ON LAMBEK'S REPRESENTATION SHEAF

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Presented by M. A. Akcoglu, F.R.S.C.

In [3], Lambek showed that certain modules, called symmetric modules, can be represented as modules of sections of a sheaf. In this note we point out that, for a large class of rings, this sheaf is a subsheaf of an inverse image of a torsion-theoretic separated presheaf of the type introduced in [1].

Throughout the following, R will denote an associative ring with unit element 1. Modules and homomorphisms will be taken from the category R-mod of unitary left R-modules. The injective hull of a left R-module M will be denoted by $E(M)$. The complete brouwerian lattice of all (hereditary) torsion theories on R-mod will be denoted by R-tors. Notation and terminology will follow [1]. In particular, for a left R-module M we will denote by $\xi(M)$ the smallest torsion theory relative to which M is torsion and by $\chi(M)$ the largest torsion theory relative to which M is torsionfree. If $\tau \in$ R-tors then we denote the τ-torsion submodule of a left R-module by $T_\tau(M)$ and the localization of M at τ by $Q_\tau(M)$. A nonzero left R-module N is τ-<u>cocritical</u> if and only if it is τ-torsionfree while every proper homomorphic image of it is τ-torsion. The ring R is said to be <u>left definite</u> if and only if every nonzero left R-module has a cocritical submodule. Such rings were studied in [1] under the name of <u>left</u> 0-<u>rings</u>. A torsion theory is said to be <u>prime</u> if and only if it is of the form $\chi(M)$ for some cocritical left R-module M. The set of all prime torsion theories in R-tors is called the <u>left spectrum</u> of R and will be denoted by R-sp.

For any (two-sided) ideal I of R let pgen($\xi(R/I)$) denote the set of all elements of R-sp relative to which R/I is torsion. Then {pgen($\xi(R/I)$) |

I an ideal of R} forms the basis for a topology on R-sp which we will call the symmetric basic order topology. Denote by spec(R) the set of all prime ideals of the ring R. If R is a left definite ring then $\chi(R/P) \in$ R-sp for all $P \in$ spec(R) and so we have a function Φ:spec(R) → R-sp given by $P \mapsto \chi(R/P)$ which is easily seen to be monic [2, p. 20]. For any ideal I of R, let $\bar{U}(I) = \{P \in \text{spec}(R) \mid I \not\subseteq P\}$. The family of all such sets forms the basis for the Zariski topology on spec(R). If $P \in$ spec(R) and if I is an ideal of R then $P \in \bar{U}(I)$ if and only if $\chi(R/P) \in \text{pgen}(\xi(R/I))$. Thus we see that if R is a left definite ring then the function Φ is a continuous map from the space spec(R), endowed with the Zariski topology, to the space R-sp, endowed with the symmetric basic order topology.

Following [3], we say that a left R-module M is symmetric if and only if for all $m \in M$ and all $a,b \in R$ the relation $abm = 0$ implies $bam = 0$. If M is a symmetric left R-module then $(0:m)$ is an ideal of R for all $m \in M$. Moreover, if $P \in$ spec(R) then $T_{\chi(R/P)}(M) = \{m \in M \mid P \in \bar{U}((0:m))\}$. In [3] Lambek constructed a sheaf representation for symmetric left R-modules in the following manner: if M is such a module then let $E_0(M)$ be the disjoint union of all modules of the form $M/T_{\chi(R/P)}(M)$, for $P \in$ spec(R). Define the function $\psi_0:E_0(M) \to \text{spec}(R)$ by setting $\psi_0(x) = P$ if and only if $x \in M/T_{\chi(R/P)}(M)$. Each $m \in M$ defines a function $\hat{m}$: spec(R) → $E_0(M)$ given by $\hat{m}: P \mapsto m+T_{\chi(R/P)}(M) \in M/T_{\chi(R/P)}(M)$. Take as a basis for a topology on $E_0(M)$ all sets of the form $\hat{m}(\bar{U}(I))$, for $m \in M$ and I an ideal of R. Then $(E_0(M),\psi_0)$ is a sheaf space over spec(R) with corresponding sheaf of sections $\Gamma E_0(M)$.

If M is a left R-module then we can define a separated presheaf $\bar{Q}(\_,M)$ on R-sp, endowed with the symmetric basic order topology, by setting $\bar{Q}(U,M) = Q_{\wedge U}(M)$. See [1] for details of this construction. Now assume that R is left definite. For any left R-module M, we can define the <u>inverse image</u> $\Phi^{*}\bar{Q}(\_,M)$ of $\bar{Q}(\_,M)$ under the continous map Φ as in [4]: let $E_2(M)$ be the disjoint union of the modules $Q_\pi(M)$ for $\pi \in$ R-sp and define the function $\psi_2 : E_2(M) \to$ R-sp by setting $\psi_2(x) = \pi$ if and only if $x \in Q_\pi(M)$. Take as a basis for a topology on $E_2(M)$ those sets of the form $\{m+T_\pi(M) \mid \pi \in \mathrm{pgen}(\xi(R/I))\}$, for $m \in M$ and I an ideal of R. Then $\psi_2 : E_2(M) \to$ R-sp is a continuous function and we can form the pullback diagram of topological spaces

$$\begin{array}{ccc} E_1(M) & \xrightarrow{\quad\theta\quad} & E_2(M) \\ {\scriptstyle\psi_1}\downarrow & & \downarrow{\scriptstyle\psi_2} \\ \mathrm{spec}(R) & \xrightarrow{\quad\Phi\quad} & \text{R-sp} \end{array}$$

in which we note that θ is monic since Φ is. Moreover, $(E_1(M),\psi_1)$ is a sheaf space [4, p. 58]. The sheaf of sections of this sheaf space is what we will call $\Phi^{*}\bar{Q}(\_,M)$.

By construction, we see that $E_1(M) = \{(e,P) \in E_2(M) \times \mathrm{spec}(R) \mid \psi_2(e) = \Phi(P)\}$ and this is just the disjoint union of the modules $Q_{\chi(R/P)}(M)$ for $P \in \mathrm{spec}(R)$. Moreover, ψ_1 is given by $x \mapsto P$ if and only if $x \in Q_{\chi(R/P)}(M)$. Thus, if M is a symmetric left R-module, we have a canonical embedding of $E_0(M)$ into $E_1(M)$ which, for each $P \in \mathrm{spec}(R)$, sends $M/T_{\chi(R/P)}(M)$ into $Q_{\chi(R/P)}(M)$ canonically. Therefore we have shown the following:

THEOREM: <u>Let</u> R <u>be a left definite ring. If</u> M <u>is a symmetric left R-module then the Lambek representation sheaf of</u> M <u>is a subsheaf of</u> $\Phi^{*}\bar{Q}(\_,M)$.

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HIGHER ORDER MODULI OF CONTINUITY BASED ON THE JACOBI TRANSLATION OPERATOR AND BEST APPROXIMATION

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Presented by G. de B. Robinson, F.R.S.C.

If ω_1 is the first order modulus of continuity with respect to the Jacobi translation, then it was shown in a number of papers that it is equivalent to a suitable K-functional defined via the corresponding derivative. In this paper, firstly a long-standing problem is solved, namely the generalization to moduli of continuity ω_r of higher order, where ω_r is defined in terms of certain iterated differences. Secondly, applications are given to best approximation; in particular, the still open Jackson-type theorem with ω_r in the Jacobi frame is solved.

1. INTRODUCTION

Let $R_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n and order (α,β), $\alpha,\beta > -1$ (cf. [14, Chap. IV]). Let X denote either the space $L^p_{(\alpha,\beta)}(-1,1)$, $1 \leq p < \infty$, or $C[-1,1]$, endowed with the norms

$$\|f\|_{(\alpha,\beta),p} := \{\int_{-1}^{1} |f(u)|^p w_{(\alpha,\beta)}(u)du\}^{1/p}, \quad w_{(\alpha,\beta)}(u) := (1-u)^\alpha(1+u)^\beta,$$

$$\|f\|_C := \sup\{|f(u)|; u \in [-1,1]\}.$$

The index (α,β) is dropped if there is no misunderstanding.

With the Jacobi transform, given for $f \in X$ by

$$f^\wedge(k) := \int_{-1}^{1} f(u)R_k(u)w(u)du \qquad (k \in \mathbb{P} := \{0,1,2,\dots\}),$$

one can define a generalized translation operator τ_t by

$$(\tau_t f)^\wedge(k) := R_k(t)f^\wedge(k) \qquad (k \in \mathbb{P}; t \in (-1,1)).$$

It was shown in [1;8] that this unique translation is a bounded, linear operator from X into itself, satisfying [1)]

$$(1.1) \qquad \|\tau_t f\|_X \leq M\|f\|_X \qquad (f \in X; t \in (-1,1)),$$

$$(1.2) \qquad \lim_{t\to 1-} \|\tau_t f - f\|_X = 0 \qquad (f \in X)$$

1) Throughout, M, m denote positive constants which may be different at each occurrence. The dependence on any parameter apart from α,β is explitly indicated by the notation $M(a,b,\dots)$.

provided $\alpha>\beta>-1$, $\alpha+\beta>-1$, or $\alpha=\beta=-1/2$, to which the matter is restricted..

Finally the (strong) Jacobi derivative D^1f of $f\in X$ is defined by

(1.3) $$D^1 f := \lim_{h\to 1-} \frac{\Delta_h f}{1-h}, \quad \Delta_h f := f-\tau_h f,$$

the limit being understood in X-norm. Higher derivatives are defined inductively, $D^r f = D^1 D^{r-1} f$, $r=2,3,\ldots$ The existence of $D^r f$ can be characterized by the Jacobi transform of f. Indeed, one has the equivalence of the following assertions (cf. [11], or [13] for the case $\alpha=\beta=0$):

(1.4) $$f\in W_X^r := \{f\in X; D^r f \text{ exists as an element of } X\},$$

(1.5) there exists $g\in X$ such that

$$\left(\frac{k(k+\alpha+\beta+1)}{2(\alpha+1)}\right)^r f^\wedge(k) = g^\wedge(k) \qquad (k\in\mathbb{P}).$$

In this event, $(D^r f)(x) = g(x)$ (a.e.).

2. MODULI OF CONTINUITY AND THE K-FUNCTIONAL

The modulus of continuity of $f\in X$ is defined by

(2.1) $$\omega_1(\eta;f;X) := \sup\{\|\Delta_h f\|_X;\ \eta\leq h<1\} \qquad (0\leq\eta<1)$$

and the K-functional of $f\in X$ for the subspace W_X^r by

(2.2) $$K(\delta,f;X,W_X^r) := \inf\{\|f-g\|_X + \delta\|D^r g\|_X;\ g\in W_X^r\} \qquad (\delta>0).$$

For η,δ fixed, $\omega_1(\eta;\cdot;X)$ and $K(\delta,\cdot;X,W_X^r)$ are seminorms on X; they are equivalent. Indeed, it is known (cf. [2;11] or [13] for $\alpha=\beta=0$) that there are constants $m, M>0$ such that for all $f\in X$, $0\leq\eta<1$

(2.3) $$m\,\omega_1(\eta;f;X) \leq K(1-\eta,f;X,W_X^1) \leq M\,\omega_1(\eta;f;X).$$

The basic question now is how to define a modulus of continuity which is equivalent to (2.2) not only for $r=1$ but also for $r\geq 2$. The following turns out to be the right choice:

(2.4) $$\omega_r(\eta;f;X) := \sup\{\|\Delta_{h_1}\cdots\Delta_{h_r} f\|_X;\eta\leq h_j<1, j=1,2,\ldots,r\},$$

defined for $f\in X$, $r\in\mathbb{N}$ (= naturals), $0\leq\eta<1$. In contrast to the classical higher order modulus of continuity defined in terms of the iterated differences $\Delta_h^r f := \Delta_h^1\Delta_h^{r-1} f$ with the same increment h in each iteration, the h_j in (2.4) are now allowed to be different. As in [13] one can prove

Proposition 1. If $f, f_1, f_2 \in X$, $g \in W_X^s$, $s \in \mathbb{N}$, $0 \leq \eta < 1$, then

(2.5) $$\omega_r(\eta;f;X) \leq M(r)\|f\|_X, \quad \lim_{\eta \to 1-} \omega_r(\eta;f;X) = 0,$$

(2.6) $$\omega_r(\eta;f_1+f_2;X) \leq \omega_r(\eta;f_1;X) + \omega_r(\eta;f_2;X),$$

(2.7) $$\omega_r(\eta;g;X) \leq M(r) \begin{cases} (1-\eta)^s \omega_{r-s}(\eta;D^s g;X) & (1 \leq s \leq r-1) \\ (1-\eta)^r \|D^r g\|_X & (s = r). \end{cases}$$

The result indicated above now reads

Theorem 1. There exist constants $m(r)$, $M(r) > 0$ such that for all $f \in X$, $0 \leq \eta < 1$

$$m(r)\omega_r(\eta;f;X) \leq K((1-\eta)^r, f; X, W_X^r) \leq M(r)\omega_r(\eta;f;X).$$

This solves a problem posed explicitely in [5] and which would have been useful in earlier papers, e.g. [2;3;10;11;13]. The authors could not follow the proof of [6] in the matter (cf. [3, p. 265]).

3. BEST APPROXIMATION BY ALGEBRAIC POLYNOMIALS

Defining the best approximation of $f \in X$ by algebraic polynomials p_n of degree $\leq n$ by

$$E_n(f;X) := \inf_{p_n} \|f-p_n\|_X,$$

it was shown in [2;11] that for $s \in \mathbb{P}$, $0 < \sigma < 1$:

(3.1) $$E_n(f;X) = O(n^{-2s-2\sigma}) \qquad (n \to \infty)$$

$$\Leftrightarrow f \in W_X^s \text{ and } \omega_1(\eta;D^s f;X) = O((1-\eta)^\sigma) \qquad (\eta \to 1-).$$

Under the restriction $\sigma < 1$ the cases $E_n(f;X) = O(n^{-j})$ with j being an <u>even</u> integer would be excluded. But this situation is covered if one employs the ω_r of (2.4). Indeed, for any positive σ one has

Theorem 2. Let $f \in X$, $r, s \in \mathbb{P}$, $\sigma > 0$, $s + \sigma < r$. The following assertions are equivalent:

(i) $$E_n(f;X) = O(n^{-2s-2\sigma}) \qquad (n \to \infty),$$

(ii) $$\omega_r(\eta;f;X) = O((1-\eta)^{s+\sigma}) \qquad (\eta \to 1-),$$

(iii) $\quad f \in W_X^s$ and $\omega_{r-s}(\eta;D^s f;X) = O((1-\eta)^\sigma) \qquad (\eta \to 1-).$

To establish this theorem one makes use of the general Butzer - Scherer theorem on best approximation (see e.g. [4]) and of the following Jackson- and Bernstein-type inequalities.

Proposition 2. For each $r \in \mathbb{N}$ there exist constants $M(r)$ such that

(3.2) $$E_n(f;X) \le M(r)n^{-2r}\|D^r f\|_X \qquad (f \in W_X^r; n \in \mathbb{N}),$$

(3.3) $$\|D^r p_n\|_X \le M(r)n^{2r}\|p_n\|_X \qquad (n \in \mathbb{N}).$$

For a proof of (3.2) for $r = 1$ see [2;11], the general case follows like the particular cases $\alpha = \beta = -1/2$ or $\alpha = \beta = 0$ (see [4;13]). For (3.3) see [12].

From (3.2) and Thm. 1 one can also derive a Jackson-type theorem.

Corollary. There exists a constant $M(r) > 0$ such that for all $f \in X$, $n \in \mathbb{N}$

$$E_n(f;X) \le M(r)\omega_r(1-n^{-2};f;X) .$$

Note that the moduli of continuity and the derivatives $D^s f$ depend on the translation operator τ_t, thus on the order (α,β) of the underlying space. So in case $X = L^1_{(\alpha,\beta)}(-1,1)$ also the norms depend on (α,β), and so does $E_n(f;L^p_{(\alpha,\beta)})$. However, if $X = C[-1,1]$, then the definition of $E_n(f;C[-1,1])$ is independent of (α,β). By Thm. 2 this implies that the statements

(3.4) $$\omega_r(\eta;f;C[-1,1]) = O((1-\eta)^\tau) \qquad (\eta \to 1-)$$

do not depend on the translation operator provided $0 < \tau < r$. This fact even enables one to characterize (3.4), defined in terms of Jacobi - differences, by a (pointwise or local) modulus of continuity with respect to the classical difference of order 2r. Indeed, defining

$$\Omega_{2r}(\delta;x;f;C[-1,1]) := \sup\{|\sum_{j=0}^{2r} (-1)^j \binom{2r}{j} f(x+jh)|; |h| \le \delta\}$$

where the supremum is taken for fixed $x \in [-1,1]$ in such a way that $x + rh \in [-1,1]$, one has

Theorem 3. The three assertions of Thm. 2 are equivalent to

(iv) $$\Omega_{2r}(\delta;x;f;C[-1,1]) = O\left(\left(\frac{\delta^2}{1-x^2}\right)^{s+\sigma}\right) \qquad (\delta \to 0+; x \in (-1,1)),$$

the large - O constant being independent of δ and x.

Let us compare this theorem in the case $r = 1$ with results of Dzjadik [7], Teljakovskiĭ [14] and Gopengauz [9], namely

$$|f(x) - p_n(x)| = O\left(\left(\frac{1-x^2}{n^2}\right)^{\sigma}\right) \qquad (n \to \infty)$$

$$\Leftrightarrow \Omega_2(\delta;x;f;C[-1,1]) = O(\delta^{2\sigma}) \qquad (\delta \to 0+)$$

for a suitable sequence of polynomials p_n and $0 < \sigma < 1$. This result shows that the factor $(1-x^2)$ has to occur either in the order of approximation (of f by p_n) or of the modulus condition.

It should be mentioned that the results of this note could be proved in the more general frame of generalized translations in the sense of [10].

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AN ANALYTIC-ALGEBRAIC APPROACH TO STATISTICAL MODELS AND INFERENCE

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This paper reports on current research in which an analytical-physical approach is taken to the definition of the statistical model and to the content of the inference base, given by the model and data together. The paper also presents some integrating results that follow from this.

The primitive statistical model with virtually no background information (except the existence of probabilities for events A on the sample space S) may be expressed $M = \{(S,A,P) : P \in P\}$ where the knowledge about the system being investigated is summarized by the collection P of possible candidates for the true probability measure P_*. With this we have recourse only to the knowledge that the empirical relative frequency of any event A in A converges to the true probability $P_*(A)$ of that event. The preceding formalizes the minimum context referred to as a random system.

Likelihood. As an example of the analytical approach, consider the following (as in Fraser [3, pp. 70, 98]). Let (S,A,μ) be the sample space S (an open subset of $\mathbb{R}^k$), the

Borel class A of events, and a support measure μ (Borel measure or an equivalent). With this the conventional statistical model is $M = \{f(\cdot \mid \theta) : \theta \in \Omega\}$, a class of continuous density functions on S. The inference problem is to assess what the model M and the data y say concerning the unknown θ_* in Ω.

For a data point y the model specifies the probability $f(y \mid \cdot)\mu(dy)$ of its occurrence, a function on Ω. As probability at a point is zero and the size of reference neighbourhood is arbitrary, the model provides only $L(y) = \{cf(y \mid \cdot) : c \in \mathbb{R}^+\}$, a ray in $\mathbb{R}^\Omega$; this is the likelihood function. The preimage of the mapping $L(\cdot)$ from S to $\mathbb{R}^\Omega/\mathbb{R}^+$ is a partition of S into sets of points that are equivalent relevant to the model. This reduction by eliminating the arbitrary is the same as that obtained in a weaker sense by invoking the sufficiency principle.

Inference Base. The analytic approach focuses initially on the requirements for the statistical model. Criteria for the model M are organized in Fraser [3, p. 3]: the purpose is to describe unknowns; the model contains real components relative to the investigation (objective); and components of the investigation are included in the model (comprehensive). This leads to the inference base $T = (M,y)$, and the problem is to determine what can be said concerning the unknowns in the base T without the inclusion of arbitrary elements.

Event Structure. Clear examples do exist where the statistical model has a distinct component, an objective probability space (S_0, A_0, P) or (S_0, A_0, μ, p) where P denotes a measure and p a density with respect to a support measure μ. The use of probability and conditional probability focuses on the nature of the information available from a data point y concerning the realized value on the probability space. We have examined the following special case in detail: $M = (S, A, \mu, p; \Phi)$ where $\Phi = \{\phi\}$ is a class of bijections $S \to S$. The response variable satisfies $y = \phi z$ where ϕ is the unknown presentation in Φ and z is the realized value on the probability space. The information from a data point y concerning a realized z can be formalized as the B-orbit of an information display: let B be the group of bijections on the response space S, and $D(\Phi,y) = \{(\phi, \phi^{-1}y) : \phi \in \Phi\}$ be the set of preimage values for z labelled by the possible presentations ϕ; then $I(\Phi,y) = BD(\Phi,y) = \{D(s\Phi, sy) : s \in B\}$ gives this information display.

Proposition. The preimage partition of the information function is the orbit space S/G, where G is the invariant group $\{s : s\Phi = \Phi, s \in B\}$.

The information obtained from a value z is given by $I(\Phi, \phi z)$ where ϕ is the unknown presentation. This is a function $S \to \text{im } I(\Phi,\cdot)$ if and only if $S/G = T$ where T consists of sets $\cup\Phi\Phi^{-1}y$. For this to hold under repeated sampling with an effective point we have that $\Phi = G\phi_0$ or with a relabelled ϕ_0 we have that the new $\Phi = G$ is a group. This is then a structural model as examined in Fraser [1,3].

Identified Form. The preceding involved an objective probability space as a component of the model. Some examples of this have been discussed in Fraser [1,3] and some risks inherent in the lack of objectiveness of the probability space [2]. We report now on some current research on the identification of distribution form. The starting point is the background information concerning a system under investigation and the examination of this by a class of bijective functions $T = \{t\}$ which allow each of the possible distribution forms to be examined on a space $\mathcal{V}e$. Three different definitions are given for a platform of functions. In each case the requirement that T be a platform is a necessary and sufficient condition for a nominal distribution form to be objective.

The preceding discussion gives the grounds for an objective probability space to be a component of a statistical model. This should not be confused with another direction on statistical inference involving the addition of a pivotal quantity as originally promoted in the writings of Fisher. The approach of this article is that such an addition is arbitrary. The objective distribution form discussed above contains functions that are formally inverses of pivotals but the resemblance is only formal. The group closure property was fundamental and obvious examples show that without this closure contradictions can occur.

Unifications. The research currently in progress is based on the preceding results and involves a direct analytical examination of the inference base and the separation of the types of information available – categorical, frequency and diffuse. Some preliminaries on the separation of categorical information may be found in Fraser [3, p. 49]. In a second direction the event information concerning realized values leads to the automatic conditioning found with identified form and structural models. The present indication is that the four necessary reduction methods [3, pp. 49, 68] will evolve as a simple consistent categorical separation of the inference base.

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ON CYCLIC BLOCK DESIGNS

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Presented by N. S. Mendelsohn, F.R.S.C.

Abstract: Two recursive constructions for cyclic block designs are presented.

1. Introduction

Combinatorial design theory [5,9] is largely concerned at present with existence problems for block designs and related configurations. A (balanced incomplete) block design $B[k,\lambda;v]$ is a pair (V,B); V is a v-set and B is a collection of k-subsets of V (called blocks) for which each 2-subset of V appears in precisely λ blocks. A cyclic block design $CB[k,\lambda;v]$ is a $B[k,\lambda;v]$ whose automorphism group contains a v-cycle. $CB(k,\lambda)$ denotes the set $\{v \mid$ a $CB[k,\lambda;v]$ exists$\}$.

The determination of $CB(k,\lambda)$ for various k and λ has been investigated since the 1890's [6,7]. In 1939, $CB(3,1)$ was determined by Peltesohn [8], and recently $CB(3,\lambda)$ has been completely determined [2]. Results on $CB(k,\lambda)$ for $k > 3$ are few and far between [3,10].

Our purpose is to present, without proof, two recursive constructions for cyclic block designs. We assume familiarity with the usual representation of cyclic block designs in terms of starter blocks [1,3,4]; we call a starter block full if it generates v blocks, extra (or short) if it generates v/k blocks. (A cyclic block design contains an extra starter block if and only if $v \equiv 0 \pmod k$).

2. First Construction

Given that $v \not\equiv 0 \pmod{k}$, $v \in CB(k,\lambda)$, $m \in CB(k,\lambda)$, and m is relatively prime to $(k-1)!$, then $vm \in CB(k,\lambda)$:

Construction A:

Let $\{S_1, \ldots, S_n\}$ be the set of starter blocks for a $CB[k,\lambda;v]$, $v \not\equiv 0 \pmod{k}$. Let $\{T_1, \ldots, T_r\}$ be the set of starter blocks for a $CB[k,\lambda;m]$, with m relatively prime to $(k-1)!$. The set of starter blocks for a $CB[k,\lambda;mv]$ is constructed as follows. For each $S_j = \{0,s_1,\ldots,s_{k-1}\}$, take the m starter blocks $\{0,s_1+iv,s_2+2iv,\ldots,s_{k-1}+(k-1)iv\}$, $0 \le i < m$, arithmetic modulo mv. Finally, for each $T_j = \{0,t_1,\ldots,t_{k-1}\}$, take the single starter block $\{0,vt_1,\ldots,vt_{k-1}\}$.

3. Second Construction

Given that $kv \in CB(k,1)$, $km \in CB(k,1)$, and m is relatively prime to $(k-1)!$, then $kmv \in CB(k,1)$:

Construction B:

Let $\{S_1, \ldots, S_n\}$ be the set of full starter blocks for a $CB[k,1;kv]$. Let $\{T_1, \ldots, T_r\}$ be the set of starter blocks for a $CB[k,1;km]$, with m relatively prime to $(k-1)!$. We construct the set of starter blocks for a $CB[k,1;kmv]$. For each $S_j = \{0,s_1,\ldots,s_{k-1}\}$, take the m starter blocks $\{0,s_1+ikv,s_2+2ikv,\ldots,s_{k-1}+(k-1)ikv\}$, $0 \le i < m$, arithmetic modulo kmv. Finally, for each $T_j = \{0,t_1,\ldots,t_{k-1}\}$, take the single starter block $\{0,vt_1,\ldots,vt_{k-1}\}$.

4. Conclusions

Previous constructions for cyclic block designs have all been *direct*; the recursive constructions given here are important for two reasons. Firstly, they provide many infinite families of cyclic block designs which were heretofore unknown (and, moreover, are not limited to small block sizes k). Secondly, they suggest the possibility that the existence of CB[k,λ;v] may be resolved by recursive techniques.

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On dihedral coverings of S^3

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It is well known [4] [5] that every closed orientable 3-manifold M^3 is a 3-fold irregular covering space of S^3 branched along some knot K. (We say simply that M^3 is a 3-fold irregular covering space of K.) However, the knot K will be very complicated even for a "simple" 3-manifold M^3. In fact, K is not knotted enough, the manifold M^3 obtained from K is quite simple. For example, if K is a 2-bridge knot, then M^3 is always a 3-sphere S^3 [3].

With a different motivation, in this paper, we study 3-fold irregular covering space of closed 3-braids and prove the following

Theorem. A 3-fold irregular covering space of S^3 branched along a closed 3-braid is a lens space L_n of type (n,1) for some integer n.

Remark. $L_0 \approx S^2 \times S^1$, $L_1 \approx S^3$ and L_2 is a projective 3-space.

Although n is an invariant of the knot type, it seems hard to determine n from a braid representation of a knot.

Outline of the proof

Let B_n be the group of n-string braids. (See [1] or [2] for the general discussion on B_n.) To each braid β with n strings, it is assigned a permutation of n letters, and this assignment is, in fact, a homomorphism ν from B_n to S_n, the symmetric group of order $n!$. An element of B_n that belongs to the kernel of ν is called a pure n-braid. From an n-braid β, we can construct a link in S^3, called the closed braid $\hat{\beta}$ of β, by identifying the initial points and end points of each of the braid strings.

Lemma 1. Let $M(\hat{\beta})$ denote the 3-fold irregular covering space of a closed 3-braid $\hat{\beta}$. Then there exists a pure 3-braid γ such that (1) the 3-fold irregular covering space $M(\hat{\gamma})$ of the closed braid $\hat{\gamma}$ is defined, and (2) $M(\hat{\gamma})$ is homeomorphic to $M(\hat{\beta})$.

Hence, we need only consider pure 3-braids.

Let γ be a pure 3-braid. Then $\pi_1(S^3 - \hat{\gamma})$ is generated by three Wirtinger generators x_1, x_2, x_3, each of which corresponds to a braid string.

Lemma 2. Suppose that there is a homomorphism ϕ from $\pi_1(S^3 - \hat{\gamma})$ onto D_3, the dihedral group of order 6. Then there exists a pure 3-braid γ_0 such that (1) $\phi(x_1)$, $\phi(x_2)$, $\phi(x_3)$ are all distinct, and (2) $M(\hat{\gamma}_0)$ is homeomorphic to $M(\hat{\gamma})$.

Now, to construct a 3-fold irregular covering space of $\hat{\gamma}_0$, we divide S^3 into two 3-cells C_1 and C_2 in such a way that (1) $\partial C_1 \cap \hat{\gamma}_0 = \partial C_2 \cap \hat{\gamma}_0$ consists of exactly six points, (2) $C_1 \cap \hat{\gamma}_0$ is isotopic to a pure 3-braid γ_0 and (3) $C_2 \cap \hat{\gamma}_0$ is isotopic to a trivial 3-braid ε. Then $M(\hat{\gamma}_0)$ is the union of 3-fold irregular covering spaces $\tilde{C}_1$ and $\tilde{C}_2$ of C_1 and C_2 branched along γ and ε, respectively. Since $\tilde{C}_1$ and $\tilde{C}_2$ are solid tori, $M(\hat{\gamma}_0)$ is a lens space of type (n,m) for some integers n, m.

To obtain more precise information on $M(\hat{\gamma}_0)$, we must consider the image of a meridian of $\tilde{C}_1$ under a "matching" homeomorphism ψ from $\partial\tilde{C}_1$ onto $\partial\tilde{C}_2$. Now, there are three mutually disjoint disks D_i $(i=1,2,3)$ in C_1, each of which bounds a braid string of γ_0. By the property (1) of γ_0 in Lemma 2, one of the three lifts, $\tilde{D}_i$, of D_i in $\tilde{C}_1$ becomes a meridian disk of $\tilde{C}_1$ and hence ∂D_i is a meridian of $\tilde{C}_1$. Since γ_0 is a 3-braid, it is not hard to see that $\partial\tilde{D}_i$ intersects only once a longitude of $\tilde{C}_2$. Therefore, $\psi(\partial\tilde{D}_i)$ represents a torus knot of type $(n,1)$ for some n, and thus $M(\hat{\gamma}_0)$ is a lens space of type $(n,1)$. This proves the theorem.

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