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SUR LES DEVIATIONS D'UN ANNEAU LOCAL

par

C. Morgenegg

Presented by P. Ribenboim, F.R.S.C.

1. ALGEBRES DIFFERENTIELLES

Soient A un anneau local, M son idéal maximal et B son corps résiduel. Une résolution X de la A-algèbre B consiste en une DGA-algèbre acyclique. Elle s'obtient comme limite inductive d'algèbres différentielles X^n construites selon la procédure décrite par Tate (6). L'algèbre X est munie de puissances divisées.

Sous réserve de la survie des bords, cf. (3), cette construction livre une résolution minimale, ie $dX \leq MX$, et le B-module gradué $\mathrm{Tor}^A(B,B)$ est muni d'une structure naturelle de Γ-algèbre. Considérons les indécomposables de cette algèbre, $Q\,\mathrm{Tor}^A(B,B)$, nous appellerons <u>n-ème déviation</u> de l'anneau local A la dimension ε_n de l'espace vectoriel $Q_n\mathrm{Tor}^A(B,B)$, contrairement à l'usage. D'après (3), elle se calcule comme suit:

<u>Proposition 1.1</u>. Les espaces vectoriels $H_n(X^n)$ et $Q_{n+1}\mathrm{Tor}^A(B,B)$ sont isomorphes.

<u>Proposition 1.2</u>. Il existe un isomorphisme naturel entre les espaces vectoriels $H_n(X^{n-1})$ et $H_n(X^n)$ pour $n \geq 3$ et un épimorphisme naturel de $H_n(X^{n-2})$ sur $H_n(X^{n-1})$ pour $n \geq 4$.

La démonstration de cette assertion et celle des assertions suivantes de cet article se trouvent dans le travail (4).

2. THEORIE HOMOLOGIQUE SIMPLICIALE

Une <u>résolution simpliciale</u> R_* de la A-algèbre B consiste en une A-algèbre simpliciale libre en tout degré, augmentée et acyclique, qui donne lieu a un isomorphisme $H_0(R_*) \cong B$. Considérons l'idéal d'augmentation I_* de la B-algèbre simpliciale $R_* \otimes_A B$ sur l'algèbre simpliciale triviale B, puis le <u>complexe cotangent</u> I_*/I_*^2. L'homologie $H(A,B,B)$ de la A-algèbre B est un B-module gradué, égal en degré n à l'espace vectoriel $H_n(I_*/I_*^2)$ et la dimension δ_n de cet espace vectoriel est appelée <u>n-ème déviation simpliciale</u> de l'anneau local A.

S'inspirant d'une méthode propre a la topologie algèbrique, on peut construire une résolution simpliciale R_* en tuant successivement des modules d'homotopie, cf. chap. IX de (1). L'algèbre simpliciale R_*^n, obtenue après n pas dans cette construction, est formée a partir de la précédente en adjoignant des variables polynomiales en le degré n. Elle apparait dans le calcul des déviations δ_n de manière explicite.

<u>Proposition 2.1.</u> Les espaces vectoriels $H_n(R_*^n)$ et $H_{n+1}(A,B,B)$ sont isomorphes.

<u>Proposition 2.2.</u> Les inclusions simpliciales $R_*^{n-2} \subseteq R_*^{n-1} \subseteq R_*^n$ induisent un isomorphisme naturel entre les espaces $H_n(R_*^{n-1})$ et $H_n(R_*^n)$ pour $n \geq 3$, et un épimorphisme naturel de $H_n(R_*^{n-2})$ sur $H_n(R_*^{n-1})$ pour $n \geq 4$.

Si le corps B est de caractéristique 0, il est connu que les déviations ε_n et δ_n sont égales en tout degré n. L'égalite subsiste pour les degrés $n \leqslant 2p$ en caractéristique $p > 0$, cf. (5) ou (2).

Les propositions précédentes sont établies indépendamment de la caracté - ristique, voici ce qu'elles donnent en caractéristique 2 .

3.APPLICATION A LA CARACTERISTIQUE 2

Relevons d'abord que l'homologie $H(R_*^n)$ d'une algèbre R_*^n est munie d' une structure naturelle de Γ -algèbre induite par la structure simpliciale de R_*^n . Par construction de l'algèbre différentielle X^n ,l'homologie $H(X^n)$ est aussi une Γ-algèbre. Il vient alors,

$$\begin{aligned} Q_5\mathrm{Tor}^A(B,B) &\cong H_4(X^2) \,/\, H_2(X^2)^2 \;, \\ H_5(A,B,B) &\cong H_4(R_*^2) \,/\, (\, H_2(R_*^2)^2 + \mathrm{Im}\,\gamma^2 \,) \;. \end{aligned}$$

Ceci conduit a un critère d'isomorphisme entre les espaces $Q_5\mathrm{Tor}^A(B,B)$ et $H_5(A,B,B)$ en termes du complexe de Koszul X^1 de la A-algèbre B . Ce critère repose sur l'isomorphisme naturel de Γ -algèbres entre $H(X^1)$ et $H(R_*^1)$, et sur le diagramme commutatif suivant,

$$\begin{tikzcd} H_2(R_*^1) \arrow[r, "\gamma^2"] \arrow[d, "\alpha"'] & H_4(R_*^1) \arrow[r, "pr."] \arrow[d, "\beta"'] & H_4(R_*^1) \,/\, H_2(R_*^1)^2 \arrow[d] \\ H_2(R_*^2) \arrow[r, "\gamma^2"] & H_4(R_*^2) \arrow[r, "pr."] & H_4(R_*^2) \,/\, H_2(R_*^2)^2 \end{tikzcd} \;,$$

ou α et β sont les homomorphismes induits par l'inclusion $R_*^1 \subsetneq R_*^2$. Désignons alors par H_i les modules d'homologie du complexe de Koszul X^1 et par ω_k des représentants d'une base de l'espace H_1 . Nous avons le résultat suivant.

<u>Proposition 3.1.</u> Les espaces vectoriels $Q_5\mathrm{Tor}^A(B,B)$ et $H_5(A,B,B)$ sont isomorphes si et seulement si les carrés divisés de H_2 sont contenus dans le sous-espace $\mathrm{Ker}\,\beta + H_2 \cdot H_2$ de H_4. L'espace $\mathrm{Ker}\,\beta$ est engendré par les classes des cycles de la forme,

$$\zeta = \Sigma\, \tau_k \cdot \omega_k \quad , \quad 1 \leqslant k \leqslant \dim H_1 \; .$$

Pour tout indice k, les éléments τ_k de X_3^1 vérifient une équation du type

$$\Sigma\, a_{ijk}\, \omega_i \cdot \omega_j = d\tau_k \quad , \quad 1 \leqslant i,j \leqslant \dim H_1 \quad ,$$

a coefficients symétriques $a_{ijk} = a_{ikj}$ dans l'anneau A .

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Algebraic Lattices

J.H.H. Chalk, F.R.S.C.

1. Let V be a real vector space $\cong \mathbb{R}^N$. A distance function on V is a continuous mapping $f : V \to \mathbb{R}$ such that

(a) $f(\underline{x}) \geq 0$ for all $\underline{x} \in V$ and $f(\underline{x}) > 0$ for some $x \in V$

(b) $f(t\underline{x}) = |t| f(\underline{x})$ for all $\underline{x} \in V$ and for all $t \in \mathbb{R}$.

For any $M > 0$, the open set defined by $f(\underline{x}) < M$ is called a star body; it is clearly symmetric in the origin. A star body is a convex star body if, further,

(c) $f(\underline{x}+\underline{x}') \leq f(\underline{x}) + f(\underline{x}')$ for all $\underline{x}, \underline{x}'$ in V.

If f satisfies (a), (b) and (c), it is called a gauge function (cf. [4]). Suppose henceforth that all star bodies considered are bounded, (i.e. $f(\underline{x}) > 0$ if $\underline{x} \neq \underline{0}$). If S is a star body, let λS $(\lambda > 0)$ denote the dilatation of S in the ratio $\lambda : 1$ (i.e. $\lambda S : f(\underline{x}) < \lambda M$ if $S : f(\underline{x}) < M$).

Let L be an algebraic number field of degree $[L : Q] = n \geq 1$ and select an order $\mathcal{O}$ in L. Then, by definition, $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank n with ring structure and containing 1 (and so $\mathbb{Z}$ itself). Fix a $\mathbb{Z}$-basis $\rho_1 = 1, \rho_2, \ldots, \rho_n$ for $\mathcal{O}$ and let

(1) $$\underline{x}^{(r)} = \rho_1^{(r)} \underline{z}_1 + \cdots + \rho_n^{(r)} \underline{z}_n, \quad (\underline{z}_k \in \mathbb{R}^m, \quad 1 \leq k \leq n)$$

where, for any $\alpha \in L$, we let $\alpha^{(r)}$ denote the Q-conjugate of $\alpha = \alpha^{(1)}$ in the field $L^{(r)}$ conjugate to $L = L^{(1)}$ over Q $(1 \leq r \leq n)$. For $r = 1$, we shall often drop the superscripts. Let

(2) $$d = |\det(\rho_s^{(r)})|^2;$$

denote the absolute value of the discriminant of $\mathcal{O}$.

Let $V = V_N$ denote the real N-dimensional vector space of points

(3) $$\underset{\sim}{X} = (\underset{\sim}{x}^{(1)}, \underset{\sim}{x}^{(2)}, \ldots, \underset{\sim}{x}^{(n)})$$

where $N = mn$ and $\underset{\sim}{x}^{(r)}$ $(1 \le r \le n)$ is as defined in (1). Observe that all elements of the $\mathcal{O}$-module $M = \mathcal{O}^m$ are given by (1) with $r = 1$ as $\underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n$ run through those of $\mathbb{Z}^m$; the corresponding elements $\underset{\sim}{x}^{(r)}$ of the conjugate module $M^{(r)}$ being determined by (1) in the same way.

__Def.__ (i) The integer algebraic lattice $\tilde{\Lambda}_0$ in V_N is the set of all $\underset{\sim}{X}$ with $\underset{\sim}{z}_k \in \mathbb{Z}^m$ $(1 \le k \le n)$.

(ii) An algebraic lattice $\tilde{\Lambda}$ is a linear transform of $\tilde{\Lambda}_0$ of the type:

$$\underset{\sim}{x}'^{(r)} = \underset{\sim}{x}^{(r)} B^{(r)}, \quad (1 \le r \le n)$$

where $B^{(r)}$ is an $m \times m$ matrix composed of entries $b^{(r)}$ of the type

$$b^{(r)} = \rho_1^{(r)}\beta_1 + \cdots + \rho_n^{(r)}\beta_n, \; (\beta_i \in \mathbb{R}, \quad 1 \le i \le n)$$

and such that

$$\Delta = \left| \prod_{1 \le r \le n} \det B^{(r)} \right| \ne 0.$$

2. We shall suppose first that L is totally real (i.e. each of the n conjugate fields $L^{(r)}$ is composed of real numbers) and indicate the modifications for the general case later. Then, if $m = 1$, $\tilde{\Lambda}_0$ is a geometric lattice Λ_1 in $\mathbb{R}^n$ (in the sense of Minkowski, [2]) of determinant $d(\Lambda_1) = d^{\frac{1}{2}}$. If $m > 1$, then $\tilde{\Lambda}_0$ is the cartesian product Λ_1^m of m copies of Λ_1 and so $\tilde{\Lambda}_0$ and consequently $\tilde{\Lambda}$ are geometric lattices in $\mathbb{R}^n$ with determinants

(4) $$d(\tilde{\Lambda}_0) = d^{m/2} \text{ and } d(\tilde{\Lambda}) = \Delta d(\tilde{\Lambda}_0),$$

respectively. Following H. Weyl ($[4]_1$, p. 140, $[4]_2$, p. 212) we shall define the successive minima of an arbitrary bounded star body $S : f(\underset{\sim}{X}) < 1$ relative to $\tilde{\Lambda}$. There is then no loss of generality in specializing $\tilde{\Lambda}$ to $\tilde{\Lambda}_0$ (cf. §3(a)).

Procedure: For $1 \le k \le m$, let $E_k = E_k$ $(\underset{\sim}{\delta}_1, \ldots, \underset{\sim}{\delta}_k)$ denote the vector space over K spanned by elements $\underline{\delta}_1, \ldots, \underline{\delta}_k$ of the $\mathcal{O}$-module $M = \mathcal{O}^m$ and put $E_0 = \{\underline{0}\}$. Then, given E_{k-1}, we may select an element $\underline{\delta}_k$ of M with $\underline{\delta}_k \notin E_{k-1}$ for which $f(\underline{X}_k)$ is minimal, where

$$\underline{X}_k = \left(\underline{\delta}_k, \underline{\delta}_k^{(2)}, \ldots, \underline{\delta}_k^{(n)}\right) \in \tilde{\Lambda}_0 ;$$

the bound being attained since $\lambda S(\lambda > 0)$ is a bounded set and $\tilde{\Lambda}_0$ is discrete. Then $f(\underline{X}) \geq f(\underline{X}_k)$ for all $\underline{X} = \left(\underline{x}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}\right) \in \tilde{\Lambda}_0$ with $\underline{x} \notin E_{k-1}$ $(\underline{x} \in M)$. Define $E_k = E_k(\underline{\delta}_1, \ldots, \underline{\delta}_{k-1}, \underline{\delta}_k)$ and set

$$\mu_k = f(\underline{X}_k). \tag{5}$$

Thus, if $0 < \mu < \mu_k$, the star body μS cannot contain k points $\underline{X}_1', \ldots, \underline{X}_k'$ of $\tilde{\Lambda}_0$ with $\underline{x}_1', \ldots, \underline{x}_k'$ linearly independent over $\mathcal{O}$, but, if $\mu > \mu_k$ then it contains at least k such points. In particular, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ are uniquely determined. Our purpose is to establish the following theorem on bounded convex star bodies in V_N from the classical theorem (our special case $n = 1$) of Minkowski on successive minima ([2]).

<u>THEOREM</u>. <u>If</u> K <u>is a convex star body with volume</u> $V(K)$, <u>then</u>

$$(\mu_1 \mu_2 \cdots \mu_m)^n V(K) \leq 2^N d(\tilde{\Lambda}_0) \qquad (N = mn). \tag{6}$$

The deduction is made possible by a remark of Weyl, which permits a restatement of Minkowski's theorem in the following form. "If $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote any positive numbers such that $\dim\{\underline{x} \in \Lambda : f(\underline{x}) < \lambda_k\} < k$, $(1 \leq k \leq m)$ where $\underline{x} \in \mathbb{R}^m$, f is a gauge function on $\mathbb{R}^m$ and Λ is a geometric lattice $\subset \mathbb{R}^m$ of determinant $d(\Lambda) \neq 0$, then

$$\lambda_1 \lambda_2 \cdots \lambda_m V(K) \leq 2^m d(\Lambda); \tag{7}$$

$V(K)$ denoting the volume of $K : f(\underline{x}) < 1$". We proceed to the proof.

<u>Proof of the Theorem</u>. If $\underline{X} = \left(\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}\right) \in \tilde{\Lambda}_0$, we define n associated points of $\tilde{\Lambda}_0$ by $\underline{X}_j = \left(\rho_j^{(1)} \underline{x}^{(1)}, \ldots, \rho_j^{(n)} \underline{x}^{(n)}\right)$, $(1 \leq j \leq n)$ with $\underline{X} = \underline{X}_1$ (since $\rho_1 = 1$). Now suppose that $\underline{X} \in \mu K$, where $0 < \mu < \mu_k$.

Then $\underline{x} = \underline{x}^{(1)} = \nu_1\underline{\delta}_1 + \cdots + \nu_{k-1}\underline{\delta}_{k-1}$ $(\nu_i \in \mathcal{O})$, by definition of μ_k. Hence

$$\underline{X} = \left(\underline{x}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}\right) = \left(\nu_1\underline{\delta}_1 + \cdots + \nu_{k-1}\underline{\delta}_{k-1}, \ldots, \nu_1^{(n)}\underline{\delta}_1^{(n)} + \cdots + \nu_{k-1}^{(n)}\underline{\delta}_{k-1}^{(n)}\right)$$
$$= \left(\nu_1\underline{\delta}_1, \ldots, \nu_1^{(n)}\underline{\delta}_1^{(n)}\right) + \cdots + \left(\nu_{k-1}\underline{\delta}_{k-1}, \ldots, \nu_{k-1}^{(n)}\underline{\delta}_{k-1}^{(n)}\right).$$

On writing

$$\nu_1 = \sum_{1\le i\le n} \rho_i z_{i1}, \ldots, \nu_{k-1} = \sum_{1\le i\le n} \rho_i z_{i,k-1} \quad (z_{ij} \in \mathbb{Z})$$

as we may, since $\nu_j \in \mathcal{O}$ $(1 \le j \le k-1)$, we have

$$\underline{X} = \sum_{1\le i\le n} z_{i1} \underline{X}_{i1} + \cdots + \sum_{1\le i\le n} z_{i,k-1} \underline{X}_{i,k-1},$$

where the $\underline{X}_{ij}$ $(1 \le i \le n)$ are the n associates of $\underline{X}_j$, for each $j \le k-1$. Hence $\underline{X}$ is a linear combination of the $n(k-1)$ points $\underline{X}_{ij}$ of $\tilde{\Lambda}_0$ with coefficients in $\mathbb{Z}$; in fact, $\dim\left\{\underline{x} \in \tilde{\Lambda}_0 : f(\underline{x}) < \mu_k\right\} \le n(k-1) < nk$ and so

$$\dim\left\{\underline{x} \in \tilde{\Lambda}_0 : f(\underline{x}) < \lambda_{n(k-1)+\ell}\right\} \le n(k-1) < n(k-1) + \ell \quad (\ell = 1, 2, \ldots, n).$$

Thus, by the remark of Weyl's, we may <u>take</u> $\lambda_{nk-n+1} = \cdots = \lambda_{nk-1} = \lambda_{nk} = \mu_k$ and so, by (7) with m replaced by N and Λ by $\tilde{\Lambda}_0$,

$$(\mu_1 \cdots \mu_m)^n V(K) \le 2^N d(\tilde{\Lambda}_0) = 2^N d^{m/2}.$$

A weaker estimate for a non-convex star body S can be deduced easily (by choosing a suitable convex star body K with $K \subset S \subset c_1(S)K$) and is stated, without proof, as a Corollary.

<u>COROLLARY</u>. If S is a non-convex star body, there exists a constant $c(S) > 0$ such that

$$(\mu_1 \cdots \mu_m)^n c(S) \leqq d(\tilde{\Lambda}_0).$$

3. <u>MODIFICATIONS</u>. (a) Let Ω denote the non-singular linear transformation of V_N represented by the block matrix $\mathrm{diag}\left(B^{(1)}, \ldots, B^{(n)}\right)$ with $|\det \Omega| = \Delta \ne 0$. Then $\tilde{\Lambda} = \Omega\tilde{\Lambda}_0$ is an algebraic lattice and $K' = \Omega K$ is a convex star body. Moreover, the successive minima μ_k' of K' are given, as in (5), by $\mu_k' = f(\Omega \underline{X}_k)$. Hence (6) can be put in the (apparently) more general form:

(8) $$(\mu_1' \dots \mu_m')^n V(K') \le 2^N d(\tilde{\Lambda}).$$

(b) In the case when L is not totally real, we let $L^{(1)}, \dots, L^{(s)}$; $L^{(s+1)}, L^{(s+t+1)}; \dots; L^{(s+t)}, L^{(s+2t)}$ $\left(L^{(s+j)} = \bar{L}^{(s+t+j)}, 1 \le j \le t\right)$ denote the s real conjugates and the t pairs of complex conjugates of L, where $N = s + 2t$. Then

$$\underline{X} = \{\underline{x}^{(1)}, \dots, \underline{x}^{(s)}, \underline{x}^{(s+1)}, \underline{x}^{(s+t+1)}, \dots, \underline{x}^{(s+2t)}\} \in \mathbb{R}^s \times \mathbb{C}^{2t} \cong \mathbb{R}^N.$$

Consider the linear transformation given by $\underline{X} = T\underline{X}'$, where $\underline{x}^{(1)} = \underline{\zeta}^{(1)}, \dots, \underline{x}^{(s)} = \underline{\zeta}^{(s)}$; $\underline{x}^{(s+j)} = \underline{\xi}^{(s+j)} + i\underline{\eta}^{(s+j)}$, $\underline{x}^{(s+j+t)} = \underline{\xi}^{(s+j)} - i\underline{\eta}^{(s+j)}$, $(1 \le j \le t)$ with $|\det T| = 2^{mt} \ne 0$ and

$$\underline{X}' = \left(\underline{\zeta}^{(1)}, \dots, \underline{\zeta}^{(s)}, \underline{\xi}^{(s+1)}, \underline{\eta}^{(s+1)}, \dots, \underline{\xi}^{(s+t)}, \underline{\eta}^{(s+t)}\right).$$

Since T^{-1} maps V_N onto $\mathbb{R}^N$, we shall <u>define</u> $V(K)$ in the natural way in terms of $V(T^{-1}K)$, i.e. $V(K) = V(T^{-1}K)|\det T|$, so that $V(K)/d(\tilde{\Lambda})$ is invariant under T. We can now apply the theorem in its extended form, as given by (8), to the pair $T^{-1}K$, $T^{-1}\tilde{\Lambda}$. Then, on applying T and using the invariance property, we obtain (8) for the pair $K, \tilde{\Lambda}$.

4. <u>REMARKS</u>. Our definition of an algebraic lattice differs from those in [3] and [4] and places more emphasis on the modules $M^{(r)}$ conjugate to M. This is inherent in our definition of a star body. In fact, they were designed to deal with the following special case. For fixed r $(1 \le r \le n)$, let $L_{rs}(\underline{x}_r)$, $1 \le s \le m$ be a set of m linear forms in m variables $\underline{x}_r = (x_{1r}, \dots, x_{mr})$ with real coefficients and suppose that the mn linear forms L_{rs} in the mn variables $\underline{X} = (\underline{x}_1, \dots, \underline{x}_n)$ have determinant $\Delta \ne 0$. Write

$$f_r(\underline{x}_r) = \max_{1 \le s \le m} |L_{rs}(\underline{x}_r)|.$$

Then our theorem assures the existence of m elements $\underline{\zeta}_1, \dots, \underline{\zeta}_m$ of $M = \mathcal{O}^m$, linearly independent over K, such that, for $k = 1, 2, \dots, m$,

$$f_r(\underline{\zeta}_\ell^{(r)}) \le \mu_k, \quad (1 \le \ell \le k,\ 1 \le r \le n)$$

with $0 < \mu_1 \le \mu_2 \le \dots \le \mu_m$ and

$$(\mu_1 \dots \mu_m)^n \le c_{m,n}\, d^{m/2} \Delta\,;$$

indeed, it assigns the value 1 to the constant $c_{m,n}$, for all m,n. This is the expected value, as stated in [1]. If $L_{1s} = \dots = L_{ns} = L_s$ say, so that f_r does not depend on r, the theorem can be found in [4], but the general case remains elusive and was responsible for this revision of Weyl's work.

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PARTITIONS INTO SMALL PRIMES

by

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Presented by P. Greiner, F.R.S.C.

ABSTRACT. Let p_n denote the n-th prime. In Theorem 1 we find estimates for

$$y(n) = 3 + \sum_{k=4}^{n} p_k.$$

We define

$$U(y,c) = \sqrt{2y}\sqrt{\log\sqrt{2y} + c \operatorname{loglog}\sqrt{2y}},$$

and in Theorem 2 find estimates for p_n in terms of $y(n)$ which imply $p_n \sim U(y(n),\frac{1}{2})$. Finally, similar estimates are used in showing that every integer $y > y_0(\varepsilon)$ can be partitioned into distinct primes p with $p < U(y,\frac{1}{2}+\varepsilon)$ (Theorems 3, 4). Numerical evidence suggests that the ε could be dropped (i.e., that $y_0(0)$ exists); however, the $\frac{1}{2}$ cannot be reduced.

THEOREM 1.

(1) $$y(n) < (n^2/2)(\log n + \operatorname{loglog} n) \quad \text{for } n \geq 4.$$

Given $0 \leq \alpha < 1$, there exists an integer $N(\alpha)$ such that

(2) $$(n^2/2)(\log n + \alpha \operatorname{loglog} n) < y(n)$$

for $n \ge N(\alpha)$, and $N(0) = 5$. Moreover, (2) is true if

$$\alpha < B(n) =_{def} 1 - \frac{2 - \log[1 - \log 2/\log n]}{\log\log n},$$

so that given $0 < \varepsilon \le 1$, we can take $\alpha = 1 - \varepsilon$ in (2), provided $n \ge n_0(\varepsilon) = N(1-\varepsilon)$.

$$\text{(3)} \qquad \sum_{k=1}^{n} p_k < (n^2/2)(\log n + \log\log n) \quad \text{for} \quad n \ge 6,$$

and lower bounds for this sum are given by those for $y(n)$.

COROLLARY. $\sum_{k=1}^{n} p_k \sim (n^2/2)(\log n + \log\log n)$.

To prove Theorem 1 we need the

LEMMA. For $n \ge 4$,

$$\frac{n^2}{2} \log\log \frac{n}{2} < \int_4^n x \log\log x \, dx < \frac{n^2}{2} \log\log n.$$

This is proved by calculating the three expressions for $n = 4,5$ and then examining their derivatives. The lemma is used in applying the Euler-Maclaurin summation formula to the following results of Rosser and Schoenfeld [2; Theorem 3]:

$$\text{(4)} \qquad k[\log k + \log\log k - 3/2] < p_k \quad \text{for} \quad k \ge 2,$$

$$\text{(5)} \qquad p_k < k[\log k + \log\log k - \tfrac{1}{2}] \quad \text{for} \quad k \ge 20.$$

These lead respectively to (2) and (1). For example, in summing (4), we find after simplification that

(6a) $$y(n) > \frac{n^2}{2}\log n + \frac{n^2}{2}\operatorname{loglog}\frac{n}{2} - n^2$$

(6b) $$> \frac{n^2}{2}\log n + \alpha\,\frac{n^2}{2}\operatorname{loglog} n,$$

inequality (6b) being equivalent to $\alpha < B(n)$. To show $N(0) = 5$, we observe that with $\alpha = 0$, (6b) holds for $n > 2 \operatorname{expexp} 2 = 3236.3\ldots$, and then obtain $y(n) > (n^2/2)\log n$ by direct computation for $5 \le n \le 3236$.

THEOREM 2. Let $\varepsilon > 0$. Then

(7) $$U\bigl(y(n),\tfrac12-\varepsilon\bigr) < p_n < U\bigl(y(n),\tfrac12+\varepsilon\bigr)$$

provided n is sufficiently large. The right-hand inequality holds for $n \ge \max\{20, n_0(\varepsilon)\}$, the left for $n \ge n_1(\varepsilon)$.

COROLLARY. $p_n \sim U\bigl(y(n),\tfrac12\bigr)$.

The quantity $n_1(\varepsilon)$ is defined later. Theorem 2 follows from the inequalities

(8) $$U\bigl(y(n),a\bigr) < U\left(\frac{n^2}{2}\log(n\log n),a\right) < n\left[\log(n\log n) - \frac32\right] < p_n$$
$$< n\left[\log(n\log n) - \frac12\right] < U\left(\frac{n^2}{2}(\log n + \alpha\operatorname{loglog} n), b\right) < U\bigl(y(n),b\bigr)$$

which we prove with $a = \frac12 - \varepsilon$ and $b = \frac12 + \varepsilon$ for all sufficiently large n. The central inequalities in (8) are just (4) and (5), while the outermost inequalities follow from (1) and (2). The remaining inequality to the left of p_n is equivalent to

(9) $$(a-\tfrac12)\log n\operatorname{loglog} n + 3\log n + (a+\tfrac12)(\operatorname{loglog} n)^2 + s(n) < \left(\operatorname{loglog} n - \frac32\right)^2,$$

where $s(n) = O(\operatorname{loglog} n)$ and $s(n) > 0$ for $n \ge 3$. Given $\varepsilon > 0$, there exists a number $n_1(\varepsilon)$ such that (9) will hold with $a = \frac12 - \varepsilon$ for $n \ge n_1(\varepsilon)$.

$n_1(\varepsilon)$ is very large, in particular much larger than 4, and so $n \geq n_1(\varepsilon)$ implies all three inequalities to the left of p_n in (8), and the left side of (7) follows. The right side of (8) similarly yields that of (7); in this argument, $b = 3/2 - \alpha$ is the optimal value of b, given α, and to conclude, we put $\alpha = 1 - \varepsilon$.

THEOREM 3. Let $n \geq 4$. Then every integer in the closed interval $[7, y(n)]$ can be partitioned into distinct primes $\leq p_n$.

Richert [1] proved in 1950 that every integer ≥ 7 can be partitioned into distinct primes. To prove Theorem 3 we give a new argument that makes explicit the size of the primes involved. Defining, for $n \geq 5$, $T_n = -2+3-5+7+11+\cdots+p_n$, we find by induction, with Bertrand's postulate, that every integer of the parity of T_n in $[-T_n, T_n]$ can be expressed in the form $\pm 2 \pm 3 \pm 5 \pm \cdots \pm p_n$ for appropriate choices of signs. Adding to these integers the sum $2+3+5+\cdots+p_n$, which also has the parity of T_n, one obtains all the evens in $[2(2+5), 2(3+7+11+\cdots+p_n)]$.

THEOREM 4. Let $0 < \varepsilon \leq 1$, $n \geq \max\{20, n_0(\varepsilon)+1\}$, and $y(n-1) < y \leq y(n)$. Then y can be partitioned into distinct primes p satisfying $p < U(y, \frac{1}{2}+\varepsilon)$.

The proof is similar to that of the right side of (7). We show that

$$\begin{aligned} p_n &< (n-1)[\log(n-1) + \log\log(n-1) - \tfrac{1}{4}] \\ &< U\left(\frac{(n-1)^2}{2}[\log(n-1) + \alpha \log\log(n-1)], c\right) < U(y(n-1), c). \end{aligned}$$

The first and third inequalities hold respectively for $n \geq 20$ and for $n \geq N(\alpha) + 1$, while the central inequality is equivalent to

$$(10)\quad \left(\log\log(n-1) - \tfrac{1}{4}\right)^2$$
$$< \left(c - \frac{3}{2} + \alpha\right)\log(n-1)\log\log(n-1) + \frac{1}{2}\log(n-1) + \alpha\left(c + \frac{1}{2}\right)[\log\log(n-1)]^2 + t(n-1),$$

where $t(n-1) = O(\log\log n)$ and $t(n-1) > 0$ for $n \geq 4$. With $c = 3/2 - \alpha$, (10) holds for $n \geq 20$. Putting $\alpha = 1 - \varepsilon$ and applying Theorem 3 yields Theorem 4.

REMARKS AND COMPUTATIONAL RESULTS. We have computed $y(n)$ and some related functions, including $U(y(n),c)$ for several values of c, for $4 \leq n \leq 100{,}000$. These computations enable us to show that $N(0) = 5$, as seen earlier, and also that $N(.1) = 5$ and $N(.156) = 140$. The latter is the largest value of $N(\alpha)$ obtainable through our computations.

Theorems 1, 2, and 4 resulted from attempts to prove the conjecture:

$$(12)\qquad p_n < U(y(n),\tfrac{1}{2})$$

for all sufficiently large n. The computations yield (12) for $49 \leq n \leq 100{,}000$, and also

$$(13)\qquad p_n < U(y(n),.7)$$

for $6 \leq n \leq 100{,}000$. On the other hand

$$(14)\qquad U(y(n),.3) < p_n$$

for $37{,}374 \leq n \leq 100{,}000$, and the two sides of (14) are much closer together than those of (13) or even of (12) for these values of n. If (13) and (14) held for all larger n, we would have $n_0(.2) = 6$ and $n_1(.2) = 37{,}374$. In general it seems that $n_1(\varepsilon)$ will be much larger than $n_0(\varepsilon)$. While one might from the computations expect $n_1(.2)$ to be larger even than $n_0(0)$, it is nevertheless the case that (14) holds for nearly all n and (12) is not known to do so.

Regarding possible improvements on Theorem 4, we have found that

$$p_n < U(y(n-1),\tfrac{1}{2}) \tag{15}$$

for $105 \le n \le 100{,}000$. If (15) held for all large n, we would obtain Theorem 4 without the ε. But the $\frac{1}{2}$ in Theorem 4 cannot be reduced, since for $n \ge 5$, a prime at least as large as p_n is needed in any partition of $y(n)$ into distinct primes.

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EXTENSION OF POSITIVE LINEAR OPERATORS IN L^p-SPACES

by

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Presented by P. G. Rooney, F.R.S.C.

Introduction: Let E and F be Banach lattices, H a linear subspace of E and let $T : H \longrightarrow F$ be a positive continuous (resp. positive contractive) linear operator. The following problems arise:

1) Give necessary and sufficient conditions for the existence of positive (resp. positive contractive) linear extensions $T_o : E \longrightarrow F$ of T.
2) Describe the scope of all such extensions. In particular, when do we have uniqueness?

We answer the first question E and F being an L^p-space and an L^q-space, respectively, with $q < p$. The second problem is also solved for such pairs of classical Banach lattices, but for positive continuous linear operators only.

If $F = L^\infty(\mu)$, μ some localizable measure, then F is a Dedekind complete AM-space with order unit. In this case the solution of problem (1) is well-known, even for non-classical Banach lattices E. We shall exclude this case.

Theorem 1: Let $p,q \in [1,+\infty[$ be such that $q < p$ and let E, F be Banach lattices with p- and q-additive norms, respectively. Then a positive continuous linear operator T of some linear subspace H of E into F with $\|T\| \leq M$, $M \geq 0$, has a positive linear extension $T_o : E \rightarrow F$ with $\|T_o\| \leq M$ iff

$$\Big\| \bigvee_{i \in I} (T(h_i))^+ \Big\| \leq M \cdot \Big\| \bigvee_{i \in I} h_i^+ \Big\|$$

for every finite family $(h_i)_{i \in I}$ in H, where $\bigvee$ denotes the supremum operation.

As an immediate consequence, for $F = H$ and T the identity mapping on H, we obtain the following result of T. Ando:

<u>Corollary 1:</u> Each closed linear sublattice H of an L^p-space, $p \in [1,+\infty[$, is the range of a positive contractive projection.

As a characterization of positively complemented subspaces of an L^p-space we have

<u>Corollary 2:</u> A closed subspace H of an L^p-space E $(p \in [1,+\infty[)$ is the range of a positive projection iff

i) H is a lattice in the ordering induced on H,

ii) there exists a constant $M \geq 0$ such that

$\Big\| \overset{H}{\bigvee_{i \in I}} (h_i \overset{H}{\vee} 0) \Big\| \leq M \cdot \Big\| \bigvee_{i \in I} h_i^+ \Big\|$ for each finite family $(h_i)_{i \in I}$ in H. Here $\overset{H}{\vee}$ denotes the supremum operation in H.

The following theorem which provides an elegant solution of the extension problem for positive linear operators in L^p-spaces is a consequence of Theorem 1:

<u>Theorem 2:</u> Let $p, q \in [1,+\infty[$ be such that $q \leq p$ and let E, F be Banach lattices with p- and q-additive norms, respectively. Then the following statements are equivalent for each posi-

tive linear operator $T: H \longrightarrow F$, where H is a linear subspace of E:

i) There exists a positive linear extension $T_0: E \rightarrow F$ of T.

ii) For each subset $A \subset H$ bounded from above in E $T(A)$ is bounded from above in F.

Let $p, q \in [1, +\infty[$, $q \leqslant p$, and let E, F be Banach lattices with p- and q-additive norms, respectively. Furthermore, let H be a linear subspace of E and $T : H \rightarrow F$ a positive linear operator possessing a positive extension $T_0: E \rightarrow F$. In order to settle the second problem, we denote by $H_{e,\varepsilon}$ (resp. $H^{e,\varepsilon}$) the set of all finite sequences $(h_i)_{i \in I}$ in H such that $\|(e - \bigwedge_{i \in I} h_i)^+\| \leqslant \varepsilon$ (resp. $\|(e - \bigvee_{i \in I} h_i)^-\| \leqslant \varepsilon$) for each $e \in E$ and $\varepsilon > 0$. Define $\hat{T} : E \times F'_+ \longrightarrow \mathbb{R} \cup \{+\infty\}$, resp. $\check{T}: E \times F'_+ \longrightarrow \mathbb{R} \cup \{-\infty\}$ by setting

$$\hat{T}(e,l) = \sup_{\varepsilon > 0} \inf \{ l(\bigwedge_{i \in I} T(h_i)) \; : \; (h_i)_{i \in I} \in H_{e,\varepsilon} \} \quad (\inf \emptyset := +\infty),$$

$$\check{T}(e,l) = \inf_{\varepsilon > 0} \sup \{ l(\bigvee_{i \in I} T(h_i)) \; : \; (h_i)_{i \in I} \in H^{e,\varepsilon} \} \quad (\sup \emptyset := -\infty).$$

With these notations we can formulate

<u>Theorem 3:</u> A linear operator $T_1: E \rightarrow F$ is a positive extension of T iff

$$\check{T}(e,l) \leqslant l(T_1(e)) < \hat{T}(e,l)$$

for each pair $(e,l) \in E \times F'_+$. Given $e \in E$, $l \in F'_+$ and $\alpha \in \mathbb{R}$ such that $\check{T}(e,l) < \alpha < \hat{T}(e,l)$ there exists a positive linear extension T_1 of T satisfying $l(T_1(e)) = \alpha$. For fixed $e \in E$ all positive linear extensions of T coincide at e iff $\check{T}(e,l) = \hat{T}(e,l)$ for each $l \in F'_+$.

As an application we give a characterization of finite-dimensional Korovkin spaces in L^p-spaces. In order to sketch the problem consider a second countable locally compact space X and a positive Radon measure μ on X. Let $\mathcal{L}^p(\mu)$ be the space of all μ-measurable real-valued functions f on X for which $|f|^p$ is μ-integrable, $p \in [1,+\infty[$.
An n-dimensional subspace $\mathcal{H}$ of $\mathcal{L}^p(\mu)$ is called a <u>Korovkin space</u> in $\mathcal{L}^p(\mu)$ whenever the following condition holds:

If $(T_i)_{i\in I}$ is an equicontinuous net of positive linear operators from $\mathcal{L}^p(\mu)$ into itself such that $(T_i(h))_{i\in I}$ converges to h with respect to the $\mathcal{L}^p$-semi-norm for all $h\in\mathcal{H}$ then $(T_i(f))_{i\in I}$ converges to f for <u>all $f\in\mathcal{L}^p(\mu)$</u> in the $\mathcal{L}^p$-topology.

For a μ-negligible set N and $x\in X$ let $\mathcal{M}_x^N(\mathcal{H})$ denote the set of all linear functionals $\sum_{i=1}^{n+1}\lambda_i\varepsilon_{x_i}$ on $\mathcal{L}^p(\mu)$ such that

$$\sum_{i=1}^{n+1}\lambda_i h(x_i) = h(x)$$

for all $h\in\mathcal{H}$, where $x_1,\dots,x_{n+1}\in X\setminus N$, $\lambda_1,\dots,\lambda_{n+1}\geq 0$ and ε_{x_i} is the evaluation functional at x_i.

<u>Theorem 4:</u> $\mathcal{H}$ is a Korovkin space in $\mathcal{L}^p(\mu)$ iff there exists a μ-negligible set $N\subset X$ such that

$$\mathcal{M}_x^N(\mathcal{H}) = \{\varepsilon_x\}$$

for all $x\in X\setminus N$.

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A BECKMAN QUARLES TYPE THEOREM FOR PLANE LORENTZ TRANSFORMATIONS

By

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Presented by J. Aczél, F.R.S.C.

1. The result we can prove is

THEOREM A: Given a fixed real number $\rho \neq 0$ and given a mapping σ of $\mathbb{R}^2$ into the set $P_0(\mathbb{R}^2)$ of all non empty subsets of $\mathbb{R}^2$ such that $\overline{AB} = \rho$ implies $\overline{A'B'} = \rho$ for all $A, B \in \mathbb{R}^2$, $A' \in A^\sigma$, $B' \in B^\sigma$. Then σ must be a Lorentz transformation. Here $\overline{AB}$ denotes the pseudo euclidean distance $(b_1-a_1)^2 - (b_2-a_2)^2$ for $A = (a_1,a_2)$, $B = (b_1,b_2)$.

2. In [1] we have proved

THEOREM B: A bijective mapping $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$\overline{AB} = \rho \text{ iff } \overline{A^\sigma B^\sigma} = \rho \text{ for all } A, B \in \mathbb{R}^2$$

for a single $\rho \neq 0$ must be a Lorentz transformation.

The following three theorems of E.M. Schröder ([4]) generalize result B (we only deal with the plane case of Schröder's theorems):

THEOREM C: A surjective mapping $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$\overline{AB} = \rho \text{ iff } \overline{A^\sigma B^\sigma} = \rho \text{ for all } A, B \in \mathbb{R}^3$$

for a single $\rho \neq 0$ must be a Lorentz transformation.

THEOREM D: Given fixed real numbers a, b such that $a \neq 0$ and $\frac{b}{a} \leq \frac{1}{4}$. Given moreover a mapping $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\overline{AB} = \rho \text{ implies } \overline{A^\sigma B^\sigma} = \rho \text{ for all } A, B \in \mathbb{R}^2$$

and $\rho \in \{a,b\}$. Then σ must be a Lorentz transformation.

THEOREM E: Given fixed real numbers a, b such that $a \neq 0$ and $\frac{b}{a} \leq \frac{1}{4}$. Given moreover a mapping $\sigma : \mathbb{R}^2 \to P_0(\mathbb{R}^2)$ such that

$$\overline{AB} = \rho \text{ implies } \overline{A'B'} = \rho \text{ for all } A, B \in \mathbb{R}^2,\ A' \in A^\sigma,\ B' \in B^\sigma$$

and $\rho \in \{a,b\}$. Then σ must be a Lorentz transformation.

E.M. Schröder has proved THEOREM E and then has established Theorems C, D as consequences of E.

Schroder's Theorem D will be used in our proof of Theorem A.

Theorem B has been generalized to the field case by F. Rado [3].

Our Theorem A, which obviously covers Theorems B, C, D, E, corresponds to a result (plane case) of Beckman, Quarles [2]:

THEOREM F: Given a fixed real number $\rho > 0$ and a mapping $\sigma : \mathbb{R}^2 \to P_0(\mathbb{R}^2)$ such that $\overline{AB} = \rho$ implies $\overline{A'B'} = \rho$ for all $A, B \in \mathbb{R}^2$, $A' \in A^\sigma$, $B' \in B^\sigma$, where $\overline{AB}$ denotes the euclidean distance of A, B. Then σ must be an euclidean isometry.

Since there exist no equilateral triangles in the pseudo euclidean geometry the proof of Beckman, Quarles cannot be carried over to this case.

A corollary of our Theorem A is:

THEOREM G: Given arbitrary functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfying the functional equation

$$\left[f\left(x+z,\ y+\frac{1}{z}\right) - f(x,y)\right]\left[g\left(x+z,\ y+\frac{1}{z}\right) - g(x,y)\right] = 1$$

for all $x,y,z \in \mathbb{R}$, $z \neq 0$. Then there exist $a \neq 0$ and $r,s \in \mathbb{R}$ such that

$$\text{I)}\quad f(x,y) = ax + r, \quad g(x,y) = \frac{1}{a}y + s$$

or

$$\text{II)}\quad f(x,y) = ay + r, \quad g(x,y) = \frac{1}{a}x + s.$$

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DUAL ORDONNE D'UN MODULE

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Pour un anneau A réticulé, totalement ordonné, quasi-complet et un A-module E ordonné filtrant, on montre, en introduisant $f^+(x) = \sup\{f(u) : 0 \leqslant u \leqslant x\}$, que l'ensemble $\mathcal{E}$ des applications régulières de E dans A est un A-module, filtrant, réticulé, un ℓ-module et un f-module qui peut être muni d'une semi-norme si E est lui-même normé.

I - Formes linéaires sur un sous-module de E.

Soit A un anneau et E un A-module. L'ensemble E^* des formes linéaires sur un sous-module de E est un A-module partiellement ordonné [1] par :

$$f_1 \subset f_2 \Longleftrightarrow \begin{cases} D_{f_1} \subset D_{f_2} \\ \forall x \in D_{f_1} : f_1(x) = f_2(x) \end{cases}$$

où D_{f_i} désigne l'ensemble de définition de f_i.

De plus, pour f et g appartenant à E^* :

$$h = f \curlyvee g \text{ existe} \Longleftrightarrow f(x_1) + g(y_1) = f(x_2) + g(y_2)$$

pour tout $x_i \in D_f$ et $y_i \in D_g$ avec $x_1 + y_1 = x_2 + y_2$.

II - Dual préordonné d'un module préordonné.

Soit A un anneau commutatif unitaire muni d'une relation d'ordre compatible avec la structure d'anneau. Soit E un A-module préordonné, la relation de préordre dans E étant compatible avec la relation d'ordre dans A. Soit E^* le dual de E et P^* l'ensemble des formes linéaires sur E positives sur le cône positif P de E.

1°) Préordre sur E^*

P^* définit un préordre $\dot{\leqslant}$ sur E^* :

$$f \mathrel{\dot{\leqslant}} g \text{ dans } E^* \Longleftrightarrow g-f \in P^* \Longleftrightarrow \forall x \in P : f(x) \leqslant g(x).$$

Au sens de ce préordre, P^* est un cône.

2°) Ordre sur E^*.

Si P engendre E, c'est-à-dire si $E = P-P$, ce qui est possible au moins dans le cas où A est totalement ordonné, alors $\leqslant$ devient un ordre.

3°) Dual préordonné.

Dans le cas où A est totalement ordonné, le dual préordonné $\tilde{\tilde{E}} = P^* - P^*$ est un sous-module de E^*.

Dans le cas où $\tilde{\tilde{E}}$ est un sous-module de E^*, il est convexe [2] et $\frac{E^*}{\tilde{\tilde{E}}}$ est un A-module préordonné dont le cône positif est $\tilde{\tilde{E}}$, ce qui prouve que $\frac{E^*}{\tilde{\tilde{E}}}$ est trivialement ordonné.

4°) Formes O-bornées dans $\tilde{\tilde{E}}$.

Tout élément de $\tilde{\tilde{E}}$ est O-borné, c'est-à-dire transforme un borné par l'ordre de E en un borné par l'ordre de A.

III - Dual ordonné d'un module sur A réticulé.

A est un anneau unitaire commutatif réticulé, un f-anneau [3] : $\forall \lambda \in A^+, \forall (a,b) \in A^2 : \lambda(a \vee b) = (\lambda a) \vee (\lambda b)$.
Nous supposons de plus que A est quasi-complet : toute partie bornée du treillis admet une borne supérieure et une borne inférieure. Enfin E est un A-module ordonné filtrant.

1°) Proposition.

Une application additive de E dans A est O-bornée si et seulement si elle est la différence de deux applications additives et positives sur le cône positif P de E (nous dirons : si et seulement si elle est régulière [5]).

Soit f une application additive O-bornée de E dans A.
Pour x appartenant à P, posons : $f^+(x) = \sup\{f(u) : 0 \leqslant u \leqslant x\}$.

- $f^+(x)$ existe car f est O-bornée et A quasi-complet ; de plus : $\forall x \in P,\ f(x) \geqslant 0$.
- On vérifie que f^+ est additive sur le cône P de E, ce qui montre que f^+ est croissante.

. Cette application f^+ peut être prolongée de façon unique en une application $\tilde{f}^+$ additive croissante de E dans A : $f \leq \tilde{f}^+$ au moins sur P, de sorte que $\tilde{f}^+ - f$ est une application additive et positive sur P.

. En écrivant f sous la forme $f = \tilde{f}^+ - (\tilde{f}^+ - f)$ on obtient le résultat : l'application f est régulière.

<u>Remarque</u>

On peut vérifier : $\forall \lambda \in A^+$, $\forall x \in P$, $f^+(\lambda x) \geq \lambda f^+(x)$. [4].
Il manque donc l'inégalité contraire pour établir l'équivalence entre formes linéaires O-bornées et régulières dans le dual $\tilde{\tilde{E}}$.

<u>2°) L'ensemble des applications régulières de E dans A</u>.

Supposons, de plus, l'anneau A totalement ordonné ; comme E est supposé filtrant, $E = P-P$. Désignons par $\mathcal{E}$ l'ensemble des applications régulières de E dans A.

a) $\mathcal{E}$ est un A-module ordonné filtrant.

b) $\mathcal{E}$ est un treillis : $f^+ = f \vee 0$ donc $f \vee g$ existe.

$$\forall x \in P \quad \begin{cases} (f \vee g)(x) = \sup\{f(u) + g(x-u) : 0 \leq u \leq x\} \\ (f \wedge g)(x) = \inf\{f(u) + g(x-u) : 0 \leq u \leq x\} \\ |f|(x) = \sup\limits_{|x'| \leq x} f(x') \end{cases}$$

c) $\mathcal{E}$ est un ℓ-module [6], c'est-à-dire :

$\forall \lambda \in A^+$, $\lambda(f \vee g) = (\lambda f) \vee (\lambda g)$,

donc $\mathcal{E}$ est aussi un f-module [6] et par suite :

$\forall \lambda \in A$, $\forall f \in \mathcal{E}$: $|\lambda f| = |\lambda| \times |f|$.

d) $\mathcal{E}$ est complet : comme $\mathcal{E}$ est filtrant, il suffit de montrer que toute partie majorée de $\mathcal{E}$ admet une borne supérieure. On utilise la méthode donnée dans [5].

e) $\mathcal{E}$ est semi-normé si E est normé :

Pour f appartenant à $\mathcal{E}$, on pose

$\|f\| = \sup |f(x)|$, $x \in P$, $\|x\| \leq 1$.

L'application $f \longmapsto \|f\|$ définit une semi-norme sur $\mathcal{E}$, sous-linéaire au sens de [8] et croissante sur le cône de $\mathcal{E}$.

3°) Conclusion.

Si A est un anneau commutatif unitaire totalement ordonné, quasi-complet et un f-anneau, si E est un A-module réticulé filtrant et normé alors l'ensemble des applications régulières de E dans A est un A-module réticulé et un ℓ-module semi-normé.

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COMMUTATOR PROPERTIES OF EXTENSION GROUPS

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Presented by H.S.M. Coxeter, F.R.S.C.

In this note let e always denote a central group extension,

$$(1) \qquad e : 0 \to A \xrightarrow{\varkappa} G \xrightarrow{\pi} Q \to 0 \quad .$$

Given Q and A, how do the group properties of G depend on the extension class $[e] \in H^2(Q,A)$? Result: It is a certain subgroup $U(e)$ of the Schur multiplicator $M(Q)$ rather than $[e]$ which determines the "commutator properties" of G, e.g. the lower central series, the size of the center $Z(G)$, the isoclinism class, verbal subgroups $\mathcal{V}G$ (defined by commutator laws). Our approach is group-theoretic with a homological flavor. The details will appear elsewhere.

NOTATION. The <u>Schur multiplicator</u> $M(Q)$ of the finite or infinite group Q is defined as $M(Q) = (R \cap F')/[R,F]$, where

$$(2) \qquad e_0 : R \hookrightarrow F \twoheadrightarrow Q$$

is any free presentation of Q and $F' = [F,F]$. Given e as in (1), the so-called <u>transgression</u> $\theta_*(e): M(Q) \to A$ can be described as the composite

$$M(Q) \hookrightarrow \frac{R}{[R,F]} \xrightarrow{\beta_*} A$$

with β_* being induced by any lifting $\beta: F \to G$ of the epimorphism $\pi: G \twoheadrightarrow Q$. We can easily construct the extensions $[e] \in H^2(Q,A)$ such that $\theta_*(e)$ is a given homomorphism $M(Q) \to A$. Our concern is $U(e) = \text{Kernel}\ \theta_*(e) \subseteq M(Q)$, this is also the image of the <u>inflation</u> $M(\pi): M(G) \to M(Q)$.

A central extension e as in (1) is called a <u>generalized representing group</u> of Q if every complex projective representation of Q can be lifted to a linear representation of G . This class contains the representing groups ("Darstellungsgruppen" in the sense of I. Schur) but is much bigger.

THEOREM A. <u>The central extension e as in (1) is a generalized representing group of Q precisely when $U(e) = 0$.</u>

Central extensions e_1 and e_2 as in (1), each term given subscripts, are called <u>isoclinic</u> if there are isomorphisms $\eta: Q_1 \to Q_2$ and $\xi: G_1' \to G_2'$ such that $\xi[g,h] = [\bar{g},\bar{h}]$ for all $g,h \in G_1$, $\bar{g},\bar{h} \in G_2$ with $\eta(\pi_1 g) = \pi_2\bar{g}$ and $\eta(\pi_1 h) = \pi_2(\bar{h})$; idea: "same commutator functions". This generalizes P. Hall's concept of group isoclinism insofar as κA may differ from $Z(G)$. The reward is that the isoclinism classification splits into two fairly separate problems: a well-rounded theory of isoclinism of extensions (Theorem B) and the study of the precise center of a central extension group (Theorem C). Isoclinism is hard to handle without theory, except in the following case. An <u>isoclinic epimorphism of extensions</u> is a morphism (α,β,γ) : $e_1 \to e_2$ of central extensions such that $G_1' \cap \operatorname{Ker}\beta = 0$ and γ is isomorphic; then $\xi = \beta|G_1',G_2'$ and γ afford an isoclinism.

THEOREM B. <u>Given central extensions $e_i : A_i \rightarrowtail G_i \twoheadrightarrow Q$ for $i := 1,2$. Then the following are equivalent:</u>

(i) <u>e_1 and e_2 are isoclinic extensions;</u>

(ii) <u>there is a central extension $e : A \rightarrowtail G \twoheadrightarrow Q$ together</u>

<u>with isoclinic epimorphisms</u> $(\alpha_i,\beta_i,\gamma_i)\colon e \to e_i$, $i := 1,2$.

(iii) $M(\eta)U(e_1) = U(e_2) \subseteq M(Q)$ <u>for some automorphism</u> η <u>of</u> Q .

This theorem shows that the isoclinism classes of extensions with factor group Q are in bijective correspondence with the $\mathrm{Aut}(Q)$-orbits of the subgroups of $M(Q)$. Together with Theorem C , it renders the problems of Jones and Wiegold [3] on group isoclinism much more accessible. For example, every finite group G of nilpotency class two admits a group H of the same order which is isoclinic but not isomorphic with G .

Let $\chi_G\colon G_{ab} \otimes Z(G) \to M(G)$ be the <u>Ganea map</u> of G , i.e. the Ganea map [2] of the central extension $Z(G) \hookrightarrow G \twoheadrightarrow G/Z(G)$.

THEOREM C. <u>Let</u> e <u>as in (1) be a central extension. Then</u>

$\pi Z(G) = \{\, x \in Z(Q) \mid q \in Q_{ab}\colon \chi_Q(q \otimes x) \in U(e) \subseteq M(Q) \,\}$.

Consequently a group Q is <u>capable</u>, i.e. isomorphic to the central factor group of some group, precisely when the adjoint $\hat{\chi}_Q\colon Z(Q) \to \mathrm{Hom}(Q_{ab}, M(Q))$ is monomorphic. This was our starting point for the detailed study of capability in [1].

Finally let Q lie in a variety $\mathcal{V}$ <u>of exponent</u> 0 , i.e. $\mathcal{V}$ contains all abelian groups. Is $G \in \mathcal{V}$ for a central extension e as in (1) ? The answer is controlled by a certain subgroup $K_{\mathcal{V}}(Q)$ of $M(Q)$ which can be reconstructed from the theorem. The quotient $M(Q)/K_{\mathcal{V}}(Q)$ is known as a varietal Schur multiplicator.

THEOREM D. <u>Under the assumptions stated, the verbal subgroup</u> $\mathcal{V}G$ <u>is the image of</u> $K_{\mathcal{V}}(Q)$ <u>under</u> $\kappa\theta_*(e)$ <u>. Thus</u> $G \in \mathcal{V} \leftrightarrow K_{\mathcal{V}}(Q) \subseteq U(e)$.

If $\mathcal{U}$ is any variety and $\mathcal{V}$ the variety of center-by-$\mathcal{U}$ groups, then

(3) $$K_{\mathcal{V}}(Q) = \mathrm{Kernel}\{\ M(\mathrm{nat}): M(Q) \to M(Q/\mathcal{U}Q)\ \}\ .$$

If $\mathcal{V}$ is the variety of abelian-by-$\mathcal{U}$ groups instead, then

(4) $$K_{\mathcal{V}}(Q) = \mathrm{Image}\{\ M(\mathrm{incl}): M(\mathcal{U}Q) \to M(Q)\ \}\ .$$

The combination of Theorem D with (3) and (4) readily leads to applications. Especially, the standard varieties of nilpotent and solvable groups (bound on class resp. length) are covered.

Theorem B overlaps with other work [3], [4], [5], but the present point of view provides unification and seems to be particularly fruitful.

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ZARISKI SURFACES*

by

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Presented by P. Ribenboim, F.R.S.C.

ABSTRACT

We define and describe a new class of algebraic surfaces in characteristic $p > 0$ which we call Zariski surfaces. An irrational surface with $p_g = 0$ exists in this class. This answers a question of Zariski.

In the classical case all unirational surfaces are rational. The situation is different in characteristic $p > 0$. This was first realized by Oscar Zariski (see On Castelnuovo's Criterion of Rationality $p_a = P_2 = 0$ of an algebraic surface, Illinois J. of Math., Vol. 2, No. 3, Sept. 1958, p. 314). Prompted by Hironaka's suggestion, I began an investigation of the type of surfaces introduced by Zariski in that paper. The research was done in 1970-71 at Harvard with the advice of Hironaka and Zariski, and then during 1974-77 it was continued under the direction of J. S. Milne and M. Hochster at the University of Michigan.

A smooth algebraic surface X defined over an algebraically closed field, k, of characteristic $p > 0$ is called a Zariski Surface (or ZS) iff there exist two elements x, y in the function field of X, denoted $k(X)$, that are algebraically independent over k and such that $k(x,y) \subseteq k(X) \subseteq k(x^{1/p}, y^{1/p})$.

The main results of the thesis are as follows. First of all, Chapter 3 answers a question posed by Zariski in 1970-71. He asked whether a ZS with vanishing geometric genus, p_g, is necessarily rational. A long counterexample

*This is the introduction to a revised version of the author's 1977 University of Michigan Ph.D. thesis. The thesis advisor was Professor J. S. Milne. The complete paper will appear in 'Dissertationes Mathematicae'.

is given in Chapter 3. Secondly, it is shown in Chapter 5 that the value of p_g is unbounded from above over any algebraically closed field of characteristic larger than or equal to 5. This together with some other results of Chapter 5 illustrates the richness of the class of ZS.

In Chapter 4 a more detailed study is made of a particularly simple subclass of ZS's which we call "generic" ZS's. (A generic ZS is a smooth minimal model of the function field of a hypersurface given by $z^p = f(x,y)$ where f has degree $p + 1$ and the hypersurface has only the simplest singularities). We determine p_a, p_g, P_2, K^2 for a generic Zariski surface as well as the rank of the Néron-Severi group ρ and the étale Betti numbers, b_i. Using "generic Zariski surfaces" we give examples of ZS's which are of general type and K3. Trivally there also exist rational ZS's.

All Zariski surfaces are unirational and consequently supersingular, i.e., $\rho = b_2$. Thus the richness of the class of ZS's in characteristic $p > 0$ is in sharp contrast to the situation in characteristic 0 where every unirational surface is well known to be rational.

The principal technical tool used in the thesis is the theory of adjoints and multiadjoints. This theory deals with the influence of singularities on differential forms and is classical. However, since no reference could be found for the results that we needed, a self-contained exposition of the facts from the theory of adjoints that we used is given in Chapter 2 of the thesis. We develop the theory of adjoints for a normal, two-dimensional hypersurface in affine or projective 3-space. The results about adjoints are proved over an arbitrary algebraically closed groundfield including fields of characteristic 0.

It seems to me that ZS's are an interesting subclass of unirational surfaces. Among the many open problems concerning ZS's, let me select two very concrete ones. The first one is to answer Zariski's question in characteristic $p > 2$. The second one, a rather puzzling problem, is whether $H^1(X,\mathcal{O}_X)$ can be nontrivial for a Zariski surface X.*

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*Added in revision: W. E. Lang gave in his 1978 Harvard thesis (see Quasi-elliptic surfaces in characteristic three, Harvard University (1978), an example of a ZS with H^1 nontrivial. He also settled by another example Zariski's question in characteristic 3.

A NOTE ON YOUNG'S RAISING OPERATOR

Glânffrwd P. Thomas

Presented by G. de B. Robinson, F.R.S.C.

Consider the following formula due to Young [4] for the calculation of the homogeneous product sum, h_λ, in terms of Schur functions;

$$h_\lambda = \sum \left| \Pi S_{rs}^{\alpha_{rs}} \right| \{\lambda\}$$

where the operation S_{rs} is defined as follows.

> S_{rs}, where $r < s$, "represents the operation of moving one letter from the s-th row up to the r-th row; and the resulting term is regarded as zero, when any row becomes less than a row below it, or when letters from the same row overlap — as, for instance, happens when $\lambda_1 = \lambda_2$ in the case of $S_{13}S_{23}$."

The following example of the above is given by Robinson [2].

$$\begin{aligned} h_{(3,2,1)} &= |1 + S_{23} + S_{13} + S_{12} + S_{12}S_{23} + S_{12}S_{13} + S_{12}S_{23} + S_{12}S_{13}| \{3,2,1\} \\ &= \{3,2,1\} + \{3,3\} + \{4,2\} + \{4,1,1\} + \{4,2\} + \{5,1\} + \{5,1\} + \{6\} . \end{aligned}$$

Calculation by other means shows that the above analysis of $h_{(3,2,1)}$ is correct, however, it will be noticed that the operator $S_{12}S_{23}$ does not appear in the above yet it is not specifically excluded by the rule Y_1. The further condition

$$Y_2 : \alpha_{rs} \leq \lambda_s \quad \text{for } s = 2,3,...$$

is also required, although neither Young nor Robinson mention this fact. (The operators produced by Robinson do in fact satisfy this above condition also.)

It will be shown (Can. J. Math.) that by using a well-established extension of the definition of a Schur function that allows for parts in non-decreasing order of magnitude, both rules Y_1 and Y_2 become unnecessary and the expression $h_\lambda = \sum \left[\Pi S_{rs}^{\alpha_{rs}} \right] \{\lambda\}$ becomes an unrestricted summation.

If we define an operator δ_{ij} to act on a sequence of integers $(a_1, a_2, \ldots, a_n)$ by replacing a_i by a_i+1 and a_j by a_j-1 leaving the other terms unaffected, the essential element of the proof is noting the fact the operation

$$\prod_{1\le i<j\le n} (1-\delta_{ij})$$

acting on the sequence $(1, 2, \ldots, n)$ gives a sum of all permutations of $(1, 2, \ldots, n)$ with sign + or - depending on whether the permutation is even or odd. For example,

$$\begin{aligned}
&\prod_{1\le i<j\le 3} (1-\delta_{ij}) \quad (1,2,3)\\
&= (1-\delta_{12})(1-\delta_{13})(1-\delta_{23}) \quad (1,2,3)\\
&= 1-\delta_{12}-\delta_{13}-\delta_{23}+\delta_{12}\delta_{13}+\delta_{12}\delta_{23}+\delta_{13}\delta_{23}-\delta_{12}\delta_{13}\delta_{23} \quad (1,2,3)\\
&= (1,2,3)-(2,1,3)-\cancel{(2,2,2)}-(1,3,2)+(3,1,2)+\cancel{(2,2,2)}\\
&\qquad +(2,3,1)-(3,2,1)\ .
\end{aligned}$$

Thus the operator $\prod_{1\le i<j\le n} (1-\delta_{ij})$ may be used to evaluate a determinant $|a_{ij}| = \Sigma \pm a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

The comparison with Littlewood's formulae for Hall-Littlewood polynomials is of particular interest. (See [1] and [3].)

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A SMALLER UNIVERSAL COVER FOR SETS OF UNIT DIAMETER

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In 1914 Lebesgue posed the problem of finding a universal cover set (UC) of least area in the plane, that is, a set having a subset congruent to any given plane set of diameter unity. It is apparent that it is sufficient to consider subsets bounded by curves of constant unit width [1].

In 1920 J. Pál [3] showed that a regular hexagon with two corners FHH', BGG' removed forms a UC of area 0.84529945... (see Figure 1).

In 1936 R. Sprague [5] showed that the portion outside the two arcs IK (centre G) and KJ (centre H) can also be removed giving a UC of area 0.84413770... .

In 1975 H. C. Hansen [2] showed that two tiny corners bounded by arcs near H' and G' can also be removed, reducing the area by an amount of order 10^{-19}. Assuming that any Reuleaux n-gon can be covered in essentially only one way by the hexagon (apart from 60° rotations) Hansen showed his UC was minimal in the set-theoretic sense.

Here I show nonetheless that a UC of yet smaller area (but not symmetrical or convex) can be constructed by removing a portion above arc V'K' (centre V) and simultaneously restoring a smaller area bounded by an arc near H (centre W) that had been excluded by Pál. All arcs mentioned have unit radius. Alternatively, the mirror image areas in symmetry axis DA may be

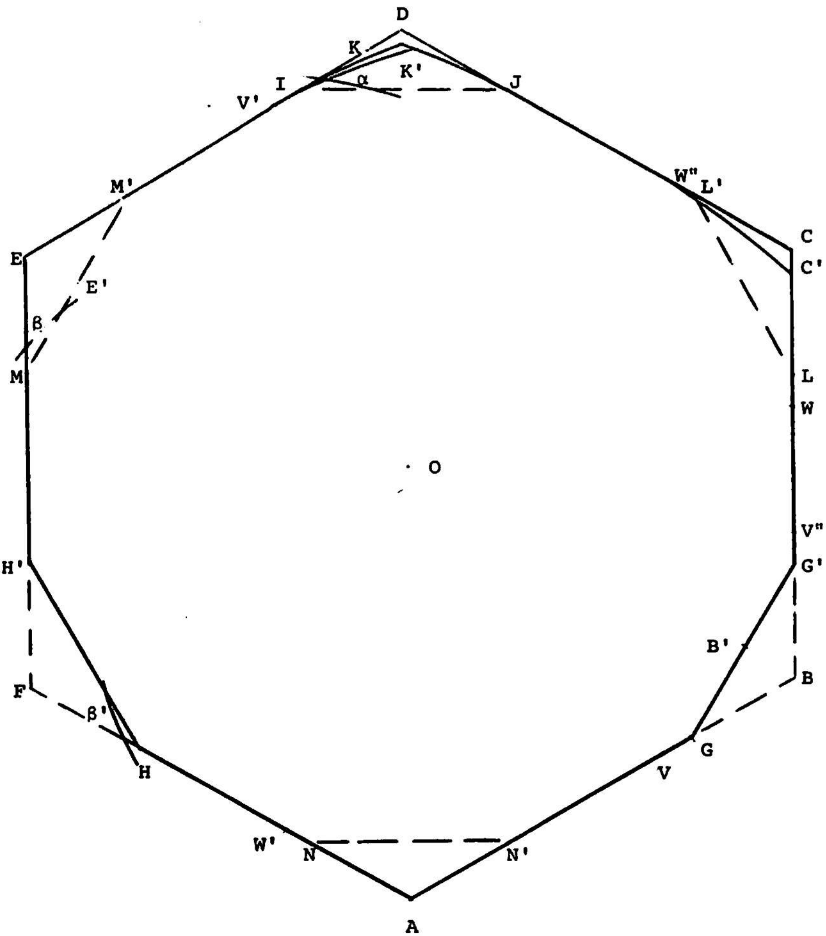

Figure 1.

adjusted. Thus two mirror image UC sets are possible, each one lacking symmetry in OA and being non convex near H or G, respectively.

Figure 1 shows regular hexagon ABCDEF circumscribed about the circle of radius 1/2 with centre O, while GG', HH', IJ, LL', MM' and NN' are portions of the sides of a congruent hexagon rotated 30°. Also V, V', V'', and W, W', W'', lie on the sides indicated at distance ε from G, I, G', and L, N, L' respectively.

Every curve c of constant width embedded in the Hansen UC meets side EI in a unique point, and side AG in the unique corresponding point defined by the unit double normal that is the diameter perpendicular to EI. We now rotate by 60° counter-clockwise about O all such c with any point above (or outside) arc V'K' (centre V). Any such c meets segment V'I because its curvature is bounded below by unity. Hence c meets VG and so lies within arc β (centre V) near M. After rotation the image curve c' lies near H within the image arc β' (centre W) and this portion of triangle FHH' will be restored to the UC. Because of its curvature, c avoids a neighbourhood of M' so c' avoids a neighbourhood of H' and so does not meet Hansen's small boundary arc near H'.

All our curves c likewise meet HA and JC in corresponding points. We now consider those c with points above arc α (centre W') near I, which therefore meet HA at or above W'. These curves all lie within an arc W"C' (centre W') and therefore are mapped by the 60° rotation on curves c' that lie below arc V'K' and so remain within the reduced area.

Since these curves c have points in upper triangle DIJ they do not meet lower triangle AN'N, so that c' avoids excluded corner triangle BG'G. These curves c also avoid a neighbourhood of N', because of their curvature, so that the c' avoid Hansen's small arc near G'. For these curves the images c' thus lie within the modified UC.

The remaining curves c to be rotated all meet HA below W'. We now rotate these a further 60° for a 120° positive rotation in all. Since these curves c have no points above arc α (centre W') near I, their images c" lie within the arc β' (centre W) near H.

With E' the midpoint of MM', we choose $\varepsilon < 0.01$ so that VE' > 1. Since c meets VG it follows that if c meets GG' it must do so on GB' where B' is the midpoint of GG'. Thus c does not meet B'G' and because of its curvature therefore does not meet V"G', a segment also of length ε. Hence c" does not meet V'I and therefore c' lies below arc V'K' (centre V). The inclusion of c" in the remaining corners of the modified UC is now easily verified.

We could adjust the symmetrically situated image areas near J and G by rotating certain curves clockwise. However, these include certain image curves that would then reoccupy the position first excluded. This creates a difficulty that has not been surmounted.

It remains to calculate the areas removed and restored, and to choose ε advantageously. We find that the perpendicular distance of K from ID is $h = (IK)^2/2 = 0.01105985\ldots$ and the area A_1 removed contains an area εh between two parallel arcs with centres V and G. A_1 also contains a double circular triangle of area approximately $0.085156\varepsilon^2$ near K; adjustments of order ε^3 must be made for deviations of the circular arcs from their tangents.

The portion A_2 restored near H has area $\varepsilon^{3/2}/3 - 3\sqrt{3}\varepsilon^2/8 - \varepsilon^{5/2}/16\cdots$. We choose $\varepsilon = 0.00056$ and conclude that the reduction in area is at least 2.00×10^{-6}. The reduced area is therefore 0.84413570... for this universal cover.

Our result confirms the opinions of earlier writers [1, 2, 3] that further reductions and complications underlie the problem of universal covers in the plane. It raises again the questions whether a universal cover of minimal area exists and is unique, convex, or symmetrical. I wish to thank B.C. Rennie whose description of this problem appears in [4].

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ON A SPECIAL TYPE OF SUBDIRECTLY IRREDUCIBLE LATTICE WITH AN APPLICATION TO PRODUCTS OF VARIETIES

by

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Abstract: We show that every lattice variety can be generated by a special type of subdirectly irreducible lattice. As an application we prove that there are only trivial solutions of the equation $X \circ Y = X \vee Y$ for lattice varieties X and Y. The special case $X = Y$ was earlier considered by A. Day.

1. Introduction. Let L be a subdirectly irreducible lattice. The interval $[a, b]$ ($a, b \in L$ and $a < b$) is critical iff $\mathrm{Con}(a, b)$ (the minimal congruence generated by $a \equiv b$) is the congruence relation making L subdirectly irreducible (that is, if C is a congruence relation of L, then either $C = \omega$ or $C \geq \mathrm{Con}(a, b)$).

R. P. Dilworth and R. Freese [2] proved that every lattice variety V is generated by those subdirectly irreducible lattices in V that have a prime critical interval, that is, a critical interval $[a, b]$ where b covers a. Going a step further, let us call such a lattice special if a is also meet - irreducible. Observe that this implies the following: (i) a is completely meet-irreducible; (ii) if $x \in L$ and $a < x$, then $b \leq x$. We shall call such an $[a, b]$ a special critical interval.

Theorem 1. Every lattice variety can be generated by special subdirectly irreducible lattices.

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We hope to convince the reader of the importance of the new condition (the meet-irreducibility of a) by applying Theorem 1 to a problem on products of lattice varieties.

Let X and Y be lattice varieties. Following H. Neumann [7] and A. I. Mal'cev [6], we define $X \circ Y$ as the class of all lattices L such that there is a congruence relation C on L with the property that all congruence classes belong to X and $L/C \in Y$. The class $X \circ Y$ is not, in general, a variety; however, it is always a quasivariety (that is, it is closed under the formation of products, ultraproducts, and sublattices).

We apply Theorem 1 to prove the following:

__Theorem 2.__ Let X and Y be incomparable lattice varieties (that is, neither $X \subseteq Y$ nor $Y \subseteq X$ hold). Then there is a (special) subdirectly irreducible lattice in $X \circ Y$ that belongs neither to X nor to Y.

We apply Theorem 2 to compare $X \circ Y$ with $X \vee Y$, the join of X and Y as varieties. Obviously, $X \vee Y \subseteq X \circ Y$ always holds.

A. Day [1] examined the equation

$$X \circ X = X$$

and showed that it has only the trivial solutions: $X = T$ (the class of one-element lattices) and $X = L$ (the class of all lattices).

Finally, we solve for lattice varieties the equation

$$X \circ Y = X \vee Y.$$

Let us call the pair X, Y __trivial__ iff X or Y is T or L (that is, $\{X, T\} \cap \{T, L\} \neq \emptyset$). It is obvious, that if X, Y is trivial, then $X \circ Y = X \vee Y$.

Theorem 3. The equation $X \circ Y = X \vee Y$ has only trivial solutions for lattice varieties.

In other words, if X and Y are lattice varieties and $\{X, Y\} \cap \{T, L\} = \emptyset$, then

$$X \vee Y \subset X \circ Y.$$

In fact, we conjecture that if $X \circ Y$ is a variety and $X \vee Y \subset X \circ Y$, then there are continuumly many lattice varieties satisfying $X \vee Y \subset Z \subset X \circ Y$. This is verified for $X = Y = D$ (the class of distributive lattices) in [4].

For undefined notation and the basic properties of lattice varieties, the reader is referred to [3].

2. Proofs. To prove Theorem 1, it is sufficient to prove that if $u < v$ in a lattice K, then there exists a special subdirectly irreducible lattice L and a homomorphism $\alpha : K \to L$ with $u\alpha \neq v\alpha$.

Let α_1 be the natural embedding $\alpha_1 : x \to (x]$ of K into $I(K)$. Let I be an ideal of K maximal with respect to containing u but not v. Let us define $a' = I$ and $b' = I \vee (v]$.

It is clear that $[a', b']$ is a prime interval and that a' is meet-irreducible. Now let C be the (unique) maximum congruence relation of $I(K)$ such that $a' \not\equiv b'$ (C). Set $L = I(K)/C$ and let α_2 be the natural homomorphism of $I(K)$ onto L. It is easily seen that L, $a = a'\alpha_2$, $b = b'\alpha_2$, and $\alpha = \alpha_1\alpha_2 : K \to L$ satisfy the requirements.

One can strengthen Theorem 1 by replacing "special subdirectly irreducible" by "finitely generated special subdirectly irreducible". (In the proof, take K to be finitely generated, and consider the sublattice of $I(K)$ generated by $(K\alpha) \vee \{a'\}$.)

To prove Theorem 2, let X and Y be incomparable lattice varieties. By Theorem 1, we can choose a special subdirectly irreducible lattice $L \in Y - X$ with a special critical interval $[a, b]$. We can further choose a finitely generated subdirectly irreducible lattice $K \in X - Y$ with a critical interval $[u, v]$; observe that K has least and greatest elements, o and i.

We set $o = a$, and if b is join-irreducible, we also set $i = b$. Hence, $L \cap K$ is either $\{a\}$ or $\{a, b\}$. We now define a lattice on the set

$$A = L \cup K$$

by requiring that L and K be sublattices of A and $i \leq b$. It follows, for $x \in K - \{a, b\}$ and $y \in L - \{a\}$, that (a) $x < y$ in A iff $a < y$ in L iff $b \leq y$ in L; and (b) $x > y$ in A iff $a > y$ in L. For example, if $x > y$ in A and $a \nleq y$, then $i \geq x \geq y \vee a \geq b$, a contradiction. The equivalence relation C on A, defined by making $K = [a, i]_A$ the only nontrivial class, is a congruence relation on A. Moreover, A/C is isomorphic to L, or to L with a and b identified in case b is join-irreducible. Thus, $A \in X \circ Y$. A is subdirectly irreducible with critical interval $[u, v]$. If $A \in X$, then L, a sublattice of A, also belongs to X, contradicting that $L \in Y - X$. The assumption $A \in Y$ similarly contradicts that $K \in X - Y$. This completes the proof of Theorem 2.

To prove Theorem 3, let $X \circ Y = X \vee Y$ and let $\{X, Y\} \cap \{T, L\} \neq \emptyset$. If X and Y are incomparable, then by Theorem 2 there exists a subdirectly irreducible lattice

$$A \in X \circ Y - (X \cup Y).$$

But by Jónsson's Lemma (see [5] or [3]) there is no subdirectly irreducible lattice in $X \vee Y - (X \cup Y)$, contradicting $X \circ Y = X \vee Y$.

If X and Y are comparable we have two cases to consider.

<u>Case 1:</u> X is contained in Y. In this case we prove that $D_n \subseteq Y$ for all $1 \leq n < \omega$. Indeed, since $Y \neq T$, the inclusion holds for $n = 1$. If $D_n \subseteq Y$, then $X \neq T$, implies that $D \subseteq X$ and so

$$D_{n+1} = D \circ D_n \subset X \circ Y = X \vee Y = Y.$$

By A. Day [1], this implies that $Y = L$, a contradiction.

<u>Case 2:</u> Y is contained in X. Using the containment: $A \circ (B \circ C) \subseteq (A \circ B) \circ C$ from [6] it follows by induction that

$$D \circ D_n \subseteq D_n \circ D \quad \text{for all } 1 < n < \omega.$$

Then we can proceed as in Case 1.

This completes the proof of Theorem 3.

Theorem 2 also follows from a result of [4]: every lattice in $X \circ Y$ can be embedded in a subdirectly irreducible lattice in $X \circ Y$. The proof of this result is technically quite complicated. It is interesting that to get Theorem 3 we could substitute the very easy construction of Theorem 1.

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ON A GOURSAT TYPE PROBLEM

by

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We consider here a Goursat type problem for a domain in which the equation is hyperbolic except on one boundary where the equation is parabolic. Let us assume the equation

$$(+)\qquad Lu = k(t).\sum_{i=1}^{n} u_{x_i x_i} + u_{tt} + r(x,t).u = 0,$$

where $x=(x_1,x_2,\dots,x_n)\in R^n$, k is non-positive for t non-positive, $k'(t)\neq 0$ for $t<0$, $n\geq 2$, $k\in C^2(\bar{D})$, $r\in C^1(\bar{D})$, in a domain D which is simply connected bounded by the hyperplane $t=0$, and by the hypersurfaces

$$S_4:\quad \sum_{i=1}^{n} x_i = n^{1/2}.\int_0^t (-k(s))^{1/2}.ds,$$

$$S_3:\quad \sum_{i=1}^{n} x_i^2 = (1+\int_0^t (-k(s))^{1/2}.ds\,)^2,$$

for $t<0$. M.H.Protter [1] has studied boundary value problems for equation (+), $k=-1$, $r=0$, $n=2$. Our paper intends to extend Protter's uniqueness results for classical solutions u in D by applying the well-known energy integral method. Denote by $v=(v_1, v_2,\dots,v_n,v_{n+1})$ the normal vector on the boundary of D, such that $v_{n+1}\geq 0$ on S_4, and $v_{n+1}\leq 0$ on S_3.

PROBLEM (R). *Find a function $u(x,t)\in C^2(\bar{D})$ which is a solution of equation (+) in D, and u assumes prescribed continuous boundary values on S_3 and on the hyperplane $S_0: t=0$, that is*

$$(++)\qquad u=f\in C^2(\bar{D})\quad \text{on } S_0\cup S_3.$$

THEOREM. *Assume the domain D described above. If the conditions $F(t)=1+2.(k/k')'>0$, and*

$$R(x,t)=r.(1-2.(k/k')')-2.(\sum_{i=1}^{n} r_{x_i}.(-k/n)^{1/2}+r_t).(k/k')\leq 0$$

hold in D, then there exists at most one classical solution

$u(x,t)\in C^2(\bar{D})$ of (R).

PROOF. For uniqueness proof, we shall show that $f=o$ on $S_o \cup S_3$ implies that $u=o$ in D. We investigate the expression

(1) $$2(Pu,Lu)=2.\iint_D Pu.Lu.dxdt = o ,$$

where

(2) $$Pu=-u \quad \text{on } S_o,$$
$$Pu=-u+(-k/n)^{1/2}.(\sum_{i=1}^{n} u_{x_i}+u_t).(-4k/k') \text{ for } t<o.$$

See [2]. By applying Green's formula we find

(3) $$2(Pu,Lu)=-\iint_D 2(r+\sum_{i=1}^{n}(-k/n)^{1/2}.r_{x_i}.(-2k/k')+(-2k/k').r_t$$
$$-2(k/k').r).u^2.dxdt+\iint_D 2.F(t).(u_t^2-k.\sum_{i=1}^{n}u_{x_i}^2$$
$$+2(-k/n)^{1/2}.\sum_{i=1}^{n}u_{x_i}u_t).dxdt+\int_{S_4}k'.u^2.(1/2).(-k)^{-1/2}.dx$$
$$+\int_{S_4}Q(u_x,u_t).dx=o,$$

where

(4) $$Q(u_x,u_t)=\sum_{i=1}^{n}k.8.(k/nk').(k/k-1)^{1/2}.u_{x_i}^2+8(-k/k')$$
$$.(k/k-1)^{1/2}.u_t^2+\sum_{i\neq j}^{n}k(-kn^{-1})^{1/2}.(-4k/k').(v_i+v_j).u_{x_i}$$
$$.u_{x_j}+2.\sum_{i=1}^{n}k.(v_i+(-k/n)^{1/2}.v_{n+1}).(-4k/k'),$$

and

(5) $$v=(n-nk)^{-1/2}.(1,1,\ldots,1,-(-nk)^{1/2})$$

on S_4.

It can be easily proved that Q is non-negative on S_4. Hence, all the integrals in (3) are non-negative in D which imply that $u=o$ on S_4 and therefore $u\equiv o$ in D, and this completes the proof.

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