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# PERPENDICULARITIES AND SEMISIMILARITIES



Erich W. Ellers*

*Presented by H.S.M. Coxeter, F.R.S.C.*

## § 1. Introduction

Recently, R.J.Plymen and C.M.Williams [3] published an essentially computational proof of the fact that every perpendicularity is induced by a semisimilarity. They assume that all spaces are finite-dimensional and of index zero. Reinhold Baer gives a conceptual proof of a similar theorem, using dualities (cf. [1], IV, 5). Especially for applications in physics it is desirable to extend the validity of this theorem to infinite-dimensional vector spaces. The purpose of this note is to show that this can be done. We shall prove a theorem that is more general than those in [1] or [3].

## § 2. Orthogonality-Preserving Semilinear Bijections

We shall introduce a number of concepts and set up the notation that we shall use. Then we shall show in our Theorem 1 that every orthogonality-preserving semilinear bijection is in fact a semisimilarity.

Let V be a vector space over a field K and J an anti-automorphism of K. The mapping $f\colon V\times V \to K$ is an inner product if it has the following property,

$$f(\alpha x_1 + x_2, \beta y_1 + y_2) = \alpha f(x_1,y_1)\beta^J + \alpha f(x_1,y_2) + f(x_2,y_1)\beta^J + f(x_2,y_1$$

for all $x_1,x_2,y_1,y_2 \in V$ and $\alpha,\beta \in K$. The inner product f is zero-symmetric if the equality $f(x,y) = 0$ is equivalent to $f(y,x) = 0$. The pair $(V,f)$, where V is a vector space and f a zero-symmetric inner product, is called a metric vector space.

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The following is well known: If f is zero-symmetric, then there is some $\varepsilon \in K\setminus\{0\}$ such that $f(x,y) = \varepsilon f(y,x)^J$, $\varepsilon\varepsilon^J = 1$, and $J^2 = I_\varepsilon$, where $I_\varepsilon$ is the inner automorphism $\xi \to \varepsilon^{-1}\xi\varepsilon$ for all $\xi \in K$. Furthermore, if (V,f) is not symplectic, then $\varepsilon$ can be chosen to be 1. If (V,f) is symplectic, then $\varepsilon = -1$ and J is the identity.

Let V be a vector space with an inner product f. If M is any subset of V, then we define $M^0 = \{x \in V;\ f(M,x) = \{0\}\}$ and ${}^0M = \{x \in V;\ f(x,M) = \{0\}\}$. Clearly, $M^0$ and ${}^0M$ are subspaces of V. In general, $M^0$ and ${}^0M$ are distinct. If f is zero-symmetric, then $M^0 = {}^0M$ and the notation $M^\perp$ is commonly used instead of $M^0$.

A metric vector space (V,f) is regular if $V^\perp = \{0\}$.

THEOREM 1. Let V and V' be vector spaces over the fields K and K', f and f' inner products for V and V', respectively. Let J and J' be the antiautomorphisms of f and f', respectively. Let $\sigma$ be a semilinear bijection of V onto V' with isomorphism $\varkappa$ of K onto K'. Assume $f(x,y) = 0$ if and only if $f'(x^\sigma,y^\sigma) = 0$.
Then there is some $\alpha \in K'\setminus\{0\}$ such that $f'(x^\sigma,y^\sigma) = f(x,y)^\varkappa\alpha$ for all $x,y \in V$.
If f and f' are zero-symmetric and if $f(V,V) \neq \{0\}$, then in addition $\alpha^{J'} = \alpha$ and $\varkappa J' = J\varkappa I_\alpha$.

Proof. We first observe that $f(x,y) = f(z,y)$ if and only if $f'(x^\sigma,y^\sigma) = f'(z^\sigma,y^\sigma)$ for all $x,y,z \in V$; namely, the following statements are equivalent: $f(x,y) = f(z,y)$, $f(x-z,y) = 0$, $f'((x-z)^\sigma,y^\sigma) = 0$, $f'(x^\sigma,y^\sigma) = f(z^\sigma,y^\sigma)$.

We can assume that there are $a,b \in V$ such that $f(b,a) \neq 0$. Then $f'(b^\sigma,a^\sigma) = f(b,a)^\varkappa\alpha$ for some $\alpha \in K'\setminus\{0\}$.

Now we shall establish that $f'(t^\sigma,a^\sigma) = f(t,a)^\tau\alpha$ for all $t \in V$. This is trivially true if $f(t,a) = 0$. Therefore, we can assume $f(t,a) \neq 0$ and consequently also $f'(t^\sigma,a^\sigma) \neq 0$. Then there is a unique $\beta \in K'\setminus\{0\}$ such that $\beta f'(t^\sigma,a^\sigma) = f'(b^\sigma,a^\sigma)$. Hence $f'((\beta^{\tau^{-1}}t)^\sigma,a^\sigma) = f'(b^\sigma,a^\sigma)$. Thus $f(\beta^{\tau^{-1}}t,a) = f(b,a)$. Consequently, $\beta f'(t^\sigma,a^\sigma) = f'(b^\sigma,a^\sigma) = f(b,a)^\tau\alpha = f(\beta^{\tau^{-1}}t,a)\alpha = \beta f(t,a)^\tau\alpha$.

Finally, we obtain $f'(x^\sigma,y^\sigma) = f(x,y)^\tau\alpha$ for all $x,y \in V$; namely, there is some $t \in V$ such that $f'(x^\sigma,y^\sigma) = f'(t^\sigma,a^\sigma) = f(t,a)^\tau\alpha = f(x,y)^\tau\alpha$.

Now we assume that $f$ and $f'$ are zero-symmetric and $f(V,V) \neq \{0\}$. Under our assumptions, $V$ is symplectic if and only if $V'$ is symplectic. Therefore $\varepsilon = \varepsilon'$; namely, $\varepsilon = \varepsilon' = -1$ in the symplectic case, and we can choose $\varepsilon = \varepsilon' = 1$ in the remaining cases as we have mentioned at the beginning of this section. The equation $f'(x^\sigma,y^\sigma) = f(x,y)^\tau\alpha$ implies now $\alpha^{J'}f(x,y)^{\tau J'} = f'(x^\sigma,y^\sigma)^{J'} = \varepsilon' f'(y^\sigma,x^\sigma) = \varepsilon' f(y,x)^\tau\alpha$ for all $x,y \in V$. For every $\lambda \in K$ there are $x$ and $y$ such that $f(x,y) = \lambda$. Then $\alpha^{J'}\lambda^{\tau J'} = \varepsilon'(\varepsilon f(x,y)^J)^\tau\alpha = \varepsilon'\varepsilon^\tau\lambda^{J\tau}\alpha$, hence $\alpha^{J'}\lambda^{\tau J'} = \lambda^{J\tau}\alpha$. For $\lambda = 1$ we get $\alpha^{J'} = \alpha$. Finally, we have $\tau J' = J\tau I_\alpha$.

For $\sigma$ = identity, Theorem 1 extends a well-known result to infinite-dimensional vector spaces.

## § 3. Perpendicularities

Let $V$ be a vector space over the field $K$. Then there is a projective geometry attached to $V$ which we denote by $\mathrm{proj}V$. The set $\mathrm{proj}V$ consists of all subspaces of $V$. The one-dimensional subspaces of $V$ are called points of $\mathrm{proj}V$ and the two-dimensional subspaces of $V$ are the lines of $\mathrm{proj}V$. Clearly, every inner product for $V$ establishes an orthogonality for the points of $\mathrm{proj}V$.

Let V and V' be vector spaces over K and K', f and f' inner products for V and V', respectively. If $\pi$ is a biject of the points of projV onto the points of projV', and if $f(A,B) = \{0\}$ is equivalent to $f'(A^{\pi},B^{\pi}) = \{0\}$ for all point A,B in projV, then $\pi$ is called a perpendicularity of projV onto projV'.

Our next lemma asserts that for a vector space V with inner product f, collinearity can be expressed in terms of orthogonality if the right radical $V^0$ of V consists of the vector only. Clearly, a similar theorem holds for left orthogonality. We shall omit the easy proof.

LEMMA 2. Let V be a vector space and f an inner product for such that $V^0 = \{0\}$.
Then the set $\{x_1,\ldots,x_k\} \subset V$ is independent if and if $\bigcap_{i=1}^{k} x_i^0 \neq \bigcap_{\substack{i=1 \\ i\neq j}}^{k} x_i^0$ for each $j = 1,\ldots,k$.

A semilinear bijection $\sigma$ of V onto V' with automorphis is called a semisimilarity if there is some $\alpha \in K'\setminus\{0\}$ such $f'(x^{\sigma},y^{\sigma}) = f(x,y)^{\kappa}\alpha$ for all $x,y \in V$.

Clearly, every semisimilarity induces a perpendicularit of the points of projV onto the points of projV'. We shall that the converse is also true.

THEOREM 3. Let V and V' be vector spaces over K and K' with $\dim V, \dim V' \geq 3$, f and f' inner products for V an V', respectively. Assume that $V^0 = \{0\}$ or ${}^0V = \{$
If $\pi$ is a perpendicularity of the points of projV onto the points of projV', then $\pi$ is induced by semisimilarity $\sigma$ of V onto V'.

Proof. The perpendicularity $\pi$ is a collineation by Lemma 2. Therefore, $\pi$ is induced by a semilinear bijection $\sigma$. Now our assertion follows from Theorem 1.

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# Complex quaternions and the Lorentz group

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Abstract: It is well known that the universal covering group of $SO(4)$ is $S^3 \times S^3$ ( $S^3$ the group of unit quaternions) and that the covering projection $\Psi: S^3 \times S^3 \to SO(4)$ is given by

$$\Psi_{p,q}(x) = p x q^{-1}, \qquad x \in \mathbb{R}^4$$

[cf. [1] sec. 8.24). In this paper an analogous result is proved for the (orthochronous) Lorentz group $SO(1,3)$. It is shown that its universal covering is the complex 3-sphere in $\mathbb{C}^4$ given by

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1$$

and that the covering projection arises from the complex algebra of quarternions.

## 1. THE COMPLEX QUATERNION ALGEBRA.

Let $E$ be a 4-dimensional complex vector space with a non-degenerate complex inner product. A vector $a \in E$ will be called a unit vector, if $(a,a) = 1$; it will be called a (complex) light vector, if $(a,a) = 0$.

A normed determinant function in an n-dimensional complex vector space is a skew symmetric $n$-linear function $\Delta$ which satisfies the Lagrange identity,

$$\Delta_E(x_1, \ldots, x_n)\, \Delta_E(y_1, \ldots, y_n) = \det((x_i, y_j)) .$$

Such a function always exists and is uniquely determined up to sign.

Now choose a unit vector $e \in E$ and denote its orthogonal complement by $E_1$. Then, if $\Delta_E$ is a normed determinant function in $E$,

$$D(y_1, y_2, y_3) = \Delta_E(e, y_1, y_2, y_3), \quad y_\nu \in E_1$$

is a normed determinant function in $E_1$. We shall use $D$ to define the complex cross-product in $E_1$ as follows: Let $a \in E_1$, $b \in E_1$. Then $a \times b$

is the unique vector which satisfies the identity

$$(a \times b, z) = D(a, b, z), \quad z \in E_1 .$$

Next we define a complex multiplication in $E$ by

$$xy = -(x,y)e + x \times y \qquad x, y \in E_1$$

and

$$ey = y, \quad x \cdot e = x \qquad x, y \in E.$$

Then $E$ becomes an associative algebra, called the <u>complex quaternion algebra</u>. The product satisfies the relation

$$(xy, xy) = (x,x)(y,y) \qquad x, y \in E.$$

Observe that (in contrast to the real case) $E$ is <u>not</u> a division algebra. In fact, the invertible elements are precisely those which satisfy $(x,x) \neq 0$.

The center of $E$ consists of the vectors $\lambda e$, $\lambda \in \mathbb{C}$.

2. <u>THE ROTATIONS</u> $\tau_a$. Let $a$ be a unit vector in $E$ and set

$$\tau_a(x) = a x a^{-1} \qquad x \in E.$$

Then $\tau_a$ is a proper complex rotation of $E$. Since $\tau_a(e) = e$, $\tau_a$ induces a proper complex rotation $\sigma_a$ in $E_1$. It is explicitly given by

(1) $$\sigma_a(y) = (2\alpha^2 - 1)y + 2\alpha\, p \times y + 2(y, p)p, \qquad y \in E_1$$

where $a$ is decomposed in the form

$$a = \alpha e + p \qquad \alpha \in \mathbb{C},\ p \in E_1 .$$

This equation shows that

$$\operatorname{tr} \sigma_a = 4\alpha^2 - 1.$$

In particular,

$$\operatorname{tr} \sigma_a = 3 \rightleftarrows \alpha = \pm 1 \rightleftarrows (p,p) = 0$$

and

$$\operatorname{tr} \sigma_a = -1 \rightleftarrows \alpha = 0 \rightleftarrows (p,p) = 1 .$$

<u>PROPOSITION I</u>: To every proper complex rotation $\varphi$ of $E_1$ there exists a unit vector $a$ such that $\sigma_a = \varphi$.

Proof: The characteristic polynomial of $\varphi$ is given by

$$f(\lambda) = (\lambda - 1) g(\lambda) \text{ where } g(\lambda) = \lambda^2 + (1 - \operatorname{tr}\varphi)\lambda + 1.$$

Consider first the case that $\operatorname{tr}\varphi \neq 3$ and $\operatorname{tr}\varphi \neq -1$. Then $g$ has two roots $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$. It follows that the corresponding eigenvectors $z_1$ and $z_2$ are linearly independent light vectors. Thus $\varphi$ has three eigenvectors $z_0$, $z_1$ and $z_2$, where $z_0$ corresponds to the eigenvalue $\lambda_0 = 1$. Since the inner product is non-degenerate, it follows that $z_0$ is not a light vector. Hence we can normalize $z_0$ such that

$$(z_0, z_0) = -\frac{1}{4}(\lambda + \lambda^{-1} + 2).$$

Now set

$$a = e \cosh\theta + z_0$$

where $\theta$ is a complex number which satisfies $e^{2\theta} = \lambda$. Then $a$ is a unit vector, and a straightforward calculation shows that $\sigma_a = \varphi$.

In the two exceptional cases ($\operatorname{tr}\varphi = -1$ and $\operatorname{tr}\varphi = 3$) one has to use the Jordan normal form of $\varphi$. It turns out that, if $\operatorname{tr}\varphi = -1$, then $\varphi = \sigma_a$ where $a$ is a unit vector in $E_1$, and if $\operatorname{tr}\varphi = 3$, then $\varphi = \sigma_a$ where $a = e + p$ and $p$ is a light vector in $E_1$.

3. THE HOMOMORPHISM $\Phi$. Let $\Gamma$ denote the group of unit vectors in $E$ and let $G$ be the group of proper rotations of $E$. Then a homomorphism $\Phi : \Gamma \times \Gamma \to G$ is defined by

$$\Phi_{a,b}(x) = a x b^{-1}, \qquad x \in E.$$

THEOREM I: The homomorphism $\Phi$ is surjective and its kernel consists of the elements $(e, e)$ and $(-e, -e)$.

Proof: If $\Phi_{a,b} = \iota$, we have $ax = xb$ for all $x \in E$. Setting $x = e$ yields $a = b$ and so we have $ax = xa$. It follows that $a = \pm e$ and so $(a, b) = (e, e)$ or $(a, b) = (-e, -e)$.

Next, let $\varphi \in G$ be given and set $\psi(x) = c^{-1}\varphi(x)$, where $c = \varphi(e)$. Then $\psi(e) = e$ and so, by Proposition I, there exists a vector $b \in E$ such that $\psi(x) = bxb^{-1}$. It follows that $\Phi_{a,b} = \varphi$ where $a = cb$.

4. THE LORENTZ GROUP. Let $F$ be a real 4-dimensional vector space with an inner product $(\ ,\ )_L$ of type $(+,-,-,-)$. The orthochronous Lorentz group is the group $SO(1,3)$ of proper isometries $\varphi$ of $F$ which satisfy the condition $(x, \varphi x)_L > 0$ for all space-like vectors $x$ (this follows, if $(e, \varphi e)_L > 0$ for a particular space-like vector $e$).

Now consider the 4-dimensional complex vector space $E = \mathbb{C} \otimes F$ with the complex inner product

$$(\alpha \otimes x, \beta \otimes y) = \alpha\beta\,(x,y)_L.$$

As usual, we shall identify every $x \in F$ with $1 \otimes x$. Then $F$ becomes a subspace of $E$ and we have $(x,y) = (x,y)_L$, $x, y \in F$.

Next, let $\Delta_F$ be a normed determinant function in $F$,

$$\Delta_F(x_1, \ldots, x_4)\Delta_F(y_1, \ldots, y_4) = -\det\left((x_i, y_j)_L\right)$$

(cf. [1], sec. 9.19) and set

$$\Delta_E(\alpha_1 \otimes x_1, \ldots, \alpha_4 \otimes x_4) = i\,\alpha_1 \cdots \alpha_4\,\Delta_F(x_1, \ldots, x_4).$$

Then $\Delta_E$ is a normed determinant function in $E$. Finally, fix space-like unit vector $e$ and consider the corresponding complex quaternion multiplication in $E$ (cf. sec. 1).

The conjugation in $E$ is the antlinear map $z \mapsto \tilde{z}$ given by $\alpha \otimes x \mapsto \bar{\alpha} \otimes x$. It satisfies the relations

(2) $(\tilde{z}_1, \tilde{z}_2) = \overline{(z_1, z_2)}$ and $\widetilde{z_1 \cdot z_2} = \tilde{z}_2 \cdot \tilde{z}_1$, $z_1, z_2 \in E$

Moreover, if $a \neq 0$, then

(3) $$(a\tilde{a}, e) > 0.$$

5. THE HOMOMORPHISM $\Psi$. Let $a$ be a unit vector in $E$ and consider the transformation $\tau_a(z) = a z \tilde{a}$, $z \in E$. Then the second relation (2) implies that

$$(\tau_a z)^{\sim} = a \tilde{z} \tilde{a} = \tau_a(\tilde{z})$$

and so $\tau_a$ restricts to a linear transformation of $F$. Since

$$(\tau_a x, \tau_a y)_L = (\tau_a x, \tau_a y) = (x, y) = (x, y)_L \qquad x, y \in F$$

$$\det \tau_a = 1 \quad \text{and} \quad (e, \tau_a e) = (e, a\tilde{a}) > 0,$$

$\tau_a$ is an orthochroneous Lorentz transformation.

THEOREM II: Let $\Psi : \Gamma \to SO(1,3)$ be the homomorphism given by $\Psi(a) = \tau_a$. Then $\Psi$ is surjective and its kernel consists of $e$ and $-e$.

Proof: To obtain the kernel of $\Psi$, simply observe that $\Psi(a) = \Phi(a, \tilde{a}^{-1})$ (where $\Phi$ is the homomorphism defined in sec. 3), and apply the second part of Theorem I. Next, let $\psi \in SO(1,3)$ be given and define $\varphi$ by $\varphi(\alpha \otimes x) = \alpha \otimes \psi(x)$. Then $\varphi$ is a proper rotation of $E$. Hence, by Theorem I, there are unit vectors $a$ and $b$ such that

$$\varphi(z) = a z b^{-1}, \qquad z \in E.$$

To show that $b^{-1} = \tilde{a}$ observe that

$$\varphi((\alpha \otimes x)^{\sim}) = \bar{\alpha} \otimes \psi(x) = (\varphi(\alpha \otimes x))^{\sim},$$

whence $(\varphi z)^{\sim} = \varphi(\tilde{z})$. This implies that

$$\tilde{b}^{-1} \tilde{z} \tilde{a} = a \tilde{z} b^{-1}, \qquad z \in E,$$

whence $b^{-1} = \varepsilon \tilde{a}$, $\varepsilon = \pm 1$.

To show that $\varepsilon = +1$ observe that, since $\psi$ orthochroeous, we have $(e, \psi e)_L > 0$. On the other hand, by equation (3), $\varepsilon(\psi e, e) = (a\tilde{a}, e) > 0$. It follows that $\varepsilon = +1$. Thus $\varphi z = a z \tilde{a}$ and hence $\psi x = a x \tilde{a}$ for $x \in F$. This shows that $\Psi$ is surjective.

The theorem shows that the homomorphism $\Psi : \Gamma \to SO(1,3)$ is a double covering.

Finally we show that $\Gamma$ is simply connected. In fact, consider the deformation

$$\varphi_t(z) = \frac{x + t\,iy}{\sqrt{(x,x)_E(1-t^2) + t^2}} \qquad z = x + iy, \quad 0 \leq t \leq 1$$

where $(\ ,\ )_E$ is a Euclidean inner product in F. Then $\varphi_0(z) = \frac{x}{|x|}$ and $\varphi_1(z) = z$. Thus the Euclidean 3-sphere $(x,x)_E = 1$, is a deformation retract of $\Gamma$ and so $\Gamma$ is simply connected. It follows that $\Gamma$ is the universal covering of the orthochronous Lorentz group.

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# Products of Reflections in $U(p,q)$

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The unitary group $U(p,q)$, $p+q = n$, is the group of $n\times n$ complex matrices fixing the hermitian form

$$f(x,y) = \sum_{i=1}^{p} \bar{\xi}_i \eta_i - \sum_{i=p+1}^{n} \bar{\xi}_i \eta_i,$$

where $x = (\xi_i)$, $y = (\eta_i) \in C^n$.

An element $r \in U(p,q)$ is called a reflection if it fixes pointwise a hyperplane $V$ of $C^n$ and has $-1$ as an eigenvalue. We say that $r$ is a positive (resp. negative) reflection if $0 \neq a \in V^{\perp}$ implies $f(a,a) > 0$ (resp. $f(a,a) < 0$). If $p,q \geq 1$ the positive and negative reflections form distinct conjugacy classes in $U(p,q)$.

We address the following question: Given $u \in U(p,q)$ with $\det(u) = \pm 1$, what is the smallest number $k = \ell(u)$ such that $u$ can be written as a product

$$u = r_1 r_2 \ldots r_k$$

with $r_1, r_2, \ldots, r_k$ positive reflections? (We take $\ell(1) = 0$). The integer $\ell(u)$ is called the length of $u$ with respect to positive reflections.

The case $q = 0$ was solved in our paper [2]. When $p,q \geq 1$ the problem is more difficult. Before the result can be described, some definitions are needed:

Definition 1. For $u \in U(p,q)$ we shall denote by $E(u)$ the 1-eigenspace of $u$, i.e., $E(u) = \mathrm{Ker}(u-1)$.

Lemma 1. If $E(u)$ has dimension $d$, then there exist positive reflections $r_1,\ldots,r_d$ such that $r_1\ldots r_d u = \tilde{u}$ satisfies $E(\tilde{u}) = 0$. Furthermore, the phase of $\det(1-\tilde{u})$ is independent of the choice of $r_1,\ldots,r_d$.

In view of Lemma 1 we can make the following:

Definition 2. For $u \in U(p,q)$ we define the invariant, $\omega(u)$, of $u$ by

$$\omega(u) = (-1)^q \frac{\det(1-\tilde{u})}{|\det(1-\tilde{u})|} .$$

If $\det(u) = \pm 1$, then it is easily seen that $\omega(u) = \pm 1, \pm i$. Thus the invariant partitions the group

$$G = \{u \in U(p,q) \mid \det(u) = \pm 1\}$$

into four parts.

Definition 3. We say that $u \in U(p,q)$ is loxodromic if $u$ has an eigenvalue $\lambda$ with $|\lambda| \neq 1$.

Definition 4. For $u \in G$ we will write

$$L(u) = \mathrm{rank}(1-u) + 1 - \mathrm{Re}\,\omega(u) .$$

We can now state a theorem concerning the lengths of some of the elements of $G$:

Theorem 1. If $u \in G$ is loxodromic, then $\ell(u) = L(u)$. For all $u \in G$, $\ell(u) = L(u) + \delta$ where $\delta = 0$ or $2$.

The remaining problem is thus to determine when $\delta = 2$. In order to describe the result we must rely on the description of conjugacy classes in $U(p,q)$ for which we refer to [1]. Each $u \in U(p,q)$ can be broken into direct sum of indecomposable elements belonging to unitary groups of perhaps smaller dimension. The indecomposable elements are classified into several types as follows:

(1) $\Delta_m(\lambda,\bar{\lambda}^{-1})$, $|\lambda| \neq 1$;

(2) $\Delta_m^{\pm}(\lambda)$, $|\lambda| = 1$.

The type (1) consists of two Jordan blocks of size $m+1$ with corresponding eigenvalues $\lambda$ and $\bar{\lambda}^{-1}$. The type (2) has a single Jordan block of size $m+1$ with eigenvalue $\lambda$; there are two conjugacy classes of such blocks and they are distinguished by the superscript + or −.

Definition 5. We say that $u \in U(p,q)$ is exceptional if the subspace $E(u)$ is negative semi-definite.

Definition 6. We say that $u \in U(p,q)$ is of the first kind if $\mathrm{Im}(1-u)$ is a positive semi-definite but not totally isotropic subspace.

Definition 7. Suppose that the indecomposable summands of $u \in U(p,q)$ are the following: $\Delta_0^+(\exp(i\alpha_j))$, $1 \le j \le r$; $\Delta_0^-(\exp(i\beta_k))$, $1 \le k \le s$; $\Delta_1^-(\exp(i\gamma))$ with multiplicity $q$; $\Delta_1^+(1)$ with multiplicity $p$; as well as blocks $\Delta_0^+(1)$ and $\Delta_0^-(1)$. Suppose further that $\alpha_j \le \gamma \le \beta_k$ for all $j$ and $k$, and that $r,s \ge 1$ if $q = 0$, and $\alpha_j \ge \pi$ if $p = q = 0$ and $\alpha_j = \beta_k$ for all $j$ and $k$. Then we say that $u$ is of the second kind.

Definition 8. If for $u \in U(p,q)$ the complex conjuagate $\bar{u}$ is of the second kind, we say that $u$ is of the third kind.

The types described in Definitions 5-8 are pictured below, where light dots denote $\Delta_0^+$ types and dark dots $\Delta_0^-$ types. (We ignore the summands $\Delta_0^{\pm}(1)$).

EXCEPTIONAL

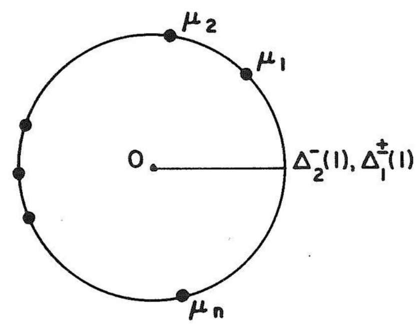


Fig. 1

FIRST KIND

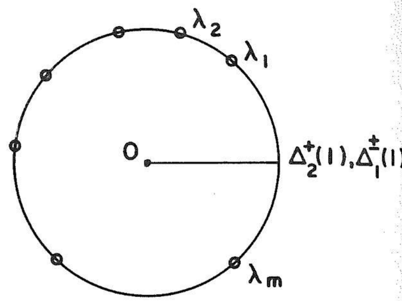


Fig. 2

SECOND KIND

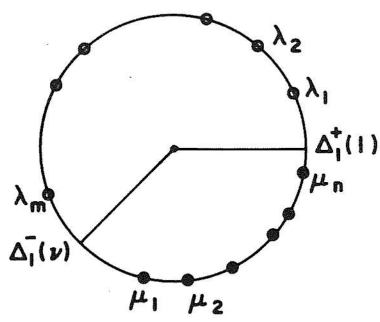


Fig. 3

THIRD KIND

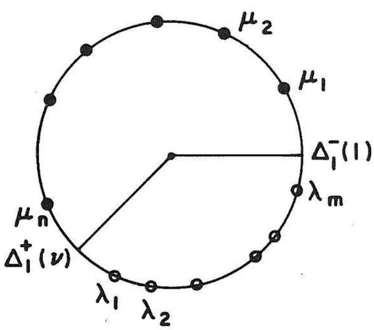


Fig. 4

Definition 9. Let u be, as above, of the second kind. Then we define

$$\theta^*(u) = \sum_{j=1}^{r} \alpha_j + \sum_{k=1}^{s} (\beta_k - 2\pi) + 2q(\gamma - 2\pi) - p\pi .$$

For the other types, $\theta^*(u)$ is defined similarly.

Definition 10. Let $u \in U(p,q)$ and let $V$ be a complement in $E(u)$ of the radical of $E(u)$. Then the restriction of u to $V^{\perp}$ is called the effective part of u.

We can now state our main theorem.

Main Theorem. Let $u \in U(p,q)$ with $p,q \geq 1$ and $\det(u) = \pm 1$. Then $\ell(u) = L(u) + \delta$ where $\delta = 0$ or $2$ and we have $\delta = 2$ precisely in the following cases:

(i) u is exceptional, $u \neq 1$;

(ii) u is of the first kind and $\theta^*(u) \neq k\pi$ for $k = 0, \pm 1, \pm 2$;

(iii) u is of the second kind, $\omega(u) = 1$, and $\theta^*(u) \neq 0$ or the effective part of u is a scalar;

(iv) u is of the second kind, $\omega(u) = i$, and $\theta^*(u) \neq \pi$;

(v) u is of the second kind, $\omega(u) = -i$, and the effective part of u is a scalar;

(vi) u is of the second kind, $\omega(u) = -1$, $\theta^*(u) \neq 2\pi$ and the effective part of u is a scalar;

(vii) u is of the third kind and $\bar{u}$ satisfies one of the conditions (iii) - (vi).

We have also determined the lengths of elements $u \in G$ with respect to all reflections (positive and negative).

The complete proofs will appear elsewhere in a paper with the same title.

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# ESPACES DE FISCHER HERMITIENS



G. DE MEUR

*Presented by G. de B. Robinson, F.R.S.C.*

B. FISCHER a classé [4] les groupes G engendrés par des classes 3-transpositions satisfaisant à : $O_3(G) \leqslant Z(G)$

$O_2(G) \leqslant Z(G)$

$G' = G''$

Dans ces conditions, $G/_{Z(G)}$ contient un sous-groupe normal simple et est isomorphe à l'un des groupes suivants :

(1) $Sym(n)$, $n \geqslant 5$

(2) $PS_{p_{2n}}(2)$, $n \geqslant 2$

(3) $O_{2n}^{\pm 1}(2)$, $n \geqslant 2$

(4) $PSU_n(2^2)$, $n \geqslant 4$

(5) $PO_n^{\pm 1}(3)$, $n \geqslant 4$ et un sous-groupe d'indice 2

(6) $Fi_{22}$, $Fi_{23}$, $Fi_{24}$.

Z(G) désigne le centre de G

$O_p(G)$ le plus grand sous-groupe normal d'ordre premier à p de G

G' le groupe dérivé de G.

Rappelons qu'une classe de 3-transpositions est une classe de conjugaison d'involutions de G, telle que le produit de 2 d'entre elles est d'ordre $\leqslant 3$. Dans la classification de Fischer on constate le rôle privilégié joué par les groupes unitaires $PSU_n(2^2)$ qui contiennent de manière naturelle des groupes symétriques, orthogonaux et symplectiques sur GF(2).
Nous nous proposons de classer les sous-groupes de Fischer de $PSU_n(2^2)$.

Ce problème est lié à la détermination des groupes de projectivités engendrés par des élations en caractéristique 2 étudiés par WAGNER [7] et Mc LAUGHLIN [6], ainsi qu'aux récents travaux de KANTOR [5].

Par sous-groupe de Fischer nous entendons les sous-groupes engendrés par un sous-ensemble de 3-transpositions.

La théorie des groupes de Fischer a été géométrisé par F. BUEKENHOUT [1]. Les involutions forment les points d'une géométrie qui est structurée par des droites de 2 ou 3 points selon que 2 involutions commutent ou engendrent $D_3$.

Un espace de Fischer est connexe s'il ne peut être partitionné en 2 sous-espaces tels que les droites joignant un point de l'un à un point de l'autre n'ont que 2 points - ou, ce qui est équivalent, le groupe ne peut être décomposé en produit direct de 2 sous-groupes.

Aux groupes classés par Fischer correspondent des espaces de Fischer, et en particulier, aux groupes Sym(n), un simplexe, aux groupes $O^{\pm}_{n}(2)$, les points extérieurs à une quadrique de PG(n-1,2), aux groupes $PS_{p_{2n}}(2)$, les points de PG(2n-1,2) muni d'une polarité symplectique, et aux groupes $PSU_{n+1}(2^2)$ les points d'une variété hermitienne de PG(n,4). Dans ce cas les involutions sont les élations ayant pour centre un point de la variété hermitienne et pour axe l'hyperplan polaire. Les droites de 3 points correspondent aux points situés sur une sécante, les droites de 2 points aux points situés sur une génératrice.

Le problème revient à déterminer les sous-espaces de Fischer de la variété hermitienne ; il peut être essentiellement ramené à la recherche des sous-espaces connexes de dimension n. Cette détermination est basée sur une double induction en les dimensions paires d'une part, impaires d'autre part.
Les deux propositions essentielles sont :

PROPOSITION 1 : L'ensemble des points correspondant aux 3-transpositions contenues dans le centralisateur C(z) d'une 3-transposition engendre avec z un sous-espace de dimension n-1 dès que $n \geqslant 5$.

PROPOSITION 2 : Le centralisateur d'une 3-transposition est connexe (ne peut être décomposé en produit direct) dès que $n \geqslant 5$.

On obtient les résultats suivants :

<u>THEOREME 1</u> : Les sous-groupes de Fischer connexes de dimension 2n de $PSU_{2n+1}(4)$ sont $Sym(2n+1).Z_3^{2n}/(2n+1,3)$

et $PSU_{2n+1}(2^2)$

<u>THEOREME 2</u> : Les sous-groupes de Fischer connexes de dimension 2n-1 de $PSU_{2n}(2^2)$ sont $Sym(2n+1)$

$Sym(2n+2)$

$Sym(2n).Z_3^{2n-1}/(2n,3)$

$O^{\pm}_{2n}(2)$

$PSp_{2n}(2)$

$PSU_{2n}(2^2)$ si $n > 2$

Pour n = 2 on a de plus un groupe possédant une classe de 126 3-transpositions isomorphe à $Z_2.PSU_4(3^2)$

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# LOCALLY COMPACT HJELMSLEV PLANES

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## 0. Introduction

Throughout this discussion, $H$ is a Hjelmslev plane — H-plane; that is an affine or projective Hjelmslev plane (briefly an AH or PH plane). The reader is referred to [7] and [6] for the definitions and basic ideas regarding such planes. $\sim$ is the neighbour relation of $H$ and $H/\sim$ is the associated ordinary plane with quotient map $\chi : H \to H/\sim$. $\wedge$ and $\vee$ are the meet and join maps defined on non-neighbouring elements and in the AH-case, $L(P,\ell)$ is the unique line through the point $P$ parallel to the line $\ell$.

0.1 Definition. *A* PH(AH) *plane* $H = \langle \mathbb{P}, L, I\rangle (\langle \mathbb{P}, L, I, \| \rangle)$ *is a topological* PH(AH) *plane* (TPH, TAH-*plane*) *if the point set* $\mathbb{P}$, *and the line set* $L$ *are endowed with topologies which render* $\vee$, $\wedge$ $(\vee, \wedge, L)$ *continuous. In addition we also assume* $\sim$ *is closed in* $\mathbb{P}^2$ *and* $L^2$. (cf. [4].)

*Examples.* (1) Ordinary Hausdorf topological planes ([8]).

(2) H-planes over topological H-rings or topological desarguesian H-planes ([4]).

(3) Ordered H-planes ([3]).

In [4] necessary and sufficient conditions were given for $H/\sim$, endowed with its quotient topologies, to be a topological plane. However, we can *now* prove

0.2 Theorem. *If* $H$ *is a topological* H-*plane, then so is* $H/\sim$. *Moreover, the quotient map* $\chi : H \to H/\sim$ *is open.*

## 1. Locally Compact H-planes

The objective of the rest of this paper is to outline some principal results concerning TH-planes whose point sets are locally compact $T_2$, i.e. locally compact H-planes. Details of the following will appear elsewhere.

For the rest of this paper $H$ is a topological H-plane.

1.1 <u>Theorem</u>. *Every locally compact* $T_2$ *H-plane is* $\sim$*-connected* ([3]).

1.2 <u>Theorem</u>. *Let* $\ell$ *be a line of a locally compact* $T_2$ *H-plane. The following are equivalent.*

(a) $\ell$ *is connected.*

(b) $\ell$ *is locally connected.*

(c) $\ell$ *is locally arcwise connected.*

(d) $\ell$ *is arcwise connected.*

(e) *For each point* $P \in \ell$, *there exists a connected subset* $K$ containing $P$ and a point $X \nsim P$.

The next result is fundamental in our investigations.

1.3 <u>Theorem</u>. *Every locally compact* $T_2$ *H-plane is a separable metric space.*

For the definition of an AH-translation plane consult [6]. If $\mathbb{R}$ is the real numbers, then we have

1.4 <u>Theorem</u>. *If* $H$ *is a locally euclidean A.H.-plane or a locally compact translation plane, then each line is homeomorphic to* $\mathbb{R}^n$ *for some integer* n.

For ordinary planes $n = 1, 2, 4$ or $8$. In topological H-planes $n$ can be any integer (cf. 2.10).

## 2. <u>H-epimorphisms and congruences</u>

In this section, we consider the important concepts of H-epimorphisms (or equivalently congruence relations) (cf. [9] and [2]).

2.1 <u>Definition</u>. (1) *Let* $H_1$ *and* $H_2$ *be H-planes. An incidence structure epimorphism* $\psi : H_1 \to H_2$ *is an H-epimorphism (or a refined neighbour property)* $\Longleftrightarrow$

$P \sim Q \Longleftrightarrow \psi(P) = \psi(Q)$ *for all points* $P$, $Q$ *of* $H$.

$\ell \sim m \Longleftrightarrow \psi(\ell) = \psi(m)$ *for all lines* $\ell$, $m$ *of* $H$.

*The equivalence relation* $\tau_\psi$ *(defined separately on* $\mathbb{P}$ *and* $L$*) by* $P\tau_\psi Q \Longleftrightarrow \psi(P) = \psi(Q)$ *and* $\ell\tau_\psi m \Longleftrightarrow \psi(\ell) = \psi(m)$ *is the kernel of* $\psi$.

(2) *An equivalence relation* $\tau$ of $H_1$ *is a congruence* $\Longleftrightarrow$ $\tau = \tau_\psi$ *for some* H-*epimorphism* $\psi : H_1 \to H_2$. $H_1/\tau$ *is the quotient plane of* $H_1$ *with quotient map* $v : H_1 \to H_1/\tau$.

We then have,

2.2 Theorem. *If* $\tau$ *is a congruence of a topological* H-*plane* $H$, *then* $H/\tau$, *endowed with the quotient topology, is a topological* H-*plane and* $v : H \to H/\tau$ *is a continuous open map.*

As a result of 1.3 we have,

2.3 The Open Mapping Theorem. *Let* $H_i$ $(i = 1,2)$ *be locally compact* H-*planes. Then, every continuous* H-*epimorphism* $\psi : H_1 \to H_2$ *is open.*

2.4 Definition ([2] and [1]). *A congruence* $\tau \neq \mathrm{id}$ *on an* H-*plane is linearly homogeneous* (LH)([2]) *if it satisfies* M(a) $P \in g,h;\ g \sim h;\ Q \in g;\ P\tau Q$ *always imply* $Q \in h$ ([1]). $\tau$ *is punctually homogeneous if* M(b), *the dual of* M(a), *holds.* $\tau$ *is homogeneous (or a minimal neighbourhood relation* [1]*) if both* M(a) *and* M(b) *hold.*

Let $A(P)$ be the induced incidence structure on the $\tau$ equivalence class of $P$ (cf. [1], Satz 1).

2.5 Theorem. *Let* $\tau$ *be a congruence on the* H-*plane* $H$. *Then,* $\tau$ *is* L.H. $\Longleftrightarrow$ (i) *Each* $A(P)$ *is an affine plane*
*and* (ii) $P \in g,h;\ g \sim h \Rightarrow \exists\, Q \in g,h$ *so that* $P\tau Q$ *and* $P \neq Q$.

This leads to the important

2.6 Definition. *Let* $H$ *be a* $T_2$ *topological* H-*plane.*

*A congruence* $\tau$ *is topologically* L.H. (T.L.H) *if*

(i)' *Each* A(P) *is a topological affine plane.*

(ii)' = [(ii) *of* 2.5].

In general, it is not clear that a L.H. congruence of a $T_2$ H-plane is a T.L.H. congruence, because the join map $\vee$ of $H$ is not defined on the (neighbour) points of $A(P)$. However,

2.7 Theorem. *In a locally compact* $T_2$ *desarguesian* H-*plane a* L.H. *congruence is a* T.L.H. *congruence.*

We also have

2.8 Theorem. *Let* $\tau$ *be a* T.L.H. *congruence of a locally compact* H-*plane. Then* (1) *Each* $A(P)$ *is a connected affine plane homeomorphic to* $H/\sim$.

(2) *If* $\tau$ *is closed, then* $H$ *is never compact.*

2.9 Definition ([1]). *An* H-*plane* $H$ *is of height* n *if there is a chain* $H = H_n \xrightarrow{\psi_{n-1}} H_{n-1} \xrightarrow{\psi_{n-2}} H_{n-2} \cdots \xrightarrow{\psi_2} H_2 \xrightarrow{\psi_1} H_1 = H/\sim$ *where each* $\tau_{\psi_i}$ *is a minimal neighbour relation.*

*An* H-*plane of height* 2 *is uniform or equivalently* $\tau = \sim$ *is a minimal neighbour relation.*

*A* TH-*plane is topologically uniform if* $\tau = \sim$ *is a* T.L.H. *congruence, i.e. each* $A(P)$ *is a topological affine plane.*

From 2.7 and 2.8 we have

2.10 Theorem. *Let* $H$ *be a locally compact* H-*plane.*

(1) *If* $H$ *is of height* n, *then* $H$ *is connected.*

(2) *If* $H$ *is a translation* AH-*plane (Moufang* PH-*plane) of height* n, *then each line is of dimension* $m \cdot n$ $(m = 1,2,4,8)$.

Finally, we examine uniform planes more closely.

2.11 Theorem. *$H$ is a locally compact topologically uniform plane. Then $\mathbb{P}$ has topological dimension* $\geq 4$. *If $H$ is also a translation (Moufang) plane, then $\mathbb{P}$ has dimension* 4, 8, 16, 32.

2.12 Theorem. *$H$ is a $T_2$ topologically uniform* AH(PH) *plane. The following are equivalent:*

(i) *$\mathbb{P}$ is locally compact with topological dimension* 4.

(ii) *$H/\sim$ is homeomorphic to the real affine (projective) plane.*

(iii) *$\mathbb{P}$ is a manifold and each neighbour class $A(P)$ is homeomorphic to the real affine plane.*

2.13 Theorem. *$H$ is a $T_2$ topologically uniform translation (Moufang) plane. The following are equivalent.*

(i) *$\mathbb{P}$ is locally compact with* $\dim(\mathbb{P}) = 8$.

(ii) *$H/\sim$ is homeomorphic to the complex affine (projective) plane.*

(iii) *Each $A(P)$ is homeomorphic to the complex affine (projective) plane.*

## 3. Compactness in H-planes

As in ordinary topological planes ([8]) the following is true.

3.1 Theorem. (1) *A locally compact* AH-*plane is never compact.*

(2) *A* PH-*plane is compact* $\Longleftrightarrow$ *its lines are compact.*

An ordinary locally compact ordered or connected locally compact projective plane is compact. Neither of these statements hold in PH-planes.

3.2 Theorem. (1) *An ordered* PH-*plane is never compact.*

(2) *If $H$ is a connected* PH-*plane, then $H$ is compact* $\Longleftrightarrow$ *$H$ is locally compact and each neighbour class is compact.*

In [4] we exhibited a topological desarguesian locally compact uniform plane of dimension 4 over $D_2$. From 3.1, 3.2, 2.7, 2.10, 2.12

we have

3.3 Theorem. *There exist connected locally compact* PH *planes which are not compact.*

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# A CONCISE PROOF OF A THEOREM OF MORI-NAGATA

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*Presented by P. Ribenboim, F.R.S.C.*

Abstract

We give a concise proof for the theorem of Mori-Nagata: the integral closure of a commutative noetherian domain is a Krull ring.

The standard proofs of this result [1], [4], [5], [9], use Cohen's theorem giving the structure of complete local rings (more precisely Nagata's result: any complete local integral domain is a Japanese ring), or recently [10], henselisation.

Using a theorem of Matijevic on noetherian rings [8] and ideas from three recent papers [2], [3], [11], we can now give a very concise and elementary proof in the spirit of noetherian ring theory.

## Notation

Let A be a commutative domain; we denote by A' the integral closure of A and by $A^g$ the <u>global transform</u> [8] of A : the ring $A^g$ is the ring of all elements x in the field of fractions K of A such that there exist $M_1, M_2, \ldots, M_s$ maximal ideals of A with the property $x M_1 M_2 \ldots M_s \subset A$. Matijevic proved [8] :

let A be a noetherian domain; then every ring B between A and $A^g$ is noetherian.

Theorem (Mori-Nagata)

Let A be a commutative noetherian domain; its integral closure $A'$ is a Krull ring.

Proof

1) Suppose A has finite dimension $n \geq 1$. We give a proof by induction on n. The result is true for $n = 1$ [9, 33.2]. Suppose it has been proved when $m < n$ and let A be a noetherian domain of dimension n. Let $B = A' \cap A^g$ ; this ring is noetherian [8], $\dim B = \dim A$ and $B' = A'$.

Let G be the set of maximal grade one ideals of B. By [7, Th. 123], $B = \bigcap_G B_P$, this representation is locally finite, $B' = \bigcap_G B'_P$ by [6, lemma 2-2] and this representation is locally finite too. So, [4, 1-4] it is enough to prove that the $B'_P$ are Krull rings.

If $P \in G$ is not a maximal ideal in B, by induction hypothesis, $B'_P$ is a Krull ring. If $P \in G$ is a maximal ideal in B, by [11, Th. 1] it is invertible. Because $PP^{-1} = B$, there exists $x \in P$ such that $xP^{-1} \not\subset P$ ; let $Q \subset P$ be a minimal prime belonging to $(x)$. Then $(xP^{-1})P \subset Q$ implies $P \subset Q$ and so $P = Q$. Since Q is a height one prime ideal [7, Th.142], $\dim B_P = 1$ and $B_P$ is a Krull ring.

2) Because the result is proved for noetherian domains of finite dimension, it is true for noetherian local rings. Since $A' = \bigcap_{G} A'_P$ where $G$ is the set of maximal grade one ideals of $A$, the result is proved for any noetherian ring.

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# RESULTS ON PROFINITE FROBENIUS GROUPS

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*Presented by P. Ribenboim, F.R.S.C.*

## 0. INTRODUCTION

A finite group $G$ is a Frobenius group, if it has one of the following properties:

1) There exists an isolated proper subgroup $H$, i.e. with the property that for all $g \in G\backslash H$, one has $H \cap H^g = (1)$.

2) There exists a proper cc-normal subgroup $K$ of $G$, i.e. with the property that for all $k \in K\backslash(1)$, one has $C_G(k) \leq K$, where $C_G(k)$ denotes the centralizer of $k$ in $G$.

3) $G = H.K$, where $K \triangleleft G$, and $H$ acts elementwise fixed point free (efpf) on $K$, i.e., $1 \neq h \in H$, $1 \neq k \in K$ imply $[h,k] \neq 1$. It turns out that $K = (G\backslash \bigcup_{g \in G} H^g) \cup \{1\}$.

The properties 1), 2), 3) were used in [2] to define and describe locally finite Frobenius groups. The goal of this article is, to extend the theory of finite Frobenius groups in a different direction, namely to profinite groups, i.e. to projective limits of finite groups.

The concepts of Sylow subgroup, Hall subgroup, order of a group, index of a subgroup are meaningful in the context of profinite groups [5].

## 1. THE RESULTS

Our main result is the following:

<u>Theorem</u>

The following conditions on a profinite group $G$ are equivalent:

1) There exists a nontrivial proper closed subgroup $H$ of $G$ such that

$F = (G \setminus H^G) \cup (1)$ is a group.

2) There exists a closed isolated Hall subgroup $H$ of $G$.

3) There exists a finite isolated Hall subgroup $H$ of $G$.

4) There exists a normal Hall cc subgroup $K$ of $G$.

5) $G = HK$, $K$ and $H$ closed subgroups of $G$ with $K$ normal and $(|H|,|K|) = 1$, and $H$ acts efpf on $K$.

6) $G$ is a projective limit of an inverse system $\{G_i | i \in I\}$ of finite Frobenius groups, where I is a poset directed from above, and the canonical maps $\varphi_{i,j}: G_i \longrightarrow G_j$ $(i \geq j)$ are surjective.

7) There exists a set $\pi$ of primes such that if $1 \neq x \in G$, either $x$ is contained in a $\pi$-subgroup of $G$ or in a $\pi'$-subgroup of $G$, and the set $K = \{x \in G | x$ is contained in a $\pi'$-subgroup$\}$ is a normal subgroup of $G$.

Corollary

Let $G$ be a profinite Frobenius group, (i.e. a profinite group satisfying the equivalent conditions of the Theorem). Then its Frobenius kernel $K$ is nilpotent.

A key fact in the proof of the theorem is the following

Proposition 1. If $H$ is a closed isolated Hall subgroup of a profinite group $G$, then it is finite and it has a unique normal complement $K$. In fact $K = (G \setminus H^G) \cup (1)$.

The proof of this, relies on the following generalization of the Zassenhaus-Schur's theorem for profinite groups due to Platonov [3].

Proposition 2. Let $K$ be a closed normal Hall subgroup of a profinite group $G$. Then $K$ has a complement in $G$ (i.e. there exists a closed subgroup $H$ of

G such that $G = HK$ and $H \cap K = (1)$), and any two complements are conjugate.

From this result one deduces the finiteness of a profinite group A, which acts continuously and efpf on a profinite group G, where $(|A|, |G|) = 1$. The other ingredients in the proof of Proposition 1, are the fact that H must be a Frobenius complement in a finite Frobenius group, and the Sylow structure of such complements [4]; an inverse limit argument then yields the result.

A sketch of the proof of the theorem is as follows. The implications 2) ⇒ 1), 2) ⇒ 4), 2) ⇒ 6), 2) ⇒ 7) and 2) ⇔ 3), are easy consequences of Proposition 1. Proposition 2, together with the observation that if $G = K \rtimes G$, then H is isolated iff H acts efpf on K, gives 4) ⇒ 2) and 5) ⇒ 2). An inverse limit argument gives 7) ⇒ 6). To prove 1) ⇒ 6), one first shows that H is finite; then one observes that for each $1 \neq h \in H$, the map $f \mapsto [h,f]$ is surjective; this implies that the finite quotients by open normal subgroups which avoid H, are finite Frobenius groups. Finally, the implication 6) ⇒ 5) is based on the following

<u>Lemma</u>. Let $G_i = H_i F_i$ be finite Frobenius groups with Frobenius kernels $F_i$ and Frobenius complements $H_i$, $i = 1,2$, and let $G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3$ be epimorphisms of groups, with $F_1 \subset \ker \alpha$ and $F_2 \subset \ker \beta$. Then $G_3$ is cyclic.

We call the groups characterized by the theorem, <u>profinite Frobenius groups</u>. In contrast to them, we exhibit several examples of pro-p-groups having some properties, which define Frobenius groups in the finite case, without being profinite Frobenius groups.

## <u>2. EXAMPLES</u>

For a prime p, $Z_p$ and $C_p$ will denote the additive group of p-adic integers and the group with p elements respectively.

As an easy example of a profinite group having an isolated subgroup H with a cc-normal complement K, but not obeying the conditions of the theorem, one may take the pro-2-group $G = \langle x,y \mid yxy^{-1} = x^{-1}, y^2 = 1 \rangle$. Take here H to be the closed subgroup generated by y, and K the closed normal subgroup generated by x.

Let $F = F(x,y)$ be the free pro-p-group on x,y. Let H be the closed subgroup of F generated by x. Then H is isolated and admits a cc-normal complement (a free pro-p-group on the space $Z_p$) which is clearly non-nilpotent. The closed subgroup of F generated by $[x,y] = x^{-1}y^{-1}xy$, is also isolated, but does not have a normal complement. The derived group F' of F is a cc-normal subgroup of F, but it has no complement.

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# A MIXED PROBLEM FOR WEAKLY HYPERBOLIC EQUATIONS OF SECOND ORDER

Kyoko Kimura

*Presented by P. Greiner, F.R.S.C.*

We consider the equation

$$L[u] = \frac{\partial^2}{\partial t^2} u - t^{2k_0} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(t,x) \frac{\partial}{\partial x_i} u\right)$$

$$+ b_0(t,x) \frac{\partial}{\partial t} u + t^{k_0 - 1} \sum_{j=1}^{n} bj(t,x) \frac{\partial}{\partial x_j} u + c(t,x)u$$

$$= f(t,x) \tag{1}$$

where $\sum_{i,j=1}^{n} a_{ij}(t,x)\, \xi_i \xi_j \geq d \sum_{j=1}^{n} \xi_j^2 \quad (d > 0)$

$$a_{ij}(t,x) = a_{ji}(t,x)$$

for all $(t,x)$ with $t \geq 0$ and $\xi = (\xi_1,\ldots,\xi_n) \in R^n$

and $k_0$ is a positive integer. $L$ is hyperbolic and degenerates on the initial surface $t = 0$. The Cauchy problem for equations of this type has been studied by many authors. Oleinik in [4] showed the well-posedness of the Cauchy problem for equations with more general degeneracy in $t$ under certain conditions on the coefficients. Now we study the mixed problem for (1) in the cylinder $[0,T] \times \Omega$, where $\Omega$ is a domain in $R^n$ with $C^\infty$ boundary $S$, under the initial conditions

$$u|_{t=0} = u_0(x) \quad , \quad \frac{\partial}{\partial t} u|_{t=0} = u_1(x) \quad , \tag{2}$$

and the Dirichlet boundary condition

$$Bu|_S \equiv u|_S = 0. \tag{3}$$

We assume that the coefficients $a_{ij}$, $b_j$, and $C$ are in $B^\infty(\bar{\Omega} \times [0,T])$. Considering the conditions in [4] for the Cauchy problem it seems to be natural to impose $t^{k_0-1}$ on the coefficients of the first derivatives in $x$ in case the coefficients of the second derivatives in $x$ have a $t^{2k_0}$ term of degeneracy.

Our result is

THEOREM: For given data $\{f(t,x), u_0(x), u_1(x)\}$ $\in \mathcal{E}_t^\infty \left(H^\infty(\Omega)\right) \times H^\infty(\Omega) \times H^\infty(\Omega)$, there exists a unique solution $u(t,x)$ in $\mathcal{E}_t^\infty \left(H^\infty(\Omega)\right)$ to the mixed problem (1), (2) and (3), provided the data satisfy the compatibility condition of infinite order.
(Here "$u(t,x) \in \mathcal{E}_t^k (E)$" means that $u(t,x)$ is k-times continuously differentiable in $t$ as an E-valued function).
The proof of the theorem is based on the following lemma.

LEMMA: Let $\varepsilon$ be a non-negative parameter. Consider the equation

$$L\varepsilon[u] \equiv \frac{\partial}{\partial t^2} u - (t+\varepsilon)^{2k_0} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(t,x)\frac{\partial}{\partial x_i} u\right)$$

$$+ b_0(t,x) \frac{\partial}{\partial t} u + (t+\varepsilon)^{k_0-1} \sum_{j=1}^{n} b_j(t,x) \frac{\partial}{\partial x_j} u + c(t,x)u$$

$$= f(t,x) \tag{4}$$

Let $m$ be a non-negative integer. Then there exists a constant $C$ and a positive integer $N$ such that for the solution

$u_\varepsilon(t,x) \in E_t^0 \left(H^{m+3}(\Omega)\right) \cap E_t^1 \left(H^{m+2}(\Omega)\right) \cap \ldots \cap E_t^{m+3} \left(L^2(\Omega)\right)$

of (4),(2) with $u_0 = u_1 = 0$ $\left(\text{say } (2)_0\right)$ the estimate

$$\sum_{p=0}^{m+3} \| \left(\frac{\partial}{\partial t}\right)^p u_\varepsilon(t,x) \|^2_{H^{m+3-p}(\Omega)} \leq C \int_0^t \| \left(\frac{\partial}{\partial \tau}\right)^{N+1} f(\tau,x) \|^2_{H^{m+2}(\Omega)} d\tau \tag{5}$$

holds for $t \in [0,T]$ provided $\left(\frac{\partial}{\partial t}\right)^p f(0,x) = 0,\ p = 0,1,\ldots N.$

Here we note that $C$ and $N$ are independent of $\varepsilon$.

In the case when $\varepsilon > 0$, $L\varepsilon$ is regularly hyperbolic; hence it is known that the problem (4),$(2)_0$ and (3) has a solution $u\varepsilon(t,x) \in E_t^\infty \left(H^\infty(\Omega)\right)$ (cf. [1]). Thus we can extract a sequence from the set $\{u\varepsilon(t,x)\}$ $\varepsilon > 0$ which converges weakly in $H^{m+3}\left(\Omega \times (0,T)\right)$. The limit function $u(t,x)$ is a unique solution of (1),$(2)_0$ and (3) when we assume that $\left(\frac{\partial}{\partial t}\right)^p f(0,x) = 0$, $p=0,1,\ldots N$.

Therefore the solution of (1),(2) and (3) can be obtained when the data satisfy the compatibility condition of infinite order.

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# A CHARACTERIZATION OF LINEAR TRANSFORMATIONS ON THE SPACE OF FOURIER-STIELTJES TRANSFORMS.[1]

by

J.M. Belley and J. Dubois

*Presented by P. G. Rooney, F.R.S.C.*

Abstract: Given an abelian locally compact group $G$ with dual group $\hat{G}$, let $B(\hat{G})$ be the space of finite linear combinations of continuous positive definite functions on $\hat{G}$ and let $\{\Phi_{\hat{z}}: \hat{z} \in \hat{G}\}$ be a family of bounded linear functionals on $B(\hat{G})$. It is shown that a necessary and sufficient condition for $\Lambda$, defined by $(\Lambda f)(\hat{z}) = \Phi_{\hat{z}}(f)$ $(f \in B(\hat{G}))$, to be a linear transformation on $B(\hat{G})$, is given by $\Lambda(G) \subseteq B(\hat{G})$ (where $G = \hat{\hat{G}}$ is viewed as a subset of $B(\hat{G})$). Corollaries are then obtained for the case where the $\Phi_{\hat{z}}$ are induced by measures on the Borel subsets of $G$.

1. Main result. Let $G$ be an abelian locally compact group with dual group $\hat{G}$; the value of the character $\hat{z} \in \hat{G}$ at the point $z \in G$ being denoted by $\langle z, \hat{z} \rangle$. Let $M(G)$ and $M(\hat{G})$ denote the space of all complex-valued (and so finite) regular countably additive measures on the Borel sets $\mathcal{B}(G)$ and $\mathcal{B}(\hat{G})$ respectively.

In the case where $G$ is taken to be the circle group $T=\{w \in \mathbb{C}; |w| = 1\}$, R.D. Mauldin [5] has given a characterization of the continuous linear operators $\Lambda$ acting on $M(T)$ as being those of the form

$$\Lambda\mu(E) = \int_T \psi(E, w)\, d\mu(w)$$

1. This research was supported by grants from the Natural Sciences and Engineering Research Council of Canada and the ministry of education of Québec.

where $\psi$ is a function defined on $\mathcal{B}(T) \times T$. Cheney and de Korvin [2, 3] tried to generalize this result to the case of a general measure space with $\psi(\cdot, z)$ a finitely additive set function.

In this paper, we attack the problem of characterizing the continuous linear operators on $M(G)$ by studying the equivalent problem on the space of Fourier-Stieltjes transforms of elements in $M(G)$. A strong motivation for doing this is the following natural problem. Given a doubly infinite matrix $\{a_{kn}: k,n=0,\pm 1,\pm 2,\ldots\}$ of complex numbers such that $\sum_n |a_{kn}|<\infty$ $(k=0,\pm 1,\pm 2, \ldots)$, then the sums $\sum_n a_{kn}\, \hat{\mu}(n) = \sum_n a_{kn} \int_T w^n d\mu(w)$ are well defined for all $\mu \in M(T)$. A question which arises naturally is to give a criterion for the matrix to have the property that for any $\mu \in M(T)$ there exists a $\nu \in M(T)$ such that $\sum_n a_{kn}\, \hat{\mu}(n) = \int_T w^k d\nu(w)$ for any $k$, (i.e. such that $\sum_n a_{kn}\, \hat{\mu}(n)$ is the Fourier-Stieltjes transform of an element of $M(T)$ for all $\mu \in M(T)$). As our general theorem shows, such a condition is given by $\sum_n e^{int} a_{kn}$ being, for any $t \in [0, 2\pi)$, a finite linear combination of (continuous) positive definite functions of $k$ on the group of integers (with discrete topology).

Recall that given $\mu \in M(G)$ and $\nu \in M(\hat{G})$, their Fourier-Stieltjes transforms are the complex-valued continuous functions defined on $\hat{G}$ and $G$ respectively by

$$\hat{\mu}(\hat{z}) = \int_G \langle z, \hat{z}\rangle \, d\mu(z) \quad \text{and} \quad \hat{\nu}(z) = \int_{\hat{G}} \langle z, \hat{z}\rangle \, d\nu(\hat{z})$$

where, by Pontryagin's duality theorem, $\langle z, \hat{z}\rangle$ also represents the character $z \in \hat{\hat{G}}$ evaluated at $\hat{z} \in \hat{G}$. Recall also that if $B(\hat{G})$ denotes the linear space generated by the continuous positive definite complex valued functions on $\hat{G}$, then, by Bochner's theorem, this space can be identified with the Fourier-Stieltjes transforms of the elements of $M(G)$.

The main theorem is the following. In this theorem $B(\hat{G})$ is viewed as a subspace of the class $C_b(\hat{G})$ of bounded continuous functions on $\hat{G}$ with the supremum norm.

Theorem. Let $\{\Phi_{\hat{z}}:\hat{z} \in \hat{G}\}$ be a family of continuous linear functionals on $B(\hat{G})$. In order that the function $\hat{z} \longmapsto \Phi_{\hat{z}}(f)$ be an element of $B(\hat{G})$ for all $f$ in $B(\hat{G})$ it is necessary and sufficient that the function $\hat{z} \longmapsto \Phi_{\hat{z}}(z)$ be an element of $B(\hat{G})$ for all $z \in G$.

As an immediate corollary we get:

Corollary. Let $\{\nu_{\hat{z}}:\hat{z} \in \hat{G}\}$ be a family of measures in $M(\hat{G})$. In order that, for all $\mu \in M(G)$,

$$\hat{z} \longmapsto \int_{\hat{G}} \hat{\mu}(\hat{w})\, d\,\nu_{\hat{z}}(\hat{w})$$

be the Fourier-Stieltjes transform of an element of $M(G)$, it is necessary and sufficient that

$$\hat{z} \longmapsto \hat{\nu}_{\hat{z}}(z)$$

be a finite linear combination of continuous positive definite functions in $\hat{z}$ for each $z \in G$.

This yields the following:

Corollary. Let $\nu$ be an element of $M(\hat{G})$. Then, for all $\mu \in M(G)$, the "convolution" $\int_{\hat{G}} \hat{\mu}(\hat{z} - \hat{w})\, d\nu(\hat{w})$ is the Fourier-Stieltjes transform of an element of $M(G)$.

2. <u>Method of proof</u>. We now give the principal ideas used to obtain the theorem. Let $\hat{z} \in \hat{G}$ be given. Since an element of $C_b(\hat{G})$ is the restriction to $\hat{G}$ of a unique continuous function on the Stone-Čech compactification $\beta\hat{G}$ of $\hat{G}$, the Hahn-Banach theorem and the Riesz representation theorem yield a measure $\nu_{\hat{z}} \in M(\beta\hat{G})$ such that $f \longmapsto \int_{\beta\hat{G}} f \, d\,\nu_{\hat{z}}$ is a norm preserving extension to $C(\beta\hat{G})$ of the linear functional $\Phi_{\hat{z}}$. Let $\mathcal{A}_{\hat{z}} = \{A \in \mathcal{B}(\beta\hat{G}) : |\nu_{\hat{z}}|\,(\overline{A} \setminus \overset{o}{A}) = 0\}$ where $|\nu_{\hat{z}}|$ is the total variation of $\nu_{\hat{z}}$ and $\overline{A}$ and $\overset{o}{A}$ denote the closure and interior of $A$ in $\beta\hat{G}$. Then $\mathcal{A}_{\hat{z}} \cap \hat{G}$ is an algebra of sets of $\hat{G}$ which separates points of $\hat{G}$ and since $\hat{G}$ is dense in $\beta\hat{G}$ we can easily show that we have a well defined <u>charge</u> $\lambda_{\hat{z}}$ (that is a complex-valued bounded finitely additive set function) on $\mathcal{A}_{\hat{z}} \cap \hat{G}$ given by $\lambda_{\hat{z}}(A \cap \hat{G}) = \nu_{\hat{z}}(A)$ and such that

$$(1) \quad \Phi_{\hat{z}}(f) = \int_{\hat{G}} f(\hat{w})\, d\lambda_{\hat{z}}(\hat{w}) \quad \text{for all} \quad f \in B(\hat{G}).$$

In (1) the integral is defined by the usual Moore-Smith method [8,pp. 401-404] and exists whenever $f$ is $\mathcal{A}_{\hat{z}} \cap \hat{G}$ - continuous (see Darst [4] for the definition of continuity with respect to an algebra and the existence theorem) which is the case for any $f \in B(\hat{G})$. We can also show that if $\{z_n\}$ and $\{\hat{z}_m\}$ are sequences in $G$ and $\hat{G}$ respectively such that the iterated limits $\alpha = \lim_n \lim_m \langle z_n, \hat{z}_m\rangle$ and $\beta = \lim_m \lim_n \langle z_n, \hat{z}_m\rangle$ exist then $\alpha = \beta$. Moreover the function defined on $G$ by $z \longmapsto \langle z, \hat{z}\rangle$ is $\mathcal{B}(G)$-continuous for every $\hat{z} \in \hat{G}$ and so by a theorem of Sinclair [7, pp 363-364] we have that for any $\mu \in M(G)$ and any $\hat{z} \in \hat{G}$

$$(2) \quad \int_{\hat{G}} \int_{G} \langle z, \hat{w}\rangle \, d\mu(z)\, d\lambda_{\hat{z}}(\hat{w}) = \int_{G} \int_{\hat{G}} \langle z, \hat{w}\rangle \, d\lambda_{\hat{z}}(\hat{w})\, d\mu(z)$$

where $\lambda_{\hat{z}}$ is the charge given in (1).

Now given $\mu \in M(G)$ there exists a Borel measurable complex valued function $g$ on $G$ with $|g| = 1$ and $d\mu = g\, d|\mu|$ [6, page 126]. Writing $g = (g_1^+ - g_1^-) + i(g_2^+ - g_2^-)$ where $g_j^{\pm}$ are non-negative functions on $G$ and using (1) and (2) we can easily deduce the theorem.

We mention here that analogous results are to appear in a paper recently accepted for publication [1] for the less difficult case where $M(G)$ consists of bounded finitely additive measures.

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## Instructions

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2. Typing should be on one side only with a margin of 1 inch on the left and 1 inch above and below. Clear typing using a black ribbon and writing of inserted symbols in black ink are *crucial*. Every symbol (including brackets) which is not on the typewriter in customary form should be inserted by hand.

3. If superscripts or subscripts are superposed or if fractions (not using a solidus) appear, then this material should be typed as display.

4. Underlining should be used in place of italics. Title should be in capitals and underlined.

5. A *short abstract* should precede the text.

6. Extra spacing should precede and follow definitions, examples, and the statements of theorems, lemma, corollaries etc. and their proofs.

7. References should be listed at the end of a paper in the form used by *Mathematical Reviews,* followed by the author's mailing address.

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