C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1979) No. 5

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BORNOLOGICAL PROPERTIES OF SPACES OF NON-ARCHIMEDEAN VALUED FUNCTIONS

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Presented by M. A. Akcoglu, F.R.S.C.

Let K be a nontrivially non-Archimedean valued field of rank one and let X be an ultraregular space. Then $X \subseteq \upsilon_K X \subseteq \upsilon_0 X \subseteq \beta_0 X$ where $\upsilon_K X$, $\upsilon_0 X$ and $\beta_0 X$ denote, respectively, the K-repletion of X, its Z-repletion and its Banaschewsky compactification.

Let P be a family of relatively compact subsets of $\upsilon_0 X$ such that $Y_P = \cup P \supseteq X$. We assume that P contains all subsets of its members and all their closures in $\upsilon_0 X$. For each $F \in P$ there is a semi-norm $\|.\|_F$ on C(X,K), the family of continuous functions from X into K, defined by

$$\|f\|_F = \sup_{x \in F} |f|(x)$$

Hence P induces a locally convex topology P_p on C(X,K).

A locally convex space(over K) is called bornological iff the absolutely convex neighborhoods are the only absolutely convex sets that absorb all bounded sets. The definition of ultrabornological spaces is obtained by replacing "bounded sets" by "absolutely convex compact sets". A locally convex space is called semi-bornological iff a linear functional that is boundedon bounded sets, is continuous.

Topological preliminaries can be found in [1], the functional analytic parts should be compared with those of the classical (real-or-complex-) case [3].

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DEFINITION 1 : An ultraregular compact space A is K-homogeneous if and only if there is a linear functional τ_0 from $C(A,K)$ into K, continuous with respect to the uniform norm topology on $C(A,K)$, such that for each proper compact subset B of A there is an $f\in C(A,K)$ with $\|f\|_B=0$ and $\tau_0(f)\neq 0$.

DEFINITION 2 : P is saturated iff a set $A\subseteq \upsilon_0 X$ belongs to P whenever (*) holds :

(*) If $(A_i)_{i=1}^{\infty}$ is a sequence of clopen subsets of $\upsilon_0 X$ and if for each $F\in P$, $A_i\cap F=\phi$ for all but finitely many i, then $A_i\cap A=\phi$ for all but finitely many i.

Similarly P is weakly saturated iff a compact K-homogeneous set $A\subseteq \upsilon_0 X$ belongs to P whenever (*) holds.

PROPOSITION 1. If P is weakly saturated , then $Y_P\supseteq\upsilon_0 X$. If $P=K(X)$ (family of compact subsets of X) or if $P=A(X)$ (family of finite subsets of X) then the following are equivalent (i) P is saturated

(ii) P is weakly saturated

(iii) $X=\upsilon_0 X$

THEOREM 1. $C(X,K,P_P)$ is bornological iff P is saturated.

THEOREM 2 : If K is complete and $\upsilon_K X=\upsilon_0 X$ or if K is spherically complete, then $C(X,K,P_P)$ is semi-bornological iff P is weakly saturated.

COROLLARY 1 : If either K is complete and $\upsilon_K X=\upsilon_0 X$ or if K is spherically complete, then the following are equivalent

(1) $C(X,K,K(X))$ is bornological

(2) $C(X,K,A(X))$ is bornological

(3) $C(X,K,K(X))$ is semi-bornological

(4) $C(X,K,A(X))$ is semi-bornological

(5) X is z-replete.

REMARK 1 : From [1, Theorem 15] we know that $\upsilon_K X = \upsilon_0 X$ whenever K is complete and has nonmeasurable cardinality.

THEOREM 3 : $C(X,K,P_p)$ is ultrabornological iff K is a locally compact field and $P = K(\upsilon_0 X)$.

The following result shows that semi-bornological spaces $C(X,K,P_p)$ are those on which each sequentially continuous linear functional is continuous.

THEOREM 4 : If τ is a linear functional from $C(X,K,P_p)$ into K, then τ is sequentially continuous iff it is bounded on bounded sets.

For the classical case a result analogous to the above one with $P = K(X)$ can be found in [2].

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A MATHEMATICAL STRUCTURE FOR A THEORY OF GROWTH

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INTRODUCTION: The study of the growth of plants has allowed us to conceive a mathematical structure which we briefly summarize here. For the question of its interpretation in plant biology, of the supporting data, of the underlying or unifying theory of growth which emerges, we refer the reader to the articles given in the references. These articles point out the pertinence of the formalism, definitions, propositions and theorems elaborated thereat and propose a mechanism of evolution (phylogenism, bio-entropy) instead of a developmental mechanism for the growth of plants. The theory is based on the concept of an algorithm: growth matrices generate hierarchies and rhythms which together with the principle of minimization of entropy (our version of the principle of optimal design) determine vortices of growth that is the paths of minimal entropy among which is the hierarchy of Fibonacci representing Fibonacci phyllotaxis. Besides a necessary and sufficient condition producing this type of phyllotaxis, the theory can explain, contrarily to the other theories, its prevalence over other types of phyllotaxis.

DEFINITIONS: A growth matrix is a square matrix $C = (c_{ij})$ of order n, whose entries are 0 or 1, where $0 < \sum c_{ij} \leq 2$ for all j and whose directed graph is strongly connected. Let $C = (c_{ij})$ be a growth matrix of order n, S a set of n symbols $a_1, a_2, \ldots, a_n$, P a function defined on S by $P(a_j) = a_1^{c_{1j}} a_2^{c_{2j}} \ldots a_n^{c_{nj}}$ for $j = 1, 2, \ldots, n$ where $a_i^{c_{kj}}$ is a_i if $c_{kj} = 1$ and vanish if $c_{kj} = 0$ (empty word); let w_0 be a word (concatenation of symbols) made with the symbols of S: the hierarchy generated by C is an ordered set (W, R) where $W = (w_0, w_1, w_2, \ldots)$ is the sequence of the generations w_t such that $w_t = P(w_{t-1})$, $t = 1, 2, 3, \ldots$, where $P(a_i a_j) = P(a_i)P(a_j)$ (a_i is a simple or a double node according if $P(a_i) = a_j$ or $P(a_i) = a_j a_k$) and $R = (r_1, r_2, r_3, \ldots)$ such that r_t is the family of all couples (a_r, a_s) where a_s is a symbol in $P(a_r)$ for every a_r in w_{t-1} (the element (a_r, a_s)

may be repeated in the family). We denote by s_t, $t = 1, 2, 3, \ldots$, the relative frequency of duplications (a duplication is the set of two couples made from a double node) in $\bigcup_{i=1}^{t} r_i$. A path in a hierarchy from w_i to w_{i+k} is a finite sequence $l_{i+1}, l_{i+2}, \ldots, l_{i+k}$ where l_{i+j} is in r_{i+j} and the first component of l_{i+j+1} is the second component of l_{i+j} for $j = 1, 2, \ldots, k-1$. The oriented segment of a straight line from the first node of the path to the last is a branch from w_i to w_{i+k}. A tree in a hierarchy is an infinite sequence $T_1, T_2, T_3, \ldots$, where T_k is a set of branches from w_i to w_j for $i = 0, 1, 2, \ldots, k-1$ and $j = 1, 2, \ldots, k$, each node of $\bigcup_{i=1}^{k} w_i$ being the extremity of exactly one branch, and such that $T_k \subset T_{k+1}$. The number of distinct T_k, denoted by X_k, is a degree of complexity. The growth function $f(t)$ associated with the growth matrix C and w_0 is defined by the number of symbols in w_t $(t = 1, 2, 3, \ldots)$. The preponderant period of the rhythm of a hierarchy is the smallest n such that a growth matrix of order n generates the hierarchy. We say that a hierarchy possesses the property of Fibonacci if every double node of w_k, $k = 1, 2, 3, \ldots$, is linked, by r_{k+1}, to a double node and a simple node of w_{k+1} and if every simple node of w_k is linked to a double node of w_{k+1}. A normal (anomalous) hierarchy possesses by definition the property of Fibonacci and w_0 contains t nodes $(2t+1)$, $t \geq 1$ $(t \geq 2)$, among which 1 (t) is (are) a double node(s). The normal hierarchy with $t=1$ is the hierarchy of Fibonacci. The principle of entropy states that the entropy E_t of a hierarchy at time t is the number $k \ln s_t p_t$ where k is a negative constant and p_t, the probability of existence of a growth, is the number $1/X_t$; s_t is the stability of the growth and $s_t p_t$ is its probability of survival. The biological principle of minimization of entropy states that evolution creates organisms having smaller entropy and an organism minimizes its entropy at each instant of its growth. The path of minimal entropy of order n is the hierarchy generated by the Frobenius growth matrix of order n $(n = 2, 3, 4, \ldots)$ $C = (c_{ij})$ where $c_{11} = c_{12} = \ldots = c_{1n-2} = 0$ and $c_{1n-1} = c_{1n} = 1$ and where $w_0 = a_{n-1}$. For $n=2$ we have the hierarchy of Fibonacci.

THEOREMS: a) Let $C^t = (c_{ij}^{(t)})$ be the tth power, t = 1, 2, 3, ... of the growth matrix C of order n and S a set of n symbols $a_1, a_2, \ldots, a_n$. If a_j, j = 1, 2, ..., n appears m_j times in w_0 then for n = 2, 3, ...the degree of complexity at time t is given by $X_t = \prod_{k=1}^{t} k \sum_{j=1}^{n} \sum_{i=1}^{n} m_j\, c_{ij}^{(k)}$

b) If C is primitive, let $f_j(t)$ (j = 1, 2, ..., n) be the growth function determined by $m_i = \delta_{ij}$, i = 1, 2, ..., n the number of a_i in w_0 and let p_{1j} be the coefficient of r^t in the expression for $f_j(t)$ in terms of the eigenvalues of C. If $x = (x_i)$ is the positive eigenvector corresponding to r, $\lim C^t/r^t = (p_{11}x, p_{12}x, \ldots, p_{1n}x)/\sum_{i=1}^{n} x_i$ and, if C is a companion matrix, $p_{1j} = (1 + r + r^2 + \ldots + r^{n-1})\ g_j(r)/g'(r)r^{n-j+1}$ where g' is the derivative of the characteristic polynomial $g(t) = t^n - \sum_{i=1}^{n} b_{n-i}t^{n-i}$ and $g_j(t) = \sum_{k=j}^{n} b_{n-k}t^{n-k}$.

c) If $\lim f(t)/f(t-1)$ exists, it is the Perron root of the growth matrix C, that is its positive eigenvalue equal to its spectral radius r in the interval $[1,2]$.

d) Given a growth matrix with spectral radius $r = \lim f(t)/f(t-1)$, let p_t and q_t be respectively the numbers of double and simple nodes in w_t of a generated hierarchy such that $q_t \neq 0$ from some t. Then $\lim p_t/q_t = (r-1)/(2-r)$ and if the system possesses at least one double node, $\lim s_t = r-1$.

e) If C is a Frobenius growth matrix cyclic of index h, with spectral radius r, the following properties are equivalent: (1) $\lim f(t)/f(t-1)$ exists; (2) the coefficients of λ_i^t, i = 2, 3, ..., h, where $|\lambda_i| = r$, $\lambda_i \neq r$ in the expression giving f(t) in terms of the eigenvalues λ_i of C, are equal to 0; (3) r=1; (4) the characteristic polynomial of C is $\lambda^n - 1$; (5) h=n; (6) C generates hierarchies with simple nodes only.

f) Under the (ontogenetic) principle of minimization of entropy, a necessary and sufficient condition for the generation of the hierarchy of Fibonacci is that the rhythm of the growth be established early enough, that is before node seventh appears.

CONCLUSION: - This structure gives a grasp on one of the most fundamental principle of relational biology, the principle of optimal design, where the functional cost chosen is the bio-entropy of the pair organism-environment.

- Some of our results concern an ergodic property for stable growth, a subject much studied by Bernardelli-Leslie-Lewis in the biology of populations.

- Some of our theorems propose a relation between an analytical representation of growth processes (in terms of spectral analysis of certain operators) and a discrete, heretofore entirely formal representation (in terms of automata theory and linguistics). We think that this kind of relationship may have important implications; we have developed it in the concrete area of phyllotaxis.

- Some of our theorems are a contribution to the theory of growth functions of L-systems of Lindenmayer-Herman-Rozenberg and represent the first introduction of Perron-Frobenius spectral theory in plant biology.

- The last result presents an homology with Adler's main result in his contact pressure theory based on an entirely different conceptual framework.

- Other results, on more general matrices, concerning the limiting behavior of $f(t+1)/f(t)$ and C^t/r^t and, on the possibility of approximating $f(t)$ by kr^t (k a constant), allowed us to analyze populations with lineage control (subm.)

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Monotone solutions to certain differential equations

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In a previous paper [1], we have derived a necessary and sufficient condition that the equation

(1) $$y^{(n)} + \rho(x)f(y) = 0$$

with

(2) $yf(y) > 0$ for $y \neq 0$, f non-decreasing and continuous, $\rho \leq 0$,

have all its solutions continue on $[0,\infty)$. Such solutions may have their first n derivatives eventually of persistent sign. The pattern of signs may take several forms, however, it is always the case that for some p, the first p derivatives have the same sign. Such functions played a large role in [1] where we used precise information about the growth of the derivatives. Here we continue the study to see when solutions exist with this precise behavior.

Precisely, introduce the classes

$$M(p,n) \equiv \{y \mid y \in C^n(a,\infty),\ y^{(i)}(x) \geq 0,\ \ 0 \leq i \leq p,\ (-1)^k y^{(p+k)}(x) \geq 0,$$
$$k = 1,\ldots,n-p\}$$

if $n-p$ is even and

$$M(n,p) \equiv \{y \mid y \in C^n(a,\ ),\ y^{(i)}(x) \leq 0,\ \ 0 \leq i \leq p,\ (-1)^{k+1} y^{(p+k)}(x) \geq 0,$$
$$k = 1,\ldots,n-p\}$$

if $n-p$ is odd.

Under the hypotheses (2) we will show that (1) possesses a solution in $M(n,p)'$ if and only if

(3) $\int^{\infty}(-\rho(x))f(\beta x^{p})x^{n-p-1}dx < \infty$ for some $\beta > 0$.

Here $n-p$ is non-zero and even, and $M(n,p)'$ requires $y^{(p)}(\infty) \neq 0$.

To see that there are natural sort of solutions we offer the lemma.

<u>Lemma 1</u>. Let $y \in C^{n}(a,\infty)$ and $y^{(n)} \geq 0$. There there is a $b > a$ and a p such that $y \in M(n,p)$ or $y^{(i)}(x) \geq 0$ $i = 0,\ldots,n$ on (b,∞).

There is a representation for the functions in $M(n,p)$. Define x_{+}^{n} by $x_{+}^{n} = x^{n}$ if $x \geq 0$, $x_{+}^{n} = 0$ if $x < 0$, $n = 0,1,2,\ldots$

<u>Lemma 2</u>. If $y \in M(n,p)$ then ($n-p$ even)

(4) $$y(x) = q(x) + \frac{\gamma x^{p}}{p} + \frac{1}{(p-1)!}\,\frac{1}{(n-p-1)!}\int_{a}^{\infty}\int_{a}^{\infty}(x-s)_{+}^{p-1}(t-s)_{+}^{n-p-1}y^{(n)}(t)\,dt\,ds\,.$$

and in particular

(5) $$y^{(p)}(x) = \gamma + \frac{1}{(n-p-1)!}\int_{a}^{\infty}(t-x)_{+}^{n-p-1}y^{(n)}(t)\,dt$$

where $q(x)$ is a polynomial of degree at most $(p-1)$.

Now consider the operator T defined by

(6) $$(Ty)(x) = \frac{\gamma x^{p}}{p!} + \int_{a}^{\infty} f(y(t))(-\rho(t))\int_{a}^{\infty}\left(\frac{(x-s)_{+}^{p-1}(t-s)_{+}^{n-p-1}ds}{(p-1)!(n-p-1)!}\right)dt$$

whenever the integral is finite. Note that if $Ty = y$, then y satisfies the differential equation (1); compare (4) with $q \equiv 0$.

If we now assume (3) then the set

$$S_{a}(\beta,p) \equiv \{y \mid y \text{ is continuous and } \theta \leq y(t) \leq \beta t^{p} \text{ on } [a,\infty)\}$$

is in the domain of T. We look for a fixed point of T in $S(\beta,p)$ by showing that $TS_{a}(\beta,p)$ is compact. We give the exact result in the Theorem.

Theorem 1. Let the conditions (2) hold for equation (1) and $n-p$ be even and positive. Then the existence of a $\beta > 0$ such that

$$\int^{\infty} f(\beta x^p) x^{n-p-1}(-\rho(x))dx < \infty,$$

is necessary and sufficient for the existence in $[a,\infty)$ of a solution of (1) which is in $M(n,p)'$ for sufficiently large a.

These results can be extended to equations of the form

$$Ly = \rho(t)f(y) \tag{7}$$

where

$$Ly = y^{(n)} + \sum_{i=0}^{n-1} a_i(t)y^{(i)} \tag{8}$$

with the sign conditions

$$Sp: \begin{cases} a_i(t) \leq 0 & 0 \leq i \leq p \\ (-1)^{i-p-1}a_i(t) \geq 0 & p+1 \leq i \leq n-i \end{cases}$$

and

$$\int_x^{\infty} t^{n-i-1}|a_i(t)|dt < \infty \quad 0 \leq i \leq n-1. \tag{9}$$

Theorem 2. Suppose f satisfies the hypothesis of Theorem 1 and $\rho(t) \geq 0$. If $n-p$ is even and L satisfies Sp, then (7) has a solution in $M'(n,p)$. Conversely, if there is a solution in $M'(n,p)$ and Sp conditions hold, then condition (3) and (9) for $0 \leq i \leq p$ are necessarily true.

Related results can be found in Kim [2] and Lovelady [3]. Lovelady considers the special case $Ly = y^{(n)}$ and $f(y) = |y|^a \operatorname{sgn}(y)$. The linear case is of special interest since condition (3) is independent of p, and therefore implies disconjugacy of the equation on $[a,\infty)$.

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Differential Equations on T^n

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Given an orientation preserving homeomorphism of the circle $f:S^1 \to S^1$ (all such f are isotopic to the identity), one has the famous Poincaré-Denjoy rotation number [1], $\rho = \lim_{n\to\infty} F^n(x)/n$, where $F:R \to R$ is the homeomorphism of the covering space $R \overset{\pi}{\to} R/Z = S^1$ induced by f. The number ρ exists and is constant; ρ exists and is rational iff f has a periodic point; ρ irrational implies that the minimal set for the discrete flow $\{f^n\}_{n=-\infty}^{\infty}$ is all of S^1 (if f is C^2) or is a Cantor type set. The embeddability of discrete flows in continuous flows, $\phi:X\times R \to X$ such that $f(x) = \phi(x,1)$, is a difficult question in general; on S^1, Foland and Utz [2] proved that f is embeddable iff ρ is rational or ρ is irrational with S^1 minimal.

Let $f:T^n \to T^n$ be isotopic to the identity, $n \geq 2$, and formally define, à la Poincaré-Denjoy, $\rho(F)(x) = \lim_{m\to\infty} F^m(x)/m$, where $F:R^n \to R^n$ is the homeomorphism of the covering space $R^n \overset{\pi}{\to} R^n/Z^n = T^n$ induced by f. If $\rho(F)(x)$ exists, it generally will not be constant (e.g., $F(x,y) = (x + \sin 2\pi y\ ,\ y)$, $\rho(F)(x) = (\sin 2\pi y\ ,\ 0)$). It is easy to show that $F(x) = x + p(x)$, where $p(x)$ is a function which is periodic of period 1 in each component of x, whence it follows that

$$\rho(F)(x) = \lim_{n\to\infty} \frac{1}{n} \Sigma_{k=0}^{n-1}\, p(F^k(x)) .$$

It is then a direct consequence of the Birkhoff ergodic theorem that $\rho(F)(x)$ exists almost everywhere relative to any measure with respect to which f is invariant. One may also define

$$\bar{\rho}(F)(x) = \limsup_{n} F^n(x)/n \equiv \limsup_{n} \frac{1}{n} \Sigma_{k=0}^{n-1}\, p(F^K(x)),$$

where the lim sup is taken componentwise; this always exists.

The following properties of $\bar{\rho}(F)(x)$ (and $\rho(F)(x)$ when it exists) are easily shown.

1. $\bar{\rho}(F)(x)$ is constant on orbits.

2. $\rho(F)(x)$ exists and belongs to Q^n (Q = rationals) for all points x representing periodic points of f.

3. If $H:R^n \to R^n$ represents a change of coordinates on T^n (so that $H(x) = Ax + p_1(x)$ with $A \in GL(n,Z)$), then $\bar{\rho}(H^{-1}\circ F\circ H)(x) = A^{-1}\bar{\rho}(F)(H(x))$.

4. If f is a rotation of T^n through an angle $\alpha \in T^n$, then $\rho(F)(x) = \alpha$.

5. $\bar{\rho}(F)(x)$ is constant on subsets of R^n representing stable sets of f on T^n (a stable set is a set of the form $W(p) = \{q \in T^n: d(f^n(q),f^n(p)) \to 0$ as $n \to \infty\}$, where d is a metric on T^n).

Immediate consequences of the above properties are

a) $\rho(F)(x)$ exists and assumes at most a finite number of values all in Q^n for Morse-Smale diffeomorphisms f.

b) If $\rho(F)(x) = \text{const} \in Q^n$, then the periods of all the periodic points (if any) are multiples of the least common multiple of the denominators of the components of $\rho(F)(x)$ all expressed in lowest terms.

In analogy to the results of Denjoy, one would like topological conclusions about the discrete flow $\{f^n\}$ on T^n based on the rationality or irrationality of some or all of the components of $\rho(F)(x)$ for various x. We present here a few results of this nature.

Definition. $\rho(F)(x)$ is said to equal α absolutely iff

$$\lim_{n\to\infty} \frac{1}{n} \Sigma_{k=0}^{n-1} |\rho(F^k(x) - \alpha| = 0.$$

$\rho(F)(x)$ is said to be rational in power iff, for some positive integer N, $\rho(F^n)(x)$ equals an integer lattice point absolutely.

If $\rho(F)(x)$ exists absolutely, it exists and has the same value. Existence in power need not imply existence.

Lemma. If $\{a_n\}$ is a bounded sequence of real numbers, then

$\lim_{n\to\infty} \frac{1}{n} \Sigma_{k=0}^{n-1} |a_n| \to 0$ iff there is a subset J of the integers, J of density 0 (i.e., $\frac{1}{n}$ cardinality $\{ J \cap \{0,1,\ldots,n-1\}\} \to 0$ as $n \to \infty$) such that $\lim_{n\notin J} a_n = 0$.

The periodic map $p:R^n \to R^n$ naturally induces a map $\bar{p}:T^n \to R^n$ such that $\bar{p}\circ\pi = p$ and $\rho(F)(x) = \lim_{n\to\infty} \frac{1}{n} \Sigma_{k=0}^{n-1} \bar{p}(f^k(m))$.≡. $\rho(f)(m)$, $m=\pi(x)$.

Proposition. If $\rho(F)(x) = \alpha$ absolutely, then there exists $q \in \omega(m)$ such that $\bar{p}(q) = \alpha$, where $\omega(p)$ is the positive limit set of p.

Proof. According to the previous lemma, there is a sequence $\{n_j\}$ such that $\lim_j \bar{p}(f^{n_j}(m)) = \alpha$. Choose a subsequence $\{n_{j_i}\}$ such that $f^{n_{j_i}}(m) \to q \in \omega(m)$.

Theorem. There exists $p \in T^n$ such that $\rho(f)(p) = k \in Z^n$ absolutely iff f has a fixed point. There exists $p \in T^n$ such that $\rho(f)(p)$ is rational in power iff f has a periodic point.

For the remaining theorems, it will be assumed that f is embeddable in a flow $\phi:T^n xR \to T^n$ arising from a continuous differential equation.

Proposition. Let $f(p) = \phi(p,1)$ on T^2 where $\phi(p,t)$ is non-singular and $\rho(F)(x)$ exists for all $p \in T^2$ ($F(x) = \Phi(x,1)$ where Φ is the lift of ϕ to R^2 and $\pi(x) = p$). Then there exists a real number k and a coordinate system (u_1,u_2) on R^2 parametrizing T^2 such that $\rho(\mathcal{F})(u) = (\alpha(u), k\alpha(u))$ for all $u \in R^2$ ($\mathcal{F}$ is F in u-coordinates).

This proposition gives necessary conditions for a homeomorphism to be embeddable in a non-singular flow.

Motivation. Since ϕ is non-singular, it has a transverse section $S^1 \subset T^2$ which lifts to a family of curves one of which $\sigma_0 \subset R^2$ we may take to go through (0,0) and (a,b). If $\rho(F)(x) \not\equiv (ca,cb)$, then it can be shown that every orbit through σ_0 intersects σ_1 (which goes through (0,1) and (a,b+1)) inducing a homeomorphism g of S^1. Making a change of variables mapping σ_0 to the u_2-axis, it can be shown that $\rho_2(F)(u)/\rho_1(F)(u) = k$ = the Poincaré-Denjoy rotation number of g in the new coordinates.

Corollary. Under the above hypotheses, if $\rho(F)(x) \not\equiv (0,0)$ and $\rho(F)(x)$ is in Q^2 for all $x \in R^2$, then f has a periodic point.

This is an immediate consequence of the fact that k is rational so that ϕ has a closed orbit on which the rotation number of the time 1 map is in Q.

Corollary. If $f(p) = \phi(p,1)$ on T^2 and all closed orbits are orbitally stable with phase, then $\rho(F)(x) \in Q^2$ for <u>some</u> x in R^2 implies there exists periodic points for f.

This is a consequence of the following easily proven lemma on T^n applied to a limit cycle of $\phi(p,t)$.

Lemma. If $f(p) = \phi(p,1)$ on T^n and $\phi(q,t)$ is orbitally stable with phase, then $\{\phi(q,t)\}$ is contained in an open set V such that $\rho(f)(m) = \rho(f)(q)$ for all m in V.

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ON A CONJECTURE OF I. FENYÖ

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1. Let G be the set of functions $f : R \to R$, (R the field of reals), such that

$$f(x)f(y) = f(xy)f(\tfrac{x+y}{2}) \tag{1}$$

holds for all $x, y \in R$. The functional equation (1) is of rank 2 (s. J. Aczél [1]) and it occurs in the theory of waves in the special case where f is the Heaviside function H. We call $F \subset R$ regular in the case where x, y are in F if and only if $xy, \frac{x+y}{2}$ are in F. Using these regular sets, I. Fenyö has constructed functions $f \in G$ as follows. Given F and given a real constant A, put

$$f(x) = A \cdot \chi_F(x) \quad , \tag{2}$$

where χ_F denotes the characteristic function of F.

Remark: Since $f = \chi_Q$, Q the field of rationals, is not a solution of (1) (notice otherwise the contradication $0 = f(2-\sqrt{3})f(2+\sqrt{3}) = f(1)f(2) = 1$), a suitable set F for (2) cannot be defined by

$$x, y \in F \text{ implies } xy, \frac{x+y}{2} \in F \quad .$$

I. Fenyö [2] has posed the problem whether all solutions of (1) are of type (2). We are going to prove in the sequel that Fenyö's conjecture is essentially true. While there are solutions of (1), which are not of type (2), these solutions differ only at $x = 0$ from a solution of type (2).

2. An immediate consequence of (1) is the following:

LEMMA 1: For all $f \in G$

$$F_f := \{x \in R \mid f(x) \neq 0\} \tag{3}$$

is regular.

LEMMA 2: For all $f \in G$ and $x \in F_f$,

$$f(x) = f(x^2) \quad . \tag{4}$$

PROOF: Consider (1), $f(x)f(x) = f(x^2)f(x)$, and apply $f(x) \neq 0$.

LEMMA 3: Suppose $f \in G$ and $a \in F_f$ such that $0 \neq a \neq 1$. Then F_f is not bounded from above.

PROOF: Case 1. Assume $|a| > 1$. Hence $a^2 > 1$ and, according to (4),

$$0 \neq f(a) = f(a^2) = f(a^4) = f(a^8) = \dots$$

Case 2. $0 \neq |a| < 1$ so $0 < a^2 < 1$. Put $b_1 = a^2$, $b = a^8$. Hence $b_1^2 - b > 0$ and

$$f(b_1+\sqrt{b_1^2-b})f(b_1-\sqrt{b_1^2-b}) = f(b)f(b_1) \neq 0$$

because of $a^2, a^8 \in F_f$. By induction, the sequence

$$b_{n+1} = b_n + \sqrt{b_n^2-b} \quad , n=1,2,3,\dots \quad ,$$

is well defined since $b_n^2 > b$, $b_{n+1} > b_n > 0$ imply $b_{n+1}^2 > b_n^2 > b$. Again by induction and (1),

$$f(b_{n+1})f(b_n-\sqrt{b_n^2-b}) = f(b)f(b_n) \quad ,$$

i.e. $f(b_{n+1}) \neq 0$. Assume the sequence $b_1 < b_2 < b_3 < \dots$ to be bounded from above and assume B to be the limit point. Then $B = B + \sqrt{B^2-b}$, i.e. $B^2 = b$, contradicting $B > b_n \geq b_1$, i.e. $B^2 > b_1^2 > b$. Since $B_n \in F_f$ for all $n=1,2,3,\dots$, the set F_f cannot be bounded from above.

Case 3. $a = -1$. Now we have $1 = (-1)(-1) \in F_f$ and hence $\frac{1+(-1)}{2} \in F_f$ since F_f is regular. Thus $\frac{1+0}{2}$ is in F_f and this case can be reduced to a case $0 \neq |a| < 1$.

LEMMA 4: Given $F \in G$. Consider $a, b \in F_f$ such that $a^2 \geq b$. Then $f(a) = f(2a^2-b)$.

PROOF: Put $x = a + \sqrt{a^2-b}$, $y = 2a - x$, so $x, y \in F_f$ since $xy = b$, $\frac{x+y}{2} = a$ are in the regular set F_f. By (1) and (4) we get

$$f(\frac{x+y}{2}) = \frac{f(x)f(y)}{f(xy)} = \frac{f(x^2)f(y^2)}{f(x^2y^2)} = f(\frac{x^2+y^2}{2})$$

and thus $f(a) = f(2a^2-b)$.

LEMMA 5: Let $f \in G$ and $x, y \in F_f$, $0 < x \leq y$. Then $f(x) = f(xy)$.

PROOF: Putting $a = y$, $b = x^2$ in LEMMA 4, we get

$$f(y) = f(2y^2-x^2). \tag{5}$$

Now (1) implies, by means of (4) and (5),

$$f(2y^2-x^2)f(x^2) = f((2y^2-x^2)x^2)\cdot f(y^2), \quad \text{i.e.,}$$

(6) $$f(x^2) = f(2x^2y^2-x^4).$$

Putting $a = xy$, $b = x^4$ in LEMMA 4 we get, since $xy, x^4 \in F_f$ and $(xy)^2 \geq x^4$,

(7) $$f(xy) = f(2(xy)^2-x^4).$$

Now (4), (6), (7) imply $f(x) = f(x^2) = f(xy)$.

We are now able to prove the following.

THEOREM: For all $f \in G$ there exists a real constant A such that

(8) $$f(x) = A \cdot \chi_F(x) \quad \text{for all } x \neq 0 \text{ in } R,$$

where F_f is denoted by F. In (8) $A \neq 0$ for $F \neq 0$.

PROOF: Nothing has to be proved in the case $F = \emptyset$. There are precisely two regular sets containing exactly one element, namely $F = \{0\}$ and $F = \{1\}$. Again, nothing has to be proved in those cases. - In the remaining case, F contains at least two elements. Since $F = \{0,1\}$ is not a regular set (for else $\frac{0+1}{2} \in F$), there exists an $a \neq 0,1$ in F. Hence by LEMMA 3, F is not bounded from above. Take any positive x,y in F and a $z \in F$ such that

$$z > \max\left(\frac{x}{y}, \frac{y}{x}\right) \quad .$$

Hence $0 < x < yz$ and $0 < y < xz$ and thus, by LEMMA 5,

$$f(x) = f(x \cdot yz), \quad f(y) = f(y \cdot xz) \quad .$$

Hence $f(x) = f(y)$. Call A the common value of $f(x)$, $0 < x \in F$. Take now a negative x in F. Then $0 < x^2 \in F$ and thus $f(x^2) = A$. From (4) we finally get $f(x) = f(x^2) = A$, thus proving our theorem.

3. Since the following function is a solution of (1)

$$f(x) = \begin{cases} 2 & \\ 1 & \text{for} \\ 0 & \end{cases} \begin{matrix} x > 0 \\ x = 0 \\ x < 0 \end{matrix} \quad ,$$

our theorem cannot generally be improved so that (8) holds also for $x = 0$. But the following can be proved:

THEOREM: Let $F \in G$. Then (8) is also true for $x = 0$ if one of the following conditions (i), (ii) is fulfilled:

(i) $0 \notin F := F_f$

(ii) $0 \in F$ and there exists an $a < 0$ in F.

If none of the conditions (i), (ii) holds, then G contains also

$$g(x) := \begin{cases} f(x) & \text{for } x \neq 0 \\ b, \text{ arbitrary} & \text{for } x = 0. \end{cases}$$

PROOF: The result is trivial in case (i). Assume now (ii). By (1)

$$f(\sqrt{|a|})f(-\sqrt{|a|}) = f(-|a|)f(0) = f(a)f(0) \neq 0, \tag{9}$$

so $\sqrt{|a|}, -\sqrt{|a|} \in F$ and thus $f(0) = A$ by (9), since $f(a) = f(\sqrt{|a|}) = f(-\sqrt{|a|}) = A$ according to (8). — If (i), (ii) are not fulfilled, then $0 \in F$ and $f(x) = 0$ for all $x < 0$. Then indeed (1) has also the solution

$$g(x) = \begin{cases} A & \text{for } x \in F\backslash\{0\} \\ 0 & \text{for } x \notin F \\ b, \text{ arbitrary} & \text{for } x = 0. \end{cases}$$

Final remark: Regular sets can easily be constructed. Examples are R, $\emptyset$, $\{0\}$, $\{1\}$, $\{r \in R | r \geq 0\}$, $\{r \in R | r > 0\}$. If S is a subfield of R, denote by qS the quadratic closure of S and by aS the algebraic closure of S in the complex field C. Then $R \cap qS$ and $R \cap aS$ are regular. If F_1, F_2 are regular then so is $F_1 \cap F_2$. The smallest regular set $F(T)$ containing $T \subset R$ is the intersection of all regular sets $F \supset T$.

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QUELQUES RESULTATS SUR UNE CLASSE D'ENSEMBLES SPECTRAUX DE DIMENSION ≤ 1

par

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Presented by P. Ribenboim, F.R.S.C.

SUMMARY. A characterization of a class of spectral spaces [3] of dimension ≤ 1 is established and some applications to a problem about spectral sets [6] posed by LEWIS et OHM [7] are given.

Les anneaux considérés sont commutatifs unitaires. Les notations ou notions non précisées sont celles de [1] , [4] , [5] .

Etant donné un anneau A, son spectre (premier), muni de la topologie de Zariski est noté Spec(A). Le spectre d'un anneau peut également être considéré comme un ensemble ordonné par inclusion. Se pose, alors, le problème de caractériser les ensembles spectraux, ensembles ordonnés isomorphes -en tant qu'ensembles ordonnés- au spectre d'un quelque anneau [4] . (Il est bien connu que le problème topologique correspondant, concernant les espaces spectraux, a été résolu par HOCHSTER [3]).

Déjà en dimension 1, la caractérisation des ensembles spectraux est un problème ouvert. On ne connaît que des conditions suffisantes pour qu'un ensemble ordonné de dimension 1 soit spectral (par exemple, voir [7]).

Dans ce travail, nous établissons une caractérisation d'une classe d'espaces spectraux de dimension ≤ 1, que nous appelons espaces principaux. Nous en déduisons une condition suffisante pour qu'une classe plus large d'espaces topologiques, ordonnés par l'ordre associé à leur topologie, soient des ensembles spectraux et en donnons quelques applications à un problème posé par LEWIS et OHM [7] .

Tous les résultats sont ici énoncés sans démonstration. Les preuves ainsi que d'autres propriétés et exemples concernant les problèmes abordés dans cette Note seront objet d'une autre publication.

Soit X un ensemble ordonné. On note $X^{(0)}$ [resp. $X^{(1)}$] l'ensemble des éléments de X de hauteur 0 [resp. de hauteur 1] et $S(x) = \{y \in X \mid y > x\}$ [resp. $G(x) = \{y \in X \mid y < x\}$] l'ensemble des spécialisations [resp. générisations] de $x \in X$. Si A, B et C sont des anneaux, on note A^n l'anneau produit de A n fois et $A \times_C B$ l'anneau produit fibré de $A \longrightarrow C$ et $B \longrightarrow C$ (au dessus de C). Soit $C = D^n$; pour $n = 0$, par convention on pose $A \times_C B = A \times B$.

DEFINITION. On appelle <u>espace principal</u> tout espace topologique noethérien sobre (c.-à-d. tout fermé irréductible possède un point générique et un seul, cf. [5, p. 264]) X tel que, si $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t$ sont ses composants irréductibles, la topologie induite par X sur chaque $\overline{x}_i - x_i$ est la topologie cofinie.

PROPOSITION. Pour un espace topologique irréductible X les affirmations suivantes sont équivalentes entre elles:

(i) X est un espace principal;

(ii) Il existe un corps k et une k-algèbre intègre noethérienne A, de dimension ≤ 1, dont les corps résiduels sont isomorphes à k et le spectre est homéomorphe à X;

(iii) Il existe un corps k et une sous-algèbre A de k(T), tels que A soit un anneau principal dont les corps résiduels sont isomorphes à k et dont le spectre est homéomorphe à X;

(iv) Pour tout corps premier Π, il existe une extension k de Π et une sous-algèbre A de k(T) tels que A soit un anneau principal dont les corps résiduels sont isomorphes à k et dont le spectre est homéomorphe à X.

Dans le cas général, on a le résultat suivant:

THEOREME A . Soit X un espace topologique. Les assertions suivantes sont équivalentes:

(i) X est un espace principal;

(ii) X est une somme amalgamée finie d'espaces principaux irréductibles au dessus de sous-espaces fermés, finis et discrets;

(iii) X est un espace spectral homéomorphe au spectre d'un anneau A réduit de la forme:

$$(\ldots((A_1 \times_{B_2} A_2) \times_{B_3} A_3) \times_{B_4} \ldots) \times_{B_t} A_t$$

où:

- A_i est une k_i-algèbre à idéaux principaux, $1 \leq i \leq t$;
- k_i est un corps, $1 \leq i \leq t$, avec

$$k_1 \subset k_2 \subset k_3 \subset \ldots \subset k_t \quad ;$$

- B_i est un produit de k_i, α_i fois par lui-même, avec $\alpha_i \in \mathbb{N}$ et $2 \leq i \leq t$;
- l'homomorphisme structural $A_i \longrightarrow B_i$ est surjectif, $2 \leq i \leq t$;
- si $A_1' = A_1$ et $A_j' = (\ldots(A_1 \times_{B_2} A_2) \times_{B_3} \ldots) \times_{B_j} A_j$, $2 \leq j \leq t-1$, l'homomorphisme structural $A_j' \longrightarrow B_{j+1}$, $1 \leq j \leq t-1$, est le composé d'un homomorphisme surjectif $A_j' \longrightarrow B_{j+1}'$, où B_{j+1}' est un produit de α_{j+1} corps égaux à certains des $k_1, k_2, \ldots, k_j$, avec l'homomorphisme d'inclusion $B_{j+1}' \hookrightarrow B_{j+1}$.

Dans ces conditions les corps résiduels de A sont des sous-corps de k_t isomorphes aux corps $k_1, k_2, \ldots, k_t$ et les espaces $\mathrm{Spec}(A_i)$, $1 \leq i \leq t$, sont homéomorphes aux composantes irréductibles de X.

Si, de plus, les homomorphismes structuraux $A_j' \longrightarrow B_{j+1}$, $1 \leq j \leq t-1$, sont surjectifs, alors A est un anneau noethérien.

REMARQUE. L'anneau réduit A décrit dans le théorème A n'est pas nécessairement noethérien, bien que son spectre soit un espace noethérien. En effet , l'anneau $k' \times_k k[T]$, où $k' \subset k$ est une extension non finie de corps, n'est pas noethérien (cf. par exemple [2]).

THEOREME B . Tout espace topologique dont les composantes connexes sont des espaces principaux est un ensemble spectral de dimension ≤ 1.

Soit X un ensemble ordonné. On dit que X est la réunion disjointe des sous-ensembles ordonnés $(X_i)_{i \in I}$ si $x \leq_X y$ équivaut à l'existence d'un et d'un seul $i \in I$ tel que $x, y \in X_i$, et $x \leq_{X_i} y$

COROLLAIRE. Soit X un ensemble ordonné de dimension ≤ 1 réunion disjointe d'ensembles ordonnés $(X_i)_{i \in I}$, tel que:

a) $\mathrm{Card}(S(x) \cap S(y))$ [resp. $\mathrm{Card}(G(x) \cap G(y))$] soit fini pour tout $x \neq y$;

b) $\mathrm{Card}(X_i^{(0)})$ [resp. $\mathrm{Card}(X_i^{(1)})$] soit fini pour tout $i \in I$;

alors, X est un ensemble spectral.

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THE TYPE STRUCTURE OF REPRESENTATIONS WHICH VANISH AT INFINITY

by

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In this note we state some results which generalize the work in [10] where the type structure of the regular representation of a locally compact group was studied. Here we consider a representation which vanishes at infinity. Before the results can be stated some notation is established.

G	:	a locally compact group,
π	:	a (weakly continuous unitary) representation of G ,
H_π	:	the Hilbert space of π ,
$M(\pi)$	:	the von Neumann algebra on H generated by $\pi(G)$,
$\omega_{\xi,\eta}$	:	for ξ,η in H_π and x in G , defined by $\omega_{\xi,\eta}(x) = (\pi(x)\xi\|\eta)$,
$C_0(G)$	:	the space of continuous complex valued functions on G which vanish at infinity.

The representation π is said to vanish at infinity if $\omega_{\xi,\eta} \in C_0(G)$, for all $\xi,\eta \in H_\pi$.

This property (as well as the weaker conditions of vanishing at infinity modulo the kernel or projective kernel of π) has received some attention recently. See, for example, [1] , [2] , [3] and [4] . In particular, in [1] , it has been shown that any separable locally compact group whose regular representation is not completely reducible possesses representations which are disjoint from the regular representation but still vanish at infinity. (In the language of [3], $B_0(G) \neq A(G)$.) The results announced here may be considered as a step towards determining what the type structure of $M(\pi)$ could be for such a π.

We call G a [Moore]-group if every irreducible representation of G is finite dimensional. We call G a compact extension of a [Moore]-group if there is a compact normal subgroup K of G such that G/K is a [Moore]-group. This class of groups was introduced by Moore in [6] and completely characterized with a set of topological group conditions by Moore [6] and Robertson [7]. In [10] it was necessary to consider the class of compact extensions of [Moore]-groups and the following characterization of this class can be gleaned from section 5 of [10]. Let G_{FC} denote the subgroup of G consisting of all elements whose conjugacy classes are precompact.

<u>Proposition</u> G is a compact extension of a [Moore]-group if and only if the index of G_{FC} in G is finite and the commutator subgroup of G_{FC} is precompact.

The basic idea of the results we are presenting is that, if π vanishes at infinity and generates a Type I, finite von Neumann algebra then the two conditions of this proposition must be satisfied by G. This was shown for the regular representation in [10]. The precise statements can now be given. The reader is referred to Sakai [8] for the classification theory of von Neumann algebras. Throughout, G is a locally compact group, π is a representation of G and $M(\pi)$ is the von Neumann algebra generated by π.

<u>Theorem 1</u> If π vanishes at infinity and $M(\pi)$ has a nonzero Type I, finite part, then G is a compact extension of a [Moore]-group.

<u>Remark</u> While establishing that G satisfies the properties of the proposition above, it is shown that $M(\pi)$ having a nonzero Type I_n part implies that the index of G_{FC} in G is less than or equal to n^2. This generalizes a theorem of Smith in [9].

For a compact normal subgroup K of G let μ_K be the regular Borel measure on G which is supported on K and, such that, μ_K restricted to K is normalized Haar measure on K. If π is extended to a representation of the finite regular Borel measures on G, then $\pi(\mu_K)$ is a central projection in $M(\pi)$ for each compact normal subgroup K of G.

<u>Theorem 2</u> If π vanishes at infinity and $M(\pi)$ has a nonzero type I, finite part, then there is a compact normal subgroup K of G such that $\pi(\mu_K)$ is the maximal type I, finite central projection in $M(\pi)$.

<u>Remark</u> The compact normal subgroup K of Theorem 2 is chosen minimal such that G/K is a [Moore]-group.

<u>Theorem 3</u> Suppose π is faithful and vanishes at infinity. Then $M(\pi)$ is a type I, finite von Neumann algebra if and only if G is a [Moore]-group.

<u>Remark</u> For the case when π is the regular representation Theorem 3 was established by Kaniuth ([5], Theorem 3).

Theorems 2 and 3 follow easily from Theorem 1. The proof of Theorem 1 is similar to the proof for the regular representation in [10]; however, there are some nontrivial modifications necessary. Complete details of this proof will appear elsewhere.

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THE NUMBER OF REPRESENTATIONS OF AN ODD INTEGER AS A SUM OF THREE PRIMES

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Presented by P. Ribenboim, F.R.S.C.

1. INTRODUCTION. Vinogradov has proved in 1937 (see [4]) that every sufficiently large odd integer is the sum of three primes. A well-known presentation of this result may be found in Estermann's book [1]. If we denote the number of such representations by $r_3(n)$, then it is shown in [1] that

$$(1) \qquad r_3(n) = \rho(n)S(n) + O(n^2/\log^4 n);$$

here $\rho(n) = \sum_{m_1+m_2+m_3=n} (\log m_1 \log m_2 \log m_3)^{-1}$ with $m_i \geq 2$ and, for n odd, $S(n) = 2 \prod_{p \geq 3} (1-c_p(n)(p-1)^{-3})$, with $c_p(n) = \sum_{h=1}^{p-1} e^{2\pi inh/p}$, the Ramanujan sum. The trivial estimate $n^2/3 \log^2 n < \rho(n) < n^2$, together with $S(n) \geq 2 \prod_{p \geq 3} (1-(p-1)^{-2}) > 0$, is all that is needed to show that, for n odd and sufficiently large, $r_3(n) > 0$, so that every large, odd integer has at least one representation as a sum of three primes.

It seems, however, that no asymptotic formula for $r_3(n)$ appears in the literature, unless one is willing to consider (1) as such. It is the purpose of this note to sketch a proof of the asymptotic formula

$$(2) \qquad r_3(n) = \frac{1}{2} \prod_{p} (1+(p-1)^{-3}) \prod_{p|n} (1-(p^2-3p+3)^{-1}) \frac{n^2}{\log^3 n} + O(\frac{n^2}{\log^4 n}),$$

valid for $n \to \infty$.

In (2) no restriction on the parity of n is indicated, because, for n even, the principal term vanishes. While it is possible to expand $\rho(n)$ into an asymptotic series, this would hardly be useful; indeed, all terms that are $O(n^2/\log^4 n)$ will be absorbed into the error term of (1). The latter comes from the minor intervals of the "circle method" (see [1] and [4]) and there are no prospects to improve this error term.

The topic of this note had been suggested as a project to a class in number theory at Temple University (Spring 1979) and the contributions of its members, A. Ayoub, R. Cavaliere, and M. van Rossum, are herewith acknowledged.

2. PRELIMINARIES. Let n stand for an odd integer, sufficiently large to justify the inequalities in which it appears, but always larger than 9. It is clear that the contribution to $\rho(n)$ of terms with two equal values of the m_i's ($i = 1,2,3$) is at most $O(n)$, hence negligible. Consequently,

$$\rho(n) = 6 \sum_{\substack{2 \le m_3 < m_2 < m_1 \\ m_1+m_2+m_3 = n}} \frac{1}{\log m_1 \log m_2 \log m_3} + O(n). \tag{3}$$

Also, $n/3 < m_1 < n-4$ and $(n-m_1)/2 < m_2 < \min(m_1, n-m_1-2)$, so that the sum in (3) equals (with a slight change in notation)

$$\sum_{m=n/3}^{(n-2)/2} (\log m)^{-1} \sum_{m_1=(n-m)/2}^{m} \{\log m_1 \log(n-m-m_1)\}^{-1} \tag{4}$$

$$+ \sum_{m=(n-1)/2}^{n-4} (\log m)^{-1} \sum_{m_1=(n-m)/2}^{n-m-2} \{\log m_1 \log(n-m-m_1)\}^{-1} = \sum\nolimits_1 + \sum\nolimits_2,$$

say.

These sums are estimated by the Euler-Maclaurin sum formula

$$(5) \qquad \sum_{m=a}^{b} f(m) = \int_a^b f(z)dz + \frac{1}{2}\{f(a) + f(b)\} + \int_a^b f'(z)(z-[z]-\tfrac{1}{2})dz$$

([z] = greatest integer not in excess of z).

3. PROOF OF THE MAIN RESULT. Let $u = n-m$ and set $f(z) = \{\log z \log(u-z)\}^{-1}$. We observe that in $\sum_1$,

$$(6) \qquad u/2 \le z \le m \le n-u \qquad \text{and} \qquad (n+2)/2 \le u = n - m \le 2n/3,$$

while in $\sum_2$

$$(7) \qquad u/2 \le z \le u-2 \qquad \text{and} \qquad 4 \le u = n-m \le (n+1)/2.$$

On hand of (6) and (7) one verifies that the second and third summands of (5), when substituted in $\sum_1$ and $\sum_2$, lead to terms at most $O(n^2/\log^4 n)$. Hence, it is legitimate to replace the inner sums in both, $\sum_1$ and $\sum_2$ by the first integral of (5).

By fairly elementary manipulations it is found that the inner sum of $\sum_1$ (up to terms that may be ignored) is equal to

$$\frac{1}{\log(n-m)}\left\{\frac{n-m}{2\log((n-m)/2)} - \frac{n-2m}{\log(n-2m)}\right\},$$

and that in $\sum_2$ to

$$\frac{n-m}{2(\log((n-m)/2))^2}.$$

We substitute these expressions in (4) and obtain

$$\sum_{m=n/3}^{(n-2)/2} \frac{1}{\log m \log(n-m)}\left\{\frac{n-m}{2\log((n-m)/2)} - \frac{n-2m}{\log(n-2m)}\right\} + \frac{1}{2}\sum_{m=(n-1)/2}^{n-4} \frac{n-m}{\log m \log^2((n-m)/2)}.$$

We now repeat the procedure. Each of the sums is evaluated by (5). As before, it turns out that the second and third terms lead to error terms absorbed by $O(n^2/\log^4 n)$, so that it is sufficient to replace each sum by the respective integral. While not entirely trivial, these can be evaluated and lead to

$$\left\{\left(\frac{7}{144}-\frac{1}{36}\right)+\frac{1}{16}\right\}\frac{n^2}{\log^3 n}+O\left(\frac{n^2}{\log^4 n}\right)=\frac{n^2}{12\log^3 n}+O\left(\frac{n^2}{\log^4 n}\right).$$

It now follows from (3) that $\rho(n) = n^2/(2\log^3 n) + O(n^2/\log^4 n)$.

4. THE SINGULAR SERIES AND END OF PROOF. As well-known (see, e.g., Theorem 271 in [3]), $c_p(n) = p-1$ if $p|n$, $c_p(n) = -1$ if $p\nmid n$. If we substitute this in the formula for $S(n)$, we obtain

$$S(n) = \prod_p (1+(p-1)^{-3}) \prod_{p|n} (1-(p^2-2p+3)^{-1}),$$

where both sides vanish for even n. If we now substitute for $\rho(n)$ and $S(n)$ their values in (1), we obtain (2).

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CONVERGENCE SPACES AND EXTENSIONS OF MAPS

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Presented by P.C. Greiner, F.R.S.C.

This is the third paper in a series of investigations answering various unsolved problems relative to convergence spaces (1)(2)(3)(4). The major goal of this present research is to investigate the extendibility of maps defined on "preconvergence spaces." For complete details and interesting results not contained in this note, we refer the reader to (5).

Let F(X) [resp. U(X)] denote the set of all filters [resp. ultrafilters] on X. Then a pair (X,q), where X is a nonempty set, and $q:F(X) \to \mathcal{P}(X)$ is a <u>preconvergence space</u> if

CS(1): for each $x \in X$, $x \in q(\dot{x})$,

CS(2): whenever $\mathcal{F}, \mathcal{G} \in F(X)$, $\mathcal{F} \subset \mathcal{G}$, then $q(\mathcal{F}) \subset q(\mathcal{G})$.

For $x \in X$, $\dot{x} = \{y \mid [y \subset X] \wedge [x \in y]\}$ (the principal ultrafilter generated by x). In (6), Kent calls such a "q" which satisfies CS(1), CS(2) a convergence function and if for each $x \in q(\mathcal{F})$, $\mathcal{F} \in F(X)$, it follows that $x \in q(\mathcal{F} \cap \dot{x})$, then (X,q) is a <u>convergence space</u> (7). For $\mathcal{F} \in F(X)$, $A \subset X$, let $\mathcal{F} \cap A$ signify that $F \cap A \neq \emptyset$ for each $F \in \mathcal{F}$ and $\mathcal{F} \perp A$ signify that there exists some $F \in \mathcal{F}$ such that $F \cap A = \emptyset$. For any preconvergence space (Y,p), $A \subset Y$, let $U(Y,A) = \{x \mid [x \in U(Y)] \wedge \exists y[[y \in A] \wedge [x \to y]]\}$

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[resp. $F(Y,A) = \{x | [x \in F(Y)]\ldots\}]$. A map $g:(X,q) \to (Y,p)$ is <u>weakly-continuous</u> [resp. <u>continuous</u>] if for each $\mathcal{U} \in U(X)$ [resp. $\mathfrak{F} \in F(X)$] such that $\mathcal{U} \to x$ [resp. $\mathfrak{F} \to x$], then $g(\mathcal{U})$ [resp. $g(\mathfrak{F})] \to g(x)$. For $A,B \subset X$, let $U_B(X,A) = \{\mathcal{U}_B | [\mathcal{U} \in U(X)] \wedge [\mathcal{U} \cap B] \wedge \exists y[[y \in A] \wedge [\mathcal{U} \to y]]\}$, where $\mathcal{U}_B$ is the filter on B generated by $\{U \cap B | U \in \mathcal{U}\}$. Let $\mathcal{C}$ be a collection of filter bases on X and $g:(X,q) \to (Y,p)$. Then $g(\mathcal{C}) = \{g(\mathfrak{F}) | \mathfrak{F} \in \mathcal{C}\}$. Now for each $\mathfrak{F} \in \mathcal{C}$, let $C(\mathfrak{F}) = \{x | [x \in X] \wedge [\mathfrak{F} \to x]\}$ and $I(\mathcal{C}) = \bigcap \{C(\mathfrak{F}) | \mathfrak{F} \in \mathcal{C}\}$. Assume that (X,q') is dense in (Z,q).

DEFINITION A map $g:(X,q') \to (Y,p)$ is <u>weakly-admissible</u> if for each $r \in R = Z - X$, $I(g(U_X(Z,\{r\}))) \neq \emptyset$.

Weak-admissibility is a minimum condition for extendibility as the next result indicates.

THEOREM 1. <u>If for</u> $g:(X,q') \to (Y,p)$ <u>there exists a weakly-continuous extension</u> $G:(Z,q) \to (Y,p)$, <u>then</u> g <u>is weakly-admissible</u>.

Now in order to guarantee that $g:X \to Y$ is extendible additional hypotheses seem to be required. A <u>remainder space</u> (R,q''), where $R = Z - X$ and q'' is the induced convergence function, is <u>U-principal</u> if the only q''-convergent ultrafilters on R are the principal ultrafilters. The space R is <u>separated from X</u> if for each $r \in R$, $\mathcal{U} \in U(Z,\{r\})$ implies that $\mathcal{U} \notin U(Z,\{x\})$ for each $x \in X$.

THEOREM 2. <u>If</u> $g:(X,q') \to (Y,p)$ <u>is weakly-admissible, weakly-continuous</u>, R <u>is</u> U-<u>principal and</u> X <u>is weakly-open in</u> Z, <u>then there exists a weakly-continuous extension</u> $G:(Z,q) \to (Y,p)$.

Other types of extensions, especially with respect to compactifications of (X,q'), are also investigated in (5). However, many of the major results are relative to pure generalizations of Taĭmanov's characterization for extendibility (8). Numerous previous Taimanov type "generalizations" have appeared in the literature, but they all can be shown to be only corollaries and not pure generalizations of his basic theorem. The following results are pure generalizations of Taĭmanov's classical result.

THEOREM 3. Let Y be a compact Hausdorff preconvergence space and $g:X \to Y$. If whenever $\mathcal{V}, \mathcal{V}' \in U(Y)$, $\mathcal{V} \cap g[X]$, $\mathcal{V}' \cap g[X]$, $\mathcal{V} \to y$, $\mathcal{V}' \to y'$ and $y \neq y'$, it follows that $cl_Z(g^{-1}(\mathcal{V})) \perp cl_Z(g^{-1}(\mathcal{V}'))$, then g is weakly-continuous.

THEOREM 4. Let Y be compact and weakly-Urysohn and $g:X \to Y$. If whenever $\mathcal{V}, \mathcal{V}' \in U(Y)$, $\mathcal{V} \cap g[X]$, $\mathcal{V}' \cap g[X]$, $\mathcal{V} \to y$, $\mathcal{V}' \to y'$ and $y \neq y'$, it follows that $cl_Z(g^{-1}(cl_Y\mathcal{V})) \perp cl_Z(g^{-1}(cl_Y\mathcal{V}'))$, then g is weakly-admissible.

THEOREM 5. Let $G:Z \to Y$ be a weakly-continuous extension of $g:X \to Y$. If $\mathcal{F}, \mathcal{G} \in F(X)$, $\mathcal{F} \cap g[X]$, $\mathcal{G} \cap g[X]$, and $cl_Y\mathcal{F} \perp cl_Y\mathcal{G}$, then $cl_Z(g^{-1}(\mathcal{F})) \perp cl_Z(g^{-1}(\mathcal{G}))$.

THEOREM 6. Let $G:Z \to Y$ be a weakly-continuous extension of $g:X \to Y$ and Y is regular Hausdorff. If $\mathcal{F}, \mathcal{G} \in F(X)$, $\mathcal{F} \cap g[X]$, $\mathcal{G} \cap g[X]$, $\mathcal{F} \to y$, $\mathcal{G} \to y'$ and $y \neq y'$, then $cl_Z(g^{-1}(cl_Y\mathcal{F})) \perp cl_Z(g^{-1}(cl_Y\mathcal{G}))$.

THEOREM 7. Let Y be a compact Hausdorff topological space. Then a weakly-continuous map $g:(X,q') \to (Y,T)$ has a unique continuous extension to any extension Z of X iff whenever $\mathcal{F}, \mathcal{G} \in F(X)$, $\mathcal{F} \cap g[X]$, $\mathcal{G} \cap g[X]$, $\mathcal{F} \to y$, $\mathcal{G} \to y'$ and $y \neq y'$, then $cl_Z(g^{-1}(cl_Y\mathcal{F})) \perp cl_Z(g^{-1}(cl_Y\mathcal{G}))$.

For general extensions Z of X, we have the following major contribution.

THEOREM 8. *Let* $g:X \to Y$ *be weakly-continuous and weakly-admissible. If* Z *is any extension of* X, *then* g *can be extended to a weakly-n-continuous map* $G:Z \to Y$. *If* Y *is Hausdorff, then* G *is unique*.

Finally numerous examples are given which show that these results are nontrivial. They also show that Theorem 7 is a pure generalization of the basic Taǐmanov result since it applies to a strictly larger domain class than the topological spaces.

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A NEW APPROACH TO THE CHIRAL ARCHIMEDEAN SOLIDS

by

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Abstract

Eight of the triangular faces of the snub cube lie in the face-planes of a regular octahedron. It is explained here why each vertex of the octahedron lies on an edge (extended) of the snub cube. The same situation arises when the snub dodecahedron is inscribed in an icosahedron.

The snub cube $s\begin{Bmatrix}3\\4\end{Bmatrix}$ and the snub dodecahedron $s\begin{Bmatrix}3\\5\end{Bmatrix}$ are the most remarkable members of a small family of uniform polyhedra $s\begin{Bmatrix}3\\q\end{Bmatrix}$ in which each vertex is surrounded by four equilateral triangles and one regular q-gon [Brückner 1900, p. 139]. Some triangles are entirely surrounded by other triangles; they lie in the face-planes of a Platonic solid $\{3, q\}$ having the same centre O as the 'snub'. (Compare the more familiar fact that the q-gons lie in the face-planes of a $\{q, 3\}$.) Although the whole $s\begin{Bmatrix}3\\q\end{Bmatrix}$ is chiral (except when $q = 2$ or 3), the 'solid angle' at any vertex A (see Figure 1) is symmetrical by reflection in the plane OAD, which separates two of the four triangles at A from the other two and therefore passes through the centre P' of the q-gon BAF ... . Thus the edge DA (extended) intersects the 'in-radius' OP' (extended) at a vertex P of $\{3, q\}$. It follows that the face ADE of $s\begin{Bmatrix}3\\q\end{Bmatrix}$ (entirely surrounded by other triangles) lies in the plane of a face PQR of $\{3, q\}$ in such a way that *its sides are formed by lines* PD, QE, RA, *making equal angles* γ *with the sides of the triangle* PQR

[Huybers 1976, pp. 19-30]. (Fig. 37 of Haussner [1906, p. 62] indicates that Cauchy was aware of this remarkable fact in the special case when $q = 3$.) Proceeding thus on each face of $\{3, q\}$, we can construct all the vertices of $s\left\{\begin{matrix}3\\q\end{matrix}\right\}$, provided we know the angle γ.

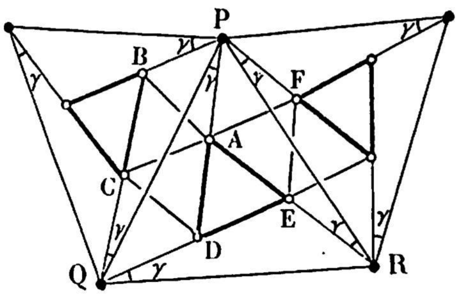

Figure 1

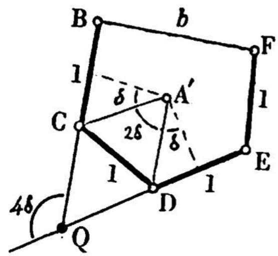

Figure 2

To express this angle in terms of q, let us examine the solid angle at Q formed by the lines QB, QP, QD. Between the planes BQP and DQP we have the dihedral angle of $\{3, q\}$, namely $180° - 2\psi$ where

$$\cos \psi = b/\sqrt{3} \quad , \quad b = 2 \cos(180°/q)$$

[Coxeter 1969, p. 157]. Figure 2 shows the vertex figure of $s\left\{\begin{matrix}3\\q\end{matrix}\right\}$: an 'isosceles' pentagon BCDEF (with sides 1, 1, 1, 1, b) whose vertices are the farther ends of all the edges at A. It is inscribed in a circle with centre A' : the intersection of two spheres with centres O and A, radii OA and AB. Each of the four congruent sides of this pentagon subtends at A' an angle 2δ whose cosine is $X/2$, X being the (greatest) root of the cubic equation

$$X^3 + 2X^2 - b^2 = 0$$

[Coxeter, Longuet-Higgins and Miller 1954, p. 423: (10.5) with $c = 0$]. Applying spherical trigonometry to the solid angle at Q, in which

$$\angle DQB = 180° - 4\delta, \quad \angle BQP = \gamma, \quad \angle PQD = 60° - \gamma$$

and the dihedral angle along QP is $180° - 2\psi$, we find

$$\begin{aligned} \cos(180°-4\delta) &= \cos\gamma\cos(60°-\gamma) + \sin\gamma\sin(60°-\gamma)\cos(180°-2\psi) \\ &= \tfrac{1}{2}\{\cos(60°-2\gamma) + \cos 60°\} - \tfrac{1}{2}\{\cos(60°-2\gamma) - \cos 60°\}\cos 2\psi \\ &= \cos(60°-2\gamma)\sin^2\psi + \tfrac{1}{2}\cos^2\psi . \end{aligned}$$

Since $\cos^2\psi = b^2/3$ and $\cos(180°-4\delta) = -\cos 4\delta = 1 - 2\cos^2 2\delta$ $= 1 - X^2/2$, this yields

$$\cos(60° - 2\gamma) = (6 - b^2 - 3X^2)/(6 - 2b^2) .$$

For instance, when $q = 3$, so that we are dealing with the icosahedron $s\begin{Bmatrix}3\\3\end{Bmatrix}$ inscribed in the tetrahedron $\{3, 3\}$, we have

$$X = \tau^{-1} = (\sqrt{5}-1)/2, \quad \cos(60°-2\gamma) = (3\sqrt{5}+1)/8, \quad \cos 2\gamma = (3\sqrt{5}-1)/8.$$

The following table includes not only the octahedron $s\begin{Bmatrix}3\\2\end{Bmatrix}$ and the icosahedron $s\begin{Bmatrix}3\\3\end{Bmatrix}$ but also the limiting case of the tessellation $s\begin{Bmatrix}3\\6\end{Bmatrix}$, which is covered by the same formula if we set $X = 1 - \varepsilon$ and ignore higher powers of ε, so that

$$b^2 = X^3 + 2X^2 = 3 - 7\varepsilon, \quad \cos(60°-2\gamma) = 13/14.$$

q	b^2	X	cos(60°−2γ)	γ
2	0	0	1	30°
3	1	0·6180339887	0·9635255	22° 14' 20"
4	2	0·8392867552	0·9433964	20° 18' 54"
5	τ^2	0·9431512592	0·9338062	19° 31' 5"
6	3	1	$0{\cdot}9\dot{2}8571\dot{4}$	19° 6' 24"

Knowing γ, we can easily derive Cartesian coordinates for the vertices of $s\begin{Bmatrix}3\\q\end{Bmatrix}$ from the known coordinates for the vertices of $\{3, q\}$. In fact, taking PQR (Figure 1) as triangle of reference and observing that PD (extended) divides QR in the ratio $\sin\gamma : \sin\gamma'$, where $\gamma' = 60° - \gamma$, we see that the vertices A, D, E have for barycentric coordinates the cyclic permutations of

$$(\sin^2 \gamma', \quad \sin\gamma' \sin\gamma, \quad \sin^2 \gamma).$$

(This is what happens for one of the two enantiomorphic varieties of $s\begin{Bmatrix}3\\q\end{Bmatrix}$; for the opposite variety we simply have to interchange γ and γ'.) In the case of the snub *cube* $s\begin{Bmatrix}3\\4\end{Bmatrix}$, Lines [1965, p. 176] denoted these barycentric coordinates (which, in this one case, can serve also as Cartesian coordinates) by (a, b, c), without noticing that they are in geometric progression.

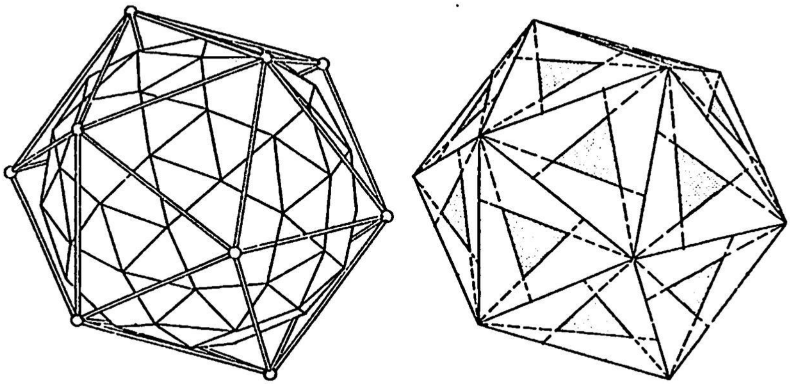

Figure 3: $s\begin{Bmatrix}3\\5\end{Bmatrix}$ inscribed in $\{3, 5\}$

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ZEROS OF QUADRATIC FORMS

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Let $f(\underline{x}) = f(x_1, x_2, \ldots, x_m) \in \mathbb{Z}[\underline{x}]$ be a quadratic form in m variables and suppose that f represents 0 over the rational integers $\mathbb{Z}$. Cassels [1] was the first* to prove, for general m, that there is a zero $\underline{x} \in \mathbb{Z}^m$ of f which is not too large compared with the 'size' of the coefficients of f, or more precisely, an $\underline{x}$ satisfying $f(\underline{x}) = 0$ and

$$(1) \qquad 0 < |\underline{x}| \leqq c_m F^{\frac{m-1}{2}}, \quad \left(|\underline{x}| = \max_{1\le i\le m} |x_i|\right)$$

with some explicit constant $c_m > 0$ depending only on m and $F = \sum_{1\le i,j\le m} |f_{ij}|$, f_{ij} being the coefficients of f. Subsequently, Davenport [3] showed that, if $m \ge 2$ and f is also non-singular, there are two linearly independent zeros $\underline{x}, \underline{y}$ over $\mathbb{Z}$ satisfying $f(\underline{x}) = f(\underline{y}) = 0$ and

$$(2) \qquad 0 < |\underline{x}||\underline{y}| \leqq c_m' F^{m-1}$$

with $c_m' > 0$ depending only on m. He remarked that the problem of obtaining bounds for 3 or more linearly independent zeros of f remained open. The best constant known for (1) is due, essentially, to Raghavan [6], who generalized the problem to algebraic number fields. If $K/\mathbb{Q}$ is a finite extension of the rational field $\mathbb{Q}$ of degree $[K:\mathbb{Q}] = n \ge 1$, define

$$\|\underline{x}\| = \max_{1\le i\le m} \max_{1\le r\le n} |x_i^{(r)}|, \quad \text{for } \underline{x} \in K^m$$

where $x^{(r)}$ denotes the $\mathbb{Q}$-conjugate of $x \in K$. Let $\mathcal{V}$ be a fixed order

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* However, see [2] for an historical note and references.

of K and suppose $f(\underline{x}) \in V[\underline{x}]$ where $\underline{x} \in V^m$. Then, on the corresponding assumption that $f(\underline{x})$ actually represents 0 over V, Raghavan proved that there is a zero $\underline{x} \in V^m$ of $f(\underline{x})$ such that

$$(3) \qquad 0 < \|\underline{x}\| \leq |d|^{m/2n}(5F)^{\frac{m-1}{2}},$$

where $|d|$ is the absolute value of the discriminant of K and

$$F = \max_{1\leq r\leq n} \sum_{1\leq i,j\leq m} |f_{ij}^{(r)}| .$$

A refinement by Cassels [2], p. 89, in the case $n = 1$, replaced the 5 on the right of (3), by 3 and so (1) is in fact true with $c_m = 3^{(m-1)/2}$. I have recently proved an extension, corresponding to (2), for the product of two elements $\underline{x}, \underline{y}$ in V^m, linearly independent over V i.e.,

$$(4) \qquad 0 < \|\underline{x}\|\|\underline{y}\| \leq c_{m,n}|d|^{m/n}(3F)^{m-1}$$

with* $f(\underline{x}) = f(\underline{y}) = 0$ and, in general, a constant $c_{m,n}$, depending only upon m and n, appreciably larger than 1. However, $c_{m,1} = 1$ and this gives a sharper form to Davenport's inequality (2). The proof of (4) follows the general pattern of Raghavan's argument for (3) and would have been a routine extension but for the requirement that $\underline{x}$ and $\underline{y}$ be linearly independent over V (or K) and not just over $\mathbb{Q}$. For this, the natural tool is a suitable theorem on the "successive minima" of a convex body and, as it turned out, neither the standard theorem of Minkowski [5] (for general lattices) nor the theorem of K. Rogers and Swinnerton-Dyer [7] (for algebraic lattices) were adequate. However, the ideas contained in the latter article

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* As the proof is rather long, it will be published elsewhere.

can be adapted to this purpose, with the consequence that $c_{m,n}$ is rather weaker than the 'expected' constant 1 when $n > 1$.

As a corollary to (4), one can deduce (from the proof itself) that, for any integer k selected from $1, 2, \ldots, m$, there is a solution of (4) with the factor $(3F)^{m-1}$ replaced by $(12F)^{m-1}$, and with

(5) $$\underline{\text{either}}\ x_k \neq 0\ \underline{\text{or}}\ y_k \neq 0\ ;$$

the x_k, y_k being the k^{th} components of $\underline{x}$, $\underline{y}$, respectively. As an application of (4) and (5), we shall take $n = 1$, $K = \mathbb{Q}$, $V = \mathbb{Z}$ and consider a problem on rational representations of a non-zero number $\kappa \in \mathbb{Q}$ by $f(\underline{x})$. Suppose then that $f(\underline{x}) \in \mathbb{Q}[\underline{x}]$ and we require effectively computable bounds for both the numerator and denominator of each of the components x_i of a suitably chosen zero $\underline{x} \in \mathbb{Q}^m$ of $f(\underline{x})-\kappa=0$, on the assumption that

(6) $$f(\underline{x}) = \kappa$$

has at least one solution of $\underline{x} \in \mathbb{Q}^m$. Without loss of generality, by clearing denominators, we can write (6) in the form

(7) $$f'(\underline{x}) = \kappa' t^2 \left(t \in \mathbb{Z}\ ,\ t \neq 0\ ,\ \underline{x} \in \mathbb{Z}^m\ ,\ \kappa' \in \mathbb{Z}\ ,\ f'(x) \in \mathbb{Z}[\underline{x}]\right)$$

and seek upper bounds for $(\underline{x}, t) = (x_1, \ldots, x_m, t)$. By (4), with $n = 1$, m replaced by $m + 1$, $|d| = 1$ and (5) with $k = m + 1$ we have two solutions $(\underline{x}_1, t_1)$, $(\underline{x}_2, t_2)$ of (7), linearly independent over $\mathbb{Q}$, which satisfy

$$0 < |(\underline{x}_1, t_1)||(\underline{x}_2, t_2)| \leq (12(F' + |\kappa'|))^m$$

and at least one of the integers t_1, t_2 is <u>not</u> zero. Of course, we can always use (1) with $c_m = 3^{(m-1)/2}$, if $f(\underline{x})$ is a form which does not represent 0 over $\mathbb{Q}$ (e.g. $f(\underline{x}) = x_1^2 + x_2^2 - 3(x_3^2 + x_4^2)$ for $m = 4$). But, by the old theorem of Meyer [4], we know that this only happens for special forms in $m \leq 4$ variables; indeed, all forms in $m \geq 5$ variables are zero forms (over $\mathbb{Q}$).

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An Affine Generalization of the Euler Line

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Let ABC be a triangle in the Euclidean plane. We denote the center of gravity of ABC by G, its circumcenter (center of the circumscribed circle) by O and its orthocenter (intersection of the three altitudes) by H. Euler (1707-1783) showed that G, O and H lie on a line and that $GH = 2OG$. There are many proofs of this theorem and the line OGH is called "the Euler line"; see for instance [1], pages 17, 18.

Euler's theorem refers to the metric of the Euclidean plane, since this metric is necessary to define the points O and H. We show here that this theorem is a special case of a much more general theorem which refers only to the <u>affine</u> structure of the plane. This affine theorem may of course mention G but not O or H.

Theorem. Let P be any point in the plane whatsoever. Let L_A be the line through the vertex A which is parallel to the line through P and the midpoint M of the opposite side BC (Figure 1). Let the lines L_B and L_C be constructed similarly. Then, the three lines L_A, L_B and L_C have a point Q in common and the points G, P and Q lie on a line and $GQ = 2PG$.

Proof. Consider the dilatation d with center G and ratio -2, and put $d(P) = Q$. Since $d(M) = A$ (Figure 1), Q lies on the line L_A. Similarly, Q lies on L_B and L_C and the remainder of the theorem follows equally easily from the properties of d.

Proof of Euler's theorem. We do not assume to know that the three altitudes of a triangle pass through a point, and choose for P the circumcenter O of ABC. Then, the line through P and M is perpendicular to the side BC (Figure 1) and the same holds for the other two sides of ABC. Hence the above theorem now states that the three altitudes of ABC pass through a point H and that the points G, O and H lie on a line and that $GH = 2OG$.

We observe that the above affine theorem generalizes immediately to arbitrary dimensions. Let $A_0, \ldots, A_n$ be a simplex in n-space and let P be an arbitrary point in n-space. Through the vertex A_i of the simplex draw the line L_i which is parallel to the line through P and the center of gravity M_i of the opposite face $A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$. Then, the $n + 1$ lines $L_0, \ldots, L_n$ have a point Q in common and the line through P and Q passes through the center of gravity G of the simplex and $GQ = nPG$. The proof of this affine theorem is entirely analogous to the proof given above for the case $n = 2$. Simply use the dilatation with center G and ratio $-n$.

However the metric theorem, that is, Euler's theorem does not extend to higher dimensions. When $n = 2$, M_i is both the center of gravity and the circumcenter of the 1-simplex $A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ and this fails in higher dimensions.

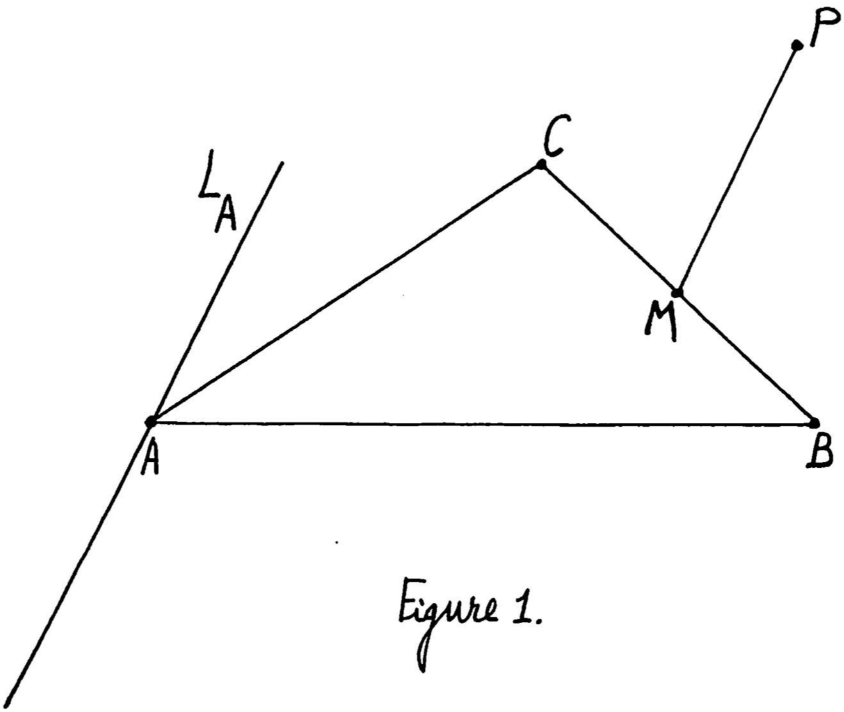

Figure 1.

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