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Mailing Addresses

1. Bantegnie, R. Université de Franche-Compté, 25030 Besançon,
   France

2. Coxeter, H.S.M. Department of Mathematics, University of Toronto,
   Toronto, Ontario, Canada M5S 1A1

3. Djoković, D.Z. Department of Pure Mathematics, University of
   Waterloo, Waterloo, Ontario, Canada N2L 3G1

4. Herrmann, R.A. Department of Mathematics, U.S. Naval Academy,
   Annapolis, Md. 21402, U.S.A.

5. Jonker, L. Department of Mathematics and Statistics, Queen's
   University, Kingston, Ontario, Canada K7L 3N6

6. Krall, A.M. McAllister Bldg., Pennsylvania State University,
   University Park, Penn. 16802, U.S.A.

7. Lelong, P. Université Pierre et Marie Curie Mathematiques,
   75230 Paris, Cedex 05, France

8. Mingarelli, A.B. Department of Mathematics, University of Ottawa,
   Ottawa, Ontario, Canada K1N 6N5

9. Morgan, C. Department of Mathematics and Statistics, Memorial
   University of Newfoundland, St. John's, Newfoundland,
   Canada A1B 3X7

10. Orecchia, F. Istituto di Matematica, dell'Università di Genova,
    16132 - Genova, Italy

11. Przeworska-Rolewicz, D. Institute of Mathematics, Polish Academy
    of Sciences, Sniadeckiah 8,
    00-950 Warszawa, Poland.

12. Rand, D. Math. Institute, University of Warwick, Coventry,
    England CV4 7AL

13. Rassias, T.M. Φ.L.A.A., Rm. 808, 279 Patision Street, Athens,
    Greece (until December 1979)

14. Yui, N. Mathematisches Institut, Universität Bonn,
    Wegelerstrasse 10, D-5300 Bonn, West Germany

15. Zinde, V.M. Mičurinskiĭ pr. 12, korpus 1, kv. 124, Moscow 117192,
    USSR

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BIFURCATIONS OF UNIMODAL MAPS OF THE UNIT INTERVAL

Leo Jonker and David Rand

Presented by F.V. Atkinson, F.R.S.C.

1. INTRODUCTION.

Let I denote a closed interval of real numbers. A unimodal map is a C^1 map $f\colon I \to I$ such that f has a unique critical point which is either a maximum or a minimum and $f(\partial I) \subset \partial I$. Throughout this note we shall generally assume that $I = [-1,1]$ and that the critical point is a maximum at 0. There is no loss of generality.

In [3] we announced results describing the non-wandering set of a unimodal map, relating the non-wandering set to the Milnor-Thurston kneading invariant $\nu(f)$. In [2] it was shown that the set of periodic points of a unimodal map f is esssntially determined by $\nu(f)$. In this note we announce results proved in [4] concerning the bifurcations of the periodic orbits and the kneading invariant of a C^1 - continuous family f_u of unimodal functions.

2. A VERSAL MODEL FOR BIFURCATIONS.

Let $\mathcal{K} \subset Z[[t]]$ be the set of all admissible kneading invariants of unimodal functions. Let $\mathcal{P} \subset \mathcal{K}$ denote the set of periodic kneading invariants and $\Pi \subset \mathcal{P}$ the set of those that are periodic but not anti-periodic. If $\nu = \sum_{i=0}^{\infty} \nu_i t^i \in \mathcal{P}$, and if n is the smallest integer such that $\nu_i = \nu_{i+n}$ for all i, we let

$$\nu^{(1)} = \sum_{k=0}^{\infty} \sum_{i=0}^{n-1} (-1)^k \nu_{kn+i} t^{kn+i}$$

Then $\nu^{(1)} \in \mathcal{P}\setminus\Pi$ and is periodic of period $2n$. Thus we can define $\nu^{(2)}, \nu^{(3)}, \ldots$ inductively, all members of $\mathcal{P}\setminus\Pi$. Conversely, if $\mu \in \mathcal{P}\setminus\Pi$, then $\mu = \nu^{(n)}$ for some $\nu \in \Pi$ and some integer $n \geq 1$. With the lexicographic ordering on $Z[[t]]$, the set $\{\mu \in \mathcal{K} \mid \nu^{(m)} < \mu < \nu^{(m+1)}\}$ is empty. Thus, with respect to the t-adic topology on $Z[[t]]$, an element of $\mathcal{P}\setminus\Pi$ is isolated in $\mathcal{K}$, an element $\nu \in \Pi$ has a neighbourhood which contains no $\mu \in \mathcal{K}$, $\mu < \nu$, while a non-periodic kneading invariant is a limit of both a monotone increasing, and a monotone decreasing sequence of kneading invariants.

We define an order reversing homeomorphism $\nu \to \nu^*$ between $\mathcal{K}$ and a subset $\mathcal{K}^*$ of $[0,1]$, and an order reversing map ζ: $[0,1] \to \mathcal{K}$ such that $\zeta^{-1}(\nu) = [\nu^*, \nu^{(1)*}]$ if $\nu \in \Pi$, $\zeta^{-1}(\nu) = (\nu^*, \nu^{(1)*}]$ if $\nu \in \mathcal{P}\setminus\Pi$, and $\zeta^{-1}(\nu) = \nu^*$ if $\nu \in \mathcal{K}\setminus\mathcal{P}$. In addition, we let ν^{**} denote the mid point of $[\nu^*, \nu^{(1)*}]$ if $\nu \in \mathcal{P}\setminus\Pi$. The map ζ serves as a versal model for the map ν that associates $\nu(f)$ to the function f; the concepts "*attracting periodic orbit class*" and "*monotone equivalence*" are defined in [3].

THEOREM 1. *For every* C^1 *continuous family* f_u, $u \in [0,1]$ *of unimodal functions there is a continuous map* ϕ: $[0,1] \to [o,1]$ *such that* $\nu(f_u) = \zeta(\phi(u))$. *Moreover, the preimages under* ϕ *of the sets* $(\mathcal{K}\setminus\mathcal{P})^*$, $(\mathcal{P}\setminus\Pi)^*$, Π^*, $(\mathcal{P}\setminus\Pi)^{**}$, $(\nu^*, \nu^{(1)}*)$, (ν^*, ν^{**}), (ν^{**}, ν^*) *are all characterized by the existence and the nature of an attracting periodic orbit class for* f_u.

THEOREM 2. *The following is true for a generic family* f_u :

(a) *If* $\phi(u_0) \in \nu^*$, $\nu \in \mathcal{P}\setminus\Pi$, *then at* $u = u_0$, 0 *is periodic under* f_{u_0} *and the kneading invariant* $\nu(f_u)$ *changes from* ν to $\nu^{(1)}$ *or vice versa*.

(b) *If* $\phi(u_0) \varepsilon \nu^*$, $\nu \varepsilon \Pi$, *then* $\nu(f_u)$ *is bigger than* ν *on one side of* u_0 *and equal to* ν *on the other*, *and a new attracting periodic orbit class arises (or disappears) by a saddle-node bifurcation* (see [1]).

(c) *If* $\phi(u_0) \varepsilon \nu^{**}$, $\nu \varepsilon \mathfrak{P}\setminus\Pi$, *then* $\nu(f_u)$ *is constant on a neighbourhood of* u *and the attracting periodic orbit class bifurcates from [resp. to] a single orbit to [resp. from] several monotone equivalent orbits via a saddle-node bifurcation or a flip bifurcation* (see [1]).

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Queen's University
Kingston, Ont. Canada

University of Warwick
Coventry, England

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Universal Vector Bundles

by

C. Morgan

Presented by P. Ribenboim, F.R.S.C.

In [1], Booth, Heath and Piccinini did a comparative study of different kinds of universal fibrations. To do this, they set up a general theory of $\mathcal{A}$-fibrations, whose main characteristics and definitions are the following. First of all, all the work is done in a convenient category $\mathcal{K}$; then, they consider a category $\mathcal{F}$ with a distinguished object F and an underlying space functor $\mathcal{F} \to \mathcal{K}$. An <u>$\mathcal{F}$-space</u> is defined to be a morphism $p : X \to A$ of $\mathcal{K}$ such that A is a CW-complex and for every $a \in A$, $p^{-1}(a) \in \text{Ob}\mathcal{F}$. The notions of <u>$\mathcal{F}$-homotopy</u> and <u>$\mathcal{F}$-homotopy equivalence</u> are easily defined. The morphisms of the category $\mathcal{F}$ are required to be $\mathcal{F}$-homotopy equivalences over a point. Given $\mathcal{F}$-spaces $q : Y \to A$ and $r : Z \to B$, we define the map $q * r : Y * Z = \bigcup_{(a,b) \in A\times B} \mathcal{F}(Y_a, Z_b) \to A \times B$ which takes $f : Y_a \to Z_b$ into (a, b), with $Y * Z$ conveniently topologized. Then a full subcategory $\mathcal{A}$ of the category of $\mathcal{F}$-spaces is defined to satisfy four axioms ([1], page 171); the objects of the category $\mathcal{A}$ are called <u>$\mathcal{A}$-fibrations</u>. An $\mathcal{A}$-fibration $p_\infty : E_\infty \to B_\infty$ is said to be:

(1) <u>Free Universal</u> if the $\mathcal{F}$-homotopy equivalence classes of $\mathcal{A}$-fibrations over a space X correspond bijectively to the free homotopy classes of maps from X into B_∞;

(2) *Grounded Universal* if a based version of (1) holds;

(3) *Aspherical Universal* if the space $\mathrm{Prin}_F E_\infty = F * E$ constructed from $c : F \to *$ and p_∞ is such that $\pi_i(\mathrm{Prin}_F E_\infty) = 0$ for every $i \geq 0$;

(4) *Extension Universal* if for every pair of CW-complexes (B, L) and every A-fibration $p : E \to B$, each F-map from $p|L$ to p_∞ extends to an F-map from p to p_∞.

It is shown in [1] that if p_∞ is a Hurewicz fibration, a principal G-bundle or an H-principal fibration in the sense of [2], then p_∞ is universal in the four senses described earlier if, and only if, it is universal in any one of these four cases. However, the four definitions are not equivalent; the easy counter-example is p_∞ = trivial fibration. All this is done by conveniently specializing the category A.

Conspicuously missing from [1] is the study of universal vector bundles. In this note, we study this missing case.

Let $\xi = (E, p, X; F, G)$ be a locally trivial fibre bundle with fibre F and structure group G over a CW-complex X; let $u : G \times F \to F$ be an effective left action. The exponential law defines a map $u' : G \to F^F$, which is monic because u is effective. The image of u' is the group $F * F = \{\tilde{g} \in F^F | \tilde{g}(y) = g \cdot y$, for all $y \in F$ and all $g \in G\}$. Notice that although G and $F * F$ are isomorphic, they are not, in general, homeomorphic. Given the bundle ξ, we construct the associated principal G-bundle $\tilde{\xi} = (\tilde{E}, \tilde{p}, X; G)$ and the associated principal fibration $\mathrm{Prin}\ p = (F * E, c*_2 p, X; F * F)$, where $c*_2 p = pr_2 \cdot c * p$. One can show that if G is homeomorphic to $F * F$, then $\tilde{\xi}$ and $\mathrm{Prin}\ p$ are equivalent principal bundles.

Lemma 1: Suppose that ξ is Free Universal. Then ξ is Aspherical Universal if, and only if, $u' : G \to F * F$ is a weak homotopy equivalence.

Let $\eta = (E, p, X; F, G)$ be a vector bundle over the CW-complex X, with fibre $F = \mathbb{R}^k$, $\mathbb{C}^k$ or $\mathbb{H}^k$ ($\mathbb{R}$ = reals, $\mathbb{C}$ = complex numbers, $\mathbb{H}$ = Caley numbers). It is known that the structure groups for such bundles will be $G = O(k)$, $U(k)$ or $Sp(k)$ if $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively [3, Theorem 7.4].

Theorem: η is universal in the four senses described if it is universal in any one of these senses.

The proof is obtained by applying Lemma 1, specializing conveniently A and using Theorems 3.1, 3.2 and 3.3 of [1].

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Department of Mathematics and Statistics,
Memorial University of Newfoundland,
St. John's, Newfoundland,
Canada. A1B 3X7

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CONVERGENCE SPACES AND NONSTANDARD COMPACTIFICATIONS

Robert A. Herrmann[1]

Presented by P.G. Rooney, F.R.S.C.

One of the major problems in the theory of convergence spaces has been to determine a meaningful characterization for the concept of the projective maximum or projective minimum object in various classes of convergence space compactifications. The goal of this present research is to obtain, from the nonstandard viewpoint, such characterizations for pseudotopological and pretopological spaces as defined by Kent in (2) (3).

We assume throughout this note that for nonempty X, the set $^*X$ is an enlargement (4) (6). A map $q: X \to P(^*X)$ (the power set of $^*X$) is a *quasimonad* (i.e. *q-monad*) map if $q(x) = \bigcup \{NF\{y\} \mid y \in A\}$ for some $A \subset {}^*X$. The operator "NF" is the "NucFil" operator as found in (4). The pair (X,q) is a q-space and there exists a one-to-one correspondance between a proper subclass of the class of all q-spaces and the class of all pseudotopological convergence spaces (5). Let $W^+ = \bigcup \{NF\{x\} \mid [x \in {}^*X] \wedge [\forall y [y \in X \to NF\{x\} \cap q(y) = \emptyset]]\} = \bigcap \{{}^*X - q(x) \mid x \in X\}$. For $p \notin X$, let $X^+ = X \cup \{p\}$ and q^+ be the q-monad map $q^+(x) = q(x)$ for each $x \in X$, and $q^+(p) = W^+ \cup \{p\}$. For the proofs of the following results see (1).

THEOREM 1. *If (X,q) is noncompact, then (X^+,q^+) is a one-point compactification.*

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Two extensions (Z,q) and (Y,q_1) of (X,q') are *isomorphic* if there exists a continuous bijection $f:(Z,q) \to (Y,q_1)$ such that f^{-1} is continuous and $f|X$ = identity.

THEOREM 2. *Let (Z,q) and (Y,q_1) be one-point compactifications of noncompact (X,q') such that $Z = X \cup \{r\}$, $Y = X \cup \{p\}$; standard $p,r \notin X$; $r \varepsilon q(r)$, $p \varepsilon q_1(p)$ and Z,Y are separated from X.* *Then Z and Y are isomorphic.*

Assume that for a space (X,q'), we have $W^- = \{NF\{x\} | x \varepsilon W^+\}$ and $|W^-| \varepsilon \omega$. The set W^- partitions W^+. Suppose that we select n disjoint subsets $W_i, i=1,\ldots,n$ such that $\cup W_i = W^+$ and consider n distinct standard points $\{p_1,\ldots,p_n\} = F$ disjoint from X. Define $Y = X \cup F$ and for each $i = 1,\ldots,n$ let $q(p_i) = W_i \cup \{p_i\}$; for each $x \varepsilon X$, let $q(x) = q'(x)$. Then (Y,q) is an n-point compactification for any noncompact (X,q').

Let E be any collection of extensions for a space (X,q). Then $Z \varepsilon E$ is *projectively larger* (resp. *smaller*) than $Y \varepsilon E$ if there exists a continuous surjection $f:Z \to Y$ (resp. $f:Y \to Z$) such that $f|X$ = identity (i.e. f is *fixed* on X). $Z \varepsilon E$ is a *projective maximum* (resp. *minimum*) if Z is projectively larger (resp. smaller) than each $Y \varepsilon E$. Recall that $A \subset {}^*X$ is *nuclear* if there exists a set $\mathcal{F} \subset P(X)$ such that $A = \mathrm{Nuc}\,\mathcal{F} = \cap\{{}^*F | F \varepsilon \mathcal{F}\}$.

THEOREM 3. *The following statements are equivalent for a noncompact pseudotopological space (X,q) and the collection $\mathcal{H}'$ of all Hausdorff convergence space compactifications of X.*

(i) *$\mathcal{H}'$ contains a projective maximum.*

(ii) *The set W^+ is nuclear.*

(ii) *$\mathcal{H}'$ contains a projective minimum.*

For noncompact (X,q), let S be any fixed set of standard points disjoint from X such that $|W^-| = |S|$. Let $\hat{X} = X \cup S$ and $H: S \to W^-$ be a bijection. If $A \subset X$, let $[A] = \{x \mid [x \in S] \wedge [H(x) \subset {}^*A]\}$. For any nonempty $\mathcal{F} \subset \mathcal{P}(X)$, let $[\mathcal{F}] = \{F \cup [F] \mid F \in \mathcal{F}\}$. Define for each $x \in X$, $\hat{q}(x) = q(x)$; for $p \in S$, let $\hat{q}(p) = H(p) \cup \mathrm{Nuc}\{<_S\{[F] \mid F \in \mathcal{F}\}>\} = \mathrm{Nuc}[\mathcal{F}]$, where $\mathrm{Fil}\{H(p)\} = \mathcal{F}$. Thus if $p \in S$, then $\hat{q}(p) = NF_X\{r\} \cup \mathrm{Nuc}\,\mathcal{G}$ for some $\mathcal{G} \subset \mathcal{P}(S)$ and $NF_X\{r\} \in W^-$. Observe that $p \in \hat{q}(p)$ for each $p \in S$.

THEOREM 4. *The following statements are equivalent for a noncompact pseudotopological space (X,q) and the collection $\mathcal{H}$ of all pseudotopological Hausdorff compactifications of X.*

(i) *$(\hat{X},\hat{q})$ is a projective maximum in $\mathcal{H}$.*

(ii) *The set W^+ is nuclear.*

(iii) *(X^+,q^+) is a projective minimum in $\mathcal{H}$.*

THEOREM 5. *The following statements are equivalent for a noncompact pretopological space (X,q) and the collection $\mathcal{N}$ of all pretopological Hausdorff compactifications of X.*

(i) *$(\hat{X},\hat{q})$ is a projective maximum in $\mathcal{H}$.*

(ii) *The set W^+ is nuclear.*

(iii) *(X^+,q^+) is a projective minimum in $\mathcal{N}$.*

(iv) *(X^+,q^+) is pretopological.*

For a deeper study of these and other concepts as well as applications to locally quasi-H-closed spaces and one-point near-compactifications, we refer the reader to (1).

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MATHEMATICS DEPARTMENT, U. S. NAVAL ACADEMY, ANNAPOLIS, MD. 21402. USA

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On the p-divisible groups arising from Fermat curves

Noriko Yui

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1. Let $C : X^m + Y^m = 1$, $m \geq 3$ be the Fermat curve of genus $g = (m-1)(m-2)/2$ defined over a finite field $k = GF(p^f)$ of characteristic $p > 0$, where $(p,m) = 1$ and p^f is the least power of p such that $p^f \equiv 1 \pmod m$. Denote by J the Jacobian variety of C which we may assume to be defined over k, and by $J(p)$ the p-divisible group of J over k. In this paper, we shall determine completely the structure of the isogeny class of $J(p)$ in the sense of Manin [1].

2. Let $K_m = \mathbb{Q}(e^{2\pi i/m})$ be the m-th cyclotomic field over $\mathbb{Q}$ of degree $\phi(m)$ where ϕ is the Euler function. The Galois group G of K_m is identified with the multiplicative group $(\mathbb{Z}/m\mathbb{Z})^\times$. We denote by H the subgroup of G defined by $H = \{ p^\nu \bmod m \mid 0 \leq \nu < f \}$. H is the decomposition group of a prime ideal $\mathfrak{p}$ over p in K_m with $N\mathfrak{p} = p^f$.

Let $\mathfrak{A}_m$ denote the set of all vectors defined as follows:

$$\mathfrak{A}_m = \left\{ \underline{a} = (a_0, a_1, a_2) \;\middle|\; \begin{array}{l} a_i \in \mathbb{Z}/m\mathbb{Z},\ a_i \not\equiv 0 \pmod m \\ a_0 + a_1 + a_2 \equiv 0 \pmod m \end{array} \right\}.$$

The cardinality of $\mathfrak{A}_m$ is easily computed to be $(m-1)(m-2) = 2g$. Now we put for each $\underline{a} \in \mathfrak{A}_m$,

$$A_H(\underline{a}) = \sum_{t \in H} \left[\sum_{i=1}^{2} \left\langle \frac{ta_i}{m} \right\rangle \right].$$

Here $< \lambda >$ denotes the "fractional part" of $\lambda \in \mathbb{R}$ defined by $< \lambda > = \lambda - [\lambda]$ where $[\lambda]$ is the "integral part" of λ. From the definition, $A_H(\underline{a})$ is a rational integer for any $\underline{a} \in \mathfrak{A}_m$ taking the value in the range $[0, f]$.

3. <u>Theorem</u>. Let $C : X^m + Y^m = 1$, $m \geq 3$ be the Fermat curve of genus g defined over k and let $J(p)$ be the p-divisible group of the Jacobian variety J of C defined over k. The notations H, $A_H(\underline{a})$ being as above, for any rational integer $0 \leq c \leq f$, let r_c denote the number of the vectors $\underline{a} \in \mathfrak{A}_m$ such that $A_H(\underline{a}) = c$. Then $J(p)$ is isogenous to a formal group of the form :

$$J(p) \sim r_0 G_{1,0} + \frac{1}{2} r_{f/2} G_{1,1} + \sum_{0<c<f/2} \frac{r_c}{f} (G_{c,f-c} + G_{f-c,c}).$$

Here the numbers c, $f-c$ are not necessarily relatively prime. If $(c, f-c) = c_0 > 1$, then we identify the formal group $G_{c,f-c}$ with $c_0 G_{c/c_0,(f-c)/c_0}$. $r_c = r_{f-c}$ for any c and moreover,

$$\sum_{0<c<f/2} r_c = \sum_{0<c<f/2} r_{f-c} = \begin{cases} g - r_0 & \text{if } 2 \nmid f \\ g - r_0 - \frac{1}{2} r_{f/2} & \text{if } 2 \mid f . \end{cases}$$

4. There exists a sharp difference to the structure of the isogeny class of $J(p)$, according as f is even or odd.

<u>Theorem</u>. With the notations and the hypothesis as above, we have

(a) f is even and $p^{f/2} + 1 \equiv 0 \pmod{m}$, if and only if $J(p) \sim g G_{1,1}$, i.e. J is supersingular.

(b) f is even but $p^{f/2} + 1 \not\equiv 0 \pmod{m}$, then $J(p)$ is isogenous to a formal group of the form:

$$J(p) \sim r_0 G_{1,0} + \frac{1}{2} r_{f/2} G_{1,1} + \sum_{0<c<f/2} \frac{r_c}{f} (G_{c,f-c} + G_{f-c,c})$$

with $0 < r_{f/2} < 2g$ and $r_c \neq 0$ for at least one $0 < c < f/2$. In particular, if $f = 2$, then $r_c = 0$ for all $0 < c < f/2$.

(c) $f = 1$, if and only if $J(p) \sim gG_{1,0}$, i.e. J is ordinary.

(d) If $f > 1$ and f is odd, then $J(p)$ is isogenous to a formal group of the form :

$$J(p) \sim r_0 G_{1,0} + \sum_{0<c<[f/2]} \frac{r_c}{f} (G_{c,f-c} + G_{f-c,c})$$

with $r_c \neq 0$ for at least one $0 < c < [f/2]$. In particular, $J(p)$ contains no factor $G_{1,1}$.

5. Examples.

p	m	g	f	J(p)
2	21	190	6	$36G_{1,0} + 6(G_{1,5} + G_{5,1}) + 33(G_{1,2} + G_{2,1}) + 19G_{1,1}$
3	11	45	5	$3(G_{1,4} + G_{4,1}) + 6(G_{2,3} + G_{3,2})$
5	24	253	2	$153G_{1,0} + 100G_{1,1}$
7	12	55	2	$28G_{1,0} + 27G_{1,1}$

11	16	105	4	$24G_{1,0} + 12(G_{1,3} + G_{3,1}) + 33G_{1,1}$
13	15	91	4	$91G_{1,1}$
17	8	21	1	$21G_{1,0}$
19	24	253	2	$100G_{1,0} + 153G_{1,1}$
23	14	78	3	$6G_{1,0} + 24(G_{1,2} + G_{2,1})$
29	13	66	3	$21G_{1,0} + 15(G_{1,2} + G_{2,1})$
31	22	210	5	$30G_{1,0} + 18(G_{1,4} + G_{4,1}) + 18(G_{2,3} + G_{3,2})$

6. The detailed version of this paper can be found in [2] which will appear in Jouranl of Algebra. These results have grown out of correspondence and conversations with Professor T. Shioda. I thank him heartily for his encouraging suggestions and criticism.

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Mathematisches Institut
Universität Bonn, Wegelerstrasse 10
D-5300 Bonn, West Germany

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THE DERIVATION OF SCHOENBERG'S STAR POLYTOPES FROM SCHOUTE'S SIMPLEX NETS

by

H.S.M. Coxeter, F.R.S.C.

Consider 6 square mirrors facing inwards on the faces of a cube, and a flat pencil of light rays reflected from these mirrors. If the plane of the rays is carefully chosen, the reflected path may close so as to form a finite polyhedron. The simplest instance is a regular tetrahedron whose 6 edges are diagonals (one each) of the 6 faces of the cube. In 4 papers on '<u>Extremum problems for the motions of a billiard ball</u>' I.J. Schoenberg has shown that this instance maximizes the minimum distance from the centre of the cube to a face of the polyhedron. He has also generalized this problem to an (n-1)-dimensional 'path' inside an n-dimensional cube. It now appears that his non-convex polytopes can be thoroughly investigated by using their connection with P.H. Schoute's 'Simplex nets', such as the (n-1)-dimensional lattice whose points have n integral coordinates with a constant sum (say zero) [1, p. 128].

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University of Toronto
Toronto, Ont. Canada

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Variations sur des thèmes de B. Grünbaum et G. C. Shephard

par Robert Bantegnie.

Presented by H.S.M. Coxeter, F.R.S.C.

Soit s un sommet d'un polyèdre convexe fermé borné. Supposons que s appartient à r faces distinctes $F_1, \ldots, F_r$ numerotées dans l'ordre obtenu en tournant autour de s; soit a_i le nombre decôtés de $F_i (1 \leq i \leq r)$.

1) $(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(r)})$ où σ est soit une permutation circulaire de $1, 2, \ldots, r$ soit l'inverse d'une telle permutation est appelé cycle de s.

2) $(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(r)})$. où σ est une permutation arbitraire est appelé pseudo-cycle de s.

Un polyèdre est dit strictement équilibré (sur les sommets) quand chacun de ses sommets a même cycle; il est dit équilibré (sur les sommets) quand chacun de ses sommets a même pseudocycle. On connait les polyèdres strictement équilibrés. Ce sont à une isomorphie près

α) les polyèdres uniformes, β) le polyèdre de Miller.

Dans un autre contexte, N. W. Johnson [1] a donné l'exemple de 6 polyèdres équilibrés qui ne sont pas strictement équilibrés. Je peux montrer qu'à une isomorphie près il n'y a pas de polyèdres possédant ces deux proprietés qui ne soient pas dans la liste de Johnson.

La méthode utilise 1) une détermination des pseudocycles possibles pour de tels polyèdres, 2) une construction des diagrammes de Schlegel de ces polyèdres (la construction est élaborée), 3) la remarque que les polyèdres de Johnson ont les diagrammes trouvés.

Une notion utile est celle de contribution (d'Euler-Lebesgue) du sommet s [2]. On peut définir aussi la notion de cycle, de pseudocycle et de contribution d'une face. Pour le problème considéré il y a dualité entre ce qui se passe pour les faces et ce qui se passe pour les sommets. On peut considèrer aussi le problème analogue concernant les pavements du plan. La détermination des pseudocycles possibles pour les pavements équilibrés se fait également. Contrairement au cas des polyèdres, on obtient une infinité de cas non-isomorphes pour les pavements équilibrés mais non-strictement équilibrés.

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Robert Bantegnie

Université de Franche-Comté, 25030 BESANÇON, FRANCE

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Studies of homomorphisms of Artin groups

by V.M. Zinde

Presented by H.S.M. Coxeter, F.R.S.C.

Artin groups were introduced by E. Brieskorn [1] in 1971 as a generalization of braid groups. They were defined as the fundamental groups of regular orbit spaces of finite irreducible Coxeter groups; their corepresentations were obtained by Brieskorn. An Artin group is denoted by one of the symbols A_n $(n \geq 2)$, B_n $(n \geq 2)$, D_n $(n \geq 3)$, E_n $(n = 6,7,8)$, F_4, G_2, H_n $(n = 3,4)$, $I_2(p)$ $(p = 5$, or $p \geq 7)$, depending on which of the above classes the corresponding Coxeter group belongs to.

For the series A_n of braid groups, V. Lin has extended a classical result of Artin to show that for $n \neq 4$ any endomorphism with a non-commutative image takes the kernel of the standard epimorphism of A_n onto the symmetric group S_{n+1} into itself. He also proved that any homomorphism of A_k into A_n for $n < k$ and $k \neq 3$ has a commutative image [2],[3].

We have proved [4] analogous results for the series B_n and D_n.

Theorem 1. For $k \neq 4$ and $n < k$, any homomorphism $\varphi : B_k \to B_n$ has a commutative image.

Denote by $I(B_n)$ the kernel of the standard epimorphism of B_n into S_n.

Theorem 2. Suppose that the endomorphism $\varphi : B_n \to B_n$ where $n \geq 5$ has a non-commutative image $\varphi(B_n)$. Then $\varphi(I(B_n)) \subset I(B_n)$ and $\varphi^{-1}(I(B_n)) = I(B_n)$.

The same theorems are true if B_n is replaced by D_n throughout.

These results are based on studies of homomorphisms of the Artin groups B_n and D_n into symmetric groups.

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Author's address: Mičurinskiĭ pr. 12, korpus 1, kv. 124, Moscow 117192,
USSR.

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Relations Between Reflections in U(n)

D.Ž. Djoković

Presented by H.S.M. Coxeter, F.R.S.C.

Let $P = (P_1,\ldots,P_m)$ and $Q = (Q_1,\ldots,Q_m)$ be point-sequences in $P(C^n)$. We say that P and Q are linked if there exist an index k $(1 \le k < m)$, a geodesic g, and a rotation ρ of g such that the following conditions hold:

(i) $P_i = Q_i$ if $i \neq k, k+1$;

(ii) $\{P_k, Q_k, P_{k+1}, Q_{k+1}\} \subset g$;

(iii) $\rho(P_k) = Q_k$ and $\rho(P_{k+1}) = Q_{k+1}$.

We say that P and Q are equivalent if there exists an integer k $(k \ge 0)$ and point-sequences

$$P^{(r)} = (P_1^{(r)},\ldots,P_m^{(r)}), \quad 0 \le r \le k$$

such that $P^{(0)} = P$, $P^{(k)} = Q$, and $P^{(r)}$ and $P^{(r+1)}$ are linked for $0 \le r < k$. If P and Q are equivalent we shall write $P \sim Q$.

We say that a point-sequence $P = (P_1,\ldots,P_m)$ is reduced if $P \sim Q = (Q_1,\ldots,Q_m)$ implies that $Q_1 \neq Q_2$.

There is a canonical bijection between the points of $P(C^n)$ and the reflections of $U(n)$. If P is a point we shall denote by R_P the corresponding reflection.

If $(P_1,P_2) \sim (Q_1,Q_2)$ then we have

$$R_{P_1} R_{P_2} = R_{Q_1} R_{Q_2}. \tag{1}$$

It follows that if $P \sim Q$ then $A_P = A_Q$ where, for instance, if $P = (P_1,\ldots,P_m)$ then

$$A_P = R_{P_1}\cdots R_{P_m}.$$

Let $G = \{A \in U(n) \mid \det A = \pm 1\}$. For $A \in G$ we denote by $\ell(A)$ the smallest integer m such that A is a product of m reflections in $U(n)$.

Theorem 1. A point-sequence $P = (P_1,\ldots,P_m)$ is reduced iff $\ell(A_P) = m$. If P and Q are reduced sequences then $P \sim Q$ iff $A_P = A_Q$. If $P = (P_1,\ldots,P_m)$ is a reduced sequence then $m \le 2n-1$ and there exist reduced sequences with $m = 2n-1$.

Theorem 2. The relations (1) together with the relations ${R_P}^2 = 1$, $P \in P(\mathbb{C}^n)$ are the defining relations for G.

Given a reduced sequence $P = (P_1,\ldots,P_m)$ we define $\Omega_P \subset P(\mathbb{C}^n)$ as follows: a point Q_1 belongs to Ω_P iff there exists a sequence $Q = (Q_1,\ldots,Q_m)$ such that $P \sim Q$. We can give a complete description of the sets Ω_P for arbitrary reduced sequences P.

University of Waterloo
Waterloo N2L 3G1

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ON HILBERT'S SIXTEENTH PROBLEM

By

Themistocles M. Rassias

Presented by G.F.D. Duff, F.R.S.C.

1. The first part of Hilbert's sixteenth problem asks us to study the topology of real algebraic varieties. In the proceedings of the International Congress of Mathematicians in 1900 Hilbert claims to have proved this; but in the revised version published a year later he says merely that he has convinced himself of it. The solution of Hilbert's sixteenth problem has resisted solution despite the continuous efforts of many mathematicians.

2. The problem considered here is to place a bound on the number of regions into which a real algebraic curve divides the real plane R^2 in terms of the degree m of the polynomial defining the curve. This bound is found to be $1 + [m(m+1)/2]$. This bound will be achieved using a curve made up of m straight lines in general position, and hence is the best possible. The notion of the circuit is introduced.
A circuit is obtained by starting out along one arc of a real branch and proceeding "naturally". A circuit is a closed path and an irreducible curve divides naturally into a number of circuits and into connected bunches of circuits. The main tool used in Euler's theorem which states that $v - e + f = 2$, where f is the number of regions determined by a closed path, e is the number of edges of the path, and v (assumed $\geqslant 1$) is the number of vertices. An initial step is to deform the circuit into a closed path having only simple crossings and for such a

closed path $e = 2V$. Similar considerations hold for a connected bunch of circuits. The final step is to give a bound on the number of circuits. In the case of a curve without singularities, Harnack's theorem says that $P+1$ is such a bound, where P is the genus. However, the number of circuits is a birational invariant and thus is P. Since one can desingularize an algebraic curve using a birational transformation, Harnack's bound also holds even if the curve has singularities. For reducible curves one treats the cases $m=1$, $m=2$ and then makes an induction on m.

Proposition 1. The number of regions into which m straight lines $\ell_1, \ell_2, \ldots, \ell_m$ can divide the plane R^2 is at most $1 + [m(m+1)/2]$.

Theorem 2. Let $f(x,y)$ be a polynomial of degree m. Then the set

$$Z_f = \{ (x,y) \in R^2 : f(x,y) = 0 \}$$

divides the plane R^2 into at most $1 + [m(m+1)/2]$ regions.

Theorem 3. Any irreducible curve of genus P, whether it has singularities or not, has at most $(P+1)$ circuits.

3. Let $Q(x,y,z)$ be a spherical harmonic. Assume the degree n of Q is positive. The set

$$V_Q = \{(x,y,z): x^2 + y^2 + z^2 = 1 \text{ and } Q(x,y,z) = 0\}$$

is called the nodal lines of Q. The nodal lines divide the unit sphere S into a certain number $M_n(Q)$ of regions, in which $Q \neq 0$.

Using Morse theory it is possible to construct spherical harmonics of degree n whose nodal lines divide S in two respectively three domains for n odd respectively even. Thus it is possible to prove that the lower bounds for Mn are actually assumed. It is also possible to prove that the upper bounds for Mn are also actually assumed. Thus we obtain a Morse theoretic generalization of Courant's nodal line theorem for spherical harmonics.

Some applications of the above results to the study of Morse-Smale index for the Jacobi operator corresponding to a variational problem can be given.

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Institute of Pure and Applied Mathematics
Pellana, Sparta
Lacony, GREECE

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ON A PROBLEM OF L. NIRENBERG

By

Themistocles M. Rassias

Presented by G.F.D. Duff, F.R.S.C.

L. NIRENBERG [1; p. 175] posed the following problem: Suppose T is a continuous map $H \to H$, H is a Hilbert space, which is expanding, i.e. $\|Tx-Ty\| \geq \|x-y\|$, and $T(0) = 0$. Suppose T maps a neighborhood of the origin onto a neighborhood of the origin. Does T map H onto H?

We are going here to give a partial answer to this problem.

Assume first that T is a linear map. We will prove that T maps H onto H. Let's assume that this is not the case and $H - T(H) \neq \emptyset$, and $y \in H - T(H)$. There exists $\lambda > 0$ so that $\frac{y}{\lambda} \in T(H)$. Therefore there exists $x \in D(T)$ $(= H)$ with $T(x) = \frac{y}{\lambda}$, thus $y = \lambda T(x)$ and so $y = T(\lambda x)$. However $\lambda x \in D(T)$. Therefore $y \in T(H)$, a contradiction. Therefore T maps H onto H. Q.E.D.

The condition given on T to map a neighborhood of the origin onto a neighborhood of the origin is essential in order for T to map H onto H. The following example explains this phenomenon.

Counterexample 1. Let H be a separable Hilbert space, and $T: H \to H$ be a linear map defined by

$$T(a_1, a_2, a_3, \ldots) = (0, 2a_1, 2a_2, \ldots.)$$

where $a = (a_1, a_2, a_3, \ldots) \in H$. It is true that $T(0) = 0$. Moreover T is continuous (uniformly) because if $a, b \in H$,

$$\|a-b\| = \left\{\sum_{i=1}^{\infty} (a_i-b_i)^2\right\}^{1/2},$$

$$\|Ta-Tb\| = \left\{\sum_{i=1}^{\infty} (2a_i-2b_i)^2\right\}^{1/2},$$

therefore $\|Ta-Tb\| > \|a-b\|$, for all a, b $\in$ H. Thus T is also expansive (in fact strictly expansive). However it is easy to see that T does not map a neighborhood of the origin onto a neighborhood of the origin. Moreover T is not locally onto and so T is not an onto map.

We will consider now the case where T is a non-linear map.

Theorem 5. Let T: $B \longrightarrow B$ be a continuous non-linear map, so that $\|Tx-Ty\| \geqslant \|x-y\|$, for all x, y $\in$ B with dim $B < \infty$, where B is a Banach space. Then T maps B onto B.

Proof. Claim that the range of T is closed.

Verification: Let $\{Tx_n\}$ be a Cauchy sequence in B. Then by the expanding property of T we obtain $\|Tx_n-Tx_m\| \geqslant \|x_n-x_m\|$ and therefore $\{x_n\}$ is a Cauchy sequence in B. Thus $\{x_n\}$ converges to some element x $\in$ B, i.e., $x_n \to x$ and by the continuity of T, it follows that $Tx = \lim_{n\to\infty} Tx_n$ which implies that the range of T is closed as Tx $\in$ B.

Claim that the range of T is open.

Verification: Let T: B $\longrightarrow$ Range of T. This is a homeomorphism and by the BROUWER's invariance of domain (e.g. E. SPANIER [2]), which is true if dim $B < \infty$, the Range of T is open, since B is open in B, and Range of T is contained in B.

Because of the fact that Range of T is both open and closed it implies that Range of T equals B and so T maps B onto B. Q.E.D.

Remark. BROUWER's Invariance of domain is not true for infinite dimensional Hilbert spaces H.

Counterexample 2. Let H be an infinite dimensional Hilbert space, and $S^{\infty} = \{ x \in H : \|x\| = 1 \}$. It is a theorem that there exists a homeomorphism

$$f : H \xrightarrow{\sim} S^{\infty}.$$

However H is an open set in H and S^{∞} is closed in H. Thus the BROUWER'S Invariance of domain does not work for Infinite dimensional Hilbert spaces.

The condition given on T to be expanding, i.e. $\|Tx-Ty\| \geq \|x-y\|$, for all $x,y \in H$ is essential in order for T to map H onto H.

The following example explains this phenomenon.

Counterexample 3. Take $H = L_2(-\pi,\pi)$ which is spanned by the orthonormal complete system

$$\left\{ \frac{e^{int}}{\sqrt{2}} \right\}. \tag{1}$$

Take a mapping $f \longmapsto T(f)$ defined by

$$[T(f)](t) = \sum_{n=-\infty}^{\infty} \frac{f_n}{1+|f_n|} \frac{e^{int}}{\sqrt{2}} \tag{2}$$

if

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{int}}{\sqrt{2}}.$$

Then T is a non-linear map because $T(cf) \neq cT(f)$, and $T(f+g) \neq T(f) + T(g)$. The mapping T is a contraction and the zero element is a fixed point, i.e. $T(0) = 0$. Since

$$\frac{|f_n|}{1+|f_n|} < 1 \tag{3}$$

the mapping of H into H is not onto.

On the other hand, given a vector $\hat{F} = \{\hat{F}_n\}$ of length $\leq k < 1$, then the equation

$$\frac{f_n}{1 + |f_n|} = \hat{F}_n \tag{4}$$

has a unique solution

$$f_n = (1 - |\hat{F}_n|)^{-1}\hat{F}_n \tag{5}$$

which defines a vector in H. Thus $\{f \in H: \|f\| < k\}$ is the image of a subset of H which contains the ball

$$\{f \in H: \|f\| < \frac{k}{1-k}\},$$

so the mapping is onto in a neighborhood of the origin.

Therefore the property on T to be expanding, i.e. $\|Tx-Ty\| \geq \|x-y\|$, for all $x, y \in H$ is essential in order for T to map H onto H.

Remark. The condition on T to satisfy the condition $\|Tf-Tg\| \geq \|f-g\|$ neither implies nor is implied by non-linearity.

Therefore all the conditions posed in NIRENBERG's problem for the non-linear case are essential. The problem then is reduced to answer NIRENBERG's problem for nonlinear mappings defined on infinite dimensional Hilbert spaces.

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Institute of Pure and Applied Mathematics
Pellana, Sparta, Lacony, Greece.

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Sur les cycles holomorphes à coefficients positifs dans C^n et un complément au théorème de E.BOMBIERI

par Pierre LELONG

Presented by P. Ribenboim, F.R.S.C.

1- Des résultats récents, notamment de E.BOMBIERI [1] , Y.T.SIU [3] et H.SKODA [4a] permettent de donner [2b] une définition globale des ensembles analytiques complexes X dans un domaine G pseudo-convexe de C^n (ou sur une variété de Stein) sous la forme

$$X = E(c,V) = \left[x \in G \ ; \ \nu(x,V) \geqslant c, \ c > 0\right]$$

où V est plurisousharmonique dans G et $\nu(x,V)$ est le nombre dit de Lelong (cf. [5]) du courant positif fermé $\theta = i\pi^{-1}d'd''V$; plus simplement $\nu(x,V)$ est en x la densité en dimension 2n-2 de la mesure positive $\sigma = (2\pi)^{-1}\Delta V$, relative au laplacien ΔV. La réunion $\Gamma(V) = \bigcup_{c>0} E(c,V)$ constitue *un cycle holomorphe à coefficients positifs dans* G : un ensemble analytique irréductible dans G sera dit un *cycle* élémentaire ; on lui associe dans $\Gamma(V)$ le coefficient $N_i = N(X_i) = \left[\inf c, \ c > 0, \ X_i \subset E(c,V)\right]$. On donne ici 3 résultats concernant l'étude à croissance des ensembles E(c,V) pour $G = C^n$; ils permettent de préciser un important théorème de Bombieri [1] sur les valeurs rationnelles d'une application méromorphe d'ordre fini .

On notera $E_k(c,V)$ la composante de E(c,V) de dimension k , $0 \leqslant k \leqslant n-1$. On associe à $E_k(c,V)$ l'indicatrice de croissance $\omega_k(r) = \left[\tau_{2k}(r)\right]^{-1} \sigma_k(r)$ si $1 \leqslant k \leqslant n-1$, où $\sigma_k(r) = \text{aire}\left[B(0,r) \cap E_k(c,V)\right]$ et $\tau_{2k}(r)$ = aire de la boule de rayon r dans C^k. Si $k = 0$ on pose $\omega_0(r) = \text{card.}\left[B(0,r) \cap E_0(c,V)\right]$. On note d'autre part $\sigma = (2\pi)^{-1}\Delta V > 0$ et $\omega_V(r)$ = indicatrice de $\sigma = \left[\tau_{2n-2}(r)\right]^{-1}\sigma(r)$.

2- Les 3 groupes de résultats (I,II,III) ont une origine différente.

I. THÉORÈME 1. - *Soient* (X_s, N_s) *les cycles élémentaires de codimension 1 contenus dans le cycle* $\Gamma(V)$ *où* V *est plurisousharmonique dans* G. *Alors le courant*

$$\theta - \sum_s N_s \left[X_s\right] \tag{1}$$

est positif fermé, où $\theta = i\pi^{-1}d'd''V$ *et* $\left[X_s\right]$ *désigne l'intégration sur* X_s.

THÉORÈME 2. - *Soit* $G = C^n$. *On a la majoration suivante de l'indicatrice* $\omega_{n-1}(r)$ *de* $E_{n-1}(c,V)$:

$$\omega_{n-1}(r) \leqslant c^{-1}\omega_V(r) \tag{2}$$

et si l'on pose $M_V(r) = \sup V(x)$ *pour* $\|x\| \leqslant r$, *on a pour* $\alpha > 1$

$$\omega_{n-1}(r) \leqslant (c \log\alpha)^{-1}\left[M_V(\alpha r) + C\right]$$

où $C = \lambda(V,0,1)$ *moyenne de* V *sur la sphère* S(0,1).

II. Le résultat suivant est une extension du théorème de E.BOMBIERI:

THÉORÈME 3. - Soit V plurisousharmonique dans C^n : l'ensemble $E(c,V)$ est contenu dans les zéros d'une fonction entière F. On a $E(c,V) \subset F^{-1}(0)$ avec

$$(3) \qquad M_F(r) \leqslant nc^{-1} M_V(\alpha r) + \varepsilon \log r + C(n, \alpha, \varepsilon)$$

où $\varepsilon > 0$, $\alpha > 1$ et $C(n, \alpha, \varepsilon)$ sont indépendants de r et $M_F(r) = \sup \log |F(x)|$ pour $\|x\| = r$.

A la différence du théorème 2, le résultat (3) fait intervenir la dimension n. La calcul utilise une amélioration de la méthode des estimées L^2 donnée récemment par H.SKODA [4b].

III. L'ensemble analytique $E(c,V)$ peut toujours être défini par $n+1$ équations $F_j(x) = 0$ dans C^n, où F_1 vérifie (3). Pour obtenir $F_2, \dots, F_{n+1}$, il a été nécessaire d'introduire une notion nouvelle ; on suppose $n \geqslant 2$.

DÉFINITION. - La valeur $c > 0$ sera dite posséder une amplitude de stabilité $\gamma > 0$ si l'on a dans $\Gamma(V)$ l'égalité des ensembles

$$E(c,V) = E(c',V) \text{ pour } c - \gamma < c' \leqslant c .$$

On obtient alors en supposant $n \geqslant 2$.

THÉORÈME 4. - Soit V une fonction plurisousharmonique dans C^n. Si c est valeur de stabilité dans $\Gamma(V)$ d'amplitude $\gamma > 0$, les $n+1$ équations $F_j = 0$ qui définissent $E(c,V)$ dans C^n peuvent être choisies de manière que F_1 vérifie (3), c'est-à-dire

$$M_1(r) \leqslant nc^{-1} M_V(\alpha r) + \varepsilon \log r + C(n, \alpha, \varepsilon)$$

et que pour $2 \leqslant j \leqslant n+1$ les indicatrices des F_j vérifient

$$(4) \qquad M_j(r) \leqslant (n-1+\varepsilon')\gamma^{-1} M_V[\alpha(r+1)] + \varepsilon \log(r+1) + C(n, \alpha, \varepsilon)$$

où $\varepsilon > 0$, $\varepsilon' > 0$, $C(n, \alpha, \varepsilon)$ sont indépendants de r.

3- Le cas algébrique. On note $S_a(C^n)$ la classe des fonctions plurisousharmoniques V dans C^n qui vérifient $\lim_{r=\infty} (\log r)^{-1} M_V(r) = a$ et on appelle classe minimale de croissance la classe $S(C^n) = \bigcup_{a>0} S_a(C^n)$.

THÉORÈME 5. - Si $V \in S_a(C^n)$, le cycle $\Gamma(V)$ est algébrique. On a $E(c,V) = \bigcap P_j^{-1}(0)$ pour $1 \leqslant j \leqslant n+1$ avec $(n \geqslant 2)$:

$$\text{degré } P_1 \leqslant n\, c^{-1}\, a$$
$$\text{degré } P_j \leqslant (n-1)\gamma^{-1}\, a \quad \text{pour } 2 \leqslant j \leqslant n+1.$$

D'où les 3 corollaires suivants qui précisent le théorème de E.Bombieri [1] concernent l'ensemble $S \subset C^n$ des points où une application f méromorphe d'ordre fini ρ prend ses valeurs dans K^N, le corps K étant algébrique de degré $[K : Q]$, avec l'hypothèse : degré de transcendance de $K(f) \geqslant n+1$ et $J[f]$ stable par les dérivations portant sur f. On pose

$a = (n+1)\rho\,[K : Q]$, le calcul de E.Bombieri consistant essentiellement à construire $V \in S_a(C^n)$ telle qu'on ait

$$S \subset E(1,V).$$

On a alors les énoncés suivants qui indiquent que S est contenu dans C^n dans un ensemble analytique Σ avec les propriétés :

1/ La composante Σ_{n-1} de dimension n-1 vérifie: degré $\Sigma_{n-1} \leqslant a = (n-1)\rho[K : Q]$.

2/ Σ est contenu dans un sous-ensemble algébrique de codimension 1 (hypersurface)

$$\Sigma' = P_1^{-1}(0) \text{ avec degré } \Sigma' \leqslant na = n(n+1)\cdot\rho\cdot[K : Q]$$

3/ L'ensemble algébrique défini par $P_j = 0$, $1 \leqslant j \leqslant n+1$, où degré P, est borné par na et pour $2 \leqslant j \leqslant n+1$ on a $(n \geqslant 2)$:

$$\text{degré } P_j \leqslant (n-1)\gamma^{-1}\, a = (n^2-1)\gamma^{-1}\cdot\rho\cdot[K : Q]$$

où $\gamma > 0$ est l'amplitude de stabilité de la valeur $c = 1$ dans le cycle algébrique $\Gamma'(V)$.

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Université Pierre et Marie Curie
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THE CONDUCTOR OF CURVES WITH REDUCED TANGENT CONE

Ferruccio Orecchia*

Presented by P. Ribenboim, F.R.S.C.

1. INTRODUCTION

Let Spec A be the local ring at a singular point of a reduced curve C over an algebraically closed field. Let M , $e(A)=s$, $\mathrm{emdim}(A)=r+1$ denote the maximal ideal, multiplicity and embedding dimension of the ring A , respectively. Suppose that the associated graded ring, G(A) , is reduced. Then the tangent cone Spec(G(A)) consists of s lines of $\mathbb{A}^{r+1}$ and the projectivized tangent cone, Proj(G(A)), of s points of $\mathbb{P}^r$. If C is contained in a smooth surface X it is a well known consequence of a result of Northcott and Matlis that the conductor of A in $\bar{A}$ is the ideal M^{s-1} (see [Ma], 13.8 and [G.V.], cor. 3.6). In this paper we show that the preceding result is a particular case of the following general fact. The conductor of a curve C at a point with reduced tangent cone depends only on the two integers e(A) and emdim(A) (if $C\subset X$, emdim(A) is constant and equal to 2). In fact the main result is: if the s points of Proj(G(A)) are in generic position (see def. 3.1) then the conductor of A in $\bar{A}$ is the ideal M^d where $d+1$ is the least degree of a hypersurface of $\mathbb{P}^r$ containing the s points (see thm. 3.2). In particular if $C\subset X$ then $r=1$ and s points of $\mathbb{P}^1$ are always in generic position so $d+1=s$ and we have the result mentioned above.

STANDING NOTATION. A is a reduced noetherian, 1-dimensional local ring with normalization $\bar{A}$ a finite A-module. M will be the

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the maximal ideal of A. For any semilocal ring B, $G(B)$ denotes the associated graded ring with respect to the Jacobson radical of B

2. THE CONDUCTOR OF REDUCED ASSOCIATED GRADED RINGS.

The results of this section are based on the following proposition which has been stated by Davis in [D] (see no. 2, remark)

PROPOSITION 2.1. The following conditions are equivalent:

1) $G(A)$ is reduced

2) the natural homomorphism $G(A) \to G(\bar{A})$ is injective.

From now on we suppose that $G(A)$ is a reduced ring. Then by prop. 2.1 we can suppose $G(A) \subset G(\bar{A})$ as rings.

Now let I be an ideal of A. $G(I) = \bigoplus_{n=0}^{\infty} (I \cap M^n / I \cap M^{n+1})$ is the homogeneous ideal of $G(A)$ generated by all the initial forms of the elements in I.

THEOREM 2.2. The ideal I is the conductor of A in $\bar{A}$ if and only if $G(I)$ is the conductor of $G(A)$ in $G(\bar{A})$.

PROPOSITION 2.3. If I is an ideal of A then for every integer n, $G(I) = [G(M)]^n$ if and only if $I = M^n$.

Finally we quote the following result (see [O], lemma 3.1) which will allow us to compute, in the next section, the conductor of $G(A)$ in its normalization. Let $P_1, \ldots, P_n$ be the minimal primes of 0 in the reduced ring D and set $D_i = D/P_i$. The canonical homomorphism $D \to D_i$ induce an embedding of rings $D \to \prod_{i=1}^{n} D_i = D'$. Then $D \subset D'$. Furthermore D' is integral over D. If the rings D_i are normal, D' is the normalization of D.

PROPOSITION 2.4. *The conductor of* D *in* D' *is the ideal* $\bigcap_{i=1}^{n} (P_i+\bigcap_{j\neq i}P_j)$.

3. THE CONDUCTOR OF CURVES WITH REDUCED TANGENT CONE.

In this section we assume that Spec A is the local ring at a singular point p of a reduced curve C over an algebraically closed field, e(A) and emdim(A) denote the multiplicity and embedding dimension of the ring A . Spec(G(A)) is the tangent cone of C at the point p and, when G(A) is reduced, Spec(G(A)) consists of s=e(A) lines containing p . Furthermore $G(\bar{A})$ is the normalization of G(A) .

DEFINITION 3.1. Let $p_1,\dots,p_h$ be points of $\mathbb{P}^r$ and let $V(d,p_1,\dots,p_h)$ be the linear system of all hypersurfaces of degree d which contain $p_1,\dots,p_h$. $\{p_1,\ ,p_h\}$ is a generic set of points of $\mathbb{P}^r$ if $\dim V = \binom{d+r}{r}-1-h$ for any integer d $(\binom{d+r}{r}-1-h<0$ if and only if $V=\phi$). This means that dim V is least possible. The points $p_1,\dots,p_s \in \mathbb{P}^r$ are *in generic position* if $S=\{p_1,\dots,p_s\}$ is a generic set of points and if any subsets consisting of s-1 points of S is a generic set

Now let endim A=r+1 , then the tangent cone Spec(G(A)) is contained in $\mathbb{A}^{r+1}$ and Proj(G(A)) consists of s=e(A) points of $\mathbb{P}^r$. The lines of Spec G(A) correspond to the minimal primes of G(A): So we can use Prop. 2.4 to compute the conductor of G(A) and we have:

THEOREM 3.2. *If* Proj(G(A)) *consists of* s *points in generic position than the conductor of* A *in its normalization is the ideal* M^d *where* $d=\mathrm{Min}\{d'\in\mathbb{N} \mid s\leq\binom{d'+r}{r}\}$ i.e. d *is the least* *degree of a hypersurface of* $\mathbb{P}^r$ *containing the points of* Proj(G(A)) .

Proof. Based on Thm. 2.2, Prop. 2.3, Prop. 2.4 and on properties of G(A) .

EXAMPLE 1. The tangent cone, at the origin, of the curve Spec $R=\mathrm{Spec}(k[X,Y,Z])/(X^2-Z^2+Z^3,ZX-Y^2)$ is reduced and consists of four distinct lines whose corresponding projective points are in generic position. Hence $(x,y,z)^2$ is the conductor of A in $\bar{A}$.

EXAMPLE 2. The preceding theorem is no longer true if the points of the projectivized tangent cone are not in generic position; in fact if $R=k[X,Y,Z]/(X,Z)\cap(Y,Z)\cap(X,Y)\cap(X-Y,Z)$ then Spec R is the tangent cone at the origin and the conductor of A in $\bar{A}$ is the ideal $(x,y)^2+(z)\neq(x,y,z)^2$ which is not a power of (x,y,z).

COROLLARY 3.3. *If* e(A)=s *and* emdim A=2 *then the conductor of* A *in* $\bar{A}$ *is the ideal* M^{s-1} .

Proof. If emdim(A)=2 then $\mathrm{Proj}(G(A))\subset\mathbb{P}^1$ and a finite number of points of $\mathbb{P}^1$ are always in generic position so we can apply thm 3.2.

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Istituto di Matematica
dell'Università di Genova
16132 - Genova

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Orthogonal Polynomials Satisfying A Fourth Order Differential Equation

by

Allan M. Krall[1,2]

Presented by F.V. Atkinson, F.R.S.C.

ABSTRACT. There are three sets of orthogonal polynomials $\{y_n\}_{n=0}^{\infty}$ which satisfy a differential equation $Ly_n = \lambda_n y_n$ of fourth order. Each set is orthogonal with respect to a Stieltjes weight function. They share many properties with the classical orthogonal polynomials, but the boundary value problems from which they arise have λ-dependent boundary conditions. Therefore a vector formulation is more appropriate in order to have a self-adjoint boundary value problem.

INTRODUCTION. In 1938 and 1940 [1], [2] H. L. Krall showed that there were exactly three sets of orthogonal polynomials which satisfied a differential equation of the form

$$\sum_{i=0}^{4} \sum_{j=0}^{i} \ell_{ij} x^j y_n^{(i)} = \lambda_n y_n$$

The differential equations specifically are:

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[1] McAllister Building, The Pennsylvania State University, University Park, Pennsylvania 16802.

[2] Supported in part by U. S. Air Force Grant AFOSR-78-3508.

Legendre type polynomials

$$(x^2-1)^2y_n^{iv} + 8x(x^2-1)y_n''' + (4\alpha+12)(x^2-1)y_n'' + 8\alpha xy_n' =$$
$$[8\alpha n + (4\alpha+12)n(n-1) + 8n(n-1)(n-2) + n(n-1)(n-2)(n-3)]y_n.$$

Laguerre type polynomials

$$x^2y^{iv} - (2x^2-4x)y_n''' + (x^2-[2R+6]x)y_n'' + ([2R+2]x-2R)y_n' = [(2R+2)n + n(n-1)]y_n$$

Jacobi type polynomials

$$(x^2-x)^2y^{iv} + 2x(x-1)([\alpha+4]x-2)y_n''' + x([\alpha^2+9\alpha+14+2M]x -$$
$$[6\alpha + 12 + 2M])y_n'' + ([(\alpha+2)(2\alpha+2+2M)]x-2M)y_n'$$
$$= [(\alpha+2)(2\alpha+2+2M)n + (\alpha^2+9\alpha+14+2M)n(n-1) +$$
$$(2\alpha+8)n(n-1)(n-2) + n(n-1)(n-2)(n-3)]y_n$$

The weight functions which make the polynomials orthogonal are:

Legendre type polynomials

$$w = (1/2)\left[\delta(x-1) + \delta(x+1) + \begin{Bmatrix}\alpha, & -1 \leq x \leq 1\\ 0, & |x| > 1\end{Bmatrix}\right]$$

Laguerre type polynomials

$$w = (1/R)\delta(x) + \begin{Bmatrix}e^{-x}, & 0 \leq x < \infty\\ 0, & -\infty < x < 0\end{Bmatrix}$$

Jacobi type polynomials

$$w = (1/M)\delta(x) + \begin{Bmatrix}(1-x)^{\alpha}, & 0 \leq x \leq 1\\ 0, & x < 0 \text{ or } x > 1\end{Bmatrix}$$

Because of lack of space we shall continue only with the Legendre type polynomials. The properties of the others are similar.

It is possible to show that

$$\int_{-1}^{1} (zLy-yLz)w\,dx = 0$$

where L represents the differential expression on the left side of the differential equation. This shows immediately that the polynomials $\{y_n\}_{n=0}^{\infty}$ are mutually orthogonal with respect to w. The polynomials themselves are given by

$$y_n = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k(2n-2k)!(\alpha+\frac{n(n-1)}{2}+2k)x^{n-2k}}{2^n k!(n-k)!(n-2k)!} .$$

Only the extra terms in the numerator involving α makes them different from the ordinary Legendre polynomials. Further $\lim_{\alpha\to\infty} y_n(x)/\alpha = P_n(x)$,

They satisfy a three term recurrence relation

$$(n+1)(\alpha+\frac{(n-1)n}{2})\,y_{n+1} - (2n+1)(\alpha+\frac{n(n+1)}{2})\,xy_n$$

$$+ n(\alpha+\frac{(n+1)(n+2)}{2})y_{n-1} = 0$$

This leads to the formula

$$\int_{-1}^{1} y_n^2\,w\,dx = \alpha(\alpha+\frac{(n-1)n}{2})(\alpha+\frac{(n+1)(n+2)}{2})/(2n+1).$$

The polynomials are related to the ordinary Legendre polynomials through

$$(\alpha - x\frac{\partial}{\partial x} + \frac{n(n+1)}{2})P_n = y_n,$$

$$P_n = \int_x^\infty \left[x^{\alpha + \frac{n(n+1)}{2}} \Big/ \xi^{\alpha + \frac{n(n+1)}{2} + 1} \right] y_n(\xi)\,d\xi.$$

They also have closely related generating functions and Rodrigues formulas.

Let $H = L^2(-1,1) \otimes R \otimes R$, with $Y = (y(x), y_1, y_{-1})^T$, $Z = (z(x), z_1, z_{-1})^T$ in H having inner product

$$(Y,Z) = \int_{-1}^{1} z(x)y(x)(\alpha/2(dx + (1/2)z_1 y_1 + (1/2)z_{-1}y_{-1}$$

Then the operator A, defined by $AY = (Ly, 8\alpha y'(1), -8\alpha y'(-1))^T$ is self-adjoint with discrete spectrum given by the bracketed terms on the right side of the Legendre type differential equation. Its eigenfunctions are $\{(y_n(x), \alpha, (-1)^n\alpha)^T\}_{n=0}^{\infty}$, which are dense in H. The resulting eigenfunction expansions converges in H to the elements they represent.

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Pennsylvania State University,
University Park, Penn. 16802.

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SOME EXTENSIONS OF THE STURM-PICONE THEOREM

by

Angelo B. Mingarelli

Presented by F.V. Atkinson, F.R.S.C.

In a recent paper [3] P. Hartman extends the classical Sturm Comparison Theorem, SCT, [4] for two differential equations of the second order to N such equations where $N \geq 3$.

In this note we extend Picone's Identity, PI, [4] for two differential equations to N such as a consequence of a more general identity which also yields the discrete versions of PI.

Of particular importance in [3] is the case $N = 3$: In this case the extension of PI provides an almost immediate proof of Hartman's extension of SCT.

<u>NOTATION</u>: Let A denote the collection of all real-valued functions defined on some finite interval I which are right-continuous and of bounded variation in I.

Δ is the forward difference operator. Hence $\Delta y_1 = y_2 - y_1$, $\Delta^2 y_1 = y_3 - 2y_2 + y_1 = \Delta(\Delta y_1)$, etc.

For brevity we shall write $x_i(t) \equiv x_i$ and omit the differentials in the integrals unless otherwise specified.

<u>Theorem 1</u>: Let p_i, $\sigma_i \in A$, $p_i > 0$ on $I = [a, b]$, and consider the Stieltjes integro-differential equations

$$p_i x_i' = c_i - \int_a^t x_i d\sigma_i \ , \qquad t \in I \tag{1}$$

$$x_i(a) = 1 \ , \quad x_i'(a) = R_i \tag{2}$$

where c_i , $R_i \in \mathbb{R}$, $i = 1,2,\ldots,N$. Then

$$\left[x_{N-1}^2 \Delta^{N-1}(p_1 r_1)\right]_a^b = \int_a^b (x'_{N-1})^2 \Delta^{N-1}(p_1) - \int_a^b x_{N-1}^2 d(\Delta^{N-1}\sigma_1)$$

$$- \sum_{k=0}^{N-1} C(N-1,k)(-1)^{N-k-1} \int_a^b p_{k+1} \frac{W^2(x_{k+1}, x_{N-1})}{x_{k+1}^2} \qquad (3)$$

where $r_i \equiv x_i'/x_i$, $W(x_i,x_j) \equiv x_i'x_j - x_i x_j'$.

The basic theory of equations of the form (1) can be found in [1] and [5].

Corollary 1.1: Let $\sigma_i \in C^1(a,b)$, $\sigma_i' = q_i$, $i = 1,2,\ldots,N$. If x_i are solutions of

$$(p_i x_i')' + q_i x_i = 0 \qquad (4)$$

then (3) holds, the only modification being that $d(\Delta^{N-1}\sigma_1) = (\Delta^{N-1}q_1)dt$.

A similar general identity is obtained for second order difference equations. We give a particular case of this identity when $N = 2$. The classical Picone Identity is obtained from Cor.1.1 upon setting $N = 2$.

Corollary 1.2: (Discrete Picone Identity). Let $c_n > 0$, $b_n \in \mathbb{R}$, and consider

$$\Delta(c_{n-1}\Delta u_{n-1}) + b_n u_n = 0 \tag{5}$$

$$\Delta(\gamma_{n-1}\Delta v_{n-1}) + \beta_n v_n = 0 \tag{6}$$

for $n = 0,1,\ldots,m$. Then

$$u_m^2\left\{c_{m-1}\,\frac{\Delta u_{m-1}}{u_{m-1}} - \frac{\gamma_{m-1}\Delta v_{m-1}}{v_{m-1}}\right\} - u_{-1}^2\left\{c_{-1}\,\frac{\Delta u_{-1}}{u_{-1}} - \gamma_{-1}\,\frac{\Delta v_{-1}}{v_{-1}}\right\}$$

$$= \sum_0^{m-1}(\beta_i - b_i)u_i^2 + \sum_0^{m-1}(c_{i-1}-\gamma_{i-1})(\Delta u_{i-1})^2 + \sum_0^{m}\gamma_{i-1}\,\frac{W^2(u_i,v_i)}{v_i v_{i-1}} \tag{7}$$

where $W(u_i,v_i) \equiv u_i v_{i-1} - u_{i-1}v_i$.

Corollary 1.2 then provides an immediate proof of the discrete version of SCT , [2]. Similarly Cor.1.1 allows, in the case $N = 3$, an immediate proof of Hartman's comparison theorem. The following result is now a consequence of theorem 1.

<u>Theorem 2</u>: Let p_i , $\sigma_i \in A$, $p_i > 0$ on I , $i = 1,2,3$. Let x_i satisfy (1)-(2) for $i = 1,2,3$ and suppose that $x_1(t) > 0$ on $[a,b]$, $x_2(b) = 0$. If

$$\Delta^2\sigma_1 \text{ is non-decreasing on } [a,b] \tag{8}$$

$$\Delta^2 p_1 \leqq 0 \tag{9}$$

$$\Delta^2(p_1(a)R_1) \leqq 0\ , \tag{10}$$

then $x_3(t)$ has at least one zero in $[a,b]$ and if $x_i(t) > 0$ on $[a,T]$, $i = 1,2,3$ then $\Delta^2 r_1 \leqq 0$ on $[a,T]$.

If we now let $\sigma_i' = q_i$, say, and $p_i \equiv 1$ in the above theorem we essentially obtain Hartman's theorem. Using the methods in [5], theorem 2 can then be reinterpreted so as to yield the discrete analog of the latter theorem.

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Department of Mathematics
Pennsylvania State University
University Park, PA.16802
U. S. A.

Present address:

Department of Mathematics
University of Ottawa
Ottawa, Ontario
Canada K1N 6N5

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INTEGRATION OF A UNIT IN LINEAR RINGS WITH RIGHT INVERTIBLE OPERATORS

D. Przeworska-Rolewicz (Warszawa)

Presented by F.V. Atkinson, F.R.S.C.

Suppose that X is a commutative linear ring over $\mathbb{C}$. Denote by $L(X)$ the set of all linear operators defined on linear subspaces of X and with ranges in X. Write: $L_0(x) = \{A \in L(x) : \operatorname{dom} A = X\}$. An operator $A \in L_0(X)$ is said to be a Volterra operator if the operator $I - \lambda A$ is invertible for all $\lambda \in \mathbb{C}$. Denote by $R(X)$ the set of all right invertible operators belonging to $L(X)$. If $D \in R(X)$ then by $\mathcal{R}_D$ we denote the set of all right inverses of D. We assume that $\mathcal{R}_D \subset L_0(X)$ (for simplicity only). $F \in L(X)$ is an initial operator for D if F is a projection onto $\ker D$ such that there exists an $R \in \mathcal{R}_D$ satisfying the condition: $FR = 0$. One can prove that any projection onto $\ker D$ is an initial operator for D. We shall assume in sequel that $\dim \ker D \neq 0$.

We say that X is a Leibniz ring (shortly: L-ring) if $D \in R(X)$ satisfies the condition:

$$D(xy) = xDy + yDx \quad \text{for all } x,y \in \operatorname{dom} D \tag{1}$$

(cf. [1]).

The following theorem is due to H. von Trotha*):

THEOREM 1. Suppose that $D \in R(X)$, X is an L-ring with a unit e and F is a multiplicative initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then we have

$$R^n z = zR^n e \quad \text{for all } z \in \ker D,\ n \in \mathbb{N}; \tag{2}$$

$$R^n e = \frac{(Re)^n}{n!} \quad \text{for all } n \in \{0\} \cup \mathbb{N}. \tag{3}$$

EXAMPLE 1. Suppose that $x = C[a,b]$, $D = \frac{d}{dt}$, $(Rx) = \int_a^b x(s)ds$, $(Fx)(t) = x(a)$ for all $x \in X$, $e(t) \equiv 1$. Then $Re = \int_a^t dt = (t-a)$ and

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*) Private communication.

Formula (3) has the following form: $R^n e = \frac{(t-a)^n}{n!}$.

This Example justifies the title of the present paper.

COROLLARY 1. Suppose that all conditions of Theorem 1 are satisfied. Then R-shifts S_h exist and are defined as follows:

(4) $S_h R^k z = \frac{z}{k!}[(R-hI)e]^k F$ for all $k \in \{0\} \cup \mathbb{N}$ and $h \in \mathbb{R}^+$ (or $h \in \mathbb{R}$)

(cf. [2]).

COROLLARY 2. Suppose that X is a commutative Banach algebra with a unit e , $D \in R(X)$, X is an L-ring, F is a bounded multiplicative initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $\|Re\| < +\infty$. Then

1^0 The series $\sum_{n=0}^{\infty} \lambda^n R^n e$ is convergent for all $\lambda \in \mathbb{C}$;

2^0 Write: $\sum_{n=0}^{\infty} \lambda^n R^n e = (I-\lambda R)^{-1} e$. Then we have

(5) $$\sum_{n=0}^{\infty} \lambda^n R^n z = z(I-\lambda R)^{-1} e \qquad \text{for all } z \in \ker D , \ \lambda \in \mathbb{C} \ ;$$

3^0 Exponential elements (i.e. eigenvectors of D) exists for all $\lambda \in \mathbb{C}$ and are of the form (5);

4^0 D-shifts exist and coincide with R-shifts defined by Formula (4). (cf. [2]).

Corollaries 1 and 2 are not true in general if the operator F is not multiplicative. However, we have the following:

THEOREM 2. Suppose that $D \in R(X)$, X is an L-ring, F is an initial operator for D corresponding to R and satisfying the following condition:

(6) $$F(zx) = zFx \quad \text{for all } z \in \ker D , \ x \in X$$

Then Formula (2) holds for all $z \in \ker D$, $n \in \mathbb{N}$ and

(7) $$R^n e = \frac{(Re)^n}{n!} - \sum_{k=2}^{n} R^{n-k} F[(Re)^k] \quad \text{for } k \le 2 \ .$$

Moreover, R-shifts exist.

COROLLARY 3. Suppose that X is a commutative Banach algebra with a unit e, $D \in R(X)$, X is an L-ring, F is a bounded initial operator for D corresponding to an $R \in \mathcal{R}_D$ and satisfying Condition (6). Then

1^0 The series $\sum_{n=0}^{\infty} \lambda^n R^n e$ is convergent for all $|\lambda| < \frac{1}{\|Re\|}$;

2^0 Write $\sum_{n=0}^{\infty} \lambda^n R^n e = (I-\lambda R)^{-1} e$ for $|\lambda| < 1/\|Re\|$. Then we have

(8) $$\sum_{n=0}^{\infty} \lambda^n R^n z = z(I-\lambda R)^{-1} e \quad \text{for } |\lambda| < 1/\|Re\| \ ;$$

3^0 Exponential elements exist for $|\lambda| < 1/\|Re\|$ and are of the form (8);

4^0 R is not a Volterra operator;

5^0 D-shifts do not exist.

THEOREM 3. Suppose that $D \in R(X)$, X is a Duhamel ring, i.e. a commutative linear ring such that

(9) $$D(xy) = xDy \quad \text{for all } x,y \in \operatorname{dom} \mathcal{D}_D$$

(cf. [1] . Suppose, moreover, that X has a unit e and F is an initial operator for D corresponding to an R and satisfying Condition (6), then Formula (2) holds and

(10) $$R^n e = (Re)^n - \sum_{k=2}^{n} R^{n-k} F[(Re)]^k \quad \text{for all } n \in \mathbb{N} \ .$$

Moreover, R-shifts exist. If F is multiplicative then

(11) $$R^n e = (Re)^n \quad \text{for all } n \in \mathbb{N} \ .$$

COROLLARY 4. Suppose that X is a commutative Banach algebra, $D \in R(X)$, X is a Duhamel ring, F is a bounded initial operator for D satisfying condition (6). Then Points 1^0, 2^0, 3^0, 4^0, 5^0 of Corollary 3 hold (even in the case when F is multiplicative).

Similar results can be obtained for QL-rings, i.e. such commutative linear rings that an operator $D \in R(X)$ satisfies the condition

$$D(xy) = xDy + yDx + d(Dx)(Dy) \quad \text{for all } x,y \in \operatorname{dom} D, \tag{12}$$

where $0 \neq d \in \mathbb{C}$ does not depend on x and y, (cf. [1]).

Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8, 00-950 Warszawa, Poland

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Institute of Mathematics,
Polish Academy of Sciences,
Warszawa, Poland

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