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A property of totally ordered abelian groups

George A. Elliott

*Presented by N.S. Mendelsohn, F.R.S.C.*

1. The most elementary ordered group is of course the cyclic one  $\mathbb{Z}$ , with positive cone  $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ . A close second is  $\mathbb{Q}$ , with positive cone  $\mathbb{Q}^+ = \{p/q \mid p, q \in \mathbb{Z}^+\}$ .

Almost equally familiar are the other totally ordered groups of rank one; these are isomorphic to the dense subgroups of  $\mathbb{Q}$  with the relative order—such as the subgroup  $\mathbb{Z} \left[ \frac{1}{2} \right]$  of dyadic rationals.

2. It is easily seen that every totally ordered group of rank one can be expressed as the inductive limit of a sequence of cyclic ordered groups

$$\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \dots$$

Here each map  $\mathbb{Z} \rightarrow \mathbb{Z}$  is understood to be positive; in this case this just means multiplication by some element of  $\mathbb{Z}^+$ .

The dyadic rationals, for example, may be expressed as the inductive limit of the sequence of cyclic ordered groups

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \dots$$

where  $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$  denotes multiplication by 2.

3. The ordered group direct sum of two or more cyclic ordered groups is of course not totally ordered, but it is still interesting; we shall call such an ordered group simplicially ordered.

An interesting fact about simplicially ordered groups is that the inductive limit of a sequence of such ordered groups

$$\mathbb{Z}^{r_1} \xrightarrow{m_1} \mathbb{Z}^{r_2} \xrightarrow{m_2} \dots$$

(where  $m_k$  is an  $r_{k+1} \times r_k$  matrix over  $\mathbb{Z}^+$ ) may still happen to be totally ordered. In fact it may be isomorphic to an arbitrary countable totally ordered abelian group.

This is the property of totally ordered abelian groups referred to in the title. It may be stated in a stronger form and for possibly uncountable groups as follows.

4. Theorem. Let  $G$  be a totally ordered abelian group, and let  $F$  be a finite subset of  $G^+$ . Then there exists an independent subset  $E$  of  $G^+$  such that the semigroup generated by  $E$  contains  $F$ .

Proof. Here I shall give the proof in the case that  $G$  has rank two. The proof in the general case is rather more complicated, and will be given in [2].

First, since any subgroup of  $G$  is totally ordered, we may assume that the underlying group of  $G$  is finitely generated, and hence isomorphic as a group to  $\mathbb{Z}^2$ .

Choose a basis  $B_1$  for  $G$ . Replacing elements of  $B_1$  by

their negatives if necessary, we may choose  $B_1$  so that  $B_1 \subset G^+$ . By induction, it is enough, given  $g \in G^+$ , to find a basis  $B_0$  of  $G$ , with  $B_0 \subset G^+$ , such that the semigroup generated by  $B_0$  contains  $B_1$  and also  $g$ .

Write  $g = ma - nb$ , where  $B_1 = \{a, b\}$  and  $m, n \in \mathbb{Z}^+$ .

Consider the two cases  $a < b$  and  $a > b$ .

If  $a < b$  then  $g = (m-n)a - n(b-a)$ . Since  $g \geq 0$ ,  $m-n < 0$  is impossible, so  $|m-n| = m-n < m$  (we may assume  $n \neq 0$ ). Set  $\{a, b-a\} = B_2$ .

If  $b < a$  then  $g = m(a-b) - (n-m)b$ . If  $n-m < 0$ , we are finished (we may set  $\{a-b, b\} = B_0$ ). If  $n-m \geq 0$  then  $|n-m| = n-m < n$  (we may assume  $m \neq 0$ ). Set  $\{a-b, b\} = B_2$ .

Thus, in either case, we have replaced  $B_1$  by a basis  $B_2$  generating a larger semigroup, such that either  $g$  belongs to this, or the sum of the absolute values of the coefficients of  $g$  with respect to  $B_2$  is strictly smaller than with respect to  $B_1$ . By induction, after a finite number of repetitions of this procedure, to obtain bases  $B_1, B_2, B_3, \dots, B_n \subset G^+$ , we may set  $B_n = B_0$ .

5. For a countable ordered group  $G$  the property stated in 4 says that  $G$  can be expressed as the inductive limit of a sequence of simplicially ordered groups  $\mathbb{Z}^{r_1} \rightarrow \mathbb{Z}^{r_2} \rightarrow \dots$  in which the maps are injective. This is stronger than the property stated in 3, in which the maps were not specified to be injective.

6. It is natural to ask if 4 can be extended to lattice-ordered groups (whose prime quotients are totally ordered). This is possible for lattice-ordered groups with at

most finitely many independent ideals, but is not possible for the Cartesian product of infinitely many cyclic ordered groups.

7. In [1], the inductive limit of a sequence of simplicially ordered groups was called a dimension group. It was shown that the abelian group  $K_0$  of the inductive limit of a sequence of semisimple matrix algebras over a fixed field is a dimension group, in a natural order, and that this ordered group is a complete invariant for Morita equivalence of such rings. (Also, additional structure was given to obtain a complete isomorphism invariant.)

8. The results described in 7, combined with 4, show that the classification of countable totally ordered abelian groups is equivalent to the classification of certain noncommutative rings. An initial study of totally ordered groups from this point of view will be described in [2], and some isomorphism and nonisomorphism theorems for rings obtained in this way.

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## PROJECTION PARTITIONS OF FUNCTION SEMIGROUPS

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*Presented by J. Aczél, F.R.S.C.*

A function on a set  $A$  is any partial transformation  $f$  of  $A$ , i.e. any mapping  $f$  of a subset  $A_1 \subset A$  onto a subset  $A_2 \subset A$ . Here  $A_1$  and  $A_2$  are called the first and second projections of  $f$  and denoted by  $\text{pr}_1 f = A_1$ ,  $\text{pr}_2 f = A_2$ .

A function semigroup is any nonempty set  $F$  of functions on a fixed set  $A$  closed under the usual composition (=superposition) of functions. On any function semigroup  $F$  we introduce the following equivalence relations  $E_1$  and  $E_2$ : for every  $f, g \in F$   $(f, g) \in E_i \Leftrightarrow \text{pr}_i f = \text{pr}_i g$ ,  $i = 1, 2$ .

Now let  $S$  be an abstract semigroup. A partition of  $S$  is called a first (second) projection partition whenever there exists an isomorphism  $i$  of  $S$  onto a function semigroup and for every  $s, t \in S$  both  $s$  and  $t$  belong to a same class of the partition precisely when  $\text{pr}_1 i(s) = \text{pr}_1 i(t)$ , i.e. when  $(i(s), i(t)) \in E_1$  ( $\text{pr}_2 i(s) = \text{pr}_2 i(t)$ , i.e. when  $(i(s), i(t)) \in E_2$ ).

In this note we find characterizations of projection partitions of abstract semigroups. By the fundamental theorem on relation algebras [1] such characterizations do exist and can be expressed in the first order predicate language. Our main results were announced without proofs in [2]. Their proofs are based on results established in [3] (see also [4]).

A partition  $\equiv$  of a semigroup  $S$  is called left regular if  $s \equiv t \Rightarrow us \equiv ut$  for all  $s, t, u \in S$ . Here  $s \equiv t$  means that  $s$  and  $t$  belong to a same partition class of  $S$ . Right regular partitions are defined dually.

THEOREM 1. A partition  $\cap$  of a semigroup  $S$  is a first projection partition if and only if it is left regular and

$$s_0 \cap s_1 t_1 \& s_1 \cap s_2 t_2 \& \dots \& s_n \cap s_0 t_0 \Rightarrow s_0 \cap s_1 \quad (1)$$

for all  $s_i, t_i \in S$  and all  $n \geq 1$ .

THEOREM 2. A partition  $\cup$  of a semigroup  $S$  is a second projection partition if and only if it is right regular,

$$s_0 \cup t_1 s_1 \& s_1 \cup t_2 s_2 \& \dots \& s_n \cup t_0 s_0 \Rightarrow s_0 \cup s_1 \quad (2)$$

for all  $s_i, t_i \in S$  and all  $n \geq 1$ , and

$$sx = sy \& s \cup t \Rightarrow tx = ty \quad (3)$$

for all  $s, t, x \in S$  and all  $y \in S^1$ .

REMARK. Since the antecedent of (1) is invariant under cyclic permutations of  $(0, 1, \dots, n)$ , (1) is equivalent to

$$s_0 \cap s_1 t_1 \& s_1 \cap s_2 t_2 \& \dots \& s_n \cap s_0 t_0 \Rightarrow s_0 \cap s_1 \cap \dots \cap s_n.$$

An analogous equivalent form exists for (2).

PROOF OF THEOREM 2. NECESSITY. Suppose that  $\cup$  is a function semigroup and the partition  $\cup$  corresponds to  $\varepsilon_2$ . If  $s \cup t$ , i.e.  $\text{pr}_2 s = \text{pr}_2 t$ , then for every  $u \in S$   $\text{pr}_2 su = u(\text{pr}_2 s) = u(\text{pr}_2 t) = \text{pr}_2 tu$ , hence  $s \cup tu$  and  $\cup$  is right regular. If the antecedent of (2) holds, then  $\text{pr}_2 s_0 = \text{pr}_2 t_1 s_1 \subset \text{pr}_2 s_1 = \text{pr}_2 t_2 s_2 \subset \text{pr}_2 s_2 = \dots = \text{pr}_2 t_0 s_0 \subset \text{pr}_2 s_0$  and  $\text{pr}_2 s_0 = \text{pr}_2 s_1$ . So  $s_0 \cup s_1$  and (2) holds. If  $s \cup t$ , then  $ts^{-1}s = t$  (here  $s^{-1}$  is the binary relation converse to  $s$ ).

So  $sx = sy$  implies  $tx = ts^{-1}sx = ts^{-1}sy = ty$  and (3) holds.

**SUFFICIENCY.** Let  $\sqsubset$  be a right regular partition of an abstract semigroup  $S$  and let (2) and (3) hold. Define the following binary relation  $\sqsubset$  on  $S$ :  $s \sqsubset t$  iff either  $s = t$  or there exist  $s_1, t_1 \in S$  such that  $s = s_0 \sqsubset t_1 s_1$ ,  $s_1 \sqsubset t_2 s_2, \dots, s_{n-1} \sqsubset t_n s_n$ ,  $s_n = t$ . We omit a straightforward verification of the following facts:

(i)  $\sqsubset$  is a quasi-order (i.e. reflexive and transitive) binary relation on  $S$ ;

(ii)  $\sqsubset$  is right regular and left negative (the latter means that  $st \sqsubset t$  for all  $s, t \in S$ );

(iii)  $sx = sy \ \& \ t \sqsubset s \Rightarrow tx = ty$  for all  $s, t, x \in S$  and  $y \in S^1$ ;

(iv) (2) holds for all  $n \geq 1$  iff  $s \sqsubset t \ \& \ t \sqsubset s \Rightarrow s \sqsubset t$  for all  $s, t \in S$  (since  $s \sqsubset t \Rightarrow s \sqsubset t \ \& \ t \sqsubset s$ , we have that (2) means the following:  $s \sqsubset t \ \& \ t \sqsubset s \Leftrightarrow s \sqsubset t$ ).

It follows from [3] (see also §8 of [4]) that there exists an isomorphism  $i$  of  $S$  onto a function semigroup such that  $s \sqsubset t \Leftrightarrow \text{pr}_2 i(s) \subset \text{pr}_2 i(t)$ . Moreover, all the functions  $i(s)$  can be full (i.e. everywhere defined). Thus  $s \sqsubset t \Leftrightarrow s \sqsubset t \ \& \ t \sqsubset s \Leftrightarrow \text{pr}_2 i(s) \subset \text{pr}_2 i(t) \ \& \ \text{pr}_2 i(t) \subset \text{pr}_2 i(s) \Leftrightarrow \text{pr}_2 i(s) = \text{pr}_2 i(t) \Leftrightarrow (i(s), i(t)) \in E_2$ . Thus  $\sqsubset$  is a second projection partition.

**OUTLINE OF A PROOF OF THEOREM 1.** It is quite analogous to the preceding proof. In the proof of sufficiency we define  $s \sqsupset t$  for  $s, t \in S$  iff either  $s = t$  or  $s = s_0 \sqsupset t_1 s_1, \dots, s_{n-1} \sqsupset t_n s_n, s_n = t$ . Then  $\sqsupset$  is a left regular and right negative quasi-order relation on  $S$ , so ([3] or §6 of [4]) there

exists an isomorphism  $i$  of  $S$  onto a function semigroup such that  $s \leq t$  iff  $pr_1 i(s) \subset pr_1 i(t)$ . Thus  $s \leq t \Leftrightarrow pr_1 s = pr_1 t \Leftrightarrow (s, t) \in E_1$ .

Elementary (i.e. expressed as formulas of the first order predicate language) axioms for  $\sqcap$  and  $\sqcup$  given in Theorems 1 and 2 are sufficiently simple. However, they are infinite in number (for both (1) and (2) are schemes of axioms, they produce separate axioms for each value of  $n$ ). A natural question is whether the infinity of the axioms is due to imperfectness of the method used by the author or it is 'in the nature of things'. An answer is provided by the following

THEOREM 3. The classes of first projection partitioned semigroups and second projection partitioned semigroups cannot be characterized by any finite sets of elementary axioms.

We omit the proof of this theorem. It is based on examples of partitioned semigroups,  $S_n$ , which satisfy any finite number of axioms from Theorem 2 and which are not second projection partitioned. Then we apply the Gödel Completeness theorem for the first order predicate calculus.

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On the representation of modular forms by theta series

by M. Eichler

*Presented by J.H.H. Chalk, F.R.S.C.*

A theta series

$$\theta(z, f, p) = \sum c(n, f, p) e^{2\pi i n z}$$

formed with a primitive definite quadratic form  $F$  in  $2k$  variables and a spherical polynomial  $p$  of degree  $\ell$  is a modular form of weight  $k+\ell$  with respect to the modular congruence group  $\Gamma_0(N)$  and the character  $\chi(a) = (D/a)$  where  $N$  and  $D$  are the level and the discriminant of  $F$ . In the 19<sup>th</sup> century Jacobi and Liouville determined the number  $c(n, F, 1)$  of representations of  $n$  by a quadratic  $F$ , in some cases, by comparing  $\theta(z, F, 1)$  with modular forms of known Fourier expansions. In the years 1936-41 Hecke proposed a systematic analytic number theory of quadratic forms, based on his famous operator theory of modular forms. For explicit results he had to assume a knowledge of the representation of the  $\theta(z, F, p)$  by a basis of modular forms of this type. But such knowledge is available only for few individual quadratic forms.

In 1952 the author described the action of Hecke's  $T(q)$  for primes  $q$  with  $(D/q) = 1$  and of the  $T(q^2)$  for  $(D/q) = 1$  on the  $\theta(z, F, p)$  by the (weighted) numbers of integral transformations  $X^t F X = nF'$  of the forms into similar other forms ( $n=q$  or  $q^2$ ). This has been explicitly carried out in the case  $k=2$  and  $D$  a square. The arithmetic analogues of the linear representations of the  $T(n)$  in the spaces of modular forms are now the Brandt matrices  $B(n)$  defined in quaternion algebras over  $\mathbb{Q}$ . By comparison of their traces it has been shown that all modular forms of square-free level  $N$ , character  $\chi(a) = 1$ , and weight  $\ell+2$  can be represented by theta series [1], [6].

The results have been extended to levels  $N=q^{2r+1}N'$  with a prime  $q$  not dividing  $N'$  by Pizer [8]. He proves that all new-forms (in the sense of Atkin and Lehner) are representable by quaternary theta series. This is no longer true for a level  $N=q^2N'$ ; the modular forms not so representable are obtained from other modular forms by the so-called process of twisting, see [9] and earlier literature quoted there.

An open question in this connection is the following: which of the theta series for norm forms  $F[x]$  of quaternion ideals are linearly dependent or independent? Without an answer our knowledge on the representation numbers  $c(n,F,p)$  is incomplete.

Recently Waldspurger [10] made a completely new approach which draws heavily on deep results of Zagier [12] who constructed modular forms with unexpected arithmetically given Fourier coefficients. Among others he showed that all modular new-forms of character  $\chi(a) = 1$  are linear combinations of theta series with level  $N$  and discriminant  $N^2$ , provided that the weight is  $k+l$  with  $k \equiv 0 \pmod{4}$ .

Even more important is Waldspurger's theorem on forms of character  $\chi(a) = (D/a)$  in case  $D$  is odd and square-free. Under this assumption all modular forms can be obtained by the  $\theta(z,F,p)$  and the functions generated from them by application of all Hecke operators. Even the  $\theta(z,F,p)$  alone suffice if and only if the Hecke operator  $T(D)$  has no real eigenvalues.

Under the last assumption all  $T(q)$  transform theta series into linear combinations of such. The same has been claimed by Freitag [5], without that assumption for the simple theta series  $\theta(z,F,1)$ , by connecting them with Siegel modular forms. At the moment there is still a gap in the proof for operators  $T(q)$  with

$(D/q) = -1$ . Such  $q$  do not exist in the case of  $D$  a square, and this approach challenges a purely arithmetic understanding of the connection between representation of numbers and of forms by forms.

In the case of quaternary quadratic forms with non-square discriminant there is again a connection between the representation numbers  $c(n, F, p)$  and the Clifford algebras and orders attached to the forms  $F$ . The Clifford algebra is a quaternion algebra over the real quadratic field  $\mathbb{Q}(\sqrt{D})$ . The Brandt matrices act as linear operators on the  $c(n, F, p)$  as do the Hecke operators [2], [10]. In general most of the  $\theta(z, F, p)$  are linearly dependent, and the traces of the actions of the Brandt matrices on them are unknown. For  $D=q$ , a prime, many linear dependencies between the  $\theta(z, F, p)$  have been determined, but not all [3]. The quadratic forms  $F[x]$  which represent 1 play a particular role. Their Clifford orders have a certain symmetry, and the theta functions in 2 variables attached to them are symmetric Hilbert modular forms. Kitaoka found that the number of classes of such forms is half the rank of modular forms of level  $q$ , weight 2, and character  $(q/a)$  [7]. If their theta series and the theta series for the adjoint forms  $\tilde{F}$  are linearly independent, they represent all modular forms of this type. The answer is positive according to Waldspurger's second-mentioned theorem if and only if the Hecke operator  $T(D)$  has no real eigenvalue.

The mean representation number  $c(n, F, 1)$  for these particular forms which represent 1 has been determined by the author in the case  $q \equiv 13 \pmod{24}$ . It is a certain sum extended

over class numbers of imaginary quadratic fields which bears some similarity with the trace formula for Brandt matrices [4].

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The irreducible subgroups of  $GL_n(2)$  with  $n \leq 6$

by Koichiro Harada

and

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*Presented by H. Zassenhaus, F.R.S.C.*

The authors have been investigating irreducible subgroups of the general linear groups of small degree over the field of two elements. The following results have, so far, been obtained.

Let  $X$  be a subgroup of  $GL_n(2)$  which acts irreducibly on an  $n$ -dimensional vector space  $V$  over  $GF(2)$ , the field of two elements. Two pairs  $(V, X_1)$  and  $(V, X_2)$  are said to be isomorphic if  $X_1 \cong X_2$  and the semi-direct products  $VX_1$  and  $VX_2$  are isomorphic. It is clear that  $VX_1 \cong VX_2$  if and only if  $X_1$  and  $X_2$  are conjugate in  $GL_n(2)$ .

If  $n = 1, 2$  or  $3$ , then correspondingly there are one, two or three isomorphism types of the pairs  $(V, X)$ . If  $n = 4$ , then there are twenty isomorphism types, of which ten are nonsolvable. If  $n = 5$ , there are only three; namely  $X$  is isomorphic to a cyclic group of order  $31$ , a Frobenius group of order  $5 \cdot 31$ , or  $GL_5(2)$ . If  $n = 6$ , then there are sixty-four isomorphism types, of which twenty-four are nonsolvable.

The method used to classify the irreducible subgroups of  $GL_n(2)$ ,  $n \leq 6$  is group theoretical and computational. Two theorems of Sims [2] and of Mizutani [1] in which all groups (in particular, all simple groups) having primitive permutation representations of degree less than

49 are classified, are useful. The detail of the classification will appear elsewhere.

The authors are at present investigating irreducible subgroups of  $GL_n(2)$  for  $n = 7$  and  $8$ .

Theorem. Let  $X$  be a nonsolvable irreducible subgroup of  $L_n(2)$  with  $n \leq 6$ . Then  $X$  is conjugate in  $L_n(2)$  to one of the following groups:

- (a)  $n = 3$ ,
  - (1)  $L_3(2)$ ,
- (b)  $n = 4$ ,
  - (1)  $A_5$ , (two conjugacy classes),
  - (2)  $S_5$ , (two conjugacy classes),
  - (3)  $Z_3 \times A_5$ ,
  - (4)  $(Z_3 \times A_5) \cdot Z_2$ ,
  - (5)  $A_6$ ,
  - (6)  $S_6$ ,
  - (7)  $A_7$ ,
  - (8)  $A_8$ ,
- (c)  $n = 5$ ,
  - (1)  $L_5(2)$ ,
- (d)  $n = 6$ ,
  - (1)  $PGL_2(7)$ ,
  - (2)  $Z_3 \times L_3(2)$ ,
  - (3)  $S_3 \times L_3(2)$ ,
  - (4)  $Z_3 A_6$ , non split extension,
  - (5)  $Z_3 S_6$ , non split extension,

- (6)  $SL_3(4)$  ,
- (7)  $SL_3(4) \cdot Z_2$  ,
- (8)  $GL_3(4)$  ,
- (9)  $\Gamma L_3(4)$  ,
- (10)  $L_3(2) \wr Z_2$  ,
- (11)  $L_2(8)$  ,
- (12)  $Aut(L_2(8))$  ,
- (13)  $GL_2(8) \cong Z_7 \times L_2(8)$  ,
- (14)  $\Gamma L_2(8)$  ,
- (15)  $A_7$  ,
- (16)  $S_7$  ,
- (17)  $A_8$  ,
- (18)  $S_8$  ,
- (19)  $U_3(3)$  ,
- (20)  $Aut(U_3(3)) \cong G_2(2)$  ,
- (21)  $U_4(2)$  ,
- (22)  $Aut(U_4(2))$  ,
- (23)  $Sp(6,2)$  , or
- (24)  $L_6(2)$  .

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CRITICAL BEHAVIOR OF THE GEOSTOCHASTIC LOGISTIC SYSTEM

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*Presented by Carl Herz, F.R.S.C.*1. INTRODUCTION.

The infinite geostochastic logistic system is a measure-valued stochastic process which describes a spatially homogeneous population subject to reproduction, spatial dispersion and a Verhulst-type competition effect. The basic mathematical construction of this stochastic process and the relevant terminology are given in [1], [2]. We briefly review the essential facts below.

Let  $\Omega$  denote the space of continuous functions from  $[0, \infty)$  into  $M_T(\mathbb{R}^d)$ , the space of tempered measures on  $\mathbb{R}^d$ . Let  $X(\dots): [0, \infty) \times \Omega \rightarrow M_T(\mathbb{R}^d)$  be defined by  $X(t, \omega) \equiv \omega(t)$  for  $\omega \in \Omega$ ,  $t \in [0, \infty)$ . Let  $\Pi(\Omega)$  denote the set of probability measures on  $\Omega$ . Let  $F: M_T(\mathbb{R}^d) \rightarrow M_T(\mathbb{R}^d)$  be defined by  $F(\mu) \equiv (a-b(\xi * \mu))\mu$  where  $a, b > 0$ ,  $\xi$  is the indicator function of a sphere of radius  $R_0$  centered at the origin and  $*$  denotes convolution. Let  $D(L)$  denote the algebra of functions of the form  $\psi(\langle X(t), \phi_1 \rangle, \dots, \langle X(t), \phi_n \rangle)$  where  $\langle \dots, \dots \rangle$  denotes the canonical bilinear form,  $\psi$  is a polynomial of  $n$  variables and  $\phi_1, \dots, \phi_n \in \mathcal{S}(\mathbb{R}^d)$ , the space of  $C^\infty(\mathbb{R}^d)$  functions rapidly decreasing at infinity. Let

$$(1.1) \quad L\psi(\mu) \equiv \langle \psi'(\mu), \Delta\mu + F(\mu) \rangle + \frac{1}{2} \gamma \langle \mu_D, \psi''(\mu) \rangle$$

where  $\mu_D$  denotes the measure on  $\mathbb{R}^d \times \mathbb{R}^d$  defined by  $\mu_D(A \times B) =$

$\mu(A \cap B)$ ,  $\psi', \psi''$  denote the first and second Fréchet derivatives, respectively, and  $\Delta$  denotes the  $d$ -dimensional Laplacian.

The geostochastic logistic martingale problem is to construct a mapping  $\mu \rightarrow P_\mu$  from  $M_T(\mathbb{R}^d)$  to  $\Pi(\Omega)$  such that

$$(1.2a) \quad P_\mu(X(0) = \mu) = 1,$$

$$(1.2b) \quad \text{for } \psi \in D(L) \text{ and } \mu \in M_T(\mathbb{R}^d),$$

$$\psi(X(t)) - \int_0^t L\psi(X(s)) ds \quad \text{is a } P_\mu\text{-martingale.}$$

The following basic existence theorem is proved in [1].

**THEOREM 1.1.** The martingale problem prescribed by (1.1) and (1.2) has a unique solution. (The solution is referred to as the geostochastic logistic system (GSLS).)

## 2. STABLE BEHAVIOR OF THE GSLS.

A probability measure  $P \in \Pi(\Omega)$  is said to be a spatially homogeneous steady state random measure for the GSLS if it is invariant under the transformation induced by spatial translations in  $R^d$  and if  $\int P_\mu P(d\mu)$  is temporally stationary.

**THEOREM 2.1.** (a) For fixed  $R_0 > 0$  there exists  $\gamma_c$  such that for  $\gamma < \gamma_c$ , the GSLS has a non-zero spatially homogeneous steady state random measure.

(b) For  $\gamma \gg \gamma_c$ ,  $\epsilon > 0$ , any compact set  $K$  and any spatially homogeneous ergodic initial measure  $\mu$ ,

$$P_\mu(X(t, K) > \epsilon) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Method of Proof. It has been shown in [1], [2] that cumulant densities exist for the GSLS,  $X(t)$ . In view of spatial homogeneity the first two cumulant densities can be written as  $m_1(t)$ ,  $v(t, x)$ ,  $x \in R^d$ . The following system of differential inequalities is established in [2].

$$(2.1a) \quad d/dt(m_1(t)) = am_1(t) - bI_x^0 v(t, x) - bVm_1^2(t),$$

$$(2.1b) \quad \partial/\partial t(v(t, x)) \leq A(m_1(t))v(t, x) + \gamma m_1(t)\delta(x)$$

where  $V = c_d R_0^d$ , the volume of a sphere of radius  $R_0$ ,  $\delta$  denotes the Dirac delta function,

$$I_y^x v(t, \cdot) \equiv \int_{\|y-x\| \leq R_0} v(t, y) dy,$$

and

$$A(m)u(x) \equiv 2(a-bVm)u(x) + 2\Delta u(x) + 2mb \int_{\substack{\|z-x\| \leq R_0 \\ \|z\| \leq \|x\|}} (u(x)-u(z)).$$

From (2.1) it can be established that there is an asymptotically stable pair  $(m_0, v_0(\cdot))$  such that  $m_1(t) \geq m_0 > 0$  and  $v(t, x) \leq v_0(x)$  for all  $x$  and  $t$ . The result then follows by

an ergodic theoretic argument.

Remark 2.1. The steady state random measure corresponds to the asymptotically stable fixed point,  $X(x) = a/bV$ , of the nonlinear diffusion equation obtained in the  $\gamma = 0$  limit. However for more complex multitype systems (c.f. [1]) multiple bifurcation in the  $\gamma$  parameters can occur leading to stationary stochastic states or behaviors having no deterministic analogues. This possibility is an important feature of a theory of nonequilibrium phase transitions.

### 3. CRITICAL BEHAVIOR OF THE GSLS.

We now investigate the behavior of the GSLS as  $R_0 \rightarrow \infty$ . In this case we let  $b(R_0) \equiv b/c_d R_0^d$ . Note that

$$F(X(t)) = aX(t) - bm_1(t)X(t) - (b(R_0)X(t, S_x^{R_0}) - bm_1(t))X(t).$$

Since for a spatially ergodic random measure  $b(R_0)X(S_x^{R_0}) \rightarrow bm_1(t)$ , it is natural to compare the limiting behavior of the GSLS with that of the martingale problem (1.1), (1.2) with  $F^*(\mu) = a\mu - bm_1(\mu)$  where  $m_1(\mu)$  denotes the mean density of the measure  $\mu$ . Clearly, in looking for stationary states of the  $F^*$ -martingale problem it suffices to consider measures,  $\mu$ , with  $m_1(\mu) = a/b$  since it must be preserved under the evolution.

THEOREM 3.1. (Dawson [3]). For  $d = 1, 2$ , the  $F^*$ -martingale problem has no non-zero stationary state and for  $\varepsilon > 0$ , compact set  $K$  and spatially invariant  $\mu$ ,  $P_\mu(X(t, K) > \varepsilon) \rightarrow 0$  as  $t \rightarrow \infty$ . For  $d \geq 3$ , the  $F^*$ -martingale problem has a unique non-zero invariant random measure with  $m_1(\mu) = a/b$ .

THEOREM 3.2. Let  $X_{SS}^{R_0}(\cdot)$  denote the steady state random measure for the GSLS with given  $R_0$  and let  $X_{SS}^*(\cdot)$  denote the steady state random measure for the  $F^*$ -martingale problem. Then for  $d > 4$ ,  $X_{SS}^{R_0}(\cdot)$  converges weakly to  $X_{SS}^*(\cdot)$  as  $R_0 \rightarrow \infty$ .

The proof of Theorem 3.2 is dependent on a refinement of the inequalities (2.1). Following a convexity argument (2.1b) can be replaced by

$$(3.1) \quad \partial/\partial t(v(t,r)) \leq 2(a-bm_1(t))v(t,r) + \partial^2/\partial r^2(v(t,r)) \\ + (d-1)/r \partial/\partial r(v(t,r)) + cm_1(t)bR_0 \partial/\partial r(v(t,r))$$

for  $r > R_0$ , where  $c$  is a constant and  $v(t,x) = v(t,r)$  where  $r = \|x\|$ . For  $\|x\| \leq R_0$ , with a matching boundary condition on  $r = R_0$ ,

$$(3.2) \quad \partial/\partial t(v(t,x)) \leq 2(a-bm_1(t))v(t,x) + \Delta v(t,x) + \gamma m_1(t)\delta(x).$$

Let  $\lambda_1(R_0)$  denote the smallest eigenvalue of  $-\Delta$  with zero boundary conditions on the boundary of the sphere  $S_{R_0}^d$ . Then  $\lambda_1(R_0) \sim R_0^{-2}$  as  $R_0 \rightarrow \infty$ . If  $a-bm_1(t) \leq \lambda_1$ , then  $I_Y^0 v(t,y) \leq I_Y^0 v^*(t,y) = 0 (R_0^2)$  for large  $R_0$ . But then

$$(3.3) \quad a-bm_1(t) \sim c I_Y^0 v(t,y)/R_0^d = 0 (R_0^{2-d}).$$

If  $d > 4$ , then  $(a-bm_1(t)) \ll \lambda_1(R_0)$  for large  $R_0$  and a self-consistent estimate is obtained. It can then be shown that the contribution to the martingale problem from the term  $F(\cdot) - F^*(\cdot)$  becomes negligible as  $R_0 \rightarrow \infty$ .

Remark 3.2. As in the case of the  $F^*$ -martingale problem (c.f. [3]) the large scale fluctuations are asymptotically Gaussian for  $d > 4$ . However in the case  $d = 3$ , it is apparent that the critical behavior is qualitatively different and it is expected that the large scale fluctuations near criticality are not Gaussian. This phenomena is a result of the nonlinear amplification of the large deviations of the random field.

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## GENERALIZED GAUGE FIELD THEORIES

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The standard theory of gauge fields [1] displays some very remarkable features, the most striking of which is the fact that the free Yang-Mills equations can be expressed in the form  $D(*F^\alpha) = 0$ , where  $D$  denotes an operator of exterior covariant differentiation which is defined in terms of a specially constructed connection, the latter depending explicitly on the field variables and on the structure constants of the Lie group on which the gauge field is based. It is well known, however, that the Yang-Mills equations can be identified with the Euler-Lagrange equations of an invariant action principle. From the point of view of the classical calculus of variations this naturally raises the following question: Under what circumstances is it possible to associate with some given invariant integral a connection which is such that the Euler-Lagrange equations can be expressed in terms of the corresponding covariant derivatives? It is this complex of ideas to which we shall address ourselves.

The field variables of our action principle are denoted by  $A_j^\alpha(x)$ , where  $x = (x^1, \dots, x^n)$  represents the local coordinates of a differentiable manifold  $X_n$ . [Latin and Greek indices range from 1 to  $n$ , and from 1 to  $r$ , respectively; the summation convention applies to both sets of indices.] It is supposed that the field variables are subject to generalized (inhomogeneous) gauge transformations of the form

$$(1) \quad A'_j{}^\alpha = A_j^\alpha - \eta_j^h A_h^\alpha + \zeta_\epsilon^\alpha(x) A_j^\epsilon u^\beta - \lambda_{\beta j}^\alpha(x) u^\beta - \lambda_{\beta j}^{\alpha h}(x) \partial_h u^\beta,$$

where, in accordance with the standard approach to gauge field theories, the  $u^\alpha$  denote a set of  $r$  position-dependent infinitesimal parameters. This is supposed to be accompanied by a point transformation

$$(2) \quad x'^j = x^j + \xi_{\alpha}^j(x) u^{\alpha}.$$

The entries in the Jacobian of (2) are identified with the coefficients  $\eta_j^h$  which appear in (1); the properties of the remaining coefficients in (1) are determined by the implications of the postulate that the integral of the variational principle for  $A_j^{\alpha}$  be invariant under the combined actions of (1) and (2). It is supposed also that  $X_n$  is endowed with a Riemannian metric  $(g_{hj})$ ; this allows for the construction of a wide class of appropriate Lagrangians.

From a detailed analysis of the resulting invariance identities satisfied by the Lagrangian and its derivatives, the following set of connection 1-forms emerges in an almost inevitable manner:

$$(3) \quad \omega_{\beta}^{\alpha} = \zeta_{\epsilon}^{\alpha}{}_{\beta} A^{\epsilon} + \psi_{\beta}^{\alpha}, \quad (A^{\alpha} = A_j^{\alpha} dx^j),$$

in which the coefficients  $\zeta_{\epsilon}^{\alpha}{}_{\beta}(x)$  must satisfy the relations

$$(4) \quad \zeta_{\beta}^{\alpha}{}_{\lambda} \zeta_{\gamma}^{\lambda}{}_{\epsilon} + \zeta_{\gamma}^{\alpha}{}_{\lambda} \zeta_{\epsilon}^{\lambda}{}_{\beta} + \zeta_{\epsilon}^{\alpha}{}_{\lambda} \zeta_{\beta}^{\lambda}{}_{\gamma} = 0, \quad \zeta_{\beta}^{\alpha}{}_{\gamma} = -\zeta_{\gamma}^{\alpha}{}_{\beta},$$

together with

$$(5) \quad D\zeta_{\beta}^{\alpha}{}_{\gamma} = d\zeta_{\beta}^{\alpha}{}_{\gamma} + \zeta_{\beta}^{\epsilon}{}_{\gamma} \omega_{\epsilon}^{\alpha} - \zeta_{\epsilon}^{\alpha}{}_{\gamma} \omega_{\beta}^{\epsilon} - \zeta_{\beta}^{\alpha}{}_{\epsilon} \omega_{\gamma}^{\epsilon} = 0,$$

while the 1-forms  $\psi_{\beta}^{\alpha}$  are subject to the conditions

$$(6) \quad d\psi_{\beta}^{\alpha} + \psi_{\epsilon}^{\alpha} \wedge \psi_{\beta}^{\epsilon} + \zeta_{\epsilon}^{\alpha}{}_{\beta} \psi^{\epsilon} = 0,$$

where  $\psi^{\alpha}$  denotes a set of 2-forms for which

$$(7) \quad d\psi^{\alpha} + \psi_{\beta}^{\alpha} \wedge \psi^{\beta} = 0.$$

Moreover, the requirements imposed on the coefficients in the gauge transformation (1) are such as to reduce the latter (possibly after a re-scaling)

to

$$(8) \quad \Lambda_j^{\prime \alpha} = \Lambda_j^{\alpha} - \eta_j^h \Lambda_h^{\alpha} - \partial_j u^{\alpha} - \Gamma_{\beta j}^{\alpha} u^{\beta}, \quad (\omega_{\beta}^{\alpha} = \Gamma_{\beta j}^{\alpha} dx^j).$$

The curvature 2-forms associated with the connection (3) can, by virtue of (4)-(7), be expressed as

$$(9) \quad \Omega_{\beta}^{\alpha} = \omega_{\beta}^{\alpha} + \omega_{\epsilon}^{\alpha} \wedge \omega_{\beta}^{\epsilon} = \zeta_{\epsilon \beta}^{\alpha} F^{\epsilon},$$

where the 2-forms

$$(10) \quad F^{\alpha} = dA^{\alpha} + \frac{1}{2} \zeta_{\epsilon \beta}^{\alpha} A^{\epsilon} \wedge A^{\beta} + \psi_{\beta}^{\alpha} \wedge A^{\beta} + \psi^{\alpha},$$

satisfy the Bianchi identities

$$(11) \quad DF^{\alpha} = 0.$$

The Lagrangian may now be considered as

$$(12) \quad L = L(g_{hj}, \zeta_{\epsilon \beta}^{\alpha}, \psi_{\beta j}^{\alpha}, \psi_j^{\alpha}, A_j^{\alpha}, A_{j,h}^{\alpha}),$$

where

$$(13) \quad \psi_{\beta}^{\alpha} = \psi_{\beta j}^{\alpha} dx^j, \quad \psi^{\alpha} = \frac{1}{2} \psi_h^{\alpha} dx^j \wedge dx^h, \quad A_{j,h}^{\alpha} = \partial_h A_j^{\alpha}.$$

We shall put

$$(14) \quad L_{\alpha}^{j h} = \partial L / \partial A_{j,h}^{\alpha}, \quad L_{\alpha}^j = \partial L / \partial A_j^{\alpha}.$$

The Euler-Lagrange expressions may then be represented in the form

$$(15) \quad E_{\alpha}^j(L) = \partial_h (L_{\alpha}^{j h}) - L_{\alpha}^j = D_h (L_{\alpha}^{j h}),$$

which is a covariant divergence. (In this context the operator  $D_h$  includes appropriate terms involving the Christoffel symbols of the metric.) The

divergences of these expressions vanish identically:

$$(16) \quad D_j [E_\alpha^j(L)] = 0.$$

Moreover, in terms of the  $(n-2)$ -forms

$$(17) \quad *L_\alpha = -\frac{1}{2} L_\alpha^j h_{jh}, \quad 2\theta_{jh} = \epsilon_{jh} \ell_3 \dots \ell_n dx^3 \wedge \dots \wedge dx^n,$$

the Euler-Lagrange equations admit the representation

$$(18) \quad D(*L_\alpha) = 0.$$

The energy-momentum tensor is defined as

$$(19) \quad T_h^j = -2(\partial L / \partial g_{jh}) g_{jh} = L_\alpha^j F_h^\alpha - \delta_h^j L,$$

and, for a fairly general class of Lagrangians, the covariant divergences of (19) vanish whenever (18) is satisfied, which represents a counterpart, within this framework, of the classical theorem of Noether.

One of several significant classes of special connections arises when  $\psi_\beta^\alpha = \zeta_\epsilon^\alpha \psi^\epsilon$  for a set of 1-forms  $\psi^\alpha$ . It then follows that  $\zeta_\epsilon^\alpha = \text{constant}$ ; because of (4) these coefficients may be identified with the structure constants of a Lie group. The standard theory of gauge fields is then recovered by the additional specializations  $\psi^\alpha = 0$ ,  $\Psi^\alpha = 0$ .

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## Composition of quadratic and alternate forms

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Composition of quadratic forms of the same dimension was first treated in a classical paper of Hurwitz [4]. The generalization to different dimensions was made by both Hurwitz [5] and Radon [7]. This theorem has had applications in a number of contexts: two samples are [1] and [8].

The methods of Hurwitz and Radon were matrix-theoretic. The applicability of group representations as an alternate method was pointed out by Jordan, von Neumann, and Wigner [6, 51-54] in the equidimensional case and by Eckmann [2] in the general case. Herstein's polished account of the group method [3, 141-144], it should be noted, applies only to the equidimensional case.

Recently I noticed that Clifford algebras seem to fit the problem even better than group representations. I shall devote this note to sketching how this works. At the same time I treat the parallel problem where one of the quadratic forms is replaced by an alternate form, and I make a modest invasion of characteristic 2.

Let  $V$  and  $W$  be finite-dimensional vector spaces over an algebraically closed field  $k$  of characteristic  $\neq 2$ ; let their dimensions be  $m$  and  $n$  respectively. Suppose that  $V$  supports a nonsingular quadratic form  $q$  and that  $W$  supports either a nonsingular quadratic form  $r$  or a nonsingular alternate form  $s$ . Suppose further that there is given a bilinear function  $V \times W \rightarrow W$ , which we shall simply write as  $vw$ , such that we always have

$r(vw) = q(v)r(w)$  or  $s(vw_1, vw_2) = q(v)s(w_1, w_2)$ , as the case may be. Our task is to describe the possibilities for  $m$  and  $n$ .

If  $T(v)$  is the map  $w \rightarrow vw$ , then the map  $v \rightarrow T(v)$  is a one-to-one linear transformation from  $V$  into a vector space of homotheties of  $W$ . We might as well use it to identify  $V$  with  $T(V)$ ; so  $V$  is now a subspace of  $\text{End}(W)$ . We have  $v^*v \in k$  for all  $v \in V$ , where  $*$  is the adjoint map on  $\text{End}(W)$  relative to  $r$  or  $s$ . An innocuous change in the bilinear map can assure us that  $V$  contains the identity linear transformation on  $W$ . We then have  $(\lambda + v^*)(\lambda + v) \in k$  for every scalar  $\lambda$ . It follows that  $v^* + v \in k$  and that  $V$  is an orthogonal direct sum of  $k1$  and the subspace  $S$  of skew elements. We now erect the Clifford algebra  $C$  on  $S$ .

The main point is that in addition to its known algebra structure  $C$  carries the involution induced by declaring the elements of  $S$  to be skew and letting nature take its course. It is routine to determine the structure of  $C$  as a  $*$ -algebra and then to catalogue the possibilities for its relation to the  $*$ -algebra  $\text{End}(W)$ . The facts are assembled in the table below.

Explanation: When  $m$  is odd (so that the dimension of  $S$  is even)  $C$  is a total matrix algebra. There are two possibilities for the involution: it may be of the transpose type or of the symplectic type. This is recorded in the second column. When  $m$  is even,  $C$  is the tensor product of the corresponding algebra for  $m-1$  with a two-dimensional algebra generated by a central  $z$  with  $z^2$  scalar. We need to know whether  $z$  is self-adjoint or skew and this appears in the third column. The fourth and fifth columns

list the smallest  $n$  that will accommodate  $C$ . The tableau repeats mod 8. (Note: of course the fourth column amounts to the same thing as the Radon-Hurwitz function.)

Let  $k$  have characteristic 2, take  $m$  odd, and assume the form on  $W$  to be alternate. Relative to the quadratic form  $q$  on  $V$  there is a distinguished one-dimensional subspace — the radical of the induced alternate form. We can arrange that  $Y$  is spanned by the identity linear transformation. The orthogonal complement  $S$  of  $Y$  then consists of self-adjoint elements, and the Clifford algebra on  $S$  can be used as above, yielding similar but slightly different results.

| $m$ | Inv. on $C$ | Center  | Best $n$ in quad. case | Best $n$ in alt. case |
|-----|-------------|---------|------------------------|-----------------------|
| 3   | Syml.       |         | 4                      | 2                     |
| 5   | Syml.       |         | 8                      | 4                     |
| 7   | Tr.         |         | 8                      | 16                    |
| 9   | Tr.         |         | 16                     | 32                    |
| 4   |             | Self-a. | 4                      | 2                     |
| 6   |             | Self-a. | 8                      | 8                     |
| 8   |             | Skew    | 8                      | 16                    |
| 10  |             | Skew    | 32                     | 32                    |

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Indecomposable representations of orders and Dynkin diagrams

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*Presented by V. Dlab, F.R.S.C.*

Let  $R$  be a complete valuation ring with quotient field  $K$ , and  $\Lambda$  an  $R$ -order in a semi-simple  $K$ -algebra  $A$ . It is known that some problems in the representation theory of  $\Lambda$  can be reduced to similar problems in other categories [2], however no attempt seems to exist to develop a general theory. The present note will survey some of the problems which can be tackled in this way. The procedure always will be twofold: First, there will be a reduction to one of those vector space categories which lately attracted much interest (categories of representations of quivers, species, partially ordered sets, subspace categories etc.); the second step is then the construction of lattices from the given data. In this way, one wants to list some, or even all, indecomposable  $\Lambda$ -lattices. In contrast to the usually applied methods, there will be little calculation inside the order or the lattices (the calculations are transferred to the frame of vector spaces). On the other hand, the results will usually not depend on the particular structure of the residue field  $\underline{k}$  of  $R$ , whereas most of the classical results were formulated only for finite  $\underline{k}$ .

We denote by  $N(\Lambda)$  a full set of non-isomorphic indecomposable  $\Lambda$ -lattices, and by  $N_d(\Lambda)$  the subset of those lattices of rank  $d$ .

Theorem I: If  $N(\Lambda)$  is infinite, then there exists an infinite chain of indecomposable  $\Lambda$ -lattices  $M_0 \subset M_1 \subset \dots \subset M_i \subset M_{i+1} \subset \dots$  such that  $M_{i+1}/M_i \simeq M_0$  for all  $i$ .

This result is a consequence of the validity of a similar statement for certain vector space categories, which turned out to be

true in the proof of the second Brauer-Thrall conjecture [4]. Moreover we can formulate an analogue of this conjecture for orders asserting that there are infinitely many natural numbers  $d$  such that  $N_d(\Lambda)$  is infinite in case  $\underline{k}$  is infinite, whereas for finite  $\underline{k}$ , the number of elements of  $N_d(\Lambda)$  grows rapidly with  $d$ . For a special type of orders, the reduction method gives a particularly satisfying theory: Namely it extends a recent result of Bäckström [1] into several different directions and shows that his rather unnatural seeming conditions have a fairly natural interpretation: He describes the Dynkin diagrams  $A_n, D_n, E_6, E_7$  and  $E_8$ . We call  $\Lambda$  a Bäckström order provided there exists a hereditary  $R$ -order  $\Gamma$  such that  $\Lambda \subseteq \Gamma$  and  $\text{rad} \Lambda = \text{rad} \Gamma$ . Note, that in case  $A$  is separable, and  $\Lambda$  is an arbitrary order in  $A$ , there is a hereditary order  $\Gamma$  with  $\Lambda \subseteq \Gamma \subseteq A$  such that  $\Lambda_0 = \Lambda + \text{rad} \Gamma$  is a Bäckström order containing  $\Lambda$ . In this way, results concerning lattices over Bäckström orders are of interest also for general orders. For any Bäckström order  $\Lambda$ , we introduce a valued graph in the following way: Let  $\Lambda \subseteq \Gamma$ , with  $\Gamma$  hereditary, and  $J = \text{rad} \Lambda = \text{rad} \Gamma$ . By Morita-equivalence, we may suppose  $\Lambda/J = \prod_{i=1}^s F_i$ , and  $\Gamma/J = \prod_{i=s+1}^t (F_i)_{n_i}$  where  $F_i$  are skewfields  $1 \leq i \leq t$ , and  $(F)_n$  denotes the full  $n \times n$ -matrix ring over  $F$ . Note that for later purposes we denote the various skew fields by  $F_i$ , the index ranging from 1 to  $t$ . For  $s+1 \leq j \leq t$ , let  $S_j$  be a simple  $\Gamma/J$ -module with endomorphism ring  $F_j$ , and we denote by  ${}_i S_j$  the  $F_i$ - $F_j$ -bimodule  ${}_i S_j = F_i \otimes_{\Lambda} S_j$ , where  $1 \leq i \leq s$ ,  $s+1 \leq j \leq t$ , and by  $d_{ij} = \dim_{F_i}({}_i S_j)$ ,  $d'_{ij} = \dim({}_i S_j)_{F_j}$  the resp. two dimensions of  ${}_i S_j$ . For  $i > s$ , and for  $j < s$ , we put  $d_{ij} = d'_{ij} = 0$ . In this way we get a valued graph  $G$  with  $t$  vertices and valuation  $(d_{ij}, d'_{ij})$ .

**Theorem II:** Let  $\Lambda$  be a Bäckström order with valued graph  $G$ .

- (i)  $N(\Lambda)$  is finite if and only if  $G$  is a Dynkin diagram, and in this case, the elements of  $N(\Lambda)$  correspond bijectively to the non-simple positive roots of  $G$ .
- (ii) If  $G$  is an extended Dynkin-diagram, then  $N(\Lambda)$  can be classified.

Bäckström [1] proved part (i) under the assumption that the residue field  $\underline{k}$  of  $R$  is finite and that all  $F_i = \underline{k}$  (note that in this case only the Dynkin diagrams  $A_n, D_n, E_6, E_7, E_8$  can occur), without, however, constructing the indecomposable representations explicitly. The reduction method, on the other hand, allows an actual classification of all indecomposable representations in both cases (i) and (ii). We should remark that every valued graph occurs as the valued graph of a Bäckström order. For example

$$\Lambda = \begin{pmatrix} R & \pi & R & R & R & R & R \\ \pi & \alpha & R & R & R & R & R \\ \pi & \pi & \alpha' & R & R & R & R \\ \pi & \pi & \pi & \alpha'' & \pi & R & R \\ \pi & \pi & \pi & \pi & \beta & R & R \\ \pi & \pi & \pi & \pi & \beta' & \pi & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi & R \end{pmatrix} \quad \begin{aligned} \alpha &\equiv \alpha' \equiv \alpha'' \pmod{\pi}, \quad \beta \equiv \beta' \pmod{\pi}, \\ \text{where } \pi &\text{ is the maximal ideal of } R, \end{aligned}$$

is a Bäckström order with diagram  $E_8 = \overset{\cdot}{\text{---}} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \overset{\cdot}{\text{---}}$ . Since  $E_8$  has 120 positive roots, there are 112 indecomposable  $\Lambda$ -lattices. There is one lattice of highest rank 105, with 6 generators, given as follows:

$$M = \begin{pmatrix} \pi & \pi & R & R & \pi & \pi & \pi & R & R & R & R & R & R & R & R & R \\ \alpha & \alpha' & \alpha'' & \alpha''' & \pi & \pi & \pi & R & R & R & R & R & R & R & R \\ \pi & \pi & \pi & \pi & \beta & \beta' & \beta'' & R & R & R & R & R & R & R & R \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & \gamma & \gamma' & \gamma'' & \gamma''' & \gamma'''' & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & R & \delta & \epsilon & \zeta & \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \delta' & \epsilon' & \zeta' \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & R & R & R \end{pmatrix} \quad \begin{aligned} &\text{subject to the following} \\ &\text{conditions} \\ \beta &\equiv \alpha + \alpha'' + \gamma'''' \pmod{\pi} \\ \beta' &\equiv \alpha' + \alpha''' + \gamma'''' \pmod{\pi} \\ \beta'' &\equiv \alpha'' + \alpha'''' + \gamma'''' \pmod{\pi} \\ \alpha' &\equiv \gamma \pmod{\pi} \\ \alpha'' &\equiv \gamma' \pmod{\pi} \\ \alpha''' &\equiv \gamma'' \pmod{\pi} \\ \alpha'''' &\equiv \gamma''' \pmod{\pi} \\ \delta &\equiv \delta' \pmod{\pi} \\ \epsilon &\equiv \epsilon' \pmod{\pi} \\ \zeta &\equiv \zeta' \pmod{\pi} \end{aligned}$$

Finally, we should state one of the reduction arguments on which the proof of theorems I and II is based, and which allows to reprove many of the known results on  $N(\Lambda)$  for arbitrary  $\Lambda$  (for example [3]).

Assume that the R-order  $\Lambda$  in  $A$  is contained in the hereditary R-order  $\Gamma$  in  $A$  and let  $I$  be a full two-sided  $\Gamma$ -ideal contained in  $\Lambda$  such that  $I \subseteq \text{rad} \Gamma \cap \text{rad} \Lambda$ . Put  $\bar{\Lambda} = \Lambda/I$ , and  $\bar{\Gamma} = \Gamma/I$ . We denote by  $\underline{C} = \underline{C}(\Lambda, \Gamma, I)$  the following category. The objects of  $\underline{C}$  consist of  $\bar{\Lambda}$ -monomorphisms  $i: X \rightarrow Y$ , where  $X$  is a finitely generated left  $\bar{\Lambda}$ -module,  $Y$  is a finitely generated projective left  $\bar{\Gamma}$ -module, such that  $\bar{\Gamma} \cdot \text{Im}(i) = Y$ . Morphisms are commutative diagrams  $X \xrightarrow{i} Y$  where  $\alpha$  is a  $\bar{\Lambda}$ -homomorphism, and  $\beta$  a  $\bar{\Gamma}$ -homomorphism.

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \alpha \downarrow & & \downarrow \beta \\ X & \xrightarrow{i'} & Y' \end{array}$$

Theorem III: The category of left  $\Lambda$ -lattices and the category  $\underline{C}(\Lambda, \Gamma, I)$  are representation equivalent.

In the case of Bäckström order  $\Lambda \subseteq \Gamma$ , with  $\text{rad} \Lambda = \text{rad} \Gamma$ , we apply this theorem using  $I = \text{rad} \Lambda$ .

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## A CANONICAL CURVE OF GENUS 6

by

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*Presented by H.S.M. Coxeter, F.R.S.C.*

Del Pezzo's quintic surface  $F$  in 5-dimensional projective space contains 10 lines which fall into 5 quadruples  $q$  of mutually skew lines; each line belongs to two  $q$ . The planes which meet all 4 lines of a  $q$  belong to a regulus and cut  $F$  in the conics of a pencil  $p$  [3, p.226; 1, pp.50, 51].

$F$  admits a group  $G$  of 120 self-projectivities isomorphic to the symmetric group  $S_5$  and imposing all  $5!$  permutations on the  $q$  and on the  $p$ . It is surprising if there is no earlier record of this, and one's surprise is all the greater because Baker, in a paper largely concerned with  $F$ , does obtain, and gives equations for, 9 harmonic inversions all leaving  $F$  invariant [1, p.185]. Of these, 6 are among the 10 inversions in pairs of planes that belong to  $G$ ; the other 3 are among those 15 harmonic inversions which belong to  $G$  and have line and solid as fundamental spaces. If Baker had only chosen a different notation among the several available he might well have recognised the whole of  $G$ .

Each line on  $F$  meets 3 others, and each triad of intersections has a Hessian duad. The 20 points of these duads are the base points of a pencil  $P$  of curves of order 10 on  $F$ , canonical of genus 6 when irreducible and non-singular. There are 3 reducible members  $\Lambda, \Gamma, \Gamma'$  of  $P$ ;  $\Lambda$  consists of the 10 lines themselves while each of  $\Gamma, \Gamma'$

consists of 5 conics, one in each pencil  $p$ .

Every curve of  $P$  is invariant under the 60 projectivities of the subgroup  $A_5$  of  $S_5$ ; every projectivity in the coset of  $A_5$  transposes  $\Gamma$  with  $\Gamma'$  while leaving  $\Lambda$  invariant. So every involution in the coset imposes the same involution  $J$  on  $P$ , pairing its members. Of the two curves in  $P$  fixed under  $J$  one is  $\Lambda$ ; the other is a non-singular irreducible curve  $W$ . One has, therefore, a canonical curve of genus 6, unique on  $F$ ; indeed it is the intersection of  $F$  and the invariant quadric of this irreducible representation of  $S_5$ . Whereas a general curve of genus 6 has 15 moduli  $W$  has none and the specialisation will endow it with properties that may be worth investigating; one thinks especially of the 210 Weierstrass points where the osculating 4-space has (at least) 6-point intersection. Moreover a general canonical curve of genus 6 has  $2^5(2^6 - 1)$  contact primes, i.e. 4-spaces touching it at 5 distinct points; how are the contact primes of  $W$  identified?

It has not been noticed before that  $W$  is the canonical model of the 4-nodal plane sextic shown, by Wiman in 1895, to admit a group of 120 Cremona self-transformations [4, p.208].

$S_5$  has 6 cyclic subgroups of order 5; let  $C_5$  be one of them. Those points that are fixed under all the projectivities of  $C_5$  consist of 4 isolated points and all the points of a line; this line proves to be a chord of  $F$ , so 6 chords of  $F$  occur, one for each subgroup of order 5. The 12 intersections of  $F$  with these chords fall into two hexads, the members of each hexad being nodes on a rational curve in  $P$ .

Every curve in  $P$  other than  $\Lambda, \Gamma, \Gamma'$  and these two rational curves is non-singular.

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Character Formulae for the Orthogonal Groups over GF(2)

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Presented by G. deB. Robinson, F.R.S.C.

1. INTRODUCTION. Given here for characters of levels 1 and 2, verified for level 3, and conjectured for higher levels are a set of explicit formulas, valid for all  $n$ , that yield the absolutely irreducible complex (AIC) characters of the orthogonal groups  $G_n = O_{2n+1}(2)$ ,  $H_n = O_{2n}(2,+)$  and  $K_n = O_{2n}(2,-)$ . The group  $G_n$  is defined by matrices over GF(2) that leave invariant the quadratic form

$$Q_{2n+1}(z) = \sum_{i=1}^n z_{2i-1} z_{2i} + z_{2n+1}^2 \quad (1.1)$$

and  $H_n, K_n$  are its subgroups fixing respectively the linear forms  $z_{2n+1}$  and  $z_{2n-1} + z_{2n} + z_{2n+1}$ . Normal subgroups  $H'_n$  of  $H_n$  and  $K'_n$  of  $K_n$ , of index 2, intersect in a group isomorphic with  $G_{n-1}$ , and have "outer" cosets  $H'_n t_n, K'_n t_n$  containing the transposition  $t_n = (z_{2n-1}, z_{2n})$  in class  $t$ . If we set

$$a_n = 2^{n+1} - 1, \quad b_n = 2^n - 1, \quad c_n = a_n b_n, \quad q_n = c_1 c_2 \cdots c_n \quad (1.2)$$

then  $G_n$  has order  $2^{n+1} q_n$ ,  $l = n^2$ , and the subgroups  $H_n$  and  $K_n$  have indices  $2^{n-1} a_n$  and  $2^{n-1} b_n$  in  $G_n$ . Groups  $K_3, G_3$ , and  $H_4$  are isomorphic respectively with the Weyl group  $E_6$  and the central factor groups of  $E_7$  and  $E_8$ , whose characters are known [1,2].

The characters of  $G_n$  induced by the trivial characters 1 and the alternating characters  $\bar{1}$  of  $H_n$  and  $K_n$  each split into two AIC characters of  $G_n$  [see 3], as follows:

$$1_H^G = Y + 1, \quad 1_K^G = X + 1 \quad (1.3)$$

$$\bar{1}_H^G = V + W, \quad \bar{1}_K^G = U + W \quad (1.4)$$

We call  $U, V, W, X, Y$  the level 1 AIC characters of  $G_n$ , and

assign level 1 to the AIC components (other than 1, I of level 0) of their restrictions to  $H_n$  or  $K_n$ . These form three pairs of associated level 1 characters, with degrees  $b_n b_{n-1}/3$ ,  $2b_n a_{n-2}/3$ ,  $4c_{n-1}/3$  for  $H_n$  and  $a_n a_{n-1}/3$ ,  $2a_n b_{n-2}/3$ ,  $4c_{n-1}/3$  for  $K_n$ .

Level  $k$  AIC characters in each group are defined to be AIC components of a product of  $k \leq n$ , but no fewer, level 1 characters. Each level  $k$  AIC character is contained in exactly one "extended" level  $k$  character, which may be the same (as with  $U$ ,  $V$ ,  $W$  for  $G_n$ ) or may contain other related lower level characters. For example, the extended characters  $X + 1$  of  $G_n$  and the restriction of  $W$  to  $H_n$  or  $K_n$  contain the AIC characters  $X$  and  $W - I$ , respectively, and the symmetrized square  $v^{[2]}$  of  $V$ , of level 2 for  $G_n$ , contains the AIC level 2 character  $v^{[2]} - Y - 1$ . Formulas for all extended characters of levels 1 and 2 are given below.

2. THE BASIC CLASS FUNCTIONS. The formulas

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Y + 1 \\ X + 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V \\ U \\ W \end{bmatrix} = \begin{bmatrix} \bar{\alpha} \\ \bar{\gamma} \\ \bar{\beta} \end{bmatrix} = I \begin{bmatrix} \alpha \\ \gamma \\ \beta \end{bmatrix} \quad (2.1)$$

define a set of basic class functions  $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}$  whose values on each class of  $G_n$  (or  $H_n$  or  $K_n$ ) have been shown to be 1, -1, or 0 times a power of 2 [3]. Values of the class functions  $\alpha_m, \beta_m$ , etc., on class  $C$  are defined to be the values of  $\alpha, \beta$ , etc., on the class of  $m^{\text{th}}$  powers of  $C$ -elements. For a class  $C$  of  $G_n$  represented in  $H_\lambda$  or  $K_\lambda$  but not in  $G_{\lambda-1}$ , the sign  $\sigma$  of  $\beta$  is 1, -1 or 0 according as  $C$  is represented in  $H_\lambda$  but not  $K_\lambda$ ,  $K_\lambda$  but not  $H_\lambda$ , or both. Values of  $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}$  are:

$$\alpha = 2^{2n-\rho}, \quad \bar{\alpha} = I\alpha = (-2)^{2n-\rho}, \quad \rho = \text{rank}(A-I), \quad A \text{ in } C \quad (2.2)$$

$$\beta = \sigma(\alpha_2)^{\frac{1}{2}} = Y - X, \quad \bar{\beta} = I\beta = V - U \quad (2.3)$$

$$\bar{y} = IY = V + U - W = (-2)^{\mu_3} \quad (2.4)$$

For  $j = 3, 5, 7, 9, 15, \dots$ ,  $\mu_j$  denotes the number of indecomposable summands  $A_i$  in a completely decomposed matrix  $S^{-1}AS$ , similar over  $GF(2)$  to a matrix  $A$  of class  $C$  in  $G_n$ , that have primitive  $j^{\text{th}}$  roots of unity as eigenvalues in an extension field of  $GF(2)$ . Additional higher level basic class functions are

$$\varepsilon = I(1-5)^{\mu_5}, \quad \zeta = (1+7)^{\mu_7}, \quad \eta = I(1-9)^{\mu_9} \quad (2.5)$$

The class function  $\zeta$  is  $\beta_2$  or  $-\beta_2$  according as an even or odd number of summands  $A_i$  are of class  $t$  type.

3. THE CHARACTER FORMULAS. Columns of the coefficient matrices in (2.1) are seen to be the AIC characters of  $H_1 \cong S_2$  and  $i_1 \cong S_3$ , respectively. Inverting these character matrices yields

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha/2 \\ \beta/2 \end{bmatrix} = \begin{bmatrix} Y+1 \\ X+1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha/6 \\ \gamma/3 \\ \beta/2 \end{bmatrix} = I \begin{bmatrix} V \\ U \\ W \end{bmatrix} \quad (3.1)$$

Rows of the first coefficient matrix in (3.1) are characters of  $H_1 \cong S_2$ , while those of the second coefficient matrix are characters of  $i_1 \cong S_3$ , and also of  $C_1 \cong S_3$ . Columns on the right of the equations are the level 1 extended characters of  $C_1$  that are respectively positive and negative on class  $t$ , while the latter are also extended level 1 characters of  $H_n$  and  $K_n$ . This structure generalises to level  $k$  characters if we can assign to each class  $C_\lambda$  of  $G_k$  an appropriate level  $k$  basic function. For  $G_2 = S_6$  the following assignment of basic functions  $\rho_\lambda^{(2)}$  works.

$$C(S_6): 1^6 \quad 1^3 3 \quad 1^2 2^2 \quad 2^3 \quad 3^2 \quad 6 \quad 1 \cdot 5 \quad ; \quad 1^4 2 \quad 1 \cdot 2 \cdot 3 \quad 1^2 4 \quad 2 \cdot 4 \quad (3.2)$$

$$\rho_\lambda^{(2)} : \alpha^2 \quad \alpha \quad \beta^2 \quad \alpha_2 \quad \gamma^2 \quad \gamma \quad \varepsilon \quad ; \quad \alpha\beta \quad \gamma\beta \quad \delta \quad \beta_2 \quad (3.3)$$

Class multipliers  $\omega_\lambda^i$  are defined on class  $C_\lambda$  of  $G$  by

$$\omega_\lambda^i(G) = \chi_\lambda^i(G) / |C_\lambda(G)|, \quad \chi_\lambda^i \text{ an AIC character} \quad (3.4)$$

Theorem 3.1. The 16 level 2 AIC characters of  $G_n$  (distinct if  $n \geq 4$ ) are each contained in a level 2 extended character given by

$$\sum_\lambda \phi_\lambda^{(2)} \omega_\lambda^i(H_2) / |H_2|, \quad \bar{i} \sum_\lambda \phi_\lambda^{(2)} \omega_\lambda^i(K_2) / |K_2| \quad (3.5)$$

of which the 9 derived from  $H_2$  are positive and the 7 derived from  $K_2$  are negative on class  $t$  of  $G_n$ .

Theorem 3.2. The 11 level 2 AIC characters of  $F_n$  that are positive on class  $t$  (distinct if  $n \geq 4$ ) are each contained in an extended level 2 character given by

$$\sum_\lambda \phi_\lambda^{(2)} \bar{i}(G_2) \omega_\lambda^i(G_2) / |G_2|, \quad i = 1, 2, \dots, 11 \quad (3.6)$$

Corresponding character formulas for  $H_n$  omit the factor  $\bar{i}(G_2)$ .

The level 2 characters negative on class  $t$  are products of these by the alternating characters  $\bar{i}(K_n)$  or  $\bar{i}(H_n)$  respectively.

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Applications of a Generalized Martin's Axiom

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*Presented by P. Ribenboim, F.R.S.C.*

Ever since the immensely fruitful Martin's Axiom was formulated in [ST], the problem of generalizing it to higher cardinals has been of interest. Recently there have been partial solutions, due to Laver (unpublished), Baumgartner (see [H]), and Shelah [S]. In this note (which partially summarizes [T<sub>2</sub>]) we do not attack the set-theoretical problems inherent in proving the consistency of such generalizations, but rather seek to demonstrate the importance of these generalizations by exhibiting some non-trivial consequences, non-trivial in the sense that they are not straightforward generalizations of standard Martin's Axiom results. As is common these days in set theory, many of these consequences are topological. Since the optimal version of generalized Martin's Axiom does not yet exist, we employ (an equivalent of) Baumgartner's version which is easy to state and surprisingly powerful.

Baumgartner's Axiom: Let  $\mathbb{P} = \langle P, \leq \rangle$  be a partial order which is  $\aleph_1$ -linked (i.e.  $P$  is the union of  $\leq \aleph_1$  compatible (in  $\mathcal{C}$ ) subcollections), and countably compact (a lower bound exists for any countable set whose finite subsets possess lower bounds). Then if  $\mathcal{D}$  is a collection of fewer than  $2^{\aleph_1}$  dense subsets of  $P$ , there is a maximal filter  $G \subseteq P$  which meets each  $D \in \mathcal{D}$ .

Theorem 1 (Baumgartner). Baumgartner's Axiom is consistent with  $2^{\aleph_0} = \aleph_1$  and  $2^{\aleph_1} = \kappa$ , where  $\kappa \geq \aleph_2$  is "anything reasonable".

Clearly  $2^{\aleph_1} = \aleph_2$  implies Baumgartner's Axiom, which is henceforth

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abbreviated as "BA", while its conjunction with the continuum hypothesis will be known as "BACH".

Essentially, what we shall obtain from BACH (plus  $2^{\aleph_1} > \aleph_2$ ) is a variety of useful combinatorial principles. Our first one is

$P_1$ : Let  $\mathcal{A}$  be a proper  $\sigma$ -ideal on a set  $Y$  of cardinality  $\aleph_1$ . If  $\mathcal{A}$  is  $\sigma$ -generated by a set of power less than  $2^{\aleph_1}$ , then there is an  $L \subseteq Y$  which is  $\mathcal{A}$ -Lusin, i.e.  $L$  is uncountable and for each  $I \in \mathcal{A}$ ,  $L \cap I$  is countable.

Theorem 2. BACH implies  $P_1$ ,

To see this, observe that  $P_1$  is equivalent to the natural generalization of the well-known combinatorial principle  $P(c)$ , and then generalize the Martin's Axiom proof of that. There are obvious applications of  $P_1$  to almost disjoint set problems. For a topological application of  $P_1$ , we consider L-spaces, i.e. hereditarily Lindelöf regular nonseparable spaces.

Theorem 3.  $P_1$  plus CH plus  $2^{\aleph_1} > \aleph_2$  implies there is an L-space which does not have a basis of cardinality  $\leq 2^{\aleph_0}$ .

The idea of the proof is that by CH, in the product  $X$  of  $\aleph_2$  copies of the two point discrete space there exists a set  $Y$  of power  $\aleph_1$  which meets every nonempty  $G_\delta$ .  $P_1$  applied to the trace of the first category ideal of  $X$  on  $Y$  yields the required space  $Z$ .

When one looks for examples of countably closed forcing in the literature, one often finds that only  $\aleph_2$  dense sets are involved, and that arguments using the forcing language are not employed. To apply BACH, it thus becomes a question of seeing whether " $\aleph_1$ -linked" is a sufficiently

strong chain condition, and whether "countably closed" can be enhanced to "countably compact". The case of most interest is the existence of a Kurepa tree, and a close analysis of the partial order in question indeed shows that it is  $\aleph_1$ -linked and countably compact. Thus

Theorem 4. BACH plus  $2^{\aleph_1} > \aleph_2$  implies there is a Kurepa tree.

Other direct consequences of BACH can be found in  $[T_2]$  but to obtain our final result, we apparently need more than the axiom BACH, but rather a particular model for it, namely the one iteratively constructed by Baumgartner in his consistency proof. The reason is that this result asserts the existence of certain stationary subsets of  $\omega_1$ . To prove that a (generic) set is stationary, it is necessary to prove it intersects all  $2^{\aleph_1}$  closed unbounded sets. This requires meeting  $2^{\aleph_1}$  dense sets, so a forcing argument appears to be needed.

Definition. An  $\omega_1$ -sequence  $\{x_\alpha : \alpha \in \omega_1\}$  in a space  $X$  is universal if for every  $x \in X$  there is a stationary  $S \subseteq \omega_1$  such that the net  $\{x_\alpha : \alpha \in S\}$  converges to  $x$ .

Theorem 5. There is a model of set theory (namely Baumgartner's) in which BACH plus  $2^{\aleph_1} > \aleph_2$  holds, and in which the product of fewer than  $2^{\aleph_1}$  spaces, each with a universal  $\omega_1$ -sequence, also has one. Hence  $\diamond$  holds.

The idea of the proof is to combine standard countably closed forcing techniques with the Martin's Axiom product arguments used in  $[T_1]$ . The "Hence" follows from

Lemma [G]. The combinatorial principle  $\diamond$  is equivalent to the proposition that there is a universal sequence in the product of  $\aleph_1$  copies of the two point discrete space.

The results on Kurepa trees and  $\diamond$  show that (some) models for Baumgartner's Axiom behave at  $\omega_1$  much like the constructible universe L (although I show in  $[T_2]$  that  $\diamond^*$  fails in Baumgartner's model). On the other hand, no commitment is made as to the size of  $2^{\aleph_1}$ , and this proves useful in the L-space construction, where BACH enables one to diagonalize with weaker hypotheses than would CH, as well as in Theorem 5, which yields generalizations of  $\diamond$  for functions from  $\kappa$  to  $\omega_1$ ,  $\kappa < 2^{\aleph_1}$ . Finally, to exhibit the manifold possible applications of generalizations of Martin's Axiom, let us mention Shelah's result that in his model for a generalized Martin's Axiom, the monadic theory of order on  $\omega_1$  is decidable.

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SUR L'ANNULATION EN 2 DES CLASSES RELATIVES DES  
CORPS ABELIENS  
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*Presented by P. Ribenboim, F.R.S.C.*

INTRODUCTION : Le but de cette note est d'étudier, pour une extension abélienne imaginaire  $L$  de  $\mathbb{Q}$ , l'annulation de la 2-composante du groupe des classes relatives. Soient  $G_L$  le groupe de Galois de  $L/\mathbb{Q}$ ,  $f_L$  le conducteur de  $L$  et  $S_L = (1/f_L) \sum_a (\frac{L}{a})^{-1}$  l'élément de Stickelberger (somme sur  $a$ ,  $1 \leq a \leq f_L$ ,  $(a, f_L) = 1$ ) (se reporter à [2], pp. 39-45, pour les propriétés générales des  $S_L$ ). On désigne par  $\mathfrak{K}(L)$  (resp.  $\mathfrak{K}^-(L), \mathfrak{K}^+(L)$ ) la 2-composante du groupe des classes (resp. relatives, réelles) au sens ordinaire de  $L$ . Alors il est bien connu que l'idéal  $I_2(L) = S_L Z_2[G_L] \cap Z_2[G_L]$  annule les  $Z_2[G_L]$ -modules  $\mathfrak{K}(L)$  et  $\mathfrak{K}^-(L)$ . Il est facile de se convaincre du fait que cet énoncé "global" (i. e. valable pour  $\mathfrak{K}(L)$  en entier) ne rend pas compte d'un certain nombre de phénomènes et que de ce fait, il conduit en général à une annulation moins fine que celle que l'on peut attendre conjecturalement pour  $\mathfrak{K}^-(L)$ . Nous allons montrer que l'on peut espérer, pour  $\mathfrak{K}^-(L)$ , une amélioration par le facteur  $\frac{1}{2}$  de l'annulation classique (sauf pour les classes relatives provenant des sous-corps de  $L$  qui sont de degré puissance de 2 sur  $\mathbb{Q}$ , et pour lesquels elle ne peut être améliorée) : nous démontrons ce fait dans un certain nombre de cas nouveaux. Ce genre de résultat doit permettre de préciser certains problèmes qui se posent lors de l'utilisation de l'idéal de Stickelberger en 2 (cf. [1] et [6] par exemple).

Dans I nous rappelons le principe de la décomposition "semi-simple" de  $\mathfrak{K}(L)$  ; dans II nous énonçons les résultats obtenus et dans III nous donnons un aperçu des démonstrations.

I DECOMPOSITION SEMI-SIMPLE de  $\mathfrak{K}(L), \mathfrak{K}^-(L)$  et  $\mathfrak{K}^+(L)$ . Soit  $L_1$  (resp.  $L_2$ ) le plus grand sous-corps de  $L$  de degré impair (resp. puissance de 2) sur  $\mathbb{Q}$ . Posons  $\Delta_L = \text{Gal}(L/L_2)$  (d'ordre impair) et  $\Gamma_L = \text{Gal}(L/L_1)$  (d'ordre puissance de 2 et qui contient la conjugaison complexe  $\sigma_{-1}$ ). On appelle  $\frac{1}{2}\Delta_L$  l'ensemble des caractères 2-adiques irréductibles de  $\Delta_L$ . Pour tout  $Z_2[G_L]$ -module  $\mathfrak{M}$ , et pour tout  $\varphi \in \frac{1}{2}\Delta_L$ , on appelle  $\mathfrak{M}^\varphi$  la  $\varphi$ -composante de  $\mathfrak{M}$  ( $\mathfrak{M}^\varphi = \mathfrak{M} e_\varphi$ , où  $e_\varphi$  est l'idempotent associé à  $\varphi$ ). On peut considérer  $\mathfrak{M}^\varphi$  comme un  $Z_2[G_L]e_\varphi$ -module donc comme un  $A_\varphi[\Gamma_L]$ -module où  $A_\varphi$  est l'anneau des entiers du corps cyclotomique associé à  $\varphi$  (cf. [2], p. 20). On peut appliquer ceci à  $\mathfrak{K}^\varphi(L)$ , qui est donc un  $A_\varphi[\Gamma_L]$ -module, et aux modules  $\mathfrak{K}^-(L)$  et  $\mathfrak{K}^+(L)/j\mathfrak{K}^+(L)$  (où  $j$  est l'extension à  $L$  des classes réelles) qui sont des  $A_\varphi[\Gamma_L]/(1+\sigma_{-1})$ -modules. L'utilisation de quelques exemples numériques (par exemple :  $L = \mathbb{Q}(\sqrt{-15})\mathbb{Q}_+^{(7)}$  de degré 6) montre que les résultats sont de nature différente selon que  $\varphi = 1$  ou non. Le cas  $\varphi = 1$  pouvant être considéré comme résolu (cf. [4], cas (ii) de la démonstration du théorème III 1),

supposons  $\varphi \neq 1$ ; on a alors dans ce cas deux réductions :

Lemme 1. Pour  $\varphi \neq 1$ , l'application  $j : \mathfrak{H}^{+\varphi}(L) \rightarrow \mathfrak{H}^{\varphi}(L)$ , restreinte à  $\mathfrak{H}^{+\varphi}(L)$  est injective (on identifiera désormais  $\mathfrak{H}^{+\varphi}(L)$  et son image dans  $\mathfrak{H}^{\varphi}(L)$ , dans le cas  $\varphi \neq 1$ ).

Lemme 2. Pour  $\varphi \neq 1$ , l'élément  $S_L^{\varphi} = S_L e_{\varphi}$  est dans  $Z_2[G_L]e_{\varphi}$  et il existe  $T_L^{\varphi} \in Z_2[G_L]e_{\varphi}$  défini modulo  $(1+\sigma_{-1})$ , tel que  $S_L^{\varphi} = T_L^{\varphi}(1-\sigma_{-1})$ .

Remarque. On observera, pour les énoncés qui vont suivre, que les images respectives  $\overline{S_L^{\varphi}}$  et  $\overline{T_L^{\varphi}}$  de  $S_L^{\varphi}$  et  $T_L^{\varphi}$  dans  $A_{\varphi}[\Gamma_L]/(1+\sigma_{-1})$  sont liées par la relation  $\overline{S_L^{\varphi}} = 2\overline{T_L^{\varphi}}$  et que toute annulation par  $\overline{T_L^{\varphi}}$  représente l'amélioration par le facteur  $\frac{1}{2}$  que nous avons évoquée.

II ENONCE DES RESULTATS. On désigne toujours par  $L$  une extension abélienne imaginaire quelconque de sous-corps réel maximal  $L_+$ .

**Théorème 1.** Le module  $\mathfrak{H}^{\varphi}(L)/\mathfrak{H}^{+\varphi}(L)$  est annulé par  $T_L^{\varphi}$ , pour tout  $\varphi \neq 1$ .

En application de ce résultat, on remarque que l'on a, pour  $\varphi \neq 1$ , la suite exacte  $1 \rightarrow \mathfrak{H}^{-\varphi}(L) \rightarrow \mathfrak{H}^{\varphi}(L) \xrightarrow{N_{L/L_+}} \mathfrak{H}^{+\varphi}(L) \rightarrow 1$ , et comme  $j : \mathfrak{H}^{+\varphi}(L) \rightarrow \mathfrak{H}^{+\varphi}(L)$ , il en résulte que  $\mathfrak{H}^{\varphi}(L)/\mathfrak{H}^{+\varphi}(L)$  et  $\mathfrak{H}^{-\varphi}(L)$  ont même ordre. Par conséquent :

**Corollaire.** Si  $\mathfrak{H}^{\varphi}(L)/\mathfrak{H}^{+\varphi}(L)$  et  $\mathfrak{H}^{-\varphi}(L)$  sont isomorphes, alors  $\mathfrak{H}^{-\varphi}(L)$  est annulé par  $T_L^{\varphi}$  ( $\varphi \neq 1$ ). C'est notamment le cas lorsque  $\mathfrak{H}^{+\varphi}(L) = (1)$ .

Remarque. Comme le montre l'exemple du corps de degré 6 de conducteur 4.277, un tel isomorphisme n'a pas toujours lieu (cf. [5], chap. IV, 2).

**Théorème 2.** Si les  $\varphi$ -composantes des 2-groupes des classes au sens ordinaire et restreint de  $L_+$  coïncident, alors  $\mathfrak{H}^{-\varphi}(L)$  est annulé par  $T_L^{\varphi}$  ( $\varphi \neq 1$ ).

Remarque. Cette condition ne portant que sur  $L_+$  et non sur  $L$ , le résultat est vrai pour toute extension quadratique imaginaire de  $L_+$  abélienne sur  $\mathbb{Q}$ .

**Théorème 3.** Le module  $\mathfrak{H}^{-\varphi}(L)$  est annulé par  $T_L^{\varphi}$  dès que la condition suivante est réalisée : il existe au moins une extension quadratique imaginaire

$F$  de  $L_+$  abélienne sur  $\mathbb{Q}$  telle que pour tout  $\xi$  premier ramifié dans  $F/L_+$ ,  $\xi$  ne soit pas totalement décomposé dans le sous-corps de  $L_+$  fixe par le noyau de  $\varphi$ , et pour laquelle on ait la propriété que  $T_F^{\varphi}$  annule  $\mathfrak{H}^{-\varphi}(F)$  ( $\varphi \neq 1$ ).

Remarque. Dans le cas particulier où  $\Gamma_L$  est cyclique,  $A_{\varphi}[\Gamma_L]/(1+\sigma_{-1})$  est isomorphe à un anneau d'entiers de corps cyclotomique, de la forme  $A_{\varphi}$ , associé à un caractère 2-adique irréductible  $\varphi$  de  $G_L$ , et dont  $\varphi$  est le caractère modéré associé (cf. [4], I, 2, b); dans ce cas, l'image de l'élément  $S_L^{\varphi}$  dans  $A_{\varphi}$  est le nombre de Bernoulli généralisé  $B_1(\chi^{-1})$  ( $\chi$  caractère de degré 1 divisant  $\varphi$ ) et celle de  $T_L^{\varphi}$  est  $\frac{1}{2}B_1(\chi^{-1})$ . Dans ce cas, le module  $\mathfrak{H}^{-\varphi}(L)$  est égal à  $\mathfrak{H}_{\varphi}$  que nous avons défini dans [3] et pour lequel nous avons conjecturé l'annulation par  $\frac{1}{2}B_1(\chi^{-1})$  ( $\varphi \neq 1$ ) ([3], IV, 2).

III DEMONSTRATIONS. Le lemme 1 se prouve en considérant la suite exacte de  $N_L$ -modules  $1 \rightarrow \text{Ker } j \rightarrow E^*(L)/E(L)^{1-\sigma-1}$ , où  $E(L)$  (resp.  $E^*(L)$ ) désigne le groupe des unités (resp. de norme 1 sur  $L_+$ ) de  $L$  (cf. [2], p. 23); il suffit alors d'étudier la composante en  $\varphi(\varphi \neq 1)$  de  $E^*(L)/E(L)^{1-\sigma-1}$ : soit  $\epsilon \in E(L)$ ; on sait que  $\epsilon^2$  est de la forme  $\epsilon_0 \xi^2$ , avec  $\epsilon_0$  réelle et  $\xi$  racine de l'unité; donc si  $\epsilon \in E^*(L)$ , on a  $N_{L/L_+} \epsilon^2 = 1$  soit  $\epsilon_0^2 = 1$ , et  $\epsilon$  est une racine de l'unité;  $E^*(L)/E(L)^{1-\sigma-1}$ , qui est un 2-groupe, peut donc être engendré par des racines de l'unité d'ordre une puissance de 2 qui sont nécessairement dans  $L_2$ . La composante en  $\varphi(\varphi \neq 1)$  de  $E^*(L)/E(L)^{1-\sigma-1}$  est donc triviale.

On trouvera la démonstration du lemme 2 et du théorème 1 dans [5] (th. IV 1).

Les théorèmes 2 et 3 proviennent d'un principe commun que nous allons décrire. On suppose partout  $\varphi \neq 1$ , et  $L$  est fixée. Soit  $F$  une extension quadratique imaginaire quelconque de  $L_+$ , abélienne sur  $\Omega$ , et soit  $\hat{\Omega}$  un corps cyclotomique de la forme  $\Omega^{(m)}$  contenant  $L$  et  $F$ . On travaille avec les éléments de Stickelberger modifiés  $\hat{S}$  (cf. [5], IV, 1) ce qui ne change pas les problèmes d'annulation car  $\hat{S}_F e_\varphi = S_F e_\varphi$ . On rappelle que l'on a la relation  $\eta_{\hat{\Omega}/F} \hat{S}_{\hat{\Omega}} = W_{\hat{\Omega}/F} \hat{S}_F$  où  $W_{\hat{\Omega}/F} = \prod_{\xi \in \hat{\Omega}} (1 - (\frac{F}{\xi})^{-1})$  (avec la convention  $(\frac{F}{\xi})^{-1} = 0$  si  $\xi \nmid F$ ) et où  $\eta_{\hat{\Omega}/F}$  est la projection  $\mathbb{Q}[G_{\hat{\Omega}}] \rightarrow \mathbb{Q}[G_F]$  (cf. [2], th. II 3.p. 42). Soit  $h_L$  une classe de  $\mathbb{H}^2(L)$ ; il existe une classe  $H$  de  $\hat{\Omega}$  telle que  $N_{\hat{\Omega}/L} H = h_L$  ([5], lemme IV 5); on peut alors trouver un idéal premier  $P_{\hat{\Omega}}$  de  $\hat{\Omega}$  totalement décomposé dans  $\hat{\Omega}/\mathbb{Q}$  tel que  $P_L = N_{\hat{\Omega}/L} P_{\hat{\Omega}}$  représente  $h_L$  (on applique convenablement le théorème de Tchebotarev). Pour ce  $P_{\hat{\Omega}}$ , on peut appliquer la version modifiée du théorème de Stickelberger ([5], lemme IV 1), à savoir que pour tout  $\omega \in \mathbb{Z}_2[G_{\hat{\Omega}}]$  tel que  $\omega \hat{S}_{\hat{\Omega}}$  soit 2-entier, alors  $P_{\hat{\Omega}}^{\omega \hat{S}_{\hat{\Omega}}} = (z)$  où  $z$  est un élément de  $\hat{\Omega}$  de module 1 ([5], lemme IV 2). En considérant  $\varphi$  comme caractère de  $\Delta_{\hat{\Omega}}$ , on applique ceci à  $\omega = e_\varphi$  (de façon plus correcte à un représentant de  $e_\varphi$  dans  $\mathbb{Z}[G_{\hat{\Omega}}]$  modulo une puissance de 2 convenable) et alors on peut écrire  $P_{\hat{\Omega}}^{\varphi(1-\sigma-1)} = (z)$ ; or  $z$  est de la forme  $\mu^{1-\sigma-1}$ ,  $\mu \in \hat{\Omega}$ , ce qui donne  $(P_{\hat{\Omega}}^{\varphi} / (\mu))^{1-\sigma-1} = (1)$ . Comme l'extension  $\hat{\Omega}$  est non ramifiée sur  $L_+$  pour les idéaux ou alors est de la forme  $\Omega^{(\ell^N)}$  ( $\ell$  premier), il en résulte qu'il existe un idéal  $\mathfrak{u}_+$  de  $\hat{\Omega}$  dont l'étendu  $(\mathfrak{u}_+)$  à  $\hat{\Omega}$  conduit à la relation:  $P_{\hat{\Omega}}^{\varphi} = (\theta)(\mathfrak{u}_+)$  dans  $\hat{\Omega}$ , avec  $\theta \in \hat{\Omega}$ . Ceci conduit aux deux relations (où  $P_L = N_{\hat{\Omega}/L} P_{\hat{\Omega}}$  et  $P_F = N_{\hat{\Omega}/F} P_{\hat{\Omega}}$ ):  $P_L^{\varphi} / L^{\varphi} = (N_{\hat{\Omega}/L} \theta) N_{\hat{\Omega}_+/L_+} \mathfrak{u}_+$  et  $P_F^{\varphi} / F^{\varphi} = (N_{\hat{\Omega}/F} \theta) N_{\hat{\Omega}_+/L_+} \mathfrak{u}_+$ , car  $N_{\hat{\Omega}/L}(\mathfrak{u}_+) = N_{\hat{\Omega}_+/L_+} \mathfrak{u}_+$  et  $N_{\hat{\Omega}/F}(\mathfrak{u}_+) = N_{\hat{\Omega}_+/F_+} \mathfrak{u}_+ = N_{\hat{\Omega}_+/L_+} \mathfrak{u}_+$  puisque  $F_+ = L_+$ . Appelons  $\mathfrak{q}_+ = N_{\hat{\Omega}_+/L_+} \mathfrak{u}_+$  cet idéal commun. On vérifie ensuite facilement que

l'on a  $n_{\Omega/L} T_{\Omega}^{\varphi} \equiv T_{\Omega}^{\varphi} W_{\Omega/L} \pmod{(1+\sigma_{-1})}$  et  $n_{\Omega/F} T_{\Omega}^{\varphi} \equiv T_{\Omega}^{\varphi} W_{\Omega/F} \pmod{(1+\sigma_{-1})}$ .

Supposons maintenant que  $h_L$  soit une classe relative dans  $L$  (i.e.

$h_L \in \mathfrak{K}^{-\varphi}(L)$ ); alors, dans  $F$ , la classe  $h_F$  de  $P_F$  est aussi une classe relative dans  $F/L_+$  et on peut supposer que  $h_F \in \mathfrak{K}^{-\varphi}(F)$  quitte à remplacer  $h_F$  par  $h_F^{e_{\varphi}}$ . Si on appelle  $h_+$  la classe dans  $L_+$  de  $G_+$ , c'est un élément de  $\mathfrak{K}^{+\varphi}(L)$  et on peut écrire :

$$h_L T_{L}^{\varphi} W_{\Omega/L} = h_+ \text{ et } h_F T_{F}^{\varphi} W_{\Omega/F} = h_+, \text{ puisque } h_L^{1+\sigma_{-1}} = h_F^{1+\sigma_{-1}} = 1.$$

A ce stade, on peut démontrer le théorème 3 : Soit  $F$  vérifiant les propriétés requises ; on a alors  $h_F = 1$ , donc à fortiori  $h_F T_{F}^{\varphi} W_{\Omega/F} = 1$  soit  $h_+ = 1$  soit  $h_L T_{L}^{\varphi} W_{\Omega/L} = 1$ . Il ne reste qu'à constater que l'hypothèse sur les corps de décomposition des idéaux ramifiés dans  $F/L_+$  entraîne précisément que  $W_{\Omega/L} e_{\varphi}$  est inversible (à condition de prendre ici pour  $m$  le conducteur de  $FL$ ). D'où le résultat.

Remarque. En pratique, il est facile de trouver de tels  $F$  : il suffit de prendre pour  $F$  l'extension  $L_+(\sqrt{\varphi})$  avec  $\varphi$  premier,  $\varphi \equiv 3(4)$ , vérifiant  $\forall (\varrho) \neq 1$  ( $\varphi$ ! de degré 1 divisant  $\varphi$ ) (ce qui est possible par le théorème de Tchebotarev).

Pour démontrer le théorème 2, choisissons un tel corps  $F$  pour lequel les premiers  $\varrho$  ramifiés dans  $F/L_+$  vérifient  $\forall (\varrho) \neq 1$ . La formule des  $\varphi$ -classes ambiges ([4], th. II 1, 2 et 3) appliquée à l'extension  $F/L_+$  montre que  $|\mathfrak{K}^{\varphi}(F/L_+)|$  est égal à l'ordre du groupe des  $\varphi$ -classes au sens restreint de  $L_+$ ; si l'on suppose ce nombre égal à  $|\mathfrak{K}^{\varphi}(L_+)|$ , il en résulte l'égalité  $\mathfrak{K}^{-\varphi}(F) = (\mathfrak{K}^{\varphi}(F))^{1-\sigma-1}$  qui permet d'écrire  $h_F$  sous la forme  $h_F = h^{1-\sigma-1}$ ,  $h \in \mathfrak{K}^{\varphi}(F)$ ; puisque  $S_F^{\varphi}$  annule  $\mathfrak{K}^{\varphi}(F)$  en entier, on obtient  $h_F T_{F}^{\varphi} T_{F}^{\varphi}(1-\sigma_{-1}) S_F^{\varphi} = 1$ , d'où encore  $h_F T_{F}^{\varphi} W_{\Omega/F} = 1$  soit  $h_+ = 1$  et, comme précédemment,  $h_L T_{L}^{\varphi} = 1$ .

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## NOMBRES TRANSCENDANTS ET FONCTIONS SIGMA DE WEIERSTRASS

Note de Michel WALDSCHMIDT.

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INTRODUCTION. Les théorèmes de Schneider [S] (chap. II, § 4) sur les fonctions  $\sigma$  et  $\sigma$  de Weierstrass peuvent être interprétés en termes de points algébriques sur une courbe elliptique ou sur l'extension d'une courbe elliptique par le groupe additif (ces groupes algébriques correspondent aux intégrales elliptiques de première ou deuxième espèce). On peut alors les déduire d'un théorème de Lang [L] (chap. III, § 4, th. 4). Si on applique ce même théorème de Lang à l'extension d'une courbe elliptique par le groupe multiplicatif (correspondant à certaines intégrales elliptiques de troisième espèce), dont la description a été donnée par Serre (cf. [A]), on obtient un énoncé de transcendance sur les valeurs de fonctions sigma [A] (chap. III).

Nous démontrons ici cet énoncé directement à partir du critère de transcendance de Schneider Lang, sans faire intervenir de groupe algébrique.

## § 1. FONCTIONS SIGMA DE WEIERSTRASS.

Soit  $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  un réseau de  $\mathbb{C}$ . Le produit canonique de Weierstrass associé à  $L$  est la fonction entière

$$\sigma(z) = z \prod_{\omega \in L, \omega \neq 0} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

Avec les notations habituelles, elle vérifie

$$\sigma(z + \omega_i) = -\sigma(z) \exp\left\{\eta_i \left(z + \frac{\omega_i}{2}\right)\right\}, \quad (i = 1, 2)$$

et

$$\sigma(mz) = (-1)^{m-1} \sigma(z)^m \psi_m^2(\mathcal{P}(z), \mathcal{P}'(z)), \quad (m \in \mathbb{Z}, m > 0),$$

où  $\psi_m(X, Y)$  est une fonction rationnelle de  $X, Y$  à coefficients dans  $\mathbb{Q}(\mathcal{E}_2, \mathcal{E}_3)$ . (voir par exemple [F], p. 205). On en déduit que, pour  $p$  et  $q$  entiers positifs,  $\omega \in L$  et

$\eta = \zeta(z + \omega) - \zeta(z)$ , le nombre

$$\sigma\left(\frac{p}{q}\omega\right) \exp\left(-\frac{p^2}{2q^2}\eta\omega\right)$$

est algébrique sur le corps  $\mathbb{Q}(\mathcal{E}_2, \mathcal{E}_3)$ . Par exemple

$$\sigma\left(\frac{\omega_1}{2}\right)^8 = -e^{\eta_1 \omega_1 / \psi_3(\mathcal{P}(\frac{\omega_1}{2}))}.$$

Dans le cas particulier  $L = \mathbb{Z} + \mathbb{Z}i$ , on vérifie que

$$\sigma\left(\frac{1}{2}\right) = 2^{5/4} \cdot \pi^{1/2} \cdot e^{\pi/8} \cdot \Gamma(1/4)^{-2}.$$

Enfin, pour  $u \in \mathbb{C}$ ,  $u \notin L$ , le nombre

$$\frac{\sigma(u + (\omega/2))}{\sigma(u)} \cdot \exp\left\{-\eta\left(\frac{u}{2} + \frac{\omega}{8}\right)\right\}$$

est algébrique sur le corps  $\mathbb{Q}(\mathcal{E}_2, \mathcal{E}_3, \mathcal{Y}^o(u))$ .

## § 2. UNE FONCTION MULTIPLICATIVEMENT PSEUDO-PÉRIODIQUE.

Soit  $u_0 \in \mathbb{C}$ ,  $u_0 \notin L$ . La fonction méromorphe

$$F(z) = \frac{\sigma(z + u_0)}{\sigma(z) \sigma(u_0)} e^{-\sum (u_0)z},$$

vérifie pour  $\omega \in L$

$$F(z + \omega) = F(z) \exp\left\{\eta u_0 - \omega \sum (u_0)\right\}.$$

Sa dérivée logarithmique est

$$\frac{F'(z)}{F(z)} = \frac{1}{2} \frac{\mathcal{Y}''(z) - \mathcal{Y}''(u_0)}{\mathcal{Y}'(z) - \mathcal{Y}'(u_0)}.$$

La fonction des trois variables  $u_0, u_1, u_2$

$$\frac{F(u_1 + u_2)}{F(u_1) \cdot F(u_2)} = \frac{\sigma(u_0 + u_1 + u_2) \sigma(u_0) \sigma(u_1) \sigma(u_2)}{\sigma(u_0 + u_1) \sigma(u_1 + u_2) \sigma(u_2 + u_0)}$$

est une fonction rationnelle de  $\mathcal{Y}^o(u_0), \mathcal{Y}^o(u_1), \mathcal{Y}^o(u_2), \mathcal{Y}'(u_0), \mathcal{Y}'(u_1), \mathcal{Y}'(u_2)$ , à coefficients dans  $\mathbb{Q}(\mathcal{E}_2, \mathcal{E}_3)$ .

Soit  $B_m(X) \in \mathbb{Q}(\mathcal{E}_2, \mathcal{E}_3)[X]$  le polynôme défini, pour  $m$  entier positif, par

$$B_m(\mathcal{Y}^o(z)) = \left[\psi_m(\mathcal{Y}^o(z), \mathcal{Y}'(z))\right]^2.$$

De la relation

$$\sum (mz) = m \sum (z) + \mathcal{Y}^o(z) \frac{B_m'(\mathcal{Y}^o(z))}{2m B_m(\mathcal{Y}^o(z))},$$

on déduit que, si  $u_0$  est un point de torsion, il existe un nombre  $\beta_0$ , algébrique sur le corps  $\mathbb{Q}(\mathcal{E}_2, \mathcal{E}_3, \mathcal{Y}^o(u_0))$ , tel que la fonction

$$F(z) e^{\beta_0 z}$$

soit algébrique sur le corps  $\mathbb{Q}(\mathcal{E}_2, \mathcal{E}_3, \mathcal{Y}^o(u_0), \mathcal{Y}^o(z))$ .

## § 3. LE THÉORÈME DE TRANSCENDANCE.

On suppose  $\varepsilon_2, \varepsilon_3, \wp(u_0)$  algébriques. On se propose d'appliquer le critère de Schneider Lang [L] (chap. III, § 1, th. 1) aux fonctions

$$\wp(z), F(z)e^{\beta z}, \wp'(z), 1/(\wp(z) - \wp(u_0)),$$

avec  $\beta \in \bar{\mathbb{Q}}$ . Si la fonction  $F(z)e^{\beta z}$  est algébrique sur  $\bar{\mathbb{Q}}(\wp(z))$ , on déduit de la pseudo-périodicité de  $F$  et de la relation de Legendre que  $u_0$  est un point de torsion. Comme le théorème de transcendance correspondant à  $\wp(z), e^z, \wp'(z)$  est connu [S] (chap. II, th. 18), on supposera que  $u_0$  n'est pas de torsion.

Soit  $u \in \mathbb{C}$ , avec  $u \notin L, u + u_0 \notin L$ . Supposons pour commencer  $u - u_0 \notin L$ . Comme il y a une infinité de  $m \in \mathbb{Z}$  tels que  $m u \notin L, m u - u_0 \notin L, m u + u_0 \notin L$ , on obtient la transcendance de l'un au moins des 2 nombres

$$\wp(u), F(u)e^{\beta u}.$$

Choisissons  $u$  tel que  $2u - u_0 \in L$  (donc  $u \notin L, u \pm u_0 \notin L$ ) et  $\wp(u) \in \bar{\mathbb{Q}}$ .

Après un calcul facile on en déduit la transcendance du nombre

$$\sigma(u_0)^2 \exp\{\eta u_0 + (u_0 + \omega)(\beta - \zeta(u_0))\},$$

ce qui montre que la restriction  $u - u_0 \notin L$  était superflue. On a ainsi démontré le résultat suivant

THÉORÈME. - Soient  $\wp$  une fonction elliptique de Weierstrass d'invariants  $\varepsilon_2, \varepsilon_3$  algébriques,  $u, u_0$  deux points algébriques de  $\wp$ , et  $\beta$  un nombre algébrique. On suppose que  $u_0$  n'est pas un point de torsion, et que  $u$  et  $u + u_0$  ne sont pas pôles de  $\wp$ . Alors le nombre

$$\frac{\sigma(u + u_0)}{\sigma(u)\sigma(u_0)} e^{(\beta - \zeta(u_0))u}$$

est transcendant.

## § 4. COROLLAIRES

Le premier corollaire montre que la dérivée de  $F$  prend des valeurs transcendentes aux points où  $F$  s'annule.

COROLLAIRE 1. - Soient  $\omega \in L, \eta = \zeta(z + \omega) - \zeta(z)$ . Alors le nombre

$$\sigma(u_0)^2 \exp\{\eta u_0 + (u_0 + \omega)(\beta - \zeta(u_0))\}$$

est transcendant.

Ainsi,  $u_0$  étant toujours un point algébrique de  $\mathcal{V}$  qui n'est pas de torsion, le nombre

$$\sigma(u_0)e^{-\frac{1}{2}u_0\zeta(u_0)}$$

est transcendant.

Enfin, en choisissant pour  $u$  dans le théorème un quotient de  $\omega$  par une puissance de 2, on obtient la transcendance des pseudo périodes de  $F(z)e^{\beta z}$ .

COROLLAIRE 2. - Avec les notations du corollaire 1, pour  $\omega \neq 0$  le nombre

$$\exp\{\omega\zeta(u_0) - \gamma u_0 + \beta\omega\}$$

est transcendant.

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## AN EXTENSION OF WILSON'S THEOREM

by

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*Presented by P. Ribenboim, F.R.S.C.*

The aim of this note is to prove the following extension of Wilson's theorem conjectured by W. Snyder in his Ph.D. thesis "A concept of Bernoulli numbers in algebraic function fields", Univ. of Maryland 1977. Snyder has found interesting applications of his conjecture to differentials in rings of characteristic  $p$ .

**THEOREM.** For any prime  $p$  and any residues  $x_i \pmod p$  we have

$$(1) \sum_{\sigma \in S_{p-1}} x_{\sigma(1)}^{x_{\sigma(1)} + x_{\sigma(2)}} \dots (x_{\sigma(1)} + \dots + x_{\sigma(p-1)}) \\ \equiv (x_1 + \dots + x_{p-1})^{p-1} \pmod p,$$

where the summation is taken over all permutations  $\sigma$  of  $\{1, 2, \dots, p-1\}$ .

Let for positive integers  $a_1, \dots, a_r$   $C(a_1, \dots, a_r)$  denote the coefficient of  $X = \prod_{i=1}^r x_i^{a_i}$  in the sum  $\sum_{\sigma \in S_n} P_{\sigma}$ , where  $n = a_1 + \dots + a_r$  and

$$P_{\sigma} = x_{\sigma(1)}^{x_{\sigma(1)} + x_{\sigma(2)}} \dots (x_{\sigma(1)} + \dots + x_{\sigma(n)}). \quad (\sigma \in S_n).$$

**LEMMA 1.** Let  $a_j > 1$  for  $j \leq s$ ,  $a_j = 1$  for  $s < j \leq r$ ,  $\sum_{i=1}^r a_i = n$ . Then

$$C(a_1, \dots, a_r) = (n-r) \sum_{i=1}^s C(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_r) \\ + (n-r+1)(r-s) C(a_1, \dots, a_{r-1}).$$

**PROOF.** We have

$$\sum_{\sigma \in S_n} P_{\sigma} = \sum_{j=1}^n \sum_{\sigma \in S_n, \sigma(n)=j} P_{\sigma} = \sum_{j=1}^n E_j.$$

The coefficient of  $X$  in  $P_{\sigma}$  is the same as in  $P_{\tau\sigma}$  where  $\tau \in S_n$  is any permutation stable on  $\{1, 2, \dots, s\}$  and fixing the set  $\{s+1, \dots, r\}$ . Hence the coefficient of  $X$  in  $E_j$  is 0 if  $j \leq s$ , is equal to the coefficient  $C_1$  of  $X$  in  $E_r$  if  $s < j \leq r$  and equal to the coefficient  $C_2$  of  $X$  in

$E_n = (x_1 + \dots + x_n) \sum_{\sigma \in S_{n-1}} P_{\sigma}$  if  $j > r$ . If  $r > s$ ,  $C_1$  is equal to  $C(a_1, \dots, a_{r-1})$ . On the other hand

$$C_2 = \sum_{i=1}^s C(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_r) + \sum_{i=s+1}^r C(a_1, \dots, a_{r-1}).$$

Hence

$$C(a_1, \dots, a_r) = \sum_{j=s+1}^r C_1 + \sum_{j=r+1}^n C_2 = (n-r) \sum_{i=1}^s C(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_r) + (n-r+1)(r-s) C(a_1, \dots, a_{r-1}).$$

In order to evaluate  $C(a_1, \dots, a_r)$  we introduce the following notation valid for all systems of  $r \geq b \geq a \geq 0$  positive real numbers  $a_i$ :  $R = \{1, 2, \dots, r\}$ ,

$$S_1(a, b, r, q) = \sum_{i=a+1}^b a_i \sum_{k=1}^r \frac{(|T_k|+1)!}{A(T_k) + a_i} \prod_{j=1, j \neq k}^q \frac{|T_j|!}{A(T_j) + 1},$$

$$S_2(a, b, r, q) = \sum_{i=a+1}^b \sum_{j=1}^q \frac{|T_j|!}{A(T_j) + 1},$$

where  $1 \leq q < r$  and the inner summation  $\sum_i^*$  in both sums is taken over all partitions (the order of summands neglected) of  $R - \{i\}$  into  $q$  non-empty sets  $T_j$  of cardinality  $|T_j|$  and  $A(T_j) = \sum_{1 \in T_j} a_1$ . Moreover we set

$$S_1(a, b, r, r) = 0, \quad S_2(a, b, r, 0) = 0 \quad (r \geq 2), \quad S_2(0, 0, 1, 0) = 0, \quad S_2(0, 1, 1, 0) = 1.$$

LEMMA 2. For any positive  $q < r$  the following identity holds

$$S_1(0, r, r, q) + S_2(0, r, r, q-1) = (A(R) + q) \sum_{j=1}^q \frac{|R_j|!}{A(R_j) + 1},$$

where  $\sum^{**}$  is taken over all partitions of  $R$  into  $q$  non-empty sets  $R_j$ .

PROOF. For  $q=r=1$  the identity holds trivially. For  $q=1$ ,  $r \geq 2$  we have

$$S_1(0, r, r, 1) + S_2(0, r, r, 0) = \sum_{i=1}^r a_i \frac{r!}{A(R)} = r! = (A(R) + 1) \frac{r!}{A(R) + 1}.$$

For  $q \geq 2$  we group together all terms in  $S_1(0, r, r, q)$  in which  $T_k \cup \{i\} = T$ . For any  $i, k$  we get  $|T| \geq 2$ . On the other hand for any  $T \subset R$  with  $|T| \geq 2$  we have

$$\begin{aligned} & \sum_{i=1}^r a_i \sum_{k=1, T_k \cup \{i\} = T}^q \frac{(|T_k|+1)!}{A(T_k) + a_i} \prod_{j=1, j \neq k}^q \frac{|T_j|!}{A(T_j) + 1} \\ &= \sum_{i \in T} a_i \sum_{T}^* \frac{|T|!}{A(T)} \prod_{j=1}^{q-1} \frac{|T_j|!}{A(T_j) + 1} = \sum_T^* |T|! \prod_{j=1}^{q-1} \frac{|T_j|!}{A(T_j) + 1}, \end{aligned}$$

where  $\sum_T^*$  is taken over all partitions of  $R - T$  into  $q-1$  non-empty sets  $T_j$ .

Hence

$$S_1(O, r, r, q) = \sum_{T \subset R, |T| \geq 2} \sum_{j=1}^{q-1} |T|! \frac{|T_j|!}{A(T_j)+1}.$$

Now setting in  $S_2(O, r, r, q-1)$ ,  $\{i\} = T$  we get

$$S_2(O, r, r, q-1) = \sum_{T \subset R, |T|=1} \sum_{j=1}^{q-1} |T|! \frac{|T_j|!}{A(T_j)+1}$$

thus

$$\begin{aligned} S_1(O, r, r, q) + S_2(O, r, r, q-1) &= \sum_{j=1}^q \frac{|R_j|!}{A(R_j)+1} \sum_{j=1}^q (A(R_j)+1) \\ &= (A(R)+q) \sum_{j=1}^q \frac{|R_j|!}{A(R_j)+1}. \end{aligned}$$

LEMMA 3. For any positive integers  $r, a_1, \dots, a_r$  with  $a_1 + \dots + a_r = n$  we have

$$(2) \quad C(a_1, \dots, a_r) = \frac{(n-r)!}{a_1! \dots a_r!} \sum_{q=1}^r (-1)^{r-q} (n+q)! \sum_{j=1}^q \frac{|R_j|!}{A(R_j)+1},$$

the inner sum being taken over all partitions of  $R$  into  $q$  non-empty subsets  $R_j$ .

PROOF by induction on  $n$ . For  $n=1$  the lemma holds trivially.

Assume that it is true for all sequences  $a'_i$  satisfying  $\sum_{i=1}^r a'_i = n-1$  and consider a sequence  $a_i$  with  $\sum_{i=1}^r a_i = n \geq 2$ . In view of symmetry we may assume that  $a_j > 1$  for  $j \leq s$ ,  $a_j = 1$  for  $j > s$ . Let us denote the right hand side of (2) by  $D(a_1, \dots, a_r)$ . By Lemma 1 and the inductive assumption we have

$$\begin{aligned} C(a_1, \dots, a_r) &= \frac{(n-r)!}{a_1! \dots a_r!} \sum_{q=1}^r (-1)^{r-q} (n+q-1)! (S_1(O, s, r, q) \\ &+ S_2(O, s, r, q-1)) + \frac{(n-r+1)!}{a_1! \dots a_r!} \sum_{q=1}^{r-1} (-1)^{r-q-1} (n+q-1)! S_2(s, r, r, q). \end{aligned}$$

On the other hand, by Lemma 2

$$D(a_1, \dots, a_r) = \frac{(n-r)!}{a_1! \dots a_r!} \sum_{q=1}^r (-1)^{r-q} (n+q-1)! (S_1(O, r, r, q) + S_2(O, r, r, q-1))$$

hence

$$\begin{aligned} \frac{a_1! \dots a_r!}{(n-r)!} (D(a_1, \dots, a_r) - C(a_1, \dots, a_r)) &= (n+r-1)! S_1(s, r, r, r) \\ &+ \sum_{q=1}^{r-1} (-1)^{r-q} (n+q-1)! (S_1(s, r, r, q) - (r+q-1) S_2(s, r, r, q)) \\ &+ (-1)^{r-1} n! S_2(s, r, r, 0). \end{aligned}$$

However  $S_1(s, r, r, r) = 0$ ,

$$S_1(s, r, r, q) = \sum_{i=s+1}^r \sum_{k=1}^i \frac{(|T_k|+1)!}{\Lambda(T_k)+1} \prod_{j=1, j \neq k}^q \frac{|T_j|!}{\Lambda(T_j)+1}$$

$$= \sum_{i=s+1}^r \sum_{k=1}^i \frac{(|T_k|+1)!}{\Lambda(T_k)+1} \prod_{j=1}^q \frac{|T_j|!}{\Lambda(T_j)+1} = (r+q-1) S_2(s, r, r, q)$$

and since  $r \geq 2$  or  $s \geq 1$   $S_2(s, r, r, 0) = 0$ . This gives

$$D(a_1, \dots, a_r) = C(a_1, \dots, a_r).$$

PROOF of the theorem. Since both sides of the congruence (1) are symmetric it is enough to show that  $a_1 + \dots + a_r = n - p - 1$  implies

$$C(a_1, \dots, a_r) \equiv \frac{(p-1)!}{a_1! \dots a_r!} \pmod{p}.$$

Now in formula (2) terms corresponding to  $q > 1$  are divisible by  $p$  since  $(n+q)! \equiv 0 \pmod{p}$  and  $\Lambda(R_j) + 1 < \Lambda(R) + 1 = p$ . Hence

$$C(a_1, \dots, a_r) \equiv \frac{(n-r)!}{a_1! \dots a_r!} (-1)^{r-1} \frac{(n+1)! r!}{n+1}$$

$$\equiv \frac{(p-1)!}{a_1! \dots a_r!} (-1)^{r-1} (p-r-1)! r! \equiv \frac{(p-1)!}{a_1! \dots a_r!} (p-2)!$$

and (3) follows from Wilson's theorem.

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TWO VOLUME DECREASING THEOREMS AND A NEW PROOF  
OF THE HOPF CONJECTURE FOR SURFACES OF GENUS ZERO

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1. INTRODUCTION. In obtaining a Schwarz lemma for a class of real harmonic mappings a scalar invariant  $C$  of the mapping was introduced by Chern and the author [1]. The assumption  $C \leq 0$  was essential in obtaining a volume decreasing result. It is our main purpose to announce Schwarz type lemmas for two other classes of mappings, viz., the projective and totally umbilical mappings. (Details will appear elsewhere.) Here, no assumption on the sign of  $C$  is made, for,  $C \geq 0$  for these classes.

Another situation in which  $C$  is nonnegative is given by the Gauss map of a surface immersed in  $E^3$  with constant mean curvature. It turns out that the invariant  $C$  is the obstruction to the resolution of the Hopf conjecture, viz., A closed orientable surface  $M$  immersed in  $E^3$  with constant mean curvature is isometric with a sphere. A recent result of Eells and Wood [3] allows us to conclude that if the genus of  $M$  is zero, then  $C$  vanishes and  $M$  is isometric with a sphere (cf. reference [8]).

2. THE ACCELERATION MAP. Let  $f : M \rightarrow N$  be a smooth mapping between smooth Riemannian manifolds of dimensions  $m$  and  $n$ , respectively. Denote by  $F$  the section of  $T^*(M) \otimes T^*(M) \otimes T(N)$  defined by

$$F(X, Y) = \tilde{\nabla}_{f_* X} f_* Y - f_* \tilde{\nabla}_X Y,$$

where  $X, Y \in T(M)$  and  $\tilde{\nabla}$  (resp.  $\tilde{\nabla}$ ) denotes the Riemannian connection on  $M$  (resp.  $N$ ).  $F$  is related to the acceleration map introduced in [1], and it is symmetric in  $X$  and  $Y$ . The mapping  $f$  is said to be harmonic if

trace  $F = 0$  (see [2]). We shall say that  $f$  is totally umbilical if  $F = \lambda g$ , where  $g$  is the metric tensor of  $M$ . It turns out that  $\lambda = \tau(f)/m$ , where  $\tau(f)$  is the tension field of  $f$ . If a path in  $(M, \nabla)$  is mapped into a path in  $(N, \tilde{\nabla})$ ,  $f$  is said to be a projective mapping [7].  $f$  is projective iff there is a smooth 1-form  $\sigma$  on  $M$  such that

$$F(X, Y) = \sigma(X)f_*Y + \sigma(Y)f_*X.$$

(Here,  $\nabla$  and  $\tilde{\nabla}$  may be taken to be any symmetric linear connections.)

3. VOLUME DECREASING THEOREMS. Let  $m = n$  and denote by  $(a_{\alpha i})$  the matrix of the mapping  $f_*$  (see [1] for notations). Then,  $A = \det(a_{\alpha i})$  is the ratio of the volume elements of  $M$  and  $N$ . Let  $(B_{i\alpha})$  be the adjoint of  $(a_{\alpha i})$  and put  $u = A^2$ . From formula (56) in [1]

$$(1) \quad \frac{1}{2} \Delta u = 2 \sum_j A_j^2 - C + A \sum_{i\alpha} B_{i\alpha} a_{\alpha k k i} + u(R - \sum_{\beta\gamma} \tilde{R}_{\beta\gamma}^{\alpha} a_{\beta k} a_{\gamma k})$$

where  $C_{kij} = \sum_{\alpha} B_{\alpha k} a_{\alpha i j}$  and  $C = \sum C_{kij} C_{ikj}$ .

For a totally umbilical mapping,  $C = \sum_j A_j^2$  and

$$(2) \quad \frac{n-1}{2} \Delta u \geq -u(R - \sum_{\beta\gamma} \tilde{R}_{\beta\gamma}^{\alpha} a_{\beta k} a_{\gamma k}),$$

whereas for a projective mapping,  $C = (n+3)u \sum_j \sigma_j^2$  and

$$(3) \quad \frac{n-1}{2(n+1)} \Delta u \geq u(R - \sum_{\beta\gamma} \tilde{R}_{\beta\gamma}^{\alpha} a_{\beta k} a_{\gamma k}).$$

The volume decreasing theorems are derived from (2) and (3) using

Lemma 1 [4]. Let  $M$  be a complete Riemannian manifold with Ricci curvature bounded below, and let  $u$  be a  $C^2$  function on  $M$ . Then, for any constants  $\alpha > 0$  and  $\delta < \sup u$ , there exists a sequence  $\{p_\nu\}$  in  $M$  such that

$$\lim_{\nu \rightarrow \infty} |u(p_\nu) - \delta |^{-1-\alpha} \| du(p_\nu) \| = 0,$$

$$\lim_{\nu \rightarrow \infty} |u(p_\nu) - \delta |^{-1-2\alpha} \Delta u(p_\nu) \leq 0,$$

$$\lim_{\nu \rightarrow \infty} u(p_\nu) = \sup u.$$

Theorem 2. Let  $M$  be a complete Riemannian manifold of dimension  $n$  with Ricci curvature bounded below, and let  $N$  be a Riemannian manifold of the same dimension whose Ricci curvature is bounded below by a positive constant  $S/n$ . If  $f : M \rightarrow N$  is a totally umbilical mapping and the scalar curvature of  $M$  is bounded above by  $S$ , then either  $f$  is totally degenerate i.e.,  $u \equiv 0$ , or else it is volume decreasing.

Theorem 3. Let  $M$  be a complete Riemannian manifold with Ricci curvature bounded below by a constant  $-B_1$ , and let  $N$  be a Riemannian manifold of the same dimension whose Ricci curvature is bounded above by a constant  $-B_2 < 0$ . If  $f : M \rightarrow N$  is a projective mapping, then either  $f$  is totally degenerate or  $B_1 > 0$  and  $f$  is volume decreasing up to a constant.

By computing  $\Delta \| f_* \| ^2$  and applying Lemma 1 a distance decreasing theorem for totally umbilical mappings of bounded dilatation (see [5]) may be obtained. Theorem 2 was obtained by Har'El [6] using other means.

Conjecture. Let  $f$  be a totally umbilical mapping. Then, the integral manifolds of the distribution  $E : M \rightarrow \cup_{p \in M} G(T_p)$ , where  $E_p = \{v \in T_p M \mid f_* v \perp \tau(f)_{f(p)}\}$  and  $G(T_p)$  is the Grassmann manifold of  $(m-1)$ -dimensional subspaces of  $T_p M$ , are mapped umbilically by  $f$ .

4. THE HOPF CONJECTURE. Let  $M$  and  $N$  be almost Hermitian manifolds with almost complex structures  $J$  and  $\tilde{J}$ , respectively. If  $f_* \circ J = \pm \tilde{J} \circ f_*$ , then  $f$  is said to be ± holomorphic. It is easy to check

Lemma 5. Let  $f : M \rightarrow N$  be a  $\pm$  holomorphic mapping between equi-dimensional Kaehler manifolds. Then,  $f$  is harmonic and  $C = 0$ .

Let  $\chi(M) = 2 - 2p$  denote the Euler characteristic of  $M$ , and let  $\deg(f)$  denote the Brouwer degree of  $f$ .

Lemma 6 [3]. If  $f : (M, g) \rightarrow (N, h)$  is a harmonic mapping between compact and orientable surfaces, and if  $\chi(M) + |\deg(f)\chi(N)| > 0$ , then  $f$  is  $\pm$  holomorphic.

Now, let  $M$  be a closed orientable surface immersed in  $E^3$  with constant mean curvature  $\mu$ . It is well-known that the Gauss map  $f$  of  $M$  is harmonic. If  $p = 0$ , it follows from Lemmas 6 and 5 that  $C = 0$  for  $f$ , and so (1) reduces to  $\Delta K^2 = -16K^2(\mu^2 - K)$ , where  $K$  is the Gaussian curvature. The r.h.s. being non-positive,  $K$  is a constant, thereby yielding the Hopf conjecture in the genus zero case.

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