C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1978) No. 1

A characterization of the rectangular distribution E. Lukacs 1

An upper bound on the rate distortion function for source coding with partial side information at the decoder J. Wolfowitz 5

The amplitude of a Petrie polygon H.S.M. Coxeter 9

A class of combinatorial quasigroups N.S. Mendelsohn 13

The nucleus in alternative rings with idempotent I.R. Hentzel, E. Kleinfeld and H.F. Smith 17

Central configurations of the n-body problem in E^4 J.I. Palmore 21

Characterizing universal fibrations P.I. Booth, P.T. Heath and R.A. Piccinini 25

How general is Lomonosov's invariant subspace theorem? H. Radjavi and P. Rosenthal 29

Projective characters of groups of Lie type L. Chastkofsky and W. Feit 33

Extremum problems for the multi-dimensional case of König and Szücs of billiard ball motions I.J. Schoenberg 37

A proof of a theorem of Coxeter J. Assion 41

Perturbations of C*-Algebras, II J. Phillips and I. Raeburn 45

An explicit construction of some discrete unitary series of representations of u(p,q) F. Lemire and J. Patera 49

Two sided quotient rings for Noetherian PI-rings B. Müller 53

The homeomeric classification of patterns B. Grünbaum and G.C. Shephard 57

Mailing Addresses

1. Assion, J. Luisenstrasse 122, 4000 Düsseldorf 1, B.R. Deutschland.

2. Booth, P. Dept. of Mathematics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1B 3X7

3. Chastkofsky, L. Dept. of Mathematics, Yale University, Box 2155, Yale Station, New Haven, Conn. 06520, U.S.A.

4. Coxeter, H.S.M. Dept. of Mathematics, University of Toronto, Toronto, Ontario, Canada. M5S 1A1

5. Feit, W. same address as (3) above

6. Grünbaum, B. Dept. of Mathematics, University of Washington, Seattle, Washington 98195, U.S.A.

7. Heath, P. same address as (2) above

8. Hentzel, I.R. Dept. of Mathematics, Iowa State University, Ames, Iowa 50010, U.S.A.

9. Kleinfeld, E. Dept. of Mathematics, University of Iowa, Iowa City, Iowa 52242, U.S.A.

10. Lemine, F. Dept. of Mathematics, University of Windsor, Windsor, Ontario, Canada. N9B 3P4

11. Lukacs, E. 3727 Van Ness Street, Washington, D.C. 20016, U.S.A.

12. Mendelsohn, N.S. Dept. of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada. R3T 2N2

13. Mueller, B.J. Dept. of Mathematics, McMaster University, Hamilton, Ontario, Canada. L8S 4K1

14. Palmore, J.I. Dept. of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, U.S.A.

15. Patera, J. Centre de Recherches Mathématiques, Université de Montréal, Case Postale 6128, Succ. A, Montréal, Québec, Canada. H3C 3J7

16. Phillips, J. Dept. of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada. B3H 3J5

17. Piccinini, R. same address as (2) above

18. Radjavi, H. Dept. of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada. B3H 3J5

19. Raeburn, I. School of Mathematics, University of South Wales, Box 1, Kensington, N.S. Wales, 2033, Australia.

20. Rosenthal, P. Dept. of Mathematics, University of Toronto, Toronto, Ontario, Canada. M5S 1A1

21. Schoenberg, I.J. Mathematics Research Center, The University of Wisconsin, 610 Walnut St., Madison, Wis. 53706, U.S.A.

22. Shephard, G.C. Dept. of Mathematics, University of East Anglia, Norwich, NR4 7TJ, England.

23. Smith, H.F. same address as (8) on preceding page

24. Wolfowitz, J. Dept. of Mathematics, University South Florida, Tampa, Florida 33620, U.S.A.

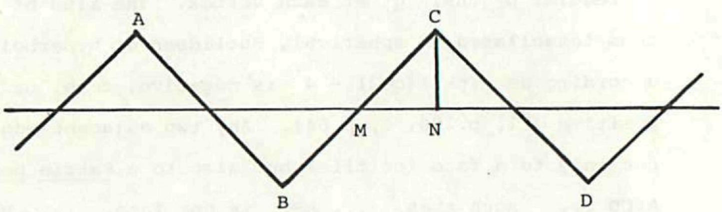
THE AMPLITUDE OF A PETRIE POLYGON

by

H.S.M. Coxeter, F.R.S.C.

The Schläfli symbol $\{p,q\}$ is used for a tessellation of regular p-gons, q at each vertex. The kind of plane thus tessellated is spherical, Euclidean or hyperbolic according as $(p-2)(q-2) - 4$ is negative, zero, or positive [1, p.200; 2, p.64]. Any two adjacent edges belong not only to a face (or tile) but also to a Petrie polygon ABCD ... such that ... ABC is one face, ... BCD is another, and so on [2, p.24]. Since the Petrie polygon is a 'regular' zigzag, the midpoints of its edges all lie on its axis (a line or, in the spherical case, a great circle) with the alternate vertices A, C, ... on one side, and B, D, ... on the other side. The distance of these vertices from the axis may reasonably be called the amplitude of the Petrie polygon; let us denote it by δ. In the spherical case with the axis as equator, δ is the latitude of the small circles AC ... and BD ... in the northern and southern hemispheres; in other words, these small circles have angular radius $(\pi/2) - \delta$, straight radius $\cos\delta$. For instance, $\delta = 0$ for a dihedron $\{p,2\}$ $(p\geq 3)$, $\delta = \frac{1}{2}\pi$ for a hosohedron $\{2,q\}$ $(q\geq 3)$, and δ has some intermediate value for each of the tessellations corresponding to Platonic solids. In the

Euclidean case, AC ... and BD ... are parallel lines and, if $AB = 2\phi$, $\delta = \phi \cos(\pi/q) = \phi \sin(\pi/p)$. In the hyperbolic case, AC ... and BD ... are the two branches of an equidistant-curve with 'altitude' δ. We seek an expression for δ as a function of p and q in the spherical and hyperbolic cases.



Let M (on the axis) be the midpoint of an edge BC; let N be the foot of the perpendicular from C to the axis. Then CMN is a right-angled triangle with $CM = \phi$, $CN = \delta$, and angle π/q at C. In the spherical case, one of the classical formulae for spherical trigonometry yields

$$\tan \delta = \tan \phi \cos(\pi/q).$$

But we know that $\cos \phi = \cos(\pi/p) \operatorname{cosec}(\pi/q)$ [2, p.21]. Hence

$$(1) \quad \tan \delta = \{\sec^2(\pi/p) \sin^2(\pi/q) - 1\}^{\frac{1}{2}} \cos(\pi/q).$$

Similarly, in the hyperbolic case, where

$$\tanh \delta = \tanh \phi \cos(\pi/q)$$

and $\cosh \phi = \cos(\pi/p) \operatorname{cosec}(\pi/q)$ [1, p.201],

(2) $$\tanh \delta = \{1 - \sec^2(\pi/p) \sin^2(\pi/q)\}^{\frac{1}{2}} \cos(\pi/q).$$

Let us apply these formulae to the 'Platonic' tessellations and to one of the infinitely many hyperbolic tessellations. In terms of the ubiquitous angles

$$\kappa = \frac{1}{2} \text{ arc sec } 3 = 35^\circ\ 16' ,$$

$$\lambda = \frac{1}{2} \text{ arc tan } 2 = 31^\circ\ 43' ,$$

$$\mu = \frac{1}{2} \text{ arc sin } \frac{2}{3} = 20^\circ\ 54'$$

[2, p. 293; 3, pp. 61, 158-159], (1) yields , for both
$\{3,3\}$ and $\{3,4\}$, $\delta = \text{arc tan } 2^{-\frac{1}{2}} = \kappa$;
for $\{4,3\}$, $\delta = \text{arc tan } 2^{-3/2} = \frac{1}{2}\pi - 2\kappa = 19^\circ\ 28'$;
for $\{3,5\}$, $\delta = \text{arc tan } \frac{1}{2} = \frac{1}{2}\pi - 2\lambda = 26^\circ\ 34'$;
and for $\{5,3\}$, $\delta = \text{arc tan } \frac{1}{2}\tau^{-2} = \lambda - \mu = 10^\circ\ 49'$.

As a hyperbolic instance let us choose $\{8,3\}$. In this case (2) yields

$$\tanh \delta = \frac{1}{2}(2^{\frac{1}{4}} - 2^{-\frac{1}{4}}) = 0.17417 ,$$

$$\delta = 0.17597.$$

The corresponding angle of parallelism $\Pi(\delta)$ is given by

$$\cos \Pi(\delta) = \tanh \delta,$$

so that $$\Pi(\delta) = \operatorname{arc} \cos 0.17417 = 79^{\circ}\ 58' .$$

This angle, being practically 80°, can be measured in Escher's Circle Limit III [4, p. 109]. His underlying tessellation {8,3} is cleverly disguised. Rows of coloured fishes swim after one another along white arcs which cut the peripheral 'absolute' circle at 80°.

REFERENCES

1. H.S.M. Coxeter, Twelve Geometric Essays (Southern Illinois University Press, Carbondale, 1968).

2. ———————, Regular Polytopes (3rd ed., Dover, New York, 1973).

3. ———————, Regular Complex Polytopes (Cambridge University Press, 1974).

4. Bruno Ernst, The Magic Mirror of M.C. Escher (Random House, New York, 1976).

University of Toronto
Toronto, Ontario Canada

Received July 16, 1978

A Class of Combinatorial Quasigroups

by N.S. Mendelsohn, F.R.S.C.

1. Introduction. In the modern theory of combinatorial designs, varieties of quasigroups which are idempotent and which are based on two-variable identities play a very prominent role. Such varieties are useful in the construction of orthogonal arrays, block designs, codes, etc.

The importance of such varieties is derived mainly from three properties viz ;

(1) If there are algebras in such a variety of orders belonging to a set of integers K, and there is a pairwise balanced block design of index 1, order v with block sizes in K, then there is an algebra of order v belonging to the variety.

(2) If an algebra A in such a variety has a 2-generated subalgebra B of order v and if B^* is a 2-generated algebra in the variety of order v then replacing B by B^* in A replaces A by an algebra A^* which is in the variety. This observation usually enables the construction of non-isomorphic designs which can be built from algebras in the variety.

(3) An important theorem of R.M. Wilson when applied to such varieties enables one to obtain asymptotically the spectrum of the variety from the construction within the variety of a few algebras of small orders.

2. <u>The basic theorem.</u> Let q be a prime power p^r. There exists an idempotent variety of quasigroups based on a finite set of two-variable identities such that the free algebra in the variety on two generators is of order q. Furthermore, the free algebra on two generators has a sharply doubly transitive automorphism group.

<u>Proof.</u> Consider the quasigroup defined on the elements of the field G.F.(q) using a binary operator $*$ where $x * y = \lambda x + (1 - \lambda)y$ where λ is a primitive element of $GF(q)$. It is obvious that the algebra so defined is an idempotent quasigroup. Put $F_1(x, y) = x * y$, $F_2(x, y) = (x * y) * y$, and recursively $F_k(x, y) = F_{k-1}(x, y) * y$. It follows easily that $F_i(x, y) = \lambda^i x + (1 - \lambda^i)y$. Hence $F_1(1, 0) = \lambda$, $F_2(1, 0) = \lambda^2$, $\cdots$, $F_{q-1}(1, 0) = \lambda^{q-1} = 1$. Thus every remaining element of the field can be expressed as a word in 1 and 0. Furthermore, each of the mappings $x \to \alpha x + \beta$ where α ranges over the non-zero elements of $GF(q)$ and β ranges over all elements of $GF(q)$ defines an automorphism of the quasigroup and in fact the set of all such mappings constitute a doubly transitive group of automorphisms. If we put $F_o(1, 0) = 0$, the elements of $GF(q)$ are, $F_o(1, 0)$, $F_1(1, 0)$, $\cdots$ $F_{q-1}(1, 0)$ and the multiplication table can be represented by q^2 equations $F_i(1, 0) * F_j(1, 0) = F_{i \circ j}(1, 0)$ where $i \circ j$ is some element of $GF(q)$. Since there is a doubly transitive group of automorphisms then 1 and 0 may be replaced by variables x and y yielding n^2 identities of the form $F_i(x, y) * F_j(x, y) = F_{i \circ j}(x, y)$. These identities are the basis of the required variety. It is clear that the free algebra on two generators is isomorphic to the quasigroup which we have constructed on $GF(q)$.

Corollary. Let q be any prime power. There exists an idempotent variety of quasigroups whose spectrum is an asymptotic subset of the set of v for which $v \equiv 1 \mod q(q-1)$ or $v \equiv q \mod q(q-1)$.

Proof. Take the variety to be the one defined in Theorem 2. If A is an algebra in this variety such that $|A| = v$, we define on A a balanced incomplete block design with block size q by taking for each pair of elements x, y in A the set of elements in the sub-algebra generated by x and y to be the block containing x and y. Hence v satisfies $v(v-1) \equiv 0 \mod q(q-1)$ and $v - 1 \equiv 0 \mod (q-1)$. By Wilson's theorem the spectrum is an asymptotic subset of those v which satisfy these two congruences. The congruences $v(v-1) \equiv 0 \mod q(q-1)$ and $v - 1 \equiv 0 \mod (q-1)$ are equivalent to $v \equiv 1 \mod q(q-1)$ or $v \equiv q \mod q(q-1)$.

Remark. If λ is any generators of $GF(q)$, which is not necessarily a primitive element, the construction of the main theorem still yields a variety with the stated properties. The varieties so obtained for different λ are in general, distinct. To prove these statements is not difficult. The proof that 0 and 1 generate the whole quasigroup is a bit tricky and is not given here.

3. Application. For a given graph G a theorem of C.C. Lindner (not yet published) states that it is possible to associate with each vertex a latin square such that two latin squares are orthogonal if and only if

the corresponding vertices in the graph are joined by an edge. The basic theorem of this paper is used to obtain information on the spectrum of such designs. The results will be published elsewhere.

4. References.

[1] N.S. Mendelsohn, The spectrum of idempotent varieties of algebras with binary operators based on two variable identities, Aequationes Mathematicae (to appear).

[2] R.M. Wilson, An existence theory for pairwise balanced designs III: proof of the existence conjectures, J. Comb. Theory A, 18(1975), 71-79.

Received July 16, 1978

N.S. Mendelsohn,
Department of Mathematics,
University of Manitoba,
Winnipeg, Canada.
R3T 2N2

C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1978) No. 1

THE NUCLEUS IN ALTERNATIVE RINGS WITH IDEMPOTENT

Irvin R. Hentzel, Erwin Kleinfeld and Harry F. Smith

Presented by J. Aczél, F.R.S.C.

It is difficult to construct examples of nonzero alternative rings R whose nucleus N is zero. Ževlakov, Slińko, Šestakov and Širšov[6] have given one such example. Since the nucleus plays such a central role in the structure theory of alternative rings, it seems reasonable to ask for some sufficient conditions on R that will guarantee $N \neq 0$. Somehow characteristic two seems to be different and it is necessary to impose characteristic not two on R, by which we understand that there should exist no elements whose additive order is two. Then any one of the following three conditions turns out to be sufficient:

(i) That R contain an idempotent $e \neq 0$,

(ii) that R be an algebra over a field, whose nil radical is finitely generated,

(iii) that R have descending chain conditions on two-sided ideals.

No doubt there must be many other sufficient conditions, but these are enough to establish our statement that indeed $N = 0$ happens only rarely for alternative rings. Perhaps the most striking of these conditions is the first, since it does not involve any structure theory, either in the statement or the proof.

Departments of Mathematics, Iowa State University, Ames, Iowa 50011 and University of Iowa, Iowa City, Iowa 52242, U.S.A.

We now give some brief clues about the proof of condition (i). A detailed account of this and other assertions will appear elsewhere. We remind the reader that in every alternative ring with idempotent e one has the Peirce decomposition of R into a direct sum of submodules

$$R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00},$$

where e acts as a left identity or annihilator, depending on whether the left subscript of the submodule is 1 or 0, and as a right identity or annihilator, depending on whether the right subscript of the submodule is 1 or 0. Further details may be found in [1]. Basic to our proof is the identity that for all a,b,c,d,f in R_{10}

(1) $\qquad (a,[cd]f+f[cd],b) = 0.$

Interestingly enough this identity fails for rings of characteristic two, but holds in rings of characteristic not two. This also has an implication for free alternative rings with idempotent and free generators a,b,c,d,f in the designated submodule, for then identity (1) results in a torsion element of order two. Next we use (1) to establish

(2) $\quad n = [a_{10}b_{10}][x_{01}y_{01}] + [x_{01}y_{01}][a_{10}b_{10}]$ lies in N.

In general however n need not be in the center of R [3]. If $N = 0$, then all elements of the type given by (2) are zero. Then one can show in successive stages that all elements of the form $x_{10}(y_{10}z_{10}) + (y_{10}z_{10})x_{10}$ must be in N, hence zero until finally R_{10} and R_{01} are in N and hence zero, at which point e must be in N, hence zero. We have reached a contradiction, so that (i) is established. Conditions (ii) and (iii) may be proved by appealing to [2], [4], [5], [6] and (i). Two further identities are worth singling out:

$$(3)\quad p = a_{10}([b_{10}c_{10}]d_{10}) - b_{10}([c_{10}d_{10}]a_{10}) + c_{10}([d_{10}a_{10}]b_{10}) - d_{10}([a_{10}b_{10}]c_{10}) \text{ lies in } N \cap R_{10},$$

$$(4)\quad (a_{10},b_{10},c_{10})(d_{10},f_{10},g_{10}) \text{ lies in the commutative center of } R.$$

By skillfully combining (3) and (4) it can be proved that in every free alternative ring with idempotent, assuming characteristic $\neq 2,3$ and at least four generators, any four elements of R_{10} satisfy a dependence relation over the center. These results also lead to a new proof of Albert's classification of simple alternative rings with idempotent [1], replacing simplicity with more general hypotheses.

REFERENCES

1. A.A.Albert, On simple alternative rings, Canadian J. Math. 4(1952), 129-135.
2. E.Kleinfeld, Simple alternative rings, Ann. of Math. 58 (1953), 544-547.
3. E.Kleinfeld, On centers of alternative algebras, to appear.
4. M.Slater, Alternative rings with d.c.c.II, J. Algebra 14 (1970), 464-484.
5. M.Slater, Alternative rings with d.c.c.III, J. Algebra 18 (1971), 179-200.
6. K.A.Ževlakov et al, Alternative algebras - Part I, Novosibirsk (1976), (Russian).

Received July 31, 1978

C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1978) No. 1

Central Configurations of the n-Body Problem in E^4

by Julian I. Palmore[1]

Presented by P. Ribenboim, F.R.S.C.

We prove the existence of many classes of central configurations in Euclidean space E^4 using topological methods.

Recall the setting of the n-body problem. Let the masses $(m_i) \in \mathbb{R}_+^n$ be fixed. The configuration space of the n-body problem is a subset of the linear space

$$M = \{(x_1,\ldots,x_n) \in (E^4)^n \mid \Sigma m_i x_i = 0\}.$$

We remove from M the fat diagonal $\Delta = \cup\Delta_{ij}$ where $\Delta_{ij} = \{x \in M \mid x_i = x_j\}$ and the union is over all $i < j$. Then $M - \Delta$ is the configuration space and the dynamics are given via a vectorfield on $T(M - \Delta)$.

The Newtonian potential appropriate to E^4 is the divergence free potential on $M - \Delta$ defined by

$$V_m(x) = -\sum_{i<j} \frac{m_i m_j}{\| x_i - x_j \|^2}$$

A configuration $(x_1,\ldots,x_n) \in M - \Delta$ is a central configuration [2] if there is a $\lambda \in \mathbb{R}\setminus\{0\}$ such that for $i = 1,\ldots,n$ we have

$$\lambda m_i x_i = -\mathrm{grad}_i V_m(x).$$

Here $\mathrm{grad}_i V_m(x)$ is the gradient of V_m by x_i. As a consequence of this definition it follows that <u>a configuration</u> $(x_1,\ldots,x_n) \in M - \Delta$, $\Sigma m_i \| x_i \|^2 = 1$ <u>is a central configuration if and only if</u> $(x_1,\ldots,x_n)$ <u>is a critical point of</u> $V_m | (S_m - \Delta)$ <u>where</u> $S_m = \{x \in M \mid \Sigma m_i \| x_i \|^2 = 1\}$ <u>and</u> $S_m - \Delta$ denotes $S_m - (S_m \cap \Delta)$.

Let x and y be two critical points of V_m in $S_m - \Delta$. Then $x \sim y$ if there is an $\alpha \in SO(4)$ such that $x = \alpha y$. A class of central configurations is an equivalence class under this equivalence relation.

---

1 Research supported in part by NSF grant MCS 78-00395.

Identify E^4 and $\mathbb{H}$, the quaternions. The subgroup of unit quaternions, S^3, acts freely on $\mathbb{H}$ and leaves invariant $M, \Delta_{ij}, \Delta, S_m$ and V_m. We denote by Q_m the quotient manifold of $S_m - \Delta$ by S^3. For each $(m_i) \in \mathbb{R}^n_+$ Q_m is homeomorphic to $\mathbb{P}_{n-2}(\mathbb{H}) - \tilde{\Delta}_{n-2}$ where $\mathbb{P}_{n-2}(\mathbb{H})$ is quaternionic projective space of (quaternion) dimension $n - 2$ and $\tilde{\Delta}_{n-2}$ is the (nontrivial) union of $n(n - 1)/2$ codimension 1 quaternionic projective subspaces.

Let $\tilde{V}_m$ denote the potential function induced on Q_m by V_m.

A class of central configurations is identified with a submanifold of Q_m of critical points of $\tilde{V}_m$ of dimension 3 which results by the residual action of $SO(3)$ on a critical point of $\tilde{V}_m$.

Clearly, every critical point of $\tilde{V}_m$ in Q_m is degenerate. We call a central configuration class nondegenerate if the associated critical points in Q_m have maximal rank: that is, the submanifold of critical points is a nondegenerate critical manifold. In this case critical point theory gives the existence of many classes of central configurations provided the homology of Q_m is sufficiently rich [1].

Let H_* be integral singular homology.

<u>Theorem A</u>. For any $n \geq 3$ and for any i, $0 \leq i \leq 3n - 6$ $H_i(\mathbb{P}_{n-2}(\mathbb{H}) - \tilde{\Delta}_{n-2}) \cong H_i(\mathbb{P}_{n-3}(\mathbb{H}) - \tilde{\Delta}_{n-3}) \oplus (n - 1)\, H_{i-3}(\mathbb{P}_{n-3}(\mathbb{H}) - \tilde{\Delta}_{n-3})$. $H_i = 0$ for $i > 3n - 6$. The homology H_* is torsion free.

Let $\beta_i = \operatorname{rank} H_i(Q_m)$ and $\chi(Q_m)$ be the Euler characteristic.

<u>Corollary A.1</u>. $\Sigma\beta_i = n!/2$ for any $n \geq 3$.

<u>Corollary A.2</u>. $\chi(Q_m) = (-1)^n(n - 2)!$ for any $n \geq 3$.

<u>Corollary A.3</u>. The Poincaré polynomial of Q_m is $\prod_{k=2}^{n-1} (1 + kt^3)$ for any $n \geq 3$.

The index of x is the index of the Hessian of $\tilde{V}_m$ at x. We denote the index of x by $\mathrm{ind}(x)$.

Theorem B. Let $x \in Q_m$ be a critical point of $\tilde{V}_m$. Then $\mathrm{ind}(x) \geqq n-2$.

By choosing a line $E^1 \subset E^4$ a subset $Y_m \subset Q_m$ is generated which is homeomorphic to $\mathbb{P}_{n-2}(\mathbb{R}) - \tilde{\Delta}_{n-2} \subset \mathbb{P}_{n-2}(\mathbb{H}) - \tilde{\Delta}_{n-2}$. A critical point $x \in Y_m$ of $\tilde{V}_m$ is called a collinear central configuration.

Theorem C. Let $x \in Y_m \subset Q_m$ be a critical point of V_m. Then $\mathrm{ind}(x) = n-2$ and x has maximal rank $= 4n - 10$.

We call a central configuration regular in E^4 if its convex hull has dimension 4.

Theorem D. If $n = 5$ there are only two classes of regular central configurations. A member of each class spans a 4-simplex and has maximal rank $= 4n - 11$.

References

1. J. I. Palmore, Measure of degenerate relative equilibria. I, Ann. of Math. 102(2) 1976, 421-429.

2. A. Wintner, The analytical foundations of celestial mechanics, Princeton Math. Series, Vol. 5, Princeton Univ. Press, 1941.

University of Illinois
Urbana, U.S.A.

Received August 10, 1978

C. R. Math. Rep. Acad. Sci. Canada - Vol. 1. (1978) No. 1

CHARACTERIZING UNIVERSAL FIBRATIONS

Peter I.Booth, Philip R.Heath and Renzo A.Piccinini

Presented by P. Ribenboim, F.R.S.C.

The notion of Universal Fibration has been defined in different ways. A fibration $p_\infty : E_\infty \longrightarrow B_\infty$ is said to be: (1) Free Universal if the fibre homotopy classes of fibrations over a space X correspond bijectively to the free homotopy classes of maps from X into B_∞ ; (2) Grounded Universal, essentially a based version of (1); (3) Aspherical Universal if the total space of the associated principal fibration (whose precise definition will be given later on) is weakly contractible, and finally, (4) Extension Universal if any partial map pair into p_∞ can be extended. Dold has proved the equivalence of (1) and a strengthened form of (3) for Principal G-bundles, while Steenrod has shown that (4) implies (1) in the same context; Allaud, using an analogue of Dold's argument and the theory of quasi-fibrations, proved that under restrictive hypothesis, (2) and (3) are equivalent for (grounded) Hurewicz fibrations.

Using techniques developed in ([1], [2] and [3]) we obtain a unified approach to the presentation and comparison of Universal Fibrations. No proofs are given here; the reader will find them in the homonymous paper to appear in the Proceedings, Algebraic Topology Conference, Vancouver 1977, Springer Lecture Notes in Mathematics # 673.

GENERAL DEFINITIONS AND MAIN THEOREM - We work in the context of the convenient category $\mathcal{K}$ of k-spaces as in [3]. Borrowing the notation of J.P.May [5], let $\mathcal{F}$ be a category with a distinguished object F together with a faithful underlying space functor $\mathcal{F} \longrightarrow \mathcal{K}$; for technical reasons we shall assume that for every object X of $\mathcal{F}$, $\mathcal{F}(F , X) \neq \emptyset$. An $\mathcal{F}$-space is a morphism $p : X \longrightarrow A$ of $\mathcal{K}$ such that A is a CW-complex and, for every $a \in A$, $p^{-1}(a) \in \mathrm{Obj}\, \mathcal{F}$; the latter condition will impose a corresponding constraint in the morphisms of $\mathcal{F}$. The notions of $\mathcal{F}$-homotopy and $\mathcal{F}$-homotopy equivalence are easy to define; we then put the following crucial condition on $\mathcal{F}$: every morphism of $\mathcal{F}$ is an $\mathcal{F}$-homotopy equivalence over a point $*$.

Given an $\mathcal{F}$-space $r : Z \longrightarrow B$ and a map $f : A \longrightarrow B$ we denote the pull-back space by $A \sqcap_f Z$; according to ([5],Lemma 1.2), the projection $r_f : A \sqcap_f Z \longrightarrow A$ is an $\mathcal{F}$-space. Given maps $q : Y \longrightarrow A$ and $r : Z \longrightarrow B$ we define $Y * Z$ to be the set $\bigcup_{a\in A, b\in B} \mathcal{F}(Y_a , Z_b)$ conveniently topologized [3], where $Y_a = q^{-1}(a)$, $Z_b = r^{-1}(b)$. Also, let $q*r : Y*Z \longrightarrow A \times B$

be the map which takes any $f : Y_a \longrightarrow Z_b$ into (a , b). An $\mathcal{A}$-fibration is an object of a non-empty, full subcategory $\mathcal{A}$ of the category of $\mathcal{F}$-spaces and $\mathcal{F}$-maps defined by the following axioms: A1) $F \longrightarrow *$ is an $\mathcal{A}$- fibration; A2) If $r : Z \longrightarrow B$ is an $\mathcal{A}$-fibration, A is a CW-complex and $f : A \longrightarrow B$ is a map, then $r_f : A \sqcap_f Z \longrightarrow A$ is an $\mathcal{A}$-fibration; A3) If $r : Z \longrightarrow B$ is an $\mathcal{A}$-fibration, $s : W \longrightarrow B$ is an $\mathcal{F}$-space and $g : Z \longrightarrow W$ is an $\mathcal{F}$-map over B which is a homeomorphism, then s is an $\mathcal{A}$-fibration; A4) If $q : Y \longrightarrow A$ and $r : Z \longrightarrow B$ are $\mathcal{A}$-fibrations, then $q*r : Y*Z \longrightarrow A \times B$ has the Covering Homotopy Property with respect to all CW-complexes.

The notions of Free, Grounded and Extension Universality in $\mathcal{A}$ are clear. As for the Aspherical case, given an $\mathcal{A}$-fibration $p_\infty : E_\infty \longrightarrow B_\infty$, we take $c : F \longrightarrow *$ and form $c*p_\infty : F*E_\infty \longrightarrow * \times B_\infty$; then we say that p_∞ is <u>Aspherical Universal</u> if $\pi_n(F*E_\infty) = 0$ for all n and all choices of base point for $F*E_\infty$.

<u>Main Theorem</u> - Every Grounded Universal $\mathcal{A}$-fibration is Free Universal; every Aspherical Universal $\mathcal{A}$-fibration is Grounded Universal. An $\mathcal{A}$-fibration is Aspherical Universal if, and only if, it is Extension Universal.

PARTICULAR CASES - (I) <u>Hurewicz Fibrations</u> - Let $\mathcal{F}_F$ be the category of spaces of the homotopy type of a fixed space F and whose morphisms are homotopy equivalences. We take $\mathcal{A}_F$ to be the category consisting of Hurewicz fibrations over CW-complexes and fibres of the homotopy type of F.

(II) <u>Principal G-bundles</u> - Let G be a topological group; $\mathcal{F}_G$ denotes the category whose objects are right G-spaces Y such that, for every $y \in Y$, $\tilde{y} : G \longrightarrow Y$, $\tilde{y}(g) = y \cdot g$ is a homeomorphism. Define $\mathcal{A}_G$ to be the category of principal G-bundles over CW-complexes.

(III) H-<u>Principal Fibrations</u> - Let H be an H-space in the sense of [4]; $\mathcal{F}_H$ is the category of spaces X with a right action of H such that, for every $y \in Y$, $\tilde{y} : H \longrightarrow Y$, $\tilde{y}(h) = y \cdot h$ is a homotopy equivalence. Take $\mathcal{A}_H$ to be the category of H-principal fibrations in the sense of [4].

<u>Theorem</u> - Let $p_\infty : E_\infty \longrightarrow B_\infty$ be an $\mathcal{A}_F$, $\mathcal{A}_G$ or $\mathcal{A}_H$ - fibration. Then p_∞ is Universal in the four senses described earlier if, and only if, it is Universal in any one of these four senses.

(IV) Trivial Fibrations - Let F and $\mathcal{F}_F$ as in (I). We take $\mathcal{A}_T$ to be the category of all trivial fibrations over CW-complexes with fibre in $\mathcal{F}_F$, that is to say, fibrations that are, to within a homeomorphism of their total spaces, projections of the product of their base space and a space in $\mathcal{F}_F$. The trivial fibration $c : F \rightarrow *$ is both Grounded and Free Universal; it is not, however, neither Aspherical Universal nor Extension Universal.

A further example shows that, if the base spaces are restricted to being simply connected CW-complexes, then a Grounded Universal Fibration is not necessarily Universal in any other sense.

BIBLIOGRAPHY

[1] Booth,P.,Heath,P. and Piccinini,R.- Section and Base-Point Functors, Math.Z.144, 181-184 (1975).

[2] Booth,P.,Heath,P. and Piccinini,R.- Restricted Homotopy Classes, An.Acad.Brasil.Cien.,49, 1-8 (1977).

[3] Booth,P.,Heath,P. and Piccinini,R.- Fibre preserving maps and Functional Spaces, to appear, Proceedings, Algebraic Topology Conference, Vancouver 1977, Springer LNM # 673.

[4] Fuchs,M.- A modified Dold-Lashof Construction that does classify H-Principal Fibrations, Math.Ann.,192, 328-340 (1971).

[5] May,J.P.- Classifying Spaces and Fibrations, Amer.Math.Soc. Memoirs # 155 (1975).

Memorial University of Newfoundland
St.John's, Nfld.,A1B 3X7
Canada

Received August 15, 1978

HOW GENERAL IS LOMONOSOV'S INVARIANT SUBSPACE THEOREM?

by Heydar Radjavi and Peter Rosenthal

Presented by P. Fillmore, F.R.S.C.

This report is a description of some recent work we have done in collaboration with several other mathematicians ([3],[4]).

In 1973 V. Lomonosov proved a theorem about existence of invariant subspaces that includes the following result: if A is a bounded linear operator on a complex Banach space, and if AB = BA for some operator B that is not a multiple of the identity and that commutes with a non-zero compact operator, then A has a non-trivial invariant subspace; (see [7],[9] and [10]). It is hard to tell whether or not an operator A satisfies this hypothesis. In fact, it appeared conceivable that all operators A might satisfy it, and thus that it would follow from Lomonosov's theorem that all operators have invariant subspaces.

In joint work with Don Hadwin and Eric Nordgren [4], we have shown that there are operators A that do not satisfy the hypothesis of Lomonosov's theorem: the unilateral weighted shifts which Shields [11] calls "quasi-analytic." Following Shields [11], we regard these operators as multiplication operators M_z on certain spaces of functions analytic in the unit disk. The commutant of M_z consists of multiplication operators M_ϕ for suitable analytic functions ϕ, so our result reduces to showing that the only compact operator K commuting with such an M_ϕ is K = 0. This requires many of the results of [11]. We first prove that there is an integer N such that $K^N = 0$ for all such K; this is accomplished by showing that K^* has many finite-dimensional invariant subspaces on which its spectrum is $\{0\}$. For any fixed such K, say K_0, we then use the fact that $K^N = 0$ for all compact operators commuting with M_ϕ to show that the Hadamard product of the restrictions of K_0^* and any other matrix on those finite-dimensional invariant subspaces is also nilpotent of order at most N. A matrix theory lemma is proven that implies that the matrix of such a K_0^* has a zero column, from which it follows that K_0^* vanishes on a dense set and therefore is 0.

The above example is perhaps surprising in view of Cowen's proof [1] that the (unweighted) unilateral shift does satisfy Lomonosov's hypothesis.

We have also worked on further generalizing Lomonosov's theorem. After a preliminary attempt we made with Nordgren and Radjabalipour [8], the following result was obtained in collaboration with C.K. Fong, E. Nordgren and M. Radjabalipour [3]: if AB = BA for some B that is not a multiple of the identity and that satisfies an equation of the form BK = KF(B), where K is a non-zero compact operator and F is an analytic function mapping a bounded open set containing $\sigma(B)$ into itself, then A has a non-trivial invariant subspace. (Lomonosov's theorem, of course, is the case where F(z) = z). Our proof of this result relies heavily on Lomonosov's work.

It is possible (though unlikely) that a generalization of Lomonosov's theorem covers all operators on Hilbert space. It seems probable that quasi-analytic shifts do not satisfy the hypothesis of the above theorem either, although this should be checked. They do, however, satisfy the hypothesis of Daughtry's theorem [2], (see also Kim, Pearcy and Shields [5]): they have rank one commutators with some compact operators, since their adjoints have point spectrum, (see [6]).

REFERENCES

1. Carl Cowen, An analytic Toeplitz operator that commutes with a compact operator, Abstract 78T-B116, Notices A.M.S. 25 (1978), p. A-434.

2. J. Daughtry, An invariant subspace theorem, Proc. Amer. Math. Soc. 49 (1975), 267-268.

3. C.K. Fong, E. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal, Extensions of Lomonosov's invariant subspace theorem, to appear.

4. D. Hadwin, E. Nordgren, H. Radjavi and P. Rosenthal, An operator not satisfying Lomonosov's hypothesis, to appear.

5. H.W. Kim, C. Pearcy, and A.L. Shields, Rank-one commutators and hyperinvariant subspaces, Mich. Math. J. 22 (1975), 193-194.

6. H.W. Kim, C. Pearcy and A.L. Shields, Sufficient conditions for rank-one commutators and hyperinvariant subspaces, Mich. Math. J. 23 (1976), 235-243.

7. V. Lomonosov, Invariant subspaces for operators commuting with compact operators, Func. Analy. Prilozen 7 (1973), 55-56 (Russian)= Funct. Anal. and Appl. 7 (1973), 213-214 (English).

8. E. Nordgren, M. Radjabalipour, H. Radjavi, and P. Rosenthal, Algebras intertwining compact operators, Acta Sci. Math. (Szeged) 39 (1977), 115-119.

9. C. Pearcy and A.L. Shields, A survey of the Lomonosov technique in the theory of invariant subspaces, pp. 220-229 of Topics in Operator Theory, Amer. Math. Soc. Surveys no. 13, Providence, 1974.

10. Heydar Radjavi and Peter Rosenthal, Invariant Subspaces, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

11. A.L. Shields, Weighted shift operators and analytic function theory, pp. 49-128 of Topics in Operator Theory, Amer. Math. Soc. Surveys no. 13, Providence, 1974.

Dalhousie University and University of Toronto

Received August 16, 1978

Projective Characters of Groups of Lie Type

by

Leonard Chastkofsky[1]

and

Walter Feit[1]

Presented by P. Ribenboim, F.R.S.C.

1. We introduce the following notation. $\mathbb{F}_q$ is the field of cardinality $q = p^n$, p a prime, and K is its algebraic closure. Let G_n, G_∞ denote a universal Chevalley group constructed over $\mathbb{F}_q$ and K respectively. We consider $G_1 \subseteq G_n \subseteq G_\infty$.

Let σ denote the Frobenius automorphism of K or $\mathbb{F}_q$. The automorphism of G_∞ or G_q is also denoted by σ.

If α, β are complex valued class functions on a group H then

$$(\alpha,\beta) = (\alpha,\beta)_H = \frac{1}{|H|} \sum_{x \in H} \alpha(x)\beta(x^{-1}).$$

2. If H is a finite group, a P.I.M. is a projective indecomposable $K[H]$ module. There is a one to one correspondence between isomorphism classes of P.I.M.'s and irreducible $K[H]$ modules such that the irreducible $K[H]$ module M corresponds to the P.I.M. P if and only if M is isomorphic to the unique irreducible submodule of P.

A well known result of Steinberg asserts that there is a set S of $K[G_\infty]$ modules, indexed by a certain subset of the weight lattice of G_∞, such that every irreducible $K[G_n]$ module is isomorphic to exactly one of $\prod_{i=1}^{n} M_i^{\sigma^i}$, where each M_i is the restriction to G_n of some module in S.

[1] The first author was supported by the Canada Council and the second author was partially supported by NSF Contract GP33591.

It is natural to ask whether there is a similar procedure for constructing the P.I.M.'s of $K[G_n]$. The answer here is not as nice and simple as it is for the irreducible modules, but there is at least an algorithm for obtaining them.

Let St_n denote the Steinberg module of $K[G_n]$ and let Γ_n denote the Brauer character afforded by St_n. Then St_n is both irreducible and projective. Furthermore the degree $\Gamma_n(1)$ is the order of a Sylow p-group of G.

Let B(S) be the set of Brauer characters afforded by the modules in S. If $\varphi_i \in B(S)$ let φ_i' be the element of B(S) of least degree such that Γ_1 is a constituent of $\varphi_i\varphi_i'$. It can be shown by looking at the highest weights of the corresponding modules that such a φ_i' always exists and is unique and that Γ_1 occurs with multiplicity 1 as a constituent of $\varphi_i\varphi_i'$. Suppose that $\varphi = \Pi\varphi_i^{\sigma^i}$ is an irreducible Brauer character of G_n with Φ the character afforded by the corresponding P.I.M. Let $\varphi' = \Pi\varphi_i'^{\sigma^i}$ and set $\Psi = \Gamma_n\bar{\varphi}'$, where $\bar{\varphi}'$ is the contragredient to φ'. Then Γ_n occurs as a constituent of $\varphi'\varphi$ with multiplicity 1. Hence $(\Psi,\varphi) = (\Gamma_n,\varphi'\varphi) = 1$. (These inner products make sense as Γ_n vanishes on p-singular elements where the Brauer characters are not defined.) Since Ψ is projective it is the sum of characters afforded by P.I.M.'s. In this sum Φ occurs exactly once. Which other characters afforded by P.I.M.'s occur?

If $\theta \neq \varphi$ is an irreducible Brauer character of G_n with Θ the corresponding projective character then the multiplicity of Θ as a summand of Ψ is $(\Psi,\theta) = (\Gamma_n,\varphi'\theta)$, and this is equal to the number of times Γ_n is a constituent of $\varphi'\theta$. There is a natural partial ordering of the irreducible Brauer characters (induced by the partial ordering of the highest weights of the corresponding modules) which will be denoted by $\prec$, such that if Γ_n is a constituent of $\varphi'\theta$ then $\theta \succ \varphi$. Hence with respect to this ordering the matrix expressing the Ψ's in terms of the Φ's is unipotent and upper triangular. Thus it is possible to solve for the Φ's in terms of the Ψ's.

3. The above ideas can be applied to the groups $Sp_4(2^n)$ and $SL_3(2^n)$. Modifications of the arguments yield similar results for the groups $Suz(2^n)$ and $SU_3(2^n)$. In these groups we are to compute $(\Gamma_n, \varphi'\theta)$. The idea is to write $\varphi' = \Pi\varphi_i'^{\sigma^i}$, $\theta = \Pi\theta_i^{\sigma^i}$ and to find the multiplication of φ_i and θ_i as Brauer characters of G_n. Then we use a graph to expand $\varphi'\theta$. In this way we have obtained formulas for the degrees of all P.I.M.'s for the above mentioned groups. For example if Φ_ϕ is the P.I.M. corresponding to the trivial Brauer character then we prove

Theorem 1. (i) $\Phi_\phi(1) = 2^{3n}(6^n - 5^n)$ for $G_n = SL_3(2^n)$ or $SU_3(2^n)$.

(ii) If $G_n = Sp_4(2^{n/2})$ for n even and $Suz(2^n)$ for n odd then

$$\Phi_\phi(1) = 2^{2n}(2^{2n} - T_n 2^n + (-1)^n),$$

where $T_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$, the n^{th} Lucas number.

We have also obtained some results about the Cartan invariant $c_{\phi\phi} = (\Phi_\phi, \Phi_\phi)$. The arguments here are much more difficult.

Theorem 2. (i) If $G_n = Sp_4(2^{n/2})$ for n even and $Suz(2^n)$ for n odd

$$c_{\phi\phi} = 2^{3n} + 2^{2n} + 2^n + (-1)^n 2^{n+1} + 2^n U_n - 2^{n+1}(2^n+1)T_n,$$

where T_n is as in Theorem 1, and $U_n = \alpha^n + \beta^n + \gamma^n$ where $(x-\alpha)(x-\beta)(x-\gamma) = x^3 - 3x^2 - x + 5$.

(ii) If n is large enough, then for each of the groups $Sp_4(2^n)$, $Suz(2^n)$, $SL_3(2^n)$ and $SU_3(2^n)$, $c_{\phi\phi}$ is larger than the order of a Sylow 2-group.

The results in Theorems 1 and 2 were proved in [1] for Suz(8). In particular, it was observed there that statement (ii) of Theorem 2 contradicts an old conjecture. Theorem 2 shows that this conjecture is badly false for an infinite class of groups.

REFERENCES

[1] P. Landrock, "A counterexample to a conjecture on the Cartan invariants of a group algebra," Bull. London Math. Soc. 5 (1973), 223-224.

Yale University
New Haven, Connecticut 06520

Received August 20, 1978

C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1978) No. 1

EXTREMUM PROBLEMS FOR THE MULTI-DIMENSIONAL CASE OF KÖNIG AND SZÜCS OF BILLIARD BALL MOTIONS

I.J. Schoenberg
Presented by H.S.M. Coxeter, F.R.S.C.

Let

(1) $\quad U_n : 0 \leq x_\nu \leq 1, \quad (\nu = 1,\dots,n),$

be the unit cube in R^n. Let (a_ν) be a point interior to U_n and

(2) $\quad L_n^1 : \quad x = \lambda_\nu u + a_\nu, \ (\nu = 1,\dots,n),$

be a rectilinear and uniform motion, where $u = t$ denotes the time. We interpret (2) as the motion of a billiard ball (b.b.); as we wish to reflect the b.b. in the usual was on striking the 2n facets $x_\nu = 0$ or 1 of U_n, we use the zigzag function $\langle x \rangle$ of period 2, defined by $\langle x \rangle = x$ in $[0,1]$, and $\langle x \rangle = 2 - x$ in $[1,2]$. The path of the b.b. within U_n may now be described by the equations

(3) $\quad \sqcap_n^1 \ : \ x_\nu = \langle \lambda_\nu u + a_\nu \rangle \ (\nu = 1,\dots,n; -\infty < u < \infty).$

A classical theorem of Kronecker (See [2]) and its generalization (See [1]) show the following: If the n components (λ_ν) are arithmetically linearly independent, then the motion (3) is <u>ergodic</u>, i.e. the path $\sqcap_n^1$ is dense in U_n. If $1 \leq k \leq n-1$, while the (λ_ν) admit precisely n-k linearly independent linear homogeneous relations with integer coefficients, then the path $\sqcap_n^1$ is contained in and is dense in a finite k-dimensional skew polytope $\sqcap_n^k$. This was shown by König and Szücs in [2] for k = 2 and n = 3. This result shows that the b.b. motions generalize naturally as follows: Let $\lambda^i = (\lambda_1^i,\dots,\lambda_n^i)$, $(i = 1,\dots,k)$, be k linearly independent vectors, where we assume that $1 \leq k \leq n-1$. We now replace (2) by

(4) $\quad L_n^k : \ x_\nu = \sum_1^k \lambda_\nu^i u_i + a_\nu, \quad (\nu = 1,\dots,n; -\infty < u_i < \infty),$

which we interpret as a k-dimensional optical signal starting from the point (a_ν) inside U_n at the time $t = 0$, and spreading uniformly within the k-flat L_n^k. As we now think of the 2n facets of U_n as mirrors, the reflected path of the signal is a finite or infinite k-dimensional skew polytope $\prod_n^k$ described by the equations

$$(5)\quad \prod_n^k : \quad x_\nu = \left\langle \sum_1^k \lambda_\nu^i u_i + a_\nu \right\rangle \quad (\nu = 1,\ldots,n;\ -\infty < u_i < \infty).$$

In order to avoid lower-dimensional problems we shall assume that the original signal (4) is in a general position.

DEFINITION 1. *We say that the signal* (4) *is in general position (G.P.), provided that*

(6) *the* n *by* k *matrix* $\|\lambda_\nu^i\|$ *has no vanishing minor of order* k.

Let $0 < \rho < \frac{1}{2}$, $x = (x_\nu)$, and consider the cube

$$(7)\qquad C_\rho^n : \quad \|x - c\|_\infty < \rho ,$$

where $c = (\frac{1}{2},\ldots,\frac{1}{2})$, and $\|x - c\|_\infty = \max_\nu (|x_\nu - \frac{1}{2}|)$.

DEFINITION 2. *We say that the path* (5) *is* ρ*-admissible, and denote it by* $\prod_n^k(\rho)$, *provided that the original signal* (4) *is in general position, and that the reflected path* $\prod_n^k$ *never penetrates into the cube* (7), *hence that*

$$(8)\qquad \prod_n^k \cap C_\rho^n = \emptyset.$$

As the opposite of the ergodic case, we study the following

PROBLEM 1. *To determine, or estimate, the quantity*

$$(9)\qquad \rho_{k,n} = \text{supremum } \rho$$

for all ρ *having a* ρ*-admissible path* $\prod_n^k(\rho)$.

THEOREM 1. *We have the inequality*

$$(10)\qquad \rho_{k,n} \geq \frac{1}{2} - \frac{k}{2n}, \qquad (1 \leq k \leq n-1).$$

Theorem 1 is established by constructing a path $\Gamma_n^k(\rho)$ for values of ρ which are as close to $\frac{1}{2} - \frac{k}{2n}$ as we wish. In [3] I have shown that the equality sign holds in (10) for the case when $k = 1$. We can now do the same for the other extreme case when $k = n-1$.

THEOREM 2. *We have that*

(11) $$\rho_{n-1,n} = \frac{1}{2} - \frac{n-1}{2n} = \frac{1}{2n}, \quad (n \geq 2).$$

The simplest case when $n = 3$, and therefore

(12) $$\rho_{2,3} = \frac{1}{6},$$

leads to what I call *Kepler's tetrahedron*. J. Kepler was the first to notice that four appropriate vertices of the cube U_3 are the vertices of a *regular* tetrahedron T. As any two facets of T intersect in a facet of U_3 forming equal angles with that facet, it should be clear that the surface of T carries a reflected signal Γ_3^2. It carries, of course, many such, but let us single out one of them and denote it by $\tilde{\Gamma}_3^2$. This signal $\tilde{\Gamma}_3^2$ is readily found to be $\frac{1}{6}$ - admissible, and it is essentially the only Γ_3^2 in G.P. which is $\frac{1}{6}$ - admissible. This is an apparently new characteristic extremum property of Kepler's tetrahedron: *Any other signal* Γ_3^2 *in general position, must penetrate into the cube* C_ρ^3, with $\rho = 1/6$.

Theorem 2 generalizes this extremum property of T: *There is an essentially unique signal* $\tilde{\Gamma}_n^{n-1}$ *which is in general position and is* $\frac{1}{2n}$ *- admissible. It is explicitly given by the equations*

(13) $$\tilde{\Gamma}_n^{n-1} : \quad \begin{aligned} x_\nu &= \langle u_\nu \rangle, \quad (\nu = 1, \ldots, n-1), \\ x_n &= \langle u_1 + u_2 + \cdots + u_{n-1} + \tfrac{n-1}{2} \rangle, \quad (-\infty < u_i < \infty). \end{aligned}$$

THEOREM 3. *We construct explicitly the signal* $\Gamma_n^k(\frac{1}{2} - \frac{k}{2n})$ *for*

(14) $$(k,n) = (2,4) \quad \text{and} \quad (k,n) = (2,6).$$

In view of Theorems 1,2, and 3, I wish to state the following

CONJECTURE 1. The value of the quantity (9) is

(15) $$\rho_{k,n} = \frac{1}{2} - \frac{k}{2n}, \qquad (1 \leq k \leq n-1).$$

The proofs of our results will appear elsewhere; they are based on a discussion of monochromes and n-chromos in R^k. This approach was already used in [3] to establish Conjecture 1 for the case when $k = 1$.

REFERENCES

1. Harald Bohr, Neuer Beweis eines allgemeinen Kronecker'schen Approximationssatzes, Kgl. Danske Videnskab. Selskab., Math.-fys. Med., 6, No.8 (1924). Also in Collected Mathematical works, vol.III, D 2.

2. D. König and A. Szücs, Mouvement d'un point abandonné à l'intérieur d'un cube, Rendiconti del Circ. Mat. di Palermo, 38 (1913), 79-90.

3. I.J. Schoenberg, Extremum problems for the motions of a billiard ball II. The L_∞ norm, Indagationes Math., 38 (1976), 263-279.

United States Military Academy
West Point, New York

and

Mathematics Research Center
University of Wisconsin - Madison

Received August 20, 1978

C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1978) No. 1

A PROOF OF A THEOREM OF COXETER

Joachim Assion

Fakultät für Mathematik, Universität Bielefeld, FR Germany

Presented by H.S.M. Coxeter, F.R.S.C.

The braid group Z_n is defined by the generators x_i for $1 \leq i \leq n-1$ and the relations

$$x_i x_j x_i = x_j x_i x_j , \quad |i-j| = 1$$

and

$$x_i x_j = x_j x_i , \quad |i-j| \neq 1.$$

For every finite factor group F_n of Z_n there exists a positive integer m such that F_n is a factor group of

$$Z_n(m) := Z_n / \langle (x_1^m)^{Z_n} \rangle.$$

A theorem of Coxeter [2] states that $Z_n(m)$ is finite if and only if $(m-2)(n-2) < 4$ holds. The purpose of this paper is to show how Burau's representation of the braid groups [1] may be used to prove Coxeter's theorem.

Let K be a commutative field and let k be a non - zero element of K. Let

$$A = \begin{pmatrix} 1-k & k \\ 1 & o \end{pmatrix}$$

and

$$X_i = \begin{pmatrix} I_{i-1} & & \\ & A & \\ & & I_{n-i-1} \end{pmatrix} , \quad 1 \leq i \leq n-1$$

where I_t is the unit $t \times t$ matrix. Then the mapping

$$x_i \longmapsto X_i$$

induces a homomorphism D of Z_n onto $\langle X_i \mid 1 \leq i \leq n-1 \rangle \leq GL(n,K)$ as Burau [1] has shown.

Let $f \in \mathrm{Hom}_K(K^2,K)$ such that $(x,y)f = x+y$. Then $Af = f$. Therefore $X_i f_n = f_n$ for all i where $f_n \in \mathrm{Hom}_K(K^n,K)$ such that $(k_1,\ldots,k_n)f_n = \sum k_i$. Hence $\ker(f_n)$ is $D(Z_n)$ - invariant. If $\{e_1,\ldots,e_n\}$ is the canonical basis of K^n and if $y_i := e_i - e_{i+1}$ then $\{y_1,\ldots,y_{n-1}\}$ is a basis of $\ker(f_n)$. With respect to this basis the following matrices are associated with the elements $Y_i := X_i | \ker(f_n)$:

$$Y_1 = \begin{pmatrix} -k & & \\ 1 & 1 & \\ & & I_{n-3} \end{pmatrix}, \quad Y_{n-1} = \begin{pmatrix} I_{n-3} & & \\ & 1 & k \\ & & -k \end{pmatrix}, \quad Y_i = \begin{pmatrix} I_{i-2} & & \\ & B & \\ & & I_{n-i-2} \end{pmatrix}$$

for $2 \leq i \leq n-2$ where

$$B = \begin{pmatrix} 1 & k & \\ & -k & \\ & 1 & 1 \end{pmatrix}.$$

Write $\overline{D}$ for this (n-1) - dimensional K - representation of Z_n.

Let $U = \langle Y_1,\ldots,Y_{n-2} \rangle$ and $W = \langle y_1,\ldots,y_{n-2} \rangle$. Then W is U - invariant and $U|W = \overline{D}(Z_{n-1})$. Therefore $\overline{D}(Z_{n-1})$ is isomorphic to a section of $\overline{D}(Z_n)$.

Now let K be the field $\mathbb{C}$ of complex numbers. Choose $k = k(m)$ as follows. Let $p_r(X) := \sum_{i=0}^{r} (-1)^i X^i \in \mathbb{Z}[X]$. Then $(1+X)p_r(X) = 1 + (-1)^r X^{r+1}$. Let $P := \{p_r \mid r \in \mathbb{N}\}$. Now choose $k \in \mathbb{C}$ such that p_{m-1} is the polynomial of lowest degree inside of P such that $p_{m-1}(k) = o$. Then $\overline{D}$ yields a $\mathbb{C}$ - representation of $Z_n(m)$ such that $o(\overline{D}(x_1)) = m$.

<u>Lemma</u>. <u>If</u> $4 < 2m \leq (m,2)n$, <u>then</u> $\overline{D}(Z_n(m))$ <u>is infinite</u>.

<u>Proof</u>. $\overline{D}(Z_n(m))$ operates on a $(n-1)$ - dimensional $\mathbb{C}$ - vector space V. Let $\{e_1,\dots,e_{n-1}\}$ be the canonical basis of V. By definition of k and by hypothesis $\sum_{i=o}^{t-1} k^i = o$ where $(2,m)t = 2m$. Moreover $t \leq n$ by hypothesis. Hence one may assume that $t = n$.

Let $z = \sum_{i=1}^{n-1}\left(\sum_{j=o}^{i-1} k^j\right)e_i$. Then $\overline{D}(Z_n(m))$ leaves z unchanged. If $\overline{D}(Z_n(m))$ were finite then, by Maschke's theorem, there would exist a complement W of $\langle z\rangle$ in V. Now, for each i, $\langle e_i\rangle$ is the eigenspace of $\overline{D}(x_i)$ corresponding to the eigenvalue $-k$. Hence all the vectors e_i would be elements of W.

<u>Theorem</u>. <u>Let</u> m <u>and</u> n <u>be positive integers such that</u> $(m-2)(n-2) \geq 4$. <u>Then</u> $\overline{D}(Z_n(m))$ <u>is infinite</u>.

<u>Proof</u>. Assume $\overline{D}(Z_3(m)) = \langle Y_1, Y_2\rangle$ is finite. If there exists an abelian subgroup of $\overline{D}(Z_3(m))$ of index at most 2 then $[Y_1^2, Y_2^2] = 1$. Hence $k = 1$, and $m = 2$ by definition of k. If such a subgroup of $\overline{D}(Z_3(m))$ does not exist then by [4; th. 26.1] the center C of $\overline{D}(Z_3(m))$ consists of scalar matrices and $\overline{D}(Z_3(m))/C$ is isomorphic either to the alternating group of degree 4 or 5 or to the symmetric group of degree 4. Hence $m \leq 5$, since no non - trivial power of $\overline{D}(x_1)$ is a scalar matrix. Therefore:

(1) If $\overline{D}(Z_3(m))$ is finite then $m \leq 5$.

Let $\overline{D}(Z_4(5)) = \langle Y_1, Y_2, Y_3\rangle$. Write $W_1 = Y_3^{Y_2Y_1}$ and $W_2 = [Y_2, Y_1^{-1}Y_3]$. If $\{e_1, e_2, e_3\}$ is the canonical basis of the $\mathbb{C}$ - vector space V where $\overline{D}(Z_4(5))$ is operating on then $\langle e_2\rangle$ is left invariant by $\langle W_1, W_2\rangle$. Write $\overline{W}_1$ resp. $\overline{W}_2$ for the elements induced by W_1 resp. W_2 on $V/\langle e_2\rangle$. Then $o(\overline{W}_1) = 5 = o(\overline{W}_2)$, $[\overline{W}_1, \overline{W}_2] \neq 1$ and

$W_1W_2^aW_1 \neq W_2^aW_1W_2^a$ for $1 \leq a \leq 4$. Hence $\langle W_1, W_2 \rangle$ is infinite by [4; th. 26.1]. Therefore:

(2) $\overline{D}(Z_4(5))$ is infinite.

Suppose there were positive integers m and n with $(m-2)(n-2) \geqslant 4$ such that $\overline{D}(Z_n(m))$ is finite. Then $m, n \geqslant 3$, and $m < 5$ by (1) and (2). By the lemma the groups $\overline{D}(Z_4(4))$ and $\overline{D}(Z_6(3))$ are infinite.

Clearly, $Z_2(m) \simeq C_m$. Moore has shown in 1897 that $Z_n(2)$ is isomorphic to the symmetric group of degree n. It is well - known [3; 6.6] that $Z_3(3) \simeq SL(2,3)$, $Z_3(5) \simeq SL(2,5) \times C_5$ and that $Z_3(4)$ is isomorphic to the centralizer of an involution of SU(3,3). Put $N = \langle (x_1^{-1}x_3)^{x_2}, x_1^{-1}x_3 \rangle$ in the $Z_4(3)$ case. Then N is a non - abelian normal subgroup of order 27 which is complemented by $\langle x_1, x_2 \rangle$. Therefore $Z_4(3) \simeq GU(3,2)$. An enumeration of the cosets of $\langle x_1, x_2, x_3 \rangle$ in $Z_5(3)$ yields $|Z_5(3)| = 3\,|Sp(4,3)|$. Since $Z_5(3)$ has Sp(4,3) and GU(4,2) as epimorphic images $Z_5(3) \simeq Sp(4,3) \times C_3$.

References

1. W. Burau. Über Zopfgruppen und gleichsinnig verdrillte Verkettungen. Abh. Math. Sem. Hamb. Univ. 11(1937), 179 - 186.

2. H. S. M. Coxeter. Factor groups of the braid group. Proc. Fourth Canad. Math. Congr. (1957), 95 - 122.

3. H. S. M. Coxeter, W. O. J. Moser. "Generators and relations for discrete groups". Springer - Verlag 1965.

4. L. Dornhoff. "Group representation theory, part A". Marcel Dekker Inc. 1971.

Received August 21, 1978

C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1978) No. 1

PERTURBATIONS OF C*-ALGEBRAS, II

by

John Phillips* and Iain Raeburn**

Presented by P. Fillmore, F.R.S.C.

INTRODUCTION

Let A and B be subalgebras of a Banach algebra C and define $||A-B|| = \sup\{||a-B_1||, ||b-A_1||: a\in A_1, b\in B_1\}$ where A_1, B_1 denote the unit balls of A, B respectively. The main question raised by Kadison and Kastler in [9] was the following: if A and B are von Neumann subalgebras of $\mathcal{B}(H)$ for some Hilbert space H and if $||A-B||$ is small, are A and B isomorphic (or unitarily equivalent). For general Banach algebras, this question has been reduced to a problem in algebra cohomology, at least in the presence of a linear homeomorphism close to the identity [8,14]. However, except when A is an injective von Neumann algebra, these hypotheses are very difficult to verify.

For this reason, other methods have been introduced to study these questions, especially in the context of C*-algebras [3,4,5,11,12]. So far, the problem has been solved for abelian C*-algebras [3,11]; ideal C*-algebras [3]; and A.F.-algebras [5,12].

Using techniques adapted from [11] together with a little sheaf cohomology we employ the Dixmier-Douady classification theory to solve the problem for certain type I C*-algebras. Complete proofs will appear elsewhere.

THE MAIN RESULTS

1. Definitions: Let $\mathbb{R}$ denote the real line and S^1 denote the unit circle. If T is a topological space, let R be the sheaf of germs of $\mathbb{R}$-valued functions on T and $\mathcal{U}$ the sheaf of germs of S^1-valued functions on T. See [15] for notation on sheaf theory. We denote by $\check{H}^n(T,\mathbb{Z})$, the n^{th} Čech

* Dalhousie University; ** University of New South Wales

cohomology group of the space T with coefficients in the integers, $\mathbb{Z}$. Moreover, as noted in [6, 10.7.13] we have $H^n(T,\mathcal{U}) \cong \check{H}^{n+1}(T,\mathbb{Z})$ for all $n>0$. As in [6, 10.9.1], if A is a separable continuous trace C*-algebra with spectrum $\hat{A}$, then A defines a unique element $\delta(A)$ in $\check{H}^3(\hat{A},\mathbb{Z})$.

2. Theorem: Let K be a Hilbert space and suppose A , B are C*-algebras on K with $||A-B|| < \frac{1}{64}$. If A is separable with continuous trace, then so is B and there is a homeomorphism $\hat{A} \to \hat{B}$ such that the induced isomorphism $\check{H}^3(\hat{A},\mathbb{Z}) \to \check{H}^3(\hat{B},\mathbb{Z})$ takes $\delta(A)$ to $\delta(B)$.

To prove this theorem, we first identify $\hat{A}$ and $\hat{B}$ with a single space T via a homeomorphism $\hat{A} \to \hat{B}$ constructed in [11]. We then carefully construct an open cover $\{T_i\}$ of T and 2-cocycles $\{u_{ijk}\}$, $\{v_{ijk}\} \in C^2(\{T_i\},\mathcal{U})$ which are representatives for $\delta(A)$, $\delta(B)$, respectively under the identification $H^2(T,\mathcal{U}) \cong \check{H}^3(T,\mathbb{Z})$. By construction we see that $|u_{ijk}-v_{ijk}| < \sqrt{2}$ so that $f_{ijk} = u_{ijk}^{-1}v_{ijk}$ is a 2-cocycle with $\arg f_{ijk} \subseteq (-\frac{\pi}{2},\frac{\pi}{2})$. Thus, we can apply Log to obtain $\{g_{ijk}\}$, an element of $C^2(\{T_i\},R)$. Since R is a fine sheaf, $\{g_{ijk}\}$ is trivial. By exponentiating we get that $\{u_{ijk}\}$ and $\{v_{ijk}\}$ are equivalent, i.e. $\delta(A) = \delta(B)$.

The relevance of this result to the isomorphism problem will be seen below.

3. Definitions: Let $K(H)$ denote the C*-algebra of compact operators on the separable Hilbert space H . A C*-algebra A is called stable [1] if $A \cong A \otimes K(H)$ (where $\otimes$ denotes the completion of the algebraic tensor product in the minimal C*-cross-norm.) Two C*-algebras A and B are said to be stably isomorphic [1] if $A \otimes K(H) \cong B \otimes K(H)$.

Using results of [6, ch. 10] and [7] it is not hard to show that if A and B are separable continuous trace C*-algebras then A is stably isomorphic to B if and only if there is a homeomorphism $\hat{A} \to \hat{B}$ carrying $\delta(A)$ to $\delta(B)$. Thus, we get:

4. Corollary: Let A and B satisfy the hypotheses of theorem 2, then A is stably isomorphic to B .

Now, in order to eliminate the "stable" conclusion we proceed by more direct but nontrivial computations to prove the following theorem.

5. Theorem: Let A and B be C*-algebras on a Hilbert space K with $||A-B|| < \frac{1}{142}$. Let X be a compact Hausdorff space such that $A \cong C(X) \otimes K(H)$. Then A is unitarily equivalent to B via a unitary in $(A \cup B)''$.

Combining this with the previous corollary, using techniques of [11] and results of [6] enables us to prove:

6. Theorem: Let A and B be C*-subalgebras of $B(K)$ with $||A-B|| < \frac{1}{426}$. If A is a separable, stable, continuous trace C*-algebra, then $B \cong A$.

In certain special cases we have been able to obtain stronger results:

7. Theorem: Let A and B be C*-algebras on K with $||A-B|| < k\ (\leq 10^{-9})$. If A is a unital continuous trace C*-algebra and $\delta(A) = 0$ then there is a unitary u in $(A \cup B)''$ with $uBu^* = A$ and $||1-u|| < 2400k + 4458\sqrt{k}$.

POSSIBLE DIRECTIONS

Since any postliminal C*-algebra has a composition series where each of the quotients has continuous trace [6, 4.5.5] our results might shed some light on this case. For example, if $0 \subseteq I \subseteq A/I$ where I and A/I have continuous trace, then $||A-B||$ small implies that we can form $0 \subseteq J \subseteq B$ with $||I-J||$ small and $||A/I - B/J||$ small [11, lemma 2.6]. Now, if we could prove a version of theorem 6 which would yield a unitary u close to 1 so that $uJu^* = I$, then we would let $\tilde{B} = uBu^*$ so that $0 \subseteq I \subseteq \tilde{B}$ and $||A/I - \tilde{B}/I||$ is small. Let $C = C^*(A,\tilde{B})$ and represent C/I on a Hilbert space K . We now apply our (hypothetical) improved version of theorem 6 to A/I and $\tilde{B}/I$

to get an isomorphism $\phi: A/I \to \tilde{B}/I$ close to the identity. Then $\psi_1 \equiv \text{id}: A/I \to C/I$ and $\psi_2 \equiv \phi \circ \text{id}: A/I \to C/I$ are two close extensions of the same C*-algebra A/I by the ideal I [cf., 13]. If these two extensions are equivalent (via a unitary in C) then $A \cong B$. This problem may be more tractable [cf., 13].

References

1. L.G. Brown, Stable Isomorphism of Hereditary Subalgebras of C*-algebras, Pac. J. Math., 71, #2, (1977), 335-348.

2. E. Christensen, Perturbations of type I von Neumann algebras, J. London Math. Soc. 9 (1975), 395-405.

3. ______________, Perturbations of operator algebras, Invent. Math. 43 (1977), 1-13.

4. ______________, Perturbations of operator algebras II, Ind. Univ. Math. J. 26 (1977), 891-904.

5. ______________, Near inclusions of C*-algebras, preprint.

6. J. Dixmier, Les C*-algèbras et leurs representations, Gauthier-Villars, Paris, 1969.

7. Dixmier, Champs continus d'espaces hilbertiens et de C*-algèbres, II, J. Math. Pures et Appl., 42 (1963), 1-20.

8. B.E. Johnson, Perturbations of Banach algebras, Proc. London Math. Soc. 34 (1977), 439-458.

9. R.V. Kadison and D. Kastler, Perturbations of von Neumann algebras I, Stability of type, Amer. J. Math. 94, No. 1 (1972), 38-54.

10. R.V. Kadison and J.R. Ringrose, Cohomology of operator algebras, II, Extended cobounding and the hyperfinite case, Ark. Mat. 9 (1971), 55-63.

11. J. Phillips, Perturbations of C*-algebras, Ind. Univ. Math. J., 23 (1974), 1167-1176.

12. ______________ and I. Raeburn, Perturbations of A.F.-algebras, Can. J. Math., to appear.

13. ______________ and I. Raeburn, On extensions of A.F.-algebras, preprint.

14. I. Raeburn and J.L. Taylor, Hochschild cohomology and perturbations of Banach algebras, J. Functional Anal., 25 (1977), 258-266.

15. F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Scott, Foresman and Co., Glenview, Ill., 1971.

Received August 25, 1978

C. R. Math. Rep. Acad. Sci. Canada - Vol. 1 (1978) No. 1

An Explicit Construction of Some Discrete Unitary Series of Representations of u(p,q)

F. Lemire* and J. Patera**

Presented by H. Zassenhaus, F.R.S.C.

Gel'fand and Graev [1] have constructed certain discrete unitary series of irreducible representations of the algebra $u(p,q)$. In this note we show that using similar methods further such series can be constructed.

Following [1] we fix a positive integer n and for $k = 1,2,\ldots,n-1$ choose pairs (i_k, i'_k) with $i_k \in \{0,\ldots,k\}$, $i'_k \in \{1,\ldots,k+1\}$ and $i_k < i'_k$. Let $H\{(i_k,i'_k)\}$ be the Hilbert space having an orthonormal basis labelled by the set of all arrays of integers $(m_{ij})_{1\leq i\leq j\leq n}$ with fixed top row $(m_{1n},\ldots,m_{nn})$ and the other components satisfying:

(1) $m_{jk} \geq m_{j+1,k}$, $\quad 1 \leq j < k \leq n$,

(2) $m_{j-1,k+1}+1 \geq m_{j,k} \geq m_{j,k+1}+1$, $\quad 1 \leq j < i_k$,

(3) $m_{j,k+1} \geq m_{jk} \geq m_{j+1,k+1}$, $\quad i_k < j < i'_k$,

(4) $m_{j+1,k+1}-1 \geq m_{jk} \geq m_{j+2,k+1}-1$, $\quad i'_k \leq j \leq k$

(by convention we set $m_{0,k+1} = +\infty$ and $m_{k+2,k+1} = -\infty$). If $\xi(m)$ is the basis vector in $H\{(i_k,i'_k)\}$ associated with the array m we define linear operators

$$E_{\mu,\mu}\,\xi(m) = (r_\mu - r_{\mu-1})\xi(m), \quad \mu = 1,2,\ldots,n ,$$

where $r_k = m_{1k} + \ldots + m_{kk}$ for $k = 1,2,\ldots,n$ and $r_0 = 0$.

$$E_{\mu,\mu-1}\,\xi(m) = a^1_{\mu-1}\,\xi(m^1_{\mu-1}) + \ldots + a^{\mu-1}_{\mu-1}\,\xi(m^{\mu}_{\mu}), \quad \mu = 2,\ldots,n ,$$

$$\text{where}\quad a^j_{\mu-1} = e^{\frac{i\pi N^j_{\mu-1}}{2}} \left[- \frac{\prod_{i=1}^{k} (m_{i\,k} - m_{j,k-1} - i + j + 1) \prod_{i=1}^{k-2} (m_{k,k-2} - m_{j,k-1} - i + j)}{\prod_{i \neq j} (m_{i,k-1} - m_{j,k-1} - i + j + 1)\,(m_{i,k-1} - m_{j,k-1} - i + j)} \right]^{1/2},$$

$(m^j_{\mu-1})$ denotes the array obtained from m replacing $m_{j,\mu-1}$ by $m_{j,\mu-1}{}^{-1}$ and $N^j_{\mu-1}$ is the number of negative factors in the expression $a^j_{\mu-1}$ (N.B. $N^j_{\mu-1}$ depends only on the indices $\{(i_k, i'_k)\}$ and not on the array m).

$$E_{\mu-1,} = E^{tr}_{\mu,\mu-1} \qquad \mu = 2,\ldots,n .$$

For certain values of the indices $\{(i_k, i'_k)\}$ [cf. 3] these operators generate an algebra isomorphic to $gl(n,C)$. In particular, when $i_k = 0$ and $i'_k = k+1$ for $k = 1,\ldots,n-1$, we have the finite dimensional representations of $gl(n,C)$ as described by Gel'fand and Tsetlin [2].

Let $\varepsilon = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, where $\varepsilon_1 = +1$, $\varepsilon_i = \pm 1$ for $i \geq 2$ and exactly p terms equal to $+1$. A set of indices $\{(i_k, i'_k)\}$ is said to be compatible with ε iff in $H\{(i_k, i'_k)\}$ we have $E^*_{\mu,\mu-1} = \varepsilon_{\mu-1}\varepsilon_\mu E_{\mu-1,\mu}$ for $\mu = 2,\ldots,n$. Observe now that if $\{(i_k, i'_k)\}$ is compatible with ε then in $H\{(i_k, i'_k)\}$ the operators $i\,E_{\mu,\mu}$ for $\mu = 1,\ldots,n$ and $i(E_{\mu,\mu-1} \pm E_{\mu-1,\mu})$ for $\varepsilon_{\mu-1}\varepsilon_\mu = \pm 1$ are all skew-Hermitian. Since these operators generate a real subalgebra $u(p,q)$ of $gl(n,C)$, we have for each such set of indices a series of unitary representations of $u(p,q)$.

A straight forward analysis of $\arg(a^j_{k-1})$ for a given $\{(i_k, i'_k)\}$ shows that there are $p+1$ sets of indices compatible with the sequence

$\underbrace{\{1,1,\ldots,1}_{p\text{ terms}},-1,\ldots,-1\}$. These yield precisely the $p+1$ series of unitary representations of $u(p,q)$ given in [1], where the various rows of the arrays correspond to the chain of subalgebras $u(p,q)\supset u(p,q-1)\supset\ldots\supset u(p,0)\supset\ldots\supset u(1,0)$. This same analysis also shows that every sequence ε admits at least one compatible set of indices (and hence series of unitary representations of $u(p,q)$) where the rows correspond to the chain of subalgebras $u(p,q)\supset u(p_{n-1},q_{n-1})\supset\ldots\supset u(p_\nu,q_\nu)\supset\ldots\supset u(1,0)$ with p_ν (resp q_ν) denoting the number of +1's in the truncated sequence $\{\varepsilon_1,\ldots,\varepsilon_\nu\}$.

In addition to the above series of unitary representation we have observed that some other sets of indices yield unitary representations of $u(p,q)$ when we place restrictions on the defining parameters $(m_{1n},\ldots,m_{nn})$. In Table I we illustrate this as well as the material above for the case $n = 3$.

References

* Department of Mathematics, University of Windsor, Windsor, Ontario, Canada
** Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec, Canada

[1] I.M. Gel'fand and M.I. Graev "Finite Dimensional Irreducible Representations of the Unitary and the Full Linear Groups and Related Special Functions" Izv. Akad. Nauk. SSSR Ser. Math 29, (1965) 1329 [AMS Transl. 64, (1967) 116 Ser. 2], MR 34 #1450.

[2] I.M. Gel'fand and M.L. Tsetlin "Finite Dimensional Representations of the Group of Unimodular Matrices" Dokl. Akad. Nauk. SSSR 71, (1950) 825, MR12 # 9.

[3] F.W. Lemire and J. Patera "Formal Analytic Continuation of Gel'fand's Finite Dimensional Representations of gl(n,C)" (to appear).

TABLE I GEL'FAND REPRESENTATIONS OF gl(3,C) and THEIR RESTRICTIONS

(i_2,i_2') \ (i_1,i_1')	(0,2) $m_{12} \geq m_{11} \geq m_{22}$	(0,1) $m_{22}-1 \geq m_{11}$	(1,2) $m_{11} \geq m_{12}+1$
(0,3) ≡ $m_{13} \geq m_{12} \geq m_{23}$ $m_{23} \geq m_{22} \geq m_{33}$	Unitary $su(3) \supset su(2)$	Not a Representation	Not a Representation
(0,2) ≡ $m_{13} \geq m_{12} \geq m_{23}$ $m_{33}-1 \geq m_{22}$	Unitary If $m_{13} = m_{23}$ $su(2,1) \supset su(2)$	Unitary $su(2,1) \supset su(1,1)$	Unitary $su(2,1) \supset su(1,1)$
(0,1) ≡ $m_{23}-1 \geq m_{12} \geq m_{33}-1$ $m_{33}-1 \geq m_{22}$	Unitary $su(2,1) \supset su(2)$	Not a Representation	Not a Representation
(1,3) ≡ $m_{12} \geq m_{13}+1$ $m_{23} \geq m_{22} \geq m_{33}$	Unitary If $m_{23} = m_{33}$ $su(2,1) \supset su(2)$	Unitary $su(2,1) \supset su(1,1)$	Unitary $su(2,1) \supset su(1,1)$
(1,2) ≡ $m_{12} \geq m_{13}+1$ $m_{33}-1 \geq m_{22}$	Unitary $su(2,1) \supset su(2)$	Never Unitary	Never Unitary
(2,3) ≡ $m_{12} \geq m_{13}+1$ $m_{13}+1 \geq m_{22} \geq m_{23}+1$	Unitary $su(2,1) \supset su(2)$	Not a Representation	Not a Representation

Received September 4, 1978