C. R. Math. Rep. Acad. Sci. Canada Vol. **40** (4) 2018, pp. 104–133

CERTAIN PROPERTIES OF TRACIAL APPROXIMATION C*-ALGEBRAS

GEORGE A. ELLIOTT, FRSC, QINGZHAI FAN, AND XIAOCHUN FANG

ABSTRACT. We show that the following properties of the C*-algebras in a class Ω are inherited by simple unital C*-algebras in the class TAΩ: (1) β-comparison ($1 \leq \beta < \infty$), (2) n-comparison, (3) trace $\mathcal{Z}$- absorption, (4) m-almost divisibility, (5) (n, m) ($m \neq 0$) comparison, and (6) tracial approximate divisibility. As an application, every unital simple C*-algebra with tracial topological rank at most k has the property of k-comparison. Also as an application, let A be an infinite-dimensional simple unital C*-algebra such that A has one of the above-listed properties. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra C*(G, A, α) also has the property under consideration.

RÉSUMÉ. On considère plusieurs propriétés d'une C*-algèbre simple à élément unité qui sont héritées par approximation traciale. Comme application on démontre que ces propriétés sont aussi héritées par la C*-algèbre produit croisé associée à une action d'un groupe fini qui possède la propriété de Rokhlin traciale.

1. **Introduction** The Elliott program for the classification of amenable C*-algebras might be said to have begun with the K-theoretical classification of AF algebras in [9]. Since then, many classes of C*-algebras have been classified by the Elliott invariant. Among them, one important class is the class of simple unital AH algebras without dimension growth (in the real rank zero case see [11], and in the general case see [12]). To axiomatize Elliott-Gong's decomposition theorem for AH algebras of real rank zero (classified by Elliott-Gong in [11]) and Gong's decomposition theorem ([20]) for simple AH algebras (classified by Elliott-Gong-Li in [12]), Huaxin Lin introduced the concepts of TAF and TAI ([27] and [28]). Instead of assuming inductive limit structure, he started with a certain abstract approximation property, and showed that C*-algebras with this abstract approximation property and certain additional properties are AH algebras without dimension growth. More precisely, Lin introduced the class of tracially approximate interval algebras (also called C*-algebras of tracial topological rank one). This axiomatization has proved to be very important in the classification of simple amenable C*-algebras. For example, it led to the classification of unital simple separable amenable C*-algebras with finite nuclear

---

Received by the editors on April 7, 2019.
AMS Subject Classification: Primary: 46L35; secondary: 46L05, 46L80.
Keywords: C*-algebras, tracial approximation, Cuntz semigroup.

dimension in the UCT class (see [21], [13], [55]). The isomorphism theorem was established first for those separable amenable C*-algebras with generalized topological tracial rank at most one (see [21]). Simple C*-algebras with generalized tracial topological rank at most one have good regularity properties. There are three regularity properties of particular interest: tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$, also called $\mathcal{Z}$-stability; finite nuclear dimension; and strict comparison of positive elements. The last property can be reformulated as an algebraic property of the Cuntz semigroup, called almost unperforation. Toms and Winter have conjectured (see e.g. [15]) that these three fundamental properties are equivalent for all separable, simple, unital, amenable C*-algebras (and this has now almost completely been proved (see [3], [24], [49], [54] and [5])).

Inspired by Lin's tracial approximation by interval algebras in [28], Elliott and Niu in [14] considered the natural notion of tracial approximation by other classes of C*-algebras. Let Ω be a class of unital C*-algebras. Then the class of unital simple separable C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by $\mathrm{TA}\Omega$, is defined as follows. A simple unital C*-algebra A is said to belong to the class $\mathrm{TA}\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any non-zero element $a \geq 0$, there are a projection $p \in A$ and a C*-subalgebra B of A with $1_B = p$ and $B \in \Omega$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in_\varepsilon B$ for all $x \in F$, and

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

The question of which properties pass from a class Ω to the class $\mathrm{TA}\Omega$ is interesting and sometimes important. In fact, the property of being of stable rank one, and the property that the strict order on projections is determined by traces, are important in the classification theorem of [21].

In this paper, we show that the following properties of unital C*-algebras in a class Ω are inherited by simple unital C*-algebras in the class $\mathrm{TA}\Omega$:

(1) β-comparison (in the sense of [24]; see below),

(2) n-comparison (in the sense of [58]; see below),

(3) tracial $\mathcal{Z}$-absorption,

(4) m-almost divisibility,

(5) (n, m) $(m \neq 0)$ comparison, and

(6) tracial approximate divisiblity.

As applications, we get a large class of C*-algebras which have the above properties. For example, every simple unital C*-algebra with tracial topological rank at most k has k-comparison (in the sense of [58]). Let Ω be a class of C*-algebras such that $rc(B) \leq n$ $(n \neq 0)$ for any $B \in \Omega$ (see 2.3 below). Then $rc(A) \leq n$ for any unital simple C*-algebra $A \in \mathrm{TA}\Omega$.

The Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by Connes in [6]. It was adapted by Herman and Ocneanu for automorphisms of UHF algebras in [22]. Rørdam [46] and Kishimoto [26] considered the Rokhlin property in a much more general C*-algebra context. More recently, Phillips and Osaka studied actions of a finite group and of the group $\mathbb{Z}$ of integers on certain simple C*-algebras which have a modified Rokhlin property (cf. 2.10) in [36] and [41].

In [19], the following result was obtained: Let Ω be a class of unital C*-algebras such that Ω is closed under passing to unital hereditary C*-subalgebras and tensoring by matrix algebras. Let $A \in \mathrm{TA}\Omega$ be an infinite dimensional simple unital C*-algebra. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property (cf. 2.10). Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ belongs to $\mathrm{TA}\Omega$.

Using the results mentioned, we obtain the following theorems: Let A be an infinite dimensional simple unital C*-algebra such that A has (n, m) $(m \neq 0)$ comparison (respectively, m-almost divisibility, tracial $\mathcal{Z}$-absorption, n-comparison, or β-comparison). Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ has (n, m) $(m \neq 0)$ comparison (respectively, m-almost divisibility, tracial $\mathcal{Z}$-absorption, n-comparison, or β-comparison).

2. **Definitions and Preliminaries**

Let a and b be positive elements of a C*-algebra A. We write $[a] \leq [b]$ if there is a partial isometry $v \in A^{**}$ with $vv^* = P_a$ such that, for every $0 \leq c \in \mathrm{Her}(a)$, $cv \in A$ and $v^*cv \in \mathrm{Her}(b)$. ($[a] \leq [b]$ implies that a is Cuntz subequivalent to b, i.e. $a \lesssim b$. If A has stable rank one then, by [7], $[a] \leq [b]$ if $a \lesssim b$ but even in this case the preorder relation $[a] \leq [b]$ is not necessarily an order relation.) We write $[a] = [b]$ if, for some v as above, $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \leq [b]$ if in addition there are n mutually orthogonal positive elements $b_1, b_2, \cdots, b_n \in \mathrm{Her}(b)$ such that $[a] \leq [b_i]$, $i = 1, 2, \cdots, n$ (see Definition 1.1 of [39], Definition 3.2 of [38], or Definition 3.5.2 of [29].)

Let A be a C*-algebra, and let $\mathrm{M}_n(A)$ denote the C*-algebra of $n \times n$ matrices with entries elements of A. Let $\mathrm{M}_\infty(A)$ denote the algebraic inductive limit of the sequence $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is the canonical embedding as the upper left-hand corner block. Let $\mathrm{M}_\infty(A)_+$ (respectively, $\mathrm{M}_n(A)_+$) denote the positive elements of $\mathrm{M}_\infty(A)$ (respectively, $\mathrm{M}_n(A)$). Given $a, b \in \mathrm{M}_\infty(A)_+$, we say that a is Cuntz subequivalent to b (written $a \lesssim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \lesssim b$ and $b \lesssim a$. We write $\langle a \rangle$ for the equivalence class of a.

The object $\mathrm{W}(A) := \mathrm{M}_\infty(A)_+/\sim$ will be called the Cuntz semigroup of A. (See [7].) Observe that any $a, b \in \mathrm{M}_\infty(A)_+$ are Cuntz equivalent to orthogonal elements $a', b' \in \mathrm{M}_\infty(A)_+$ (i.e., $a'b' = 0$), and so $\mathrm{W}(A)$ becomes an ordered semigroup when equipped with the addition operation

$$\langle a \rangle + \langle b \rangle = \langle a + b \rangle$$

whenever $ab = 0$, and the order relation

$$\langle a\rangle \leq \langle b\rangle \Leftrightarrow a \lesssim b.$$

We denote by A_+ the positive cone of A and by $\mathrm{T}(A)$ the set of tracial states of A. We define the dimension function (or rank function) d_τ associated with $\tau \in \mathrm{T}(A)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$ for $a \in \mathrm{M}_k(A)_+$, where τ is regarded as an unnormalized trace on $\mathrm{M}_k(A)$. We shall say that a separable C*-algebra A has strict comparison if for $a,\ b \in \mathrm{M}_k(A)_+$, with $d_\tau(a) < d_\tau(b)$ for any $\tau \in \mathrm{T}(A)$, then we have $a \lesssim b$. (Thus, we are suppressing the question of quasitraces).

Let A be a unital C*-algebra. We say $\mathrm{W}(A)$ is almost unperforated if whenever $(k+1)x \leq ky$ for some $k \in \mathbb{N}$, it follows that $x \leq y$ (see [47]).

Note that if A is a unital separable simple exact C*-algebra, then A has strict comparison if and only if $\mathrm{W}(A)$ is almost unperforated (see [47]).

The property of n-comparison was introduced by Winter in [58]. Let A be a unital C*-algebra. We say A has n-comparison, if, whenever $x,\ y_0,\ y_1,\ y_2,\ \cdots,\ y_n$ are elements in $\mathrm{W}(A)$ such that $x <_s y_j$ for all $j = 0,\ 1,\ \cdots,\ n$, then $x \leq y_0 + y_1 + \cdots + y_n$. Here, $x <_s y$ means $(k+1)x \leq ky$ for some natural number k. It follows immediately from the definitions that $\mathrm{W}(A)$ is almost unperforated if and only if A has 0-comparison in the sense of Winter.

Kirchberg and Rørdam introduced a weaker comparison property and also a property of a C*-algebra called β-comparison in [24]. Let A be a simple, unital, and stably finite C*-algebra. We say that A has local weak comparison, if there is a constant $\gamma(A) \in [1, \infty)$ such that the following implication holds for all positive elements a and b in A: if

$$\gamma(A) \sup_{\tau\in\mathrm{T}(A)} d_\tau(a) < \inf_{\tau\in\mathrm{T}(A)} d_\tau(b),$$

then $a \lesssim b$.

If $\mathrm{M}_n(A)$ has local weak comparison for all n, and $\sup_n \gamma(\mathrm{M}_n(A)) < \infty$, then we say that A has weak comparison.

Let A be a unital C*-algebra and let $1 \leq \beta < \infty$. We say that A has β-comparison if for all $x, y \in \mathrm{W}(A)$ and all integers $k,\ l \geq 1$ with $k > \beta l$, the inequality $kx \leq ly$ implies $x \leq y$. It follows immediately from the definitions that $\mathrm{W}(A)$ is almost unperforated if and only if A has 1-comparison in the sense of Kirchberg and Rørdam.

It is an open problem if Kirchberg's and Rørdam's weak comparison and β-comparison, Winter's m-comparison, and strict comparison all agree for simple unital C*-algebras. (Somewhat confusingly, it is known that m-comparison for a particular m does agree with β-comparison for $\beta = m + 1$.)

Let A be a stably finite unital C*-algebra. Recall that a positive element $a \in A$ is called purely positive if a is not Cuntz equivalent to a projection. This is equivalent to saying that 0 is an accumulation point of $\sigma(a)$ (recall that $\sigma(a)$ denotes the spectrum of a).

Given a in $\mathrm{M}_\infty(A)_+$ and $\varepsilon > 0$, we denote by $(a-\varepsilon)_+$ the element of $\mathrm{C}^*(a)$ corresponding (via the functional calculus) to the function $f(t) = \max(0, t-\varepsilon)$, $t \in \sigma(a)$. By the functional calculus, it follows in a straightforward manner that $((a-\varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+$.

THEOREM 2.1. *([1], [23], [48].) Let A be a stably finite* C**-algebra.*

(1) *Let $a,\ b \in A_+$ and $\varepsilon > 0$ be such that $\|a - b\| < \varepsilon$. Then there is a contraction d in A with $(a-\varepsilon)_+ = dbd^*$.*

(2) *Let $a,\ p$ be positive elements in $\mathrm{M}_\infty(A)$ with p a projection. If $p \lesssim a$, then there is b in $\mathrm{M}_\infty(A)_+$ such that $bp = 0$ and $b + p \sim a$.*

(3) *The following conditions are equivalent: $(1)'$ $a \lesssim b$, $(2)'$ for any $\varepsilon > 0$, $(a-\varepsilon)_+ \lesssim b$, and $(3)'$ for any $\varepsilon > 0$, there is $\delta > 0$, such that $(a-\varepsilon)_+ \lesssim (b-\delta)_+$.*

(4) *Let a be a purely positive element of A (i.e., a is not Cuntz equivalent to a projection). Let $\delta > 0$, and let $f \in C_0(0,1]$ be a non-negative function with $f = 0$ on $(\delta, 1)$, $f > 0$ on $(0, \delta)$, and $\|f\| = 1$. We have $f(a) \neq 0$ and $(a-\delta)_+ + f(a) \lesssim a$.*

DEFINITION 2.2. ([51].) Let A be a unital stably finite exact C*-algebra. Define the radius of comparison for A to be $\mathrm{rc}(A) = \inf\{m/n : \langle a\rangle \leq \langle b\rangle$ whenever $n\langle a\rangle + m\langle 1\rangle \leq n\langle a\rangle\}$. We say that the C*-algebra A has (n,m) comparison whenever $n\langle a\rangle + m\langle 1\rangle \leq n\langle a\rangle$ implies $\langle a\rangle \leq \langle b\rangle$, for any $a, b \in \mathrm{M}_\infty(A)_+$.

DEFINITION 2.3. ([44].) Let $m \in \mathbb{N}$. We say that A is m-almost divisible if for each $a \in \mathrm{M}_\infty(A)_+$, $k \in \mathbb{N}$, and each $\varepsilon > 0$, there exists $b \in \mathrm{M}_\infty(A)_+$ such that $k\langle b\rangle \leq \langle a\rangle$ and $\langle (a-\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b\rangle$.

Let Ω be a class of unital C*-algebras. In this paper, we shall study the class of simple unital C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by $\mathrm{TA}\Omega$.

DEFINITION 2.4. ([14], [28].) A simple unital C*-algebra A is said to belong to the class $\mathrm{TA}\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any non-zero element $a \geq 0$, there exist a non-zero projection $p \in A$ and a C*-subalgebra B of A with $1_B = p$ and $B \in \Omega$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in_\varepsilon B$ for all $x \in F$, and

(3) $[1-p] \leq [a]$.

Let $\mathfrak{I}^k$ denote the class of all unital C*-algebras which are unital hereditary subalgebras of C*-algebras of the form $C(X) \otimes F$, where X is a k-dimensional finite CW complex and F is a finite dimensional C*-algebra. A is said to have tracial topological rank at most k if $A \in \mathrm{TA}\mathfrak{I}^k$.

LEMMA 2.5. *([14].) If the class Ω is closed under tensoring with matrix algebras and under passing to unital hereditary* C**-subalgebras, then the class* $\mathrm{TA}\Omega$ *is closed under tensoring with matrix algebras and under passing to unital hereditary* C**-subalgebras.*

The following lemma is obvious, and we omit the proof.

Lemma 2.6. *The property of m-almost divisibility (respectively, n-comparison, or β-comparison) is preserved under tensoring with matrix algebras and under passing to unital hereditary* C*-*subalgebras.*

Definition 2.7. ([23].) We say a unital C*-algebra A is tracially $\mathcal{Z}$-absorbing if $A \neq \mathbb{C}$ and, for any finite set $F \subseteq A$, $\varepsilon > 0$, non-zero positive element $a \in A$, and $n \in \mathbb{N}$, there is a completely positive order zero contraction $\psi : M_n \to A$, where "order zero" means preserving orthogonality, i.e., $\psi(e)\psi(f) = 0$ for all $e, f \in \mathrm{M}_n$ with $ef = 0$, such that the following properties hold:

(1) $1 - \psi(1) \precsim a$, and

(2) for any normalized element $x \in \mathrm{M}_n$ (i.e., with $\|x\| = 1$) and any $y \in F$ we have $\|\psi(x)y - y\psi(x)\| < \varepsilon$.

Definition 2.8. ([2].) A finite dimensional C*-algebra is completely noncommutative if it has no commutative direct summands, equivalently, no abelian central projections.

Definition 2.9. ([31].) Let A be a unital C*-algebra. We say that A is tracially approximately divisible if for any $\varepsilon > 0$, any $n \in \mathbb{N}$, any finite subset $F = \{a_1,\ a_2,\ \cdots,\ a_n\} \subseteq A$, and any $y \in A_+$ with $\|y\| = 1$, there exist a projection $e \in A$, a completely noncommutative finite dimensional C*-algebra D, and an injective unital homomorphism $\varphi : D \to eAe$ such that

(1) $\|\varphi(b)a_k - a_k\varphi(b)\| < \varepsilon$ for $k = 1,\ 2,\ \cdots,\ n$ and all $b \in D$ with $\|b\| \leq 1$,

(2) $1 - e$ is Murray-von Neumann equivalent to a projection in $\overline{yAy}$, and

(3) $\|eye\| > 1 - \varepsilon$.

Definition 2.10. ([41].) Let A be a simple unital C*-algebra, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the tracial Rokhlin property if, for any finite set $F \subseteq A$, any $\varepsilon > 0$, and any non-zero positive element $b \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g,\ h \in G$,

(2) $\|e_g d - d e_g\| < \varepsilon$ for all $g \in G$ and all $d \in F$,

(3) with $e = \Sigma_{g\in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary C*-subalgebra of A generated by b, and

(4) $\|ebe\| \geq \|b\| - \varepsilon$.

Theorem 2.11. *([23].) Let A be a separable simple unital amenable* C*-*algebra. If A is tracially $\mathcal{Z}$-absorbing, then A is $\mathcal{Z}$-absorbing.*

3. The Main Results

Theorem 3.1. *Let Ω be a class of stably finite unital* C*-*algebras which are tracially $\mathcal{Z}$-absorbing. Then A is tracially $\mathcal{Z}$-absorbing for any simple unital* C*-*algebra $A \in \mathrm{TA}\Omega$.*

PROOF. We need to show that for any finite set $F = \{a_1,\ a_2,\ \cdots,\ a_k\} \subseteq A$, any $\varepsilon > 0$, any non-zero positive element $a \in A$, and any $n \in \mathbb{N}$, there is an order zero contraction $\psi : \mathrm{M}_n \to A$ such that the following properties hold:

(1) $1 - \psi(1) \precsim a$, and

(2) for any normalized element $x \in \mathrm{M}_n$ and any $y \in F$, we have $\|\psi(x)y - y\psi(x)\| < \varepsilon$.

Since $A \in \mathrm{TA}\Omega$, for $F = \{a_1,\ a_2,\ \cdots,\ a_k\}$, any $\varepsilon' > 0$, any non-zero positive element $a \in A$, by Theorem 3.3 of [17], there are a C*-subalgebra B of A and a projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

(1) $\|a_i p - p a_i\| < \varepsilon'$, for all $1 \le i \le k$,

(2) $p a_i p \in_{\varepsilon'} B$, for all $1 \le i \le k$, $pap \in_{\varepsilon'} B$, and

(3) $2[1-p] \le [a]$.

We divide the proof into two cases.

Firstly, let us assume $1 - p = 0$. We may assume that $\|a\| = 1$. Then $1_B = 1_A$ and there exist $a'_1,\ a'_2,\ \cdots,\ a'_k,\ a' \in B$ such that $\|a_i - a'_i\| < \varepsilon/3$, for all $1 \le i \le k$, and $\|a - a'\| < \varepsilon/3$.

Since $B \in \Omega$, and $(a' - \varepsilon/2)_+ \neq 0$, there is a completely positive order zero contraction $\psi : \mathrm{M}_n \to B \subseteq A$ with the following properties:

(1) $1 - \psi(1) \precsim (a' - \varepsilon/2)_+ \lesssim a$, and

(2) for any normalized element $x \in \mathrm{M}_n$ and any a'_i, we have $\|\psi(x)a'_i - a'_i\psi(x)\| < \varepsilon/3$.

Therefore, we have $\|\psi(x)a_i - a_i\psi(x)\| < \varepsilon$, for any normalized element $x \in \mathrm{M}_n$ and any $a_i \in F$.

Secondly, let us assume $p \neq 1$. Since, by (3), $2[1-p] \le [a]$, there are non-zero and mutually orthogonal projections $p_1,\ p_2 \in A$ such that $[p_1 + p_2] \le [a]$.

Since $A \in \mathrm{TA}\Omega$, for $G = \{a_1,\ a_2,\ \cdots,\ a_k,\ p_1\}$, and any $\varepsilon' > 0$, by Theorem 3.3 of [17], there are a C*-subalgebra B of A and a non-zero projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

(1)′ $\|a_i p - p a_i\| < \varepsilon'$, $\|p_1 p - p p_1\| < \varepsilon'$ for all $1 \le i \le k$,

(2)′ $p a_i p \in_{\varepsilon'} B$, $p p_1 p \in_{\varepsilon'} B$ for all $1 \le i \le k$, and

(3)′ $[1-p] \le [p_2]$, $2[1-p] \le [p_1]$.

By (1)′ and (2)′, we have

$$\|(1-p)p_1(1-p) - ((1-p)p_1(1-p))^2\| < 2\varepsilon', \|pp_1p - (pp_1p)^2\| < 2\varepsilon'.$$

When ε' is small enough, by (2)′, and by Lemma 2.5.5 of [29], there exist $a'_1,\ a'_2,\ \cdots,\ a'_k \in B$ and projections $p'_1 \in B$ and $p''_1 \in (1-p)A(1-p)$ such that

$$\|p'_1 - pp_1p\| < 4\varepsilon', \|p''_1 - (1-p)p_1(1-p)\| < 4\varepsilon', \|pa_ip - a'_i\| < \varepsilon',$$

and

$$\|p_1 - p'_1 - p''_1\| < 8\varepsilon'.$$

We claim that p'_1 is a non-zero projection. Otherwise, if $\varepsilon' < 1/8$, then $2[1-p] \le [p_1] = [p'_1 + p''_1] = [p''_1] \le [1-p]$, which contradicts the stable finiteness of A (which holds by Theorem 4.1 of [14], since C*-algebras in Ω are stably finite).

Since $B \in \Omega$, for $G = \{a'_1,\ a'_2,\ \cdots,\ a'_k\} \subseteq B$, $\varepsilon' > 0$ as specified, the non-zero positive element $p'_1 \in B$, and $n \in \mathbb{N}$, there is a completely positive order zero contraction $\psi : \mathrm{M}_n \to B \subseteq A$ with the following properties:

$(1)''$ $p - \psi(1) \precsim p'_1$, and

$(2)''$ for any normalized element $x \in \mathrm{M}_n$ and any $a'_i \in G$, we have

$$\|\psi(x)a'_i - a'_i\psi(x)\| < \varepsilon'.$$

By $(1)'$,

$$\|a_i - pa_ip - (1-p)a_i(1-p)\| < \varepsilon',$$

and, trivially,

$$\|a'_i + a''_i - pa_ip - (1-p)a_i(1-p)\| < 3\varepsilon'.$$

Therefore, for any normalized element $x \in \mathrm{M}_n$, and any $i = 1, 2, \cdots, k$, we have

$$\begin{aligned}
&\|\psi(x)a_i - a_i\psi(x)\| \\
&\leq \|\psi(x)a_i - \psi(x)(pa_ip + (1-p)a_i(1-p))\| \\
&+\|\psi(x)(pa_ip + (1-p)a_i(1-p)) - \psi(x)(a'_i + (1-p)a_i(1-p))\| \\
&+\|\psi(x)(a'_i + (1-p)a_i(1-p)) - (a'_i + (1-p)a_i(1-p))\psi(x)\| \\
&+\|(a'_i + (1-p)a_i(1-p))\psi(x) - (pa_ip + (1-p)a_i(1-p))\psi(x)\| \\
&+\|(pa_ip + (1-p)a_i(1-p))\psi(x) - a_i\psi(x)\| \\
&< 2\varepsilon' + \varepsilon' + 2\varepsilon' + \varepsilon' + 3\varepsilon' < 13\varepsilon' < \varepsilon.
\end{aligned}$$

This yields (2) when $\varepsilon' = \varepsilon/13$.

We also have (by $(3)'$, $(1)''$, and the choice of p'_1, which by Lemma 2.2 of [45] implies $[p'] \leq [p_1]$)

$$\begin{aligned}
\langle 1 - \psi(1)\rangle &= \langle (1-p) + (p - \psi(1))\rangle \\
= \langle 1-p\rangle + \langle p - \psi(1)\rangle &\leq \langle p_2\rangle + \langle p'_1\rangle \leq \langle p_2\rangle + \langle p_1\rangle \leq \langle a\rangle.
\end{aligned}$$

□

THEOREM 3.2. *Let Ω be a class of stably finite unital* C*-*algebras which have β-comparison (in the sense of [24]), for some $1 \leq \beta < \infty$. Then A has β-comparison for any simple unital* C*-*algebra $A \in \mathrm{TA}\Omega$.*

PROOF. By Lemma 2.6, enlarging the class Ω, we may suppose it is closed under passing to matrix algebras and unital hereditary C*-subalgebras (i.e., Morita equivalent C*-algebras).

Let $a, b \in \mathrm{M}_\infty(A)_+$. By Theorem 2.1 (3), we need to show that $\langle (a - 2\varepsilon)_+\rangle \leq \langle b\rangle$ for any $\varepsilon > 0$, and any integers $k,\ l \geq 1$ such that $k > \beta l$, $k\langle a\rangle \leq l\langle b\rangle$.

We may assume that $a, b \in \mathrm{M}_n(A)_+$ for some integer n. By Lemma 2.5, we may assume that $a, b \in A_+$.

We divide the proof into three cases.

Firstly, we suppose that b is not Cuntz equivalent to a projection.

Given $\varepsilon > 0$, and $k, l \geq 1$ as above, with in particular $k > l$, since $k\langle a\rangle \leq l\langle b\rangle$, by Theorem 2.1 (3), there exists $\delta > 0$ such that $k\langle(a-\varepsilon/2)_+\rangle \leq l\langle(b-\delta)_+\rangle$. Hence, by Theorem 2.1 (1), there exists $v = (v_{i,j}) \in \mathrm{M}_k(A)$, $1 \leq i \leq k$, $1 \leq j \leq k$ such that

$$v(\mathrm{diag}((b-\delta)_+ \otimes 1_l,\ 0 \otimes 1_{k-l}))v^* = (a-\varepsilon)_+ \otimes 1_k.$$

By Theorem 2.1 (4), there is a non-zero positive element d orthogonal to b such that $(b-\delta/2)_+ + d \lesssim b$.

With $F = \{a,\ b,\ v_{i,j} :\ 1 \leq i \leq k,\ 1 \leq j \leq k\}$, and any sufficiently small $\varepsilon' > 0$, since $A \in \mathrm{TA}\Omega$, there are a C*-subalgebra B of A and a projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon'$ for any $x \in F$,

(2) $pxp \in_{\varepsilon'} B$ for any $x \in F$, and

(3) $[1-p] \leq [d]$.

By (1) and (2), there are $a',\ b',\ {v_{i,j}}' \in B$ and $a'',\ b'',\ {v_{i,j}}'' \in (1-p)A(1-p)$ such that

$$\|a - a' - a''\| < 3\varepsilon',$$

$$\|b - b' - b''\| < 3\varepsilon',$$

and

$$\|v_{i,j} - {v_{i,j}}' - {v_{i,j}}''\| < 3\varepsilon',\ 1 \leq i \leq k,\ 1 \leq j \leq k.$$

Write

$$v' = ({v_{i,j}}') \in \mathrm{M}_k(B),\ v'' = ({v_{i,j}}'') \in \mathrm{M}_k((1-p)A(1-p)).$$

Then $\|v - v' - v''\| < 3\varepsilon' k^2$. Hence,

$$\|v'(\mathrm{diag}((b'-\delta)_+ \otimes 1_l,\ 0 \otimes 1_{k-l}))v'^* - (a'-\varepsilon)_+ \otimes 1_k)\| < \gamma,$$

where γ can be made arbitrarily small by suitable choice of ε'.

By Theorem 2.1 (1), we have

$$k\langle(a'-\varepsilon-\gamma)_+\rangle \leq l\langle(b'-\delta)_+\rangle.$$

Since $B \in \Omega$, this implies $\langle(a'-\varepsilon-\gamma)_+\rangle \leq \langle(b'-\delta)_+\rangle$. Therefore, if ε', and so γ, are small enough, then

$$\begin{aligned}\langle(a-2\varepsilon)_+\rangle &\leq \langle(a-\varepsilon-\gamma-3\varepsilon')_+\rangle \\ &\leq \langle(a'-\varepsilon-\gamma)_+\rangle + \langle a''\rangle \\ &\leq \langle(b'-\delta)_+\rangle + \langle a''\rangle \\ &\leq \langle(b'-\delta)_+\rangle + \langle(1-p)\rangle \\ &\leq \langle(b'-\delta)_+\rangle + \langle d\rangle \leq \langle(b-\delta/2)_+\rangle + \langle d\rangle \leq \langle b\rangle.\end{aligned}$$

(Here, and elsewhere, by $x \oplus y$ for $x, y \in \mathrm{M}_\infty(A)_+$ we mean $x + y'$ where $y' \in \mathrm{M}_\infty(A)_+$, $y' \sim y$, $y'x = 0$.)

Secondly, we suppose that b is Cuntz equivalent to a projection and a is not Cuntz equivalent to a projection. Choose a projection p_0 such that b is Cuntz equivalent to p_0. We may assume that $b = p_0$.

By Theorem 2.1 (4), there exists a non-zero positive element d orthogonal to a such that $\langle (a-\delta)_+ + d\rangle \leq \langle a\rangle$. Since $k\langle a\rangle \leq l\langle p_0\rangle$, we have $k\langle (a-\delta)_+ + d\rangle \leq l\langle p_0\rangle$. Hence, as above, by Theorem 2.1 (1), on increasing δ slightly, and decreasing d slightly, there exists $v = (v_{i,j}) \in \mathrm{M}_k(A),\ 1 \leq i \leq k,\ 1 \leq j \leq k$, such that

$$v(\mathrm{diag}(p_0 \otimes 1_l,\ 0 \otimes 1_{k-l})v^* = ((a-\delta)_+ + d) \otimes 1_k.$$

Since $A \in \mathrm{TA}\Omega$, for $F = \{a,\ p_0,\ d,\ v_{i,j} :\ 1 \leq i \leq k,\ 1 \leq j \leq k\}$, and any sufficiently small $\varepsilon' > 0$, by Theorem 3.3 of [17], there are a C*-subalgebra B of A and a projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon'$ for any $x \in F$,
(2) $pxp \in_{\varepsilon'} B$ for any $x \in F$, and
(3) $[1-p] \leq [(d - 3\varepsilon')_+]$, $2[1-p] \leq [p_0]$.

By (1) and (2),

$$\|pp_0p - (pp_0p)^2\| < 2\varepsilon', \|(1-p)p_0(1-p) - ((1-p)p_0(1-p))^2\| < 2\varepsilon'.$$

When ε' is small enough, by (1) and (2), and by Lemma 2.5.5 of [29], there are $a',\ d',\ {v_{i,j}}' \in B$ and $a'',\ d'',\ {v_{i,j}}'' \in (1-p)A(1-p)$, and projections $p_0' \in B$ and $p_0'' \in (1-p)A(1-p)$, such that

$$\|a - a' - a''\| < 3\varepsilon',$$
$$\|d - d' - d''\| < 3\varepsilon',$$
$$\|p_0 - p_0' - p_0''\| < 4\varepsilon',$$

and

$$\|v_{i,j} - {v_{i,j}}' - {v_{i,j}}''\| < 3\varepsilon',\ 1 \leq i \leq k,\ 1 \leq j \leq k.$$

By Lemma 2.5.12 of [29], we may suppose, choosing ε' small enough, that $a'd' = a''d'' = 0$.

We claim that p_0' is a non-zero projection. Otherwise, $2[1-p] \leq [p_0] = [p_0' + p_0''] = [p_0''] \leq [1-p]$, which contradicts the stable finiteness of A (since C*-algebras in Ω are stably finite (cf. proof of 3.1)).

Write

$$v' = ({v_{i,j}}') \in \mathrm{M}_k(B),\ v_k'' = ({v_{i,j}}'') \in \mathrm{M}_k((1-p)A(1-p)).$$

Then $\|v - v' - v''\| < 3\varepsilon' k^2$. Hence,

$$\|v'(\mathrm{diag}(p_0' \otimes 1_l,\ 0 \otimes 1_{k-l}))v'^* - ((a' - \delta)_+ + d') \otimes 1_k\| < \gamma,$$

where, as above, γ can be made arbitrarily small by suitable choice of ε'.

Hence, by Theorem 2.1 (1),

$$k\langle (a'-\delta-\gamma)_+ + d'\rangle \le l\langle p_0'\rangle,$$

where we have relabeled $(d'-\gamma)_+$, which we may suppose non-zero, as d'. Since $B\in\Omega$, we have

$$\langle (a'-\delta-\gamma)_+ + d'\rangle \le \langle p_0'\rangle.$$

Since $(1-p)A(1-p)\in \mathrm{TA}\Omega$, for $G=\{a'',\ p_0'',\ d'',\ {v_{i,j}}'' :\ 1\le i\le k,\ 1\le j\le k\}$, for any sufficiently small $\varepsilon''>0$, there are a C*-subalgebra C of $(1-p)A(1-p)$ and a projection $r\in(1-p)A(1-p)$ with $C\in\Omega$ and $1_C=r$ such that

$(1)'$ $\|xr-rx\|<\varepsilon''$ for any $x\in G$,

$(2)'$ $rxr\in_{\varepsilon''} C$ for any $x\in G$, and

$(3)'$ $[1-p-r]\le[d']$.

By $(1)'$ and $(2)'$,

$$\|pp_0''p-(pp_0''p)^2\|<2\varepsilon'',$$

$$\|(1-p-r)p_0''(1-p-r)-((1-p-r)p_0''(1-p-r))^2\|<2\varepsilon''.$$

When ε'' is small enough, by $(1)'$ and $(2)'$, and by Lemma 2.5.5 of [29], there are a''', d''', ${v_{i,j}}'''\in C$, a'''', d'''', ${v_{i,j}}''''\in(1-p-r)A(1-p-r)$, and projections $p_0'''\in C$ and $p_0''''\in(1-p-r)A(1-p-r)$, such that

$$\|a''-a'''-a''''\|<3\varepsilon'',$$

$$\|d''-d'''-d''''\|<3\varepsilon'',$$

$$\|p_0''-p_0'''-p_0''''\|<4\varepsilon'',$$

and

$$\|{v_{i,j}}''-{v_{i,j}}'''-{v_{i,j}}''''\|<3\varepsilon'',\ 1\le i\le k,\ 1\le j\le k.$$

Also by Lemma 2.5.12 of [29], we may suppose, choosing ε'' small enough, that $a'''d'''=a''''d''''=0$.

Write

$$v'''=({v_{i,j}}''')\in \mathrm{M}_k(C),\ v''''=({v_{i,j}}'''')\in \mathrm{M}_k((1-p-r)A(1-p-r)).$$

Then $\|v''-v'''-v''''\|<3\varepsilon''k^2$. Hence,

$$\|v'''(\mathrm{diag}(p_0'''\otimes 1_l,\ 0\otimes 1_{k-l}))v'''^*-((a'''-\delta)_+ + d''')\otimes 1_k)\|<\gamma',$$

where, as above, γ' can be made arbitrarily small by suitable choice of ε''.

By Theorem 2.1 (1),

$$k\langle (a'''-\delta-\gamma')_+ + d'''\rangle \le l\langle p_0'''\rangle,$$

where, as above, we have relabeled $(d''' - \gamma')_+$, which we may suppose still non-zero, as d'''.

Since $C \in \Omega$, we have

$$\langle (a''' - \gamma' - \delta)_+ + d'''\rangle \leq \langle p_0'''\rangle.$$

Therefore, we have

$$\begin{aligned}\langle (a-2\varepsilon)_+\rangle \quad & \leq \langle (a - 3\varepsilon' - 3\varepsilon - \gamma - \gamma' - \delta)_+\rangle\\ & \leq \langle (a' - \gamma - \delta)_+\rangle + \langle (a''' - \gamma' - \delta)_+\rangle + \langle a''''\rangle\\ & \leq \langle (a' - \gamma - \delta)_+\rangle + \langle (1-p-r)\rangle + \langle (a''' - \gamma' - \delta)_+\rangle\\ & \leq \langle (a' - \gamma - \delta)_+\rangle + \langle d'\rangle + \langle (a''' - \gamma' - \delta)_+\rangle\\ & \leq \langle p_0'\rangle + \langle p_0'''\rangle \leq \langle p_0\rangle.\end{aligned}$$

Thirdly, we suppose that both a and b are Cuntz equivalent to projections.

Choose projections $p,\ q$ such that a is Cuntz equivalent to p and b is Cuntz equivalent to q. We may assume that $a = p,\ b = q$.

Since $A \in \mathrm{TA}\Omega$, for $F = \{p,\ q\}$, for any $1/4 > \varepsilon > 0$, there are a projection $r \in A$ and a C*-subalgebra $B \subseteq A$ with $B \in \Omega, 1_B = r$ such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$, and

(2) $rxr \in_{\varepsilon} B$ for all $x \in F$.

By (1) and (2),

$$\|rpr - (rpr)^2\| < 2\varepsilon', \|(1-r)p(1-r) - ((1-r)p(1-r))^2\| < 2\varepsilon',$$

and

$$\|rqr - (rqr)^2\| < 2\varepsilon', \|(1-r)q(1-r) - ((1-r)q(1-r))^2\| < 2\varepsilon'.$$

When ε' is small enough, by (1) and (2), and by Lemma 2.5.5 of [29], there are projections $p_1, q_1 \in B$ and $p_2, q_2 \in (1-r)A(1-r)$ such that

$$\|p - p_1 - p_2\| < 4\varepsilon,$$

$$\|q - q_1 - q_2\| < 4\varepsilon.$$

Therefore, since $\varepsilon < 1/4$, we have

$$p \sim p_1 + p_2,$$

$$q \sim q_1 + q_2.$$

We also have

$$k\langle p_1\rangle \leq l\langle q_1\rangle,\ k\langle p_2\rangle \leq l\langle q_2\rangle.$$

Since $B \in \Omega$, we have $\langle p_1 \rangle \leq \langle q_1 \rangle$.

If $p_1 \sim q_1$, then we have $k\langle p_1 \rangle \leq l\langle q_1 \rangle = l\langle p_1 \rangle \leq k\langle p_1 \rangle$. So $\bigoplus_{n=1}^{k} p_1$ is equivalent to a proper subprojection of itself, and this contradicts the stable finiteness of A (since C*-algebras in Ω are stably finite (cf. proof of 3.1)).

Therefore, there is a non-zero projection s such that $p_1 \oplus s \sim q_1$.

Since $(1-r)A(1-r) \in \mathrm{TA}\Omega$, for $G = \{p_2,\ q_2,\ s\}$, for any $\varepsilon' > 0$, there are a projection $m \in (1-r)A(1-r)$ and a C*-subalgebra $C \subseteq (1-r)A(1-r)$ with $C \in \Omega,\ 1_C = m$ such that

$(1)'$ $\|xm - mx\| < \varepsilon'$ for all $x \in G$,

$(2)'$ $mxm \in'_{\varepsilon} C$ for all $x \in G$, and

$(3)'$ $[1-r-m] \leq [s]$.

By $(1)'$ and $(2)'$,

$$\|rp_2r-(rp_2r)^2\| < 2\varepsilon', \|(1-r-m)p_2(1-r-m)-((1-r-m)p_2(1-r-m))^2\| < 2\varepsilon',$$

and

$$\|rq_2r-(rq_2r)^2\| < 2\varepsilon', \|(1-r-m)q_2(1-r-m)-((1-r-m)q_2(1-r-m))^2\| < 2\varepsilon'.$$

When ε' is small enough, by $(1)'$ and $(2)'$, and by Lemma 2.5.5 of [29], there are projections $p_3,\ q_3 \in C$ and $p_4,\ q_4 \in (1-m-r)A(1-m-r)$ such that

$$\|p_2 - p_3 - p_4\| < 4\varepsilon,$$

$$\|q_2 - q_3 - q_4\| < 4\varepsilon.$$

Therefore, since $\varepsilon < 1/4$, we have

$$p_2 \sim p_3 + p_4,$$

$$q_2 \sim q_3 + q_4.$$

We also have

$$k\langle p_3 \rangle \leq l\langle q_3 \rangle,\ k\langle p_4 \rangle \leq l\langle q_4 \rangle.$$

Since $C \in \Omega$, we have $p_3 \lesssim q_3$. By $(3')$, $[p_4] \leq [1-m-r] \leq [s]$, and therefore, by $(3)'$ and since $\varepsilon' < 1/4$,

$$\begin{aligned}
\langle p \rangle &= \langle p_1 \rangle + \langle p_2 \rangle = \langle p_1 \rangle + \langle p_3 \rangle + \langle p_4 \rangle \\
&\leq \langle p_1 \rangle + \langle q_3 \rangle + \langle p_4 \rangle \\
&\leq \langle p_1 \rangle + \langle s \rangle + \langle q_3 \rangle \\
&\leq \langle q_1 \rangle + \langle q_3 \rangle \leq \langle q \rangle.
\end{aligned}$$

□

Theorem 3.3. *Let Ω be a class of stably finite unital* C*-*algebras which have Winter's n-comparison. Then A has Winter's n-comparison for any simple unital* C*-*algebra $A \in \mathrm{TA}\Omega$.*

Proof. As in the proof of Theorem 3.2, we may suppose that Ω is closed under Morita equivalence.

Let $a, b_1, b_2, \cdots, b_n \in \mathrm{M}_\infty(A)_+$. By Theorem 2.1 (3), we need only to show that $\langle (a-2\varepsilon)_+\rangle \leq \langle b_1\rangle + \langle b_2\rangle + \cdots + \langle b_n\rangle$ for any $\varepsilon > 0$, if $(k_i+1)\langle a\rangle \leq k_i\langle b_i\rangle,\ 0 \leq i \leq n$. Note that k_i can be chosen to be the same for all b_i, as, with $k = (k_0+1)(k_1+1)\cdots(k_n+1)-1$, one has $(k+1)\langle a\rangle \leq k\langle b_i\rangle$ for all $0 \leq i \leq n$.

We may assume that $a, b_1, b_2, \cdots, b_n \in \mathrm{M}_k(A)_+$ for some sufficiently large integer k. By Lemma 2.5, we may assume that $a, b_1, b_2, \cdots, b_n \in A_+$.

We divide the proof into three cases.

Firstly, let us suppose that b_i is not Cuntz equivalent to a projection for some $0 \leq i \leq n$. We may assume that b_0 is a purely positive element. Hence, as the proof of Theorem 3.2, by Theorem 2.1 (1), there exist $v_k = (v_{i,j}^k),\ 0 \leq k \leq n,\ 1 \leq i \leq k+1,\ 1 \leq j \leq k+1$ such that

$$v_0(\mathrm{diag}((b_0-\delta)_+ \otimes 1_k,\ 0))v_0^* = (a-\varepsilon)_+ \otimes 1_{k+1},$$

$$v_i(\mathrm{diag}(b_i \otimes 1_k,\ 0))v_i^* = (a-\varepsilon)_+ \otimes 1_{k+1},$$

where $1 \leq i \leq n$.

By Theorem 2.1 (4), for any $1 > \delta > 0$, there is a non-zero positive element d orthogonal to b_0 such that $(b_0-\delta/2)_+ + d \lesssim b_0$.

Since $A \in \mathrm{TA}\Omega$, for $F = \{a,\ b_0,\ b_1,\ \cdots,\ b_n,\ d,\ v_{i,j}^k :\ 1 \leq i \leq k+1,\ 1 \leq j \leq k+1,\ 0 \leq k \leq n\}$, and any sufficiently small $\varepsilon' > 0$, there are a C*-subalgebra B of A and a non-zero projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon'$ for any $x \in F$,

(2) $pxp \in_{\varepsilon'} B$ for any $x \in F$, and

(3) $[1-p] \leq [d]$.

By (1) and (2), there are $a',\ b_0',\ b_1',\ \cdots,\ b_n',\ {v_{i,j}^k}' \in B,\ \ 1 \leq i \leq k+1,\ 1 \leq j \leq k+1,\ 0 \leq k \leq n$, and $a'',\ b_0'',\ b_1'',\ \cdots,\ b_n'',\ {v_{i,j}^k}'' \in (1-p)A(1-p)$, such that

$$\|a - a' - a''\| < 3\varepsilon',$$

$$\|b_i - b_i' - b_i''\| < 3\varepsilon',\ 0 \leq i \leq n,$$

and

$$\|v_{i,j}^k - {v_{i,j}^k}' - {v_{i,j}^k}''\| < 3\varepsilon',\ 1 \leq i \leq k+1,\ 1 \leq j \leq k+1,\ 0 \leq k \leq n.$$

Write

$$v_k' = ({v_{i,j}^k}') \in \mathrm{M}_{k+1}(B),\ v_k'' = ({v_{i,j}^k}'') \in \mathrm{M}_{k+1}((1-p)A(1-p)).$$

Then $\|v_k - v'_k - v''_k\| < 3(k+1)^2\varepsilon'$. Hence,

$$\|v'_0(\mathrm{diag}((b'_0-\delta)_+ \otimes 1_k,\ 0)){v'_0}^* - (a'-\varepsilon)_+ \otimes 1_{k+1}\| < \gamma,$$

$$\|v'_i(\mathrm{diag}((b'_i-3\varepsilon')_+ \otimes 1_k,\ 0)){v'_i}^* - (a'-\varepsilon)_+ \otimes 1_{k+1}\| < \gamma,\ 1 \le i \le n,$$

where, as above, γ can be made arbitrarily small by suitable choice of ε'.

By Theorem 2.1 (1),

$$(k+1)\langle (a'-\varepsilon-\gamma)_+\rangle \le k\langle (b'_0-\delta)_+\rangle,$$

$$(k+1)\langle (a'-\varepsilon-\gamma)_+\rangle \le k\langle (b'_i-3\varepsilon')_+\rangle,\ 1 \le i \le n.$$

Since $B \in \Omega$, we have $\langle (a'-\varepsilon-\gamma)_+\rangle \le \langle (b'_0-\delta)_+\rangle + \langle (b'_i-3\varepsilon')_+\rangle + \cdots + \langle (b'_n-3\varepsilon')_+\rangle$. Therefore, if ε', and so γ, are small enough, then

$$\begin{aligned}
&\langle (a-2\varepsilon)_+\rangle\\
&\le \langle (a-\varepsilon-3\varepsilon'-2\gamma)_+\rangle\\
&\le \langle (a'-\varepsilon-2\gamma)_+\rangle + \langle a''\rangle\\
&\le \langle (b'_0-\delta)_+\rangle + \langle (b'_i-3\varepsilon')_+\rangle + \cdots + \langle (b'_n-3\varepsilon')_+\rangle\\
&\le \langle b_0\rangle + \langle b_1\rangle + \cdots + \langle b_n\rangle.
\end{aligned}$$

Secondly, let us suppose that each b_i $(0 \le i \le n)$ is Cuntz equivalent to a projection and a is not Cuntz equivalent to a projection. Choose projections $p_0,\ p_1,\ \cdots,\ p_n$ such that b_i is Cuntz equivalent to p_i for all $0 \le i \le n$. We may assume that $b_i = p_i$ for all $0 \le i \le n$.

By Theorem 2.1 (4), there is a non-zero positive element d orthogonal to a such that $\langle (a-\delta)_+ + d\rangle \le \langle a\rangle$. Since $(k+1)\langle a\rangle \le k\langle p_i\rangle$, we have $(k+1)\langle (a-\delta)_+ + d\rangle \le k\langle p_i\rangle$ for all $0 \le i \le n$.

Hence, by Theorem 2.1 (1), as earlier, on increasing δ slightly, and decreasing d slightly, there exist $v_k = (v^k_{i,j}) \in \mathrm{M}_{k+1}(A)$, $1 \le k \le n,\ 1 \le i \le k+1,\ 1 \le j \le k+1$, such that

$$v_i(\mathrm{diag}(p_i \otimes 1_k,\ 0))v_i^* = ((a-\delta)_+ + d) \otimes 1_{k+1},\ 0 \le i \le n.$$

Since $A \in \mathrm{TA}\Omega$, for $F = \{a,\ b_0,\ b_1,\ \cdots,\ b_n,\ d,\ v^k_{i,j}:\ 1 \le i \le k+1,\ 1 \le j \le k+1,\ 0 \le k \le n\}$, for any sufficiently small $\varepsilon' > 0$, there are a C*-subalgebra B of A and a projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon'$ for any $x \in F$,

(2) $pxp \in_{\varepsilon'} B$ for any $x \in F$, and

(3) $[1-p] \le [(d-3\varepsilon')_+]$.

By (1) and (2),

$$\|pp_ip - (pp_ip)^2\| < 2\varepsilon', \|(1-p)p_i(1-p) - ((1-p)p_i(1-p))^2\| < 2\varepsilon'$$

for $1 \leq i \leq k+1$.

By (1) and (2), there are $a', d', {v_{i,j}^k}' \in B$ and $a'', d'', {v_{i,j}^k}'' \in (1-p)A(1-p)$, such that

$$\|a - a' - a''\| < 3\varepsilon',$$

$$\|d - d' - d''\| < 3\varepsilon',$$

and

$$\|v_{i,j}^k - {v_{i,j}^k}' - {v_{i,j}^k}''\| < 3\varepsilon',\ 1 \leq i \leq k+1,\ 1 \leq j \leq k+1,\ 0 \leq k \leq n.$$

As earlier, we may suppose that $a'd' = a''d'' = 0$.

When ε' is small enough, by (1) and (2), and by Lemma 2.5.5 of [29], there are projections $p_0',\ p_1'\ \cdots,\ p_n' \in B$ and $p_0'',\ p_1''\ \cdots,\ p_n' \in (1-p)A(1-p)$ such that

$$\|p_i - p_i' - p_i''\| < 4\varepsilon',\ 0 \leq i \leq n.$$

Write

$$v_k' = ({v_{i,j}^k}') \in \mathrm{M}_{k+1}(B),\ v_k'' = ({v_{i,j}^k}'') \in \mathrm{M}_{k+1}((1-p)A(1-p)).$$

Then $\|v_k - v_k' - v_k''\| < 4\varepsilon' k^2$. Hence,

$$\|v_i'(\mathrm{diag}(p_i' \otimes 1_k,\ 0))v_i'^* - ((a'-\delta)_+ + d') \otimes 1_{k+1}\| < \gamma,$$

$$\|v_i''(\mathrm{diag}(p_i'' \otimes 1_k,\ 0))v_i''^* - (a''-\delta)_+ + d'') \otimes 1_{k+1}\| < \gamma,\ 1 \leq i \leq n,$$

where, as above, γ can be made arbitrarily small by suitable choice of ε'.

Hence, by Theorem 2.1 (1), where as earlier, we have relabeled $(d'-\gamma)_+$ and $(d''-\gamma)_+$, which we may suppose non-zero, as d' and d'',

$$(k+1)\langle(a'-\delta-\gamma)_+ + d'\rangle \leq k\langle p_i'\rangle,$$

and

$$(k+1)\langle(a''-\delta-\gamma)_+ + d''\rangle \leq k\langle p_i''\rangle,$$

for all $0 \leq i \leq n$. Since $B \in \Omega$, this implies $\langle(a'-\gamma-\delta)_+ + d'\rangle \leq \langle p_0'\rangle + \langle p_1'\rangle + \cdots + \langle p_n'\rangle$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{a'',\ p_0'',\ p_1'',\ \cdots,\ p_k'',\ d',\ {v_{i,j}^k}'':\ 1 \leq i \leq k+1,\ 1 \leq j \leq k+1,\ 0 \leq k \leq n\}$, for sufficiently small $\varepsilon'' > 0$, there are a C*-subalgebra C of $(1-p)A(1-p)$ and a projection $r \in A$ with $C \in \Omega$ and $1_C = r$ such that

$(1)'$ $\|xr - rx\| < \varepsilon''$ for any $x \in G$,

$(2)'$ $rxr \in_{\varepsilon''} C$ for any $x \in G$, and

$(3)'$ $[1-r-p] \leq [d']$.

By $(1)'$ and $(2)'$,

$$\|rp_i''r - (rp_i''r)^2\| < 2\varepsilon'', \|(1-r-p)p_i''(1-r-p) - ((1-r-p)p_i''(1-r-p))^2\| < 2\varepsilon''$$

for $1 \le i \le k+1$.

When ε'' is small enough, by $(1)'$ and $(2)'$, and by Lemma 2.5.5 of [29], there are projections p_0''', p_1''' $\cdots$, p_n''', ${v^k_{i,j}}'''$ $\in C$, and p_0'''', p_1'''', $\cdots, p_n''''$, ${v^k_{i,j}}''''$ $\in (1-r-p)A(1-r-p)$, such that

$$\|p_i'' - p_i''' - p_i''''\| < 4\varepsilon'',\ 0 \le i \le n.$$

Also by $(1)'$ and $(2)'$, there are a''', d''', ${v^k_{i,j}}'''$ $\in C$, and a'''', d'''' ${v^k_{i,j}}''''$ $\in (1-r-p)A(1-r-p)$, such that

$$\|a'' - a''' - a''''\| < 3\varepsilon'', \|d - d' - d''\| < 3\varepsilon''.$$

and

$$\|{v^k_{i,j}}'' - {v^k_{i,j}}''' - {v^k_{i,j}}''''\| < 3\varepsilon'',\ 1 \le i \le k+1,\ 1 \le j \le k+1,\ 0 \le k \le n.$$

As earlier, we may suppose that $a'''d''' = a''''d'''' = 0$.

Write

$$v_k''' = ({v^k_{i,j}}''') \in \mathrm{M}_{k+1}(C),\ v_k'''' = ({v^k_{i,j}}'''') \in \mathrm{M}_{k+1}((1-r-p)A(1-r-p)).$$

Then $\|{v_k}'' - {v_k}''' - {v_k}''''\| < 3\varepsilon''(k+1)^2$. Hence,

$$\|v_i'''(\mathrm{diag}(p_i''' \otimes 1_k,\ 0))v_i'''^{*} - ((a''' - \delta)_+ + d''') \otimes 1_{k+1}\| < \gamma',\ 0 \le i \le n,$$

where, as above, γ' is arbitrarily small for suitable choice of ε''.

Hence, by Theorem 2.1 (1),

$$(k+1)\langle (a''' - \delta - \gamma')_+ \rangle \le k\langle p_i''' \rangle,$$

for all $0 \le i \le n$. Since $C \in \Omega$, we have

$$\langle (a''' - \delta - \gamma')_+ \rangle \le \langle p_0''' \rangle + \langle p_1''' \rangle + \cdots + \langle p_n''' \rangle.$$

Therefore, if ε'', and so γ', are small enough, we have

$$\begin{aligned}
\langle (a - 2\varepsilon)_+ \rangle &\le \langle (a - 2\delta - \gamma - \gamma' - 2\varepsilon' - 2\varepsilon'')_+ \rangle \\
&\le \langle (a' - \delta - \gamma)_+ \rangle + \langle (a''' - \delta - \gamma')_+ \rangle + \langle a'''' \rangle \\
&\le \langle (a' - \delta - \gamma)_+ \rangle + \langle d' \rangle\ + \langle (a''' - \delta - \gamma')_+ \rangle \\
&\le \langle p_0' \rangle + \langle p_1' \rangle + \cdots + \langle p_n' \rangle + \langle (a''' - \delta - 2\gamma')_+ \rangle \\
&\le \langle p_0' \rangle + \langle p_1' \rangle + \cdots + \langle p_n' \rangle + \langle p_0''' \rangle + \langle p_1''' \rangle + \cdots + \langle p_n''' \rangle \\
&\le \langle p_0 \rangle + \langle p_1 \rangle + \cdots + \langle p_n \rangle.
\end{aligned}$$

Thirdly, let us suppose that a and b_i $(0 \le i \le n)$ are not purely positive elements. There are projections q, p_0, p_1, $\cdots$, p_n such that a is Cuntz equivalent to q and b_i is Cuntz equivalent to p_i for all $0 \le i \le n$. We may assume that $a = q$, $b_i = p_i$, $0 \le i \le n$. By Theorem 2.2 of [18], we have $q \le \langle p_0 \rangle + \langle p_1 \rangle + \cdots + \langle p_n \rangle$. □

Corollary 3.4. *Let* Ω *be a class of stably finite unital* C^**-algebras such that for any* $B \in \Omega$, $\mathrm{W}(B)$ *is almost unperforated. Then* $\mathrm{W}(A)$ *is almost unperforated for any simple unital* C^**-algebra* $A \in \mathrm{TA}\Omega$.

Proof. This is a special case of Theorem 3.2 and also of Theorem 3.3. □

Recall that if A is a unital simple exact C*-algebra, then A has strict comparison if and only if $\mathrm{W}(A)$ is almost unperforated ([47]). One has the following corollary.

Corollary 3.5. *Let* Ω *be a class of stably finite unital* C^**-algebras such that for any* $B \in \Omega$, $\mathrm{W}(B)$ *is almost unperforated. Then* A *has strict comparison for any simple separable exact unital* C^**-algebra* $A \in \mathrm{TA}\Omega$.

Theorem 3.6. *([43].) Let* A *be a unital separable infinite dimensional* C^**-algebra with nuclear dimension* n $(n \in \mathbb{N})$*. Then* A *has Winter's* n*-comparison.*

Corollary 3.7. *Let* A *be a unital separable simple infinite dimensional* C^**-algebra with tracial topological rank at most* n $(n \in \mathbb{N})$*. (See section 2.) Then* A *has at most Winter's* n*-comparison.*

Proof. By Definition 2.4, we have $A \in \mathrm{TA}\mathfrak{I}^n$, and any C*-algebra in $\mathfrak{I}^n$ has nuclear dimension at most n, the conclusion follows by Theorem 3.6 and Theorem 3.3. □

Let Ω be a class of C*-algebras that have (k, m) $(m \neq 0)$ comparison. In the following theorem we will prove that A has (k, m) $(m \neq 0)$ comparison for any simple unital C*-algebra $A \in \mathrm{TA}\Omega$. However, the $(k, 0)$ comparison property is not inherited by simple unital C*-algebras in the class $\mathrm{TA}\Omega$.

Theorem 3.8. *Let* Ω *be a class of stably finite unital* C^**-algebras which have* (k, m) $(m \neq 0)$ *comparison. Then* A *has* (k, m) *comparison for any simple unital* C^**-algebra* $A \in \mathrm{TA}\Omega$.

Proof. As in 3.2 (and 3.3), we may suppose that Ω is closed under Morita equivalence.

Let $a, b \in \mathrm{M}_\infty(A)_+$. By Theorem 2.1 (3), we need only to show that $\langle (a - 2\varepsilon)_+\rangle \leq \langle b\rangle$ for any $\varepsilon > 0$, if $k\langle a\rangle + m\langle 1\rangle \leq k\langle b\rangle$ for some $k,\ m \in \mathbb{N}$. We may assume that $\|a\| = \|b\| = 1$.

We may assume that $a, b \in \mathrm{M}_k(A)_+$ for some sufficiently large integer k. By Lemma 2.5, we may assume that $a, b \in A_+$.

We divide the proof into three cases.

Firstly, let us suppose that b is not Cuntz equivalent to a projection.

For $\varepsilon > 0$, since $k\langle a\rangle + m\langle 1\rangle \leq k\langle b\rangle$, there are $\delta > 0$ and $v_n = (v_{i,j}^n) \in \mathrm{M}_{k+m}(A)$, $1 \leq i \leq k+m$, $1 \leq j \leq k+m$, such that

$$\lim_{n\to\infty} v_n(\mathrm{diag}((b-\delta)_+ \otimes 1_k,\ 0 \otimes 1_m)){v_n}^* = \mathrm{diag}((a-\varepsilon)_+ \otimes 1_k,\ 1 \otimes 1_m).$$

There is a sufficiently large integer $N \in \mathbb{N}$ that

$$\|v_N(\mathrm{diag}((b-\delta)_+ \otimes 1_k,\ 0 \otimes 1_m))v_N{}^* - \mathrm{diag}((a-\varepsilon)_+ \otimes 1_k,\ 1 \otimes 1_m)\| < \varepsilon.$$

Hence (as implicitly deduced several times earlier) by Theorem 2.1 (1), there is $v = (v_{i,j}) \in \mathrm{M}_{k+m}(A),\ 1 \le i \le k+m,\ 1 \le j \le k+m$, such that

$$v(\mathrm{diag}((b-\delta)_+ \otimes 1_k,\ 0 \otimes 1_m))v^* = \mathrm{diag}((a-2\varepsilon)_+ \otimes 1_k,\ 1 \otimes 1_m).$$

By Theorem 2.1 (4), for any $1 > \delta > 0$, there is a non-zero positive element d orthogonal to b such that $(b-\delta/2)_+ + d \precsim a$.

Since $A \in \mathrm{TA}\Omega$, for $F = \{a,\ b,\ v_{i,j}:\ 1 \le i \le k+m,\ 1 \le j \le k+m\}$, for any sufficiently small $\varepsilon' > 0$, there are a C*-subalgebra B of A and a non-zero projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon'$ for any $x \in F$,

(2) $pxp \in_{\varepsilon'} B$ for any $x \in F$, and

(3) $[1-p] \le [d]$.

By (1) and (2), there are $a',\ b',\ v_{i,j}{}' \in B$ and $a'',\ b'',\ v_{i,j}{}'' \in (1-p)A(1-p)$ such that

$$\|a - a' - a''\| < 3\varepsilon',$$
$$\|b - b' - b''\| < 3\varepsilon',$$

and

$$\|v_{i,j} - v_{i,j}{}' - v_{i,j}{}''\| < 3\varepsilon',\ 1 \le i \le k+m,\ 1 \le j \le k+m.$$

By Lemma 2.5.12 of [29], we may suppose, choosing ε'' small enough, that $d'b' = d''b'' = 0$.

Write $v' = (v_{i,j}{}') \in \mathrm{M}_{k+m}(B)$, $v'' = (v_{i,j}{}'') \in \mathrm{M}_{k+m}((1-p)A(1-p))$. Then $\|v - v' - v''\| < 3\varepsilon'(k+m)^2$. Hence,

$$\|v'((\mathrm{diag}((b'-\delta)_+ \otimes 1_k,\ 0 \otimes 1_m))v'^* - \mathrm{diag}((a-2\varepsilon)_+ \otimes 1_k,\ p \otimes 1_m)\| < \gamma,$$

where as earlier, γ can be made arbitrarily small by suitable choice of ε'.

By Theorem 2.1 (1), if $\gamma < 1$, then (since $p = p^2$)

$$k\langle(a' - 2\varepsilon - \gamma)_+\rangle + m\langle p\rangle \le k\langle(b'-\delta)_+\rangle.$$

Since $B \in \Omega$, this implies $\langle(a' - 2\varepsilon - \gamma)_+\rangle \le \langle(b'-\delta)_+\rangle$. Therefore, if ε', and so γ, are small enough, then

$$\begin{aligned}\langle(a-3\varepsilon)_+\rangle &\le \langle(a - 2\varepsilon - \gamma - 3\varepsilon')_+\rangle\\ &\le \langle(a' - 2\varepsilon - \gamma)_+\rangle + \langle a''\rangle\\ &\le \langle(b'-\delta)_+\rangle + \langle a''\rangle\\ &\le \langle(b'-\delta)_+\rangle + \langle(1-p)\rangle\\ &\le \langle(b'-\delta)_+\rangle + \langle d\rangle \le \langle(b-\delta/2)_+\rangle + \langle d\rangle \le \langle b\rangle.\end{aligned}$$

Secondly, let us suppose that b is Cuntz equivalent to a projection and a is not Cuntz equivalent to a projection. Choose a projection p_0 such that b is Cuntz equivalent to p_0. We may assume that $b = p_0$. By Theorem 2.1 (4), there is a non-zero positive element d orthogonal to b such that $\langle (a-\delta)_+ + d\rangle \leq \langle a\rangle$. Since $k\langle a\rangle + m\langle 1\rangle \leq k\langle p_0\rangle$, we have $k\langle (a-\delta)_+ + d\rangle + m\langle 1\rangle \leq k\langle p_0\rangle$. Hence, by Theorem 2.1 (1), as earlier on increasing δ and decreasing d slightly, there is $v = (v_{i,j}) \in \mathrm{M}_{k+m}(A),\ 1\leq i\leq k+m,\ 1\leq j\leq k+m$, such that

$$v(\mathrm{diag}(p_0\otimes 1_k,\ 0\otimes 1_m)v^* = \mathrm{diag}(((a-\delta)_+ + d)\otimes 1_k,\ 1\otimes 1_m).$$

Since $A\in \mathrm{TA}\Omega$, for $F = \{a,\ p_0,\ d,\ v_{i,j}:\ 1\leq i\leq k+m,\ 1\leq j\leq k+m\}$, for any sufficiently small $\varepsilon' > 0$, there are a C*-subalgebra B of A and a projection $p\in A$ with $B\in\Omega$ and $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon'$ for any $x\in F$,

(2) $pxp \in_{\varepsilon'} B$ for any $x\in F$, and

(2) $[1-p]\leq [d]$.

By (1) and (2), we have

$$\|pp_0p - (pp_0p)^2\| < 2\varepsilon', \|(1-p)p_0(1-p) - ((1-p)p_0(1-p))^2\| < 2\varepsilon'.$$

When ε' is small enough, by (1) and (2), and by Lemma 2.5.5 of [29], there are $a',\ d',\ {v_{i,j}}'\in B$, and $a'',\ d'',\ {v_{i,j}}''\in (1-p)A(1-p)$ and projections $p_0'\in B$ and $p_0''\in (1-p)A(1-p)$ such that

$$\|a - a' - a''\| < 2\varepsilon',$$

$$\|d - d\,' - d\,''\| < 2\varepsilon',$$

$$\|p_0 - p_0' - p_0''\| < 4\varepsilon',$$

and

$$\|v_{i,j} - {v_{i,j}}' - {v_{i,j}}''\| < 3\varepsilon',\ 1\leq i\leq k+m,\ 1\leq j\leq k+m.$$

By Lemma 2.5.12 of [29], we may suppose, choosing ε' small enough, that $a'd' = a''d'' = 0$.

Write $v' = ({v_{i,j}}')\in \mathrm{M}_{k+m}(B)$ and $v'' = ({v_{i,j}}'')\in \mathrm{M}_{k+m}((1-p)A(1-p))$.

Then $\|v - v' - v''\| < 3\varepsilon'(k+m)^2$. Hence,

$$\|v'(\mathrm{diag}(p_0'\otimes 1_k,\ 0\otimes 1_m))v'^* - \mathrm{diag}(((a'-\delta)_+ + d\,')\otimes 1_k,\ p\otimes 1_m)\| < \gamma,$$

where γ is small if ε' is .

Hence, by Theorem 2.1 (1), on replacing $(d'-\gamma)_+$ by d' which we may suppose still non-zero, we have

$$k\langle ((a'-\delta-\gamma)_+ + d\,')\rangle + m\langle p\rangle \leq k\langle p_0'\rangle.$$

Since $B\in\Omega$, this implies $\langle (a'-\gamma-\delta)_+ + d\,'\rangle \leq \langle p_0'\rangle$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{a'',\ p_0'',\ d'',\ {v_{i,j}}'' :\ 1 \leq i \leq k+m,\ 1 \leq j \leq k+m\}$, any sufficiently small $\varepsilon'' > 0$, there exist a C*-subalgebra C of $(1-p)A(1-p)$ and a projection $r \in A$ with $C \in \Omega$ and $1_C = r$ such that

$(1)'$ $\|xr - rx\| < \varepsilon''$ for any $x \in G$,

$(2)'$ $rxr \in_{\varepsilon''} C$ for any $x \in G$, and

$(3)'$ $[1-p-r] \leq [d']$.

By $(1)'$ and $(2)'$, we have

$$\|rp_0''r - (rp_0''r)^2\| < 2\varepsilon'', \|(1-r)p_0''(1-r) - ((1-r)p_0''(1-r))^2\| < 2\varepsilon''.$$

When ε'' is small enough, by $(1)'$ and $(2)'$, and by Lemma 2.5.5 of [29], there are a''', d''', ${v_{i,j}}''' \in C$, and a'''', d'''', ${v_{i,j}}'''' \in (1-p-r)A(1-p-r)$ and projections $p_0''' \in C$ and $p_0'''' \in (1-r-p)A(1-r-p)$ such that

$$\|a'' - a''' - a''''\| < 3\varepsilon'',$$

$$\|d'' - d''' - d''''\| < 3\varepsilon'',$$

$$\|p_0'' - p_0''' - p_0''''\| < 4\varepsilon'',$$

and

$$\|{v_{i,j}}'' - {v_{i,j}}''' - {v_{i,j}}''''\| < 4\varepsilon'',\ 1 \leq i \leq k+m,\ 1 \leq j \leq k+m.$$

By Lemma 2.5.12 of [29], we may suppose, choosing ε'' small enough, that $a'''d''' = a''''d'''' = 0$.

Write $v''' = ({v_{i,j}}''') \in \mathrm{M}_{k+m}(C),\ v'''' = ({v_{i,j}}'''') \in \mathrm{M}_{k+m}((1-r)A(1-r))$.

Then $\|v'' - v''' - v''''\| < 4\varepsilon''(k+m)^2$. Hence, we have (as above),

$$\|v'''(\mathrm{diag}(p_0''' \otimes 1_k,\ 0 \otimes 1_m))v'''^* - \mathrm{diag}(((a''' - \delta)_+ + d''') \otimes 1_k,\ r \otimes 1_m)\| < \gamma',$$

where γ' is small if ε'' is.

Hence, by Theorem 2.1 (1), on replacing $(d''' - \delta)_+$ which we may suppose non-zero, by d''', we have

$$k\langle((a''' - \delta - \gamma')_+ + d'''))\rangle + m\langle r\rangle \leq k\langle p_0'''\rangle,$$

since $C \in \Omega$, this implies $\langle(a''' - \gamma' - \delta)_+ + d'''\rangle \leq \langle p_0'''\rangle$.

Therefore, if ε'', and so γ', are small enough, then

$$\begin{aligned}\langle(a-2\varepsilon)_+\rangle \quad &\leq \langle(a - 3\varepsilon' - 3\varepsilon'' - \gamma - \gamma' - 2\delta)_+\rangle \\ &\leq \langle(a' - \gamma - \delta)_+\rangle + \langle(a''' - \gamma' - \delta)_+\rangle + \langle a''''\rangle \\ &\leq \langle(a' - \gamma - \delta)_+\rangle + \langle(a''' - \gamma' - \delta)_+\rangle \\ &+\langle(1-p-r)\rangle \leq \langle(a' - \gamma - \delta)_+\rangle + \langle d'\rangle \\ &+\langle(a''' - \gamma' - \delta)_+\rangle \leq \langle p_0'\rangle + \langle p_0'''\rangle \leq \langle p_0\rangle.\end{aligned}$$

Thirdly, let us suppose that both a and b are Cuntz equivalent to projections.

Choose projections $p,\ q$ such that a is Cuntz equivalent to p and b is Cuntz equivalent to q. We may assume that $a = p,\ b = q$.

Since $A \in \mathrm{TA}\Omega$, with $F = \{p,\ q\}$, for any $1/4 > \varepsilon > 0$, by Theorem 3.3 of [17], there are a projection $r \in A$ and a C*-subalgebra $B \subseteq A$ with $B \in \Omega$ and $1_B = r$ such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(2) $rxr \in_\varepsilon B$ for all $x \in F$, and

(3) $2[1 - r] \leq [p]$, $2[1 - r] \leq [q]$.

By (1) and (2), we have

$$\|rqr - (rqr)^2\| < 2\varepsilon, \|(1-r)q(1-r) - ((1-r)q(1-r))^2\| < 2\varepsilon,$$

and

$$\|rpr - (rpr)^2\| < 2\varepsilon, \|(1-r)p(1-r) - ((1-r)p(1-r))^2\| < 2\varepsilon.$$

When ε' is small enough, by (1) and (2), and by Lemma 2.5.5 of [29], there are projections $p_1,\ q_1 \in B$ and $p_2,\ q_2 \in (1-r)A(1-r)$ such that

$$\|p - p_1 - p_2\| < 4\varepsilon,$$

$$\|q - q_1 - q_2\| < 4\varepsilon.$$

Therefore, since $4\varepsilon < 1$,

$$p \sim p_1 + p_2,$$

$$q \sim q_1 + q_2.$$

We claim that q_1 is a non-zero projection; otherwise, $2[1-p] \leq [q] = [q_1+q_2] = [q_2] \leq [1-p]$, but this contradicts the stable finiteness of A (since C*-algebras in Ω are stably finite).

We have

$$k\langle p_1\rangle + m\langle p\rangle \leq k\langle q_1\rangle,\ k\langle p_2\rangle + m\langle(1-p)\rangle \leq k\langle q_2\rangle.$$

Since $B \in \Omega$, we have $p_1 \lesssim q_1$.

If $p_1 \sim q_1$, then we have $k\langle p_1\rangle \leq k\langle p_1\rangle + m\langle 1\rangle \leq k\langle q_1\rangle = k\langle p_1\rangle$. So kp_1 is equivalent to a proper subprojection of itself, but this contradicts the stable finiteness of A (since C*-algebras in Ω are stably finite).

Therefore, there is a non-zero projection s such that $p_1 \oplus s \sim q_1$. Since $A \in \mathrm{TA}\Omega$, for $G = \{p_2,\ q_2,\ s\}$, for any $\varepsilon > 0$, there are a projection $t \in A$ and a C*-subalgebra $C \subseteq A$ with $C \in \Omega,\ 1_C = t$ such that

$(1)'$ $\|xt - tx\| < \varepsilon$ for all $x \in G$,

$(2)'$ $txt \in_\varepsilon C$ for all $x \in G$, and

$(3)'$ $[1 - t] \leq [s]$.

By $(1)'$ and $(2)'$, we have

$$\|tq_2t-(tq_2t)^2\|<2\varepsilon, \|(1-t)q_2(1-t)-((1-t)q_2(1-t))^2\|<2\varepsilon,$$

and

$$\|tp_2t-(tp_2t)^2\|<2\varepsilon, \|(1-t)p_2(1-t)-((1-t)p_2(1-t))^2\|<2\varepsilon.$$

When ε' is small enough, by $(1)'$ and $(2)'$, and by Lemma 2.5.5 of [29], there are projections $p_3,\ q_3\in C$ and $p_4,\ q_4\in(1-t)A(1-t)$ such that

$$\|p_2-p_3-p_4\|<4\varepsilon,$$

$$\|q_2-q_3-q_4\|<4\varepsilon.$$

Therefore, as $4\varepsilon<1$,

$$p_2\sim p_3+p_4,$$

$$q_2\sim q_3+q_4.$$

We have

$$k\langle p_3\rangle+m\langle t\rangle\leq k\langle q_3\rangle,\ k\langle p_4\rangle+m\langle(1-t)\rangle\leq k\langle q_4\rangle.$$

Since $C\in\Omega$, we have $p_3\lesssim q_3$. By $(3)'$, $[p_4]\leq[1-t]\leq[s]$, and therefore, by $(3)'$ and since $4\varepsilon<1$,

$$\begin{aligned}\langle p\rangle&=\langle p_1\rangle+\langle p_2\rangle=\langle p_1\rangle+\langle p_3\rangle+\langle p_4\rangle\\&\leq\langle p_1\rangle+\langle q_3\rangle+\langle p_4\rangle\leq\langle p_1\rangle+\langle q_3\rangle+\langle s\rangle\\&\leq\langle q_1\rangle+\langle q_3\rangle\leq\langle q\rangle.\end{aligned}$$

□

COROLLARY 3.9. *Let Ω be a class of* C*-*algebras such that $rc(B)=n$ $(n\neq 0)$ for any $B\in\Omega$. Then $rc(A)\leq n$ for any unital simple* C*-*algebra $A\in\mathrm{TA}\Omega$.*

PROOF. This follows from 3.8. □

THEOREM 3.10. *Let Ω be a class of stably finite unital* C*-*algebras which are m-almost divisible. Then A is m-almost divisible for any simple unital* C*-*algebra $A\in\mathrm{TA}\Omega$.*

PROOF. As earlier (by Lemma 2.6), we may assume that Ω is closed under Morita equivalence.

We need to show that there is $b\in\mathrm{M}_\infty(A)_+$ such that $k\langle b\rangle\leq\langle a\rangle$ and $\langle(a-\varepsilon)_+\rangle\leq(k+1)(m+1)\langle b\rangle$ for any given $a\in A_+$, $\varepsilon>0$, and $k\in\mathbb{N}$. We may assume that $\|a\|=1$.

We divide the proof into two cases.

Firstly, let us suppose that a is Cuntz equivalent to a projection. Choose a projection p such that a is Cuntz equivalent to p. We may assume that $a = p$.

Since $A \in \mathrm{TA}\Omega$, with $F = \{p\}$, for any $\varepsilon' > 0$ with ε' sufficiently small, there are a C*-subalgebra B of A and a non-zero projection $r \in A$ with $B \in \Omega$ and $1_B = r$ such that

(1) $\|rp - pr\| < \varepsilon'$, and

(2) $rpr \in_{\varepsilon'} B$.

By (1) and (2), we have

$$\|rpr - (rpr)^2\| < 2\varepsilon', \ \|(1-r)p(1-r) - ((1-r)p(1-r))^2\| < 2\varepsilon'.$$

When ε' is small enough, by (1) and (2), and by Lemma 2.5.5 of [29], there are a projection $p_1 \in B$ and a projection $p_2 \in (1-p)A(1-p)$ such that $\|p - p_1 - p_2\| < 4\varepsilon'$. Therefore, $p \sim p_1 + p_2$.

Since $p_1 \in B$ and $B \in \Omega$, there is $b_1 \in B_+$ such that $k\langle b_1\rangle \leq \langle p_1\rangle$ and $\langle p_1\rangle = \langle (p-\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b_1\rangle$.

If $\langle p_1\rangle = (k+1)(m+1)\langle b_1\rangle$, this contradicts $k\langle b_1\rangle \leq \langle p_1\rangle$ since B is a stably finite C*-algebra (cf. the proof 3.1).

So by Theorem 2.1 (2), there is $c \neq 0$ such that $\langle p_1\rangle + \langle c\rangle = (k+1)(m+1)\langle b_1\rangle$.

Since $(1-r)A(1-r) \in \mathrm{TA}\Omega$, for $F = \{p_2\}$ and $\varepsilon' > 0$, there are a C*-subalgebra D of $(1-r)A(1-r)$ and a non-zero projection $t \in (1-r)A(1-r)$ with $D \in \Omega$ and $1_D = t$ such that

$(1)'$ $\|tp_2 - p_2t\| < \varepsilon'$,

$(2)'$ $tp_2t \in_{\varepsilon'} B$, and

$(3)'$ $[1-r-t] \leq [c]$.

By $(1)'$ and $(2)'$, we have

$$\|tp_2t - (tp_2t)^2\| < 2\varepsilon', \ \|(1-r-t)p_2(1-r-t) - ((1-r-t)p_2(1-r-t))^2\| < 2\varepsilon'.$$

When ε' is small enough, by $(1)'$ and $(2)'$, and by Lemma 2.5.5 of [29], there are projections $p_3 \in D$ and $p_4 \in (1-r-t)A(1-r-t)$ such that $\|p_2 - p_3 - p_4\| < 4\varepsilon'$.

Therefore, if $\varepsilon' < 1/4$, $p_2 \sim p_3 + p_4$. Since $p_3 \in D$ and $D \in \Omega$, there is $b_3 \in D$ such that $k\langle b_3\rangle \leq \langle p_3\rangle$ and $\langle p_3\rangle = \langle (p_3 - \varepsilon)_+\rangle \leq (k+1)(m+1)\langle b_3\rangle$. Then we have

$$\begin{aligned} k\langle b_1\rangle + k\langle b_3\rangle \quad & \leq \langle p_1\rangle + \langle p_3\rangle + \langle p_4\rangle = \langle p\rangle \\ & \leq \langle p_1\rangle + \langle c\rangle + \langle p_3\rangle \\ & \leq (k+1)(m+1)\langle b_1\rangle + (k+1)(m+1)\langle b_3\rangle. \end{aligned}$$

Secondly, let us suppose that a is not Cuntz equivalent to a projection.

We need to show that there is $b \in \mathrm{M}_\infty(A)_+$ such that $k\langle b\rangle \leq \langle a\rangle$ and $\langle (a-\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b\rangle$, for any given $a \in A$, $\varepsilon > 0$ and $k \in \mathbb{N}$. We may assume that $\|a\| = 1$.

By Theorem 2.1 (4), there is a non-zero positive element d orthogonal to a such that $\langle (a-\varepsilon)_+ + d\rangle \leq \langle a\rangle$.

Since $A \in \mathrm{TA}\Omega$, for $G = \{(a-\varepsilon)_+,\ d,\ a\}$, for small enough $\varepsilon' > 0$, there are a C*-subalgebra C of A and a non-zero projection $r \in A$ with $C \in \Omega$ and $1_C = r$ such that

(1) $\|xr - rx\| < \varepsilon'/3$ for any $x \in G$,

(2) $rxr \in_{\varepsilon'/3} C$ for any $x \in G$, and

(3) $[1-p] \leq [(d-3\varepsilon')_+]$.

By (1) and (2), there are $a_1,\ d_1 \in C$ and $a_2,\ d_2 \in (1-r)A(1-r)$ such that

$$\|a - a_1 - a_2\| < \varepsilon',$$

$$\|d - d_1 - d_2\| < \varepsilon',$$

$$\|(a-\varepsilon)_+ - (a_1-\varepsilon)_+ - (a_2-\varepsilon)_+\| < \varepsilon'.$$

As before, we may suppose $a_1 d_1 = a_2 d_2 = 0$.

Therefore, by Theorem 2.1 (1), if we ensure $\varepsilon' < \varepsilon$, then $\langle (a_1 - 2\varepsilon)_+ + (a_2 - 2\varepsilon)_+ \rangle \leq \langle (a-\varepsilon)_+ \rangle$.

Since $(a_1-3\varepsilon)_+,\ (d_1-\varepsilon)_+ \in C$ and $C \in \Omega$, by hypothesis, for some $0 < \delta' \leq \varepsilon$, there exists $b_1 \in \mathrm{M}_\infty(B)_+$ such that $k\langle b_1\rangle \leq \langle (a_1-3\varepsilon)_+ + (d_1-\varepsilon)_+\rangle$ and $\langle (a_1-4\varepsilon)_+ + (d_1-2\varepsilon)_+\rangle \leq \langle (((a_1-3\varepsilon)_+ + (d_1-\varepsilon)_+) - \delta')_+\rangle \leq (k+1)(m+1)\langle b_1\rangle$.

Since $(1-r)A(1-r) \in \mathrm{TA}\Omega$, with $F = \{a_2\}$ and small enough $\varepsilon'' > 0$, there are a C*-subalgebra E of $(1-r)A(1-r)$ and a non-zero projection $t \in (1-r)A(1-r)$ with $E \in \Omega$ and $1_E = t$ such that

(1$'$) $\|ta_2 - a_2 t\| < \varepsilon''$,

(2$'$) $ta_2t \in_{\varepsilon''} E$, and

(3$'$) $[1-r-t] \leq [(d_1 - 2\varepsilon)_+]$.

By (1$'$) and (2$'$), there are $a_3 \in E$ and $a_4 \in (1-r-t)A(1-r-t)$ such that

$$\|a_2 - a_3 - a_4\| < 3\varepsilon''.$$

Then, if $3\varepsilon'' \leq \varepsilon$, by Theorem 2.1 (1) we have $(a_3-3\varepsilon)_+ + (a_4-3\varepsilon)_+ \precsim (a_2-2\varepsilon)_+$.

Since $(a_3-3\varepsilon)_+ \in E$ and $E \in \Omega$, there exists $b_3 \in \mathrm{M}_\infty(E)_+$ such that $k\langle b_3\rangle \leq \langle (a_3-3\varepsilon)_+\rangle$ and $\langle (a_3-4\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b_3\rangle$.

Therefore, we have

$$\begin{aligned}
&k\langle b_1 \oplus b_2\rangle = k\langle b_1\rangle + k\langle b_3\rangle\\
&\leq \langle (a_1-3\varepsilon)_+\rangle + \langle (d_1-\varepsilon)_+\rangle + \langle (a_3-3\varepsilon)_+\rangle + \langle (a_4-3\varepsilon)_+\rangle\\
&\leq \langle (a_1-3\varepsilon)_+\rangle + \langle (d_1-\varepsilon)_+\rangle + \langle (a_2-2\varepsilon)_+\rangle\\
&\leq \langle (a-\varepsilon)_+\rangle + \langle (d_1-\varepsilon)_+\rangle\\
&\leq \langle (a-\varepsilon)_+\rangle + \langle d\rangle \leq \langle a\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\langle (a-20\varepsilon)_+\rangle \le \langle (a_1-10\varepsilon)_+\rangle + \langle (a_2-10\varepsilon)_+\rangle \\
&\le \langle (a_1-10\varepsilon)_+\rangle + \langle (a_3-4\varepsilon)_+\rangle + \langle (a_4-4\varepsilon)_+\rangle \\
&\le \langle (((a_1-4\varepsilon)_+\rangle + \langle (a_3-4\varepsilon)_+\rangle + \langle 1-p-r\rangle \\
&\le \langle (((a_1-4\varepsilon)_+\rangle + \langle (a_3-4\varepsilon)_+\rangle + \langle (d_1-2\varepsilon)_+\rangle \\
&\le (k+1)(m+1)\langle b_1\rangle + (k+1)(m+1)\langle b_3\rangle.
\end{aligned}$$

□

Theorem 3.11. *Let Ω be a class of unital* C*-*algebras which are tracially approximately divisible. Then A is tracially approximately divisible for any simple unital* C*-*algebra $A \in \mathrm{TA}\Omega$.*

Proof. We need to show that for any $\varepsilon > 0$, any $n \in \mathbb{N}$, every $F = \{a_1,\ a_2,\ \cdots,\ a_n\} \subseteq A$, and every $a \in A_+$ with $\|a\| = 1$, there is a projection $e \in A$, a completely noncommutative finite dimensional C*-algebra D, and an injective unital homomorphism $\varphi : D \to eAe$ such that

(1) $\|\varphi(b)a_k - a_k\varphi(b)\| < \varepsilon$ for $k = 1,\ 2,\ \cdots,\ n$ and all $b \in D$ with $\|b\| \le 1$,

(2) $[1-e] \le [a]$, and

(3) $\|eae\| > 1-\varepsilon$.

Since $A \in \mathrm{TA}\Omega$, by Theorem 3.3 of [17], for any non-zero positive element $A \in A$, there are a C*-subalgebra C of A and a projection $p \in A$ with $C \in \Omega$ and $1_C = p$ such that $2[1-p] \le [a]$.

Since $2[1-p] \le [a]$, there are non-zero and mutually orthogonal projections $p_1,\ p_2$ in A such that $[p_1 + p_2] \le [a]$.

Since $A \in \mathrm{TA}\Omega$, for any $\varepsilon' > 0$ with $13\varepsilon' < \varepsilon$, there are a C*-subalgebra B of A and a non-zero projection $p \in A$ with $B \in \Omega$ and $1_B = p$ such that

$(1)'$ $\|a_ip - pa_i\| < \varepsilon'$, $\|p_1p - pp_1\| < \varepsilon'$ for all $1 \le i \le n$,

$(2)'$ $pa_ip \in_{\varepsilon'} B$, $pp_1p \in_{\varepsilon'} B$ for all $1 \le i \le n$, and

$(3)'$ $2[1-p] \le [p_2]$, $2[1-p] \le [p_1]$.

By $(1)'$ and $(2)'$, we have

$$\|pp_1p - (pp_1p)^2\| < 2\varepsilon', \|(1-p)p_1(1-p) - ((1-p)p_1(1-p))^2\| < 2\varepsilon'.$$

When ε' is small enough, by $(1)'$ and $(2)'$, and by Lemma 2.5.5 of [29], there are $a_1',\ a_2',\ \cdots,\ a_n' \in B$ and $a_1'',\ a_2'',\ \cdots,\ a_n'' \in (1-p)A(1-p)$ and projections $p_1' \in B$ and $p_1'' \in (1-p)A(1-p)$ such that

$$\|p_1 - p_1' - p_1''\| < 4\varepsilon',\ \|pa_ip - a_i'\| < \varepsilon',\ \|(1-p)a_i(1-p) - a_i''\| < \varepsilon'.$$

We assert that p_1' is a non-zero projection. Otherwise, $2[1-p] \le [p_1] = [p_1' + p_1''] = [p_1''] \le [1-p]$, which contradicts the stable finiteness of A (since C*-algebras in Ω are stably finite (cf. proof of 3.1)).

Since $B \in \Omega$, for $G = \{a'_1,\ a'_2,\ \cdots,\ a'_n\} \subseteq B$, for any $\varepsilon' > 0$, there are a projection $e \in B$, a completely noncommutative finite dimensional C*-algebra D, and an injective unital homomorphism $\psi : D \to eBe \subseteq A$ such that

(1)″ $\|\psi(b)a'_k - a'_k\psi(b)\| < \varepsilon$ for $k = 1,\ 2,\ \cdots,\ n$ and all $b \in D$ with $\|b\| \leq 1$,
(2)″ $[p - e] \leq [p'_1]$, and
(3)″ $\|ep''e\| > 1 - \varepsilon'$.

We have

$$\|a_i - pa_ip - (1-p)a_i(1-p)\| < 2\varepsilon',$$

and

$$\|a'_i + a''_i - pxp - (1-p)x(1-p)\| < 2\varepsilon',$$

for all $1 \leq i \leq n$. Therefore, for any $b \in D$ with $\|b\| \leq 1$,

$$\begin{aligned}
&\|\psi(b)a_i - a_i\psi(b)\| \\
&+\|\psi(b)a_i - \psi(b)(pa_ip + (1-p)a_i(1-p))\| \\
&+\|\psi(b)(pa_ip + (1-p)a_i(1-p)) - \psi(b)(a'_i + (1-p)a_i(1-p))\| \\
&+\|\psi(b)(a'_i + (1-p)a_i(1-p)) - (a'_i + (1-p)a_i(1-p))\psi(b)\| \\
&+\|(a'_i + (1-p)a_i(1-p))\psi(b) - (pa_ip + (1-p)a_i(1-p))\psi(b)\| \\
&+\|(pa_ip + (1-p)a_i(1-p))\psi(b) - a_i\psi(b)\| \\
&< 3\varepsilon' + \varepsilon' + 3\varepsilon' + \varepsilon' + 3\varepsilon' < \varepsilon.
\end{aligned}$$

We also have (by (3)′, (2)″, and the choice of p'_1), if $4\varepsilon' < 1$,

$$\begin{aligned}
&1 - \psi(1) = 1 - e \\
&= (1-p) + (p-e) \\
&\precsim p_2 \oplus p'_1 \precsim p_2 + p_1 \precsim a.
\end{aligned}$$

□

THEOREM 3.12. *([19].) Let Ω be a class of unital* C**-algebras which is closed under passing to unital hereditary* C**-subalgebras and closed under tensoring with matrix algebras. Let $A \in \mathrm{TA}\Omega$ be an infinite dimensional simple unital* C**-algebra. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property (*2.10*). Then the crossed product algebra* $\mathrm{C}^*(G, A, \alpha)$ *belongs to* $\mathrm{TA}\Omega$.

COROLLARY 3.13. *Let A be an infinite dimensional simple unital* C**-algebra with tracial $\mathcal{Z}$-absorption (respectively, (n,m) $(m \neq 0)$ comparison, m-almost divisibility, tracial approximate divisibility, n-comparison, or β-comparison). Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property (*2.10*). Then the crossed product* C**-algebra* $\mathrm{C}^*(G, A, \alpha)$ *has tracial $\mathcal{Z}$-absorption (respectively, (n,m) $(m \neq 0)$ comparison, m-almost divisibility, tracial approximate divisibility, n-comparison, or β-comparison).*

Acknowledgements: The research of the first author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The research of the second author was supported by grants from the National Natural Sciences Foundation of China (No. 11501357, No. 11571008). The research of the third author was supported by a grant from the National Natural Sciences Foundation of China (No. 11871375).

References

1. P. Ara, F. Perera, and A. Toms, *K-theory for operator algebras. Classification of C^*-algebras*, Aspects of operator algebras and applications, 1–71, Contemp. Math., 534, American Mathematical Society, Providence, RI, 2011.

2. B. Blackadar, A. Kumjian, and M. Rørdam, *Approximately central matrix units and the structure of non-commutative tori*, K-Theory, **6** (1992), 267–284.

3. J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, and W. Winter, *Covering dimension of C^*-algebras and 2-coloured classification*, Memoirs of the American Mathematical Society, **257** (1233), (2019).

4. N. P. Brown, F. Perera, and A. Toms, *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras*, J. Reine Angew. Math., **621** (2008), 195–211.

5. J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter, *Nuclear dimension of simple C^*-algebras*, arXiv:1901.05853.

6. A. Connes, *Outer conjugacy class of automorphisms of factors*, Ann. Sci. Ecole Norm. Sup., **8** (1975), 383–420.

7. K. T. Coward, G. A. Elliott, and C. Ivanescu, *The Cuntz semigroup as an invariant for C^*-algebras*, J. Reine Angew. Math., **623** (2008), 161–193.

8. J. Cuntz and G. K. Pedersen, *Equivalence and traces on C^*-algebras*, J. Funct. Anal., **33** (1979), 135–164.

9. G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite dimensional algebras*, J. Algebra, **38** (1976), 29–44.

10. G. A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math., **443** (1993), 179–219.

11. G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero, II*, Ann. of Math., **144** (1996), 497–610.

12. G. A. Elliott, G. Gong, and L. Li, *On the classification of simple inductive limit C^*-algebras II: The isomorphism theorem*, Invent. Math., **168** (2007), 249–320.

13. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, *On the classification of simple amenable C^*-algebras with finite decomposition rank, II*, arXiv: 1507.03437.

14. G. A. Elliott and Z. Niu, *On tracial approximation*, J. Funct. Anal., **254** (2008), 396–440.

15. G. A. Elliott and A. Toms, *Regularity properties in the classification program for separable amenable C^*-algebras*, Bull. Amer. Math. Soc., **45** (2008), 229–245.

16. Q. Fan, *Certain properties of the K_0-monoid preserved by tracial approximation*, Canad. Math. Bull., **56** (2013), 337–343.

17. Q. Fan, *Some C^*-algebra properties preserved by tracial approximation*, Israel J. Math., **195** (2013), 545–563.

18. Q. Fan, *K_0-monoid properties preserved by tracial approximation*, J. Operator Theory, **69** (2013), 535–543.

19. Q. Fan and X. Fang, *Crossed products by finite group actions with a certain tracial Rokhlin property*, Acta Math. Scientia, **38** (2018), 829–842.

20. G. Gong, *On the classification of simple inductive limit C^*-algebras. I: The reduction theorem*, Doc. Math., **7** (2002), 255–461.

21. G. Gong, H. Lin, and Z. Niu, *Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras*, Preprint, arXiv: 1501.00135.

22. R. Herman and A. Ocneanu, *Stability for integer actions on UHF C^*-algebras*, J. Funct. Anal., **59** (1984), 132–144.

23. I. Hirshberg and J. Orovitz, *Tracially $\mathcal{Z}$-absorbing C^*-algebras*, J. Funct. Anal., **265** (2013), 765–785.

24. E. Kirchberg and M. Rørdam, *Central sequence C^*-algebras and tensorial absorption of the Jiang-Su algebra*, J. Reine Angew. Math., **695** (2014), 175–214.

25. A. Kishimoto and A. Kumijian, *Crossed products of Cuntz algebras by quasi-free automorphisms*, Operator Algebras and Their Applications (Waterloo, ON, 1994/1995), Fields Institute Communications, **13** (1997), 173–192.

26. A. Kishimoto, *The Rohlin property for shifts on UHF algebras*, J. Reine Angew. Math., **465** (1995), 183–196.

27. H. Lin, *Tracially AF* C^**-algebras*, Trans. Amer. Math. Soc., **353** (2001), 683–722.

28. H. Lin, *The tracial topological rank of C^*-algebras*, Proc. London Math. Soc., **83** (2001), 199–234.

29. H. Lin, *An Introduction to the Classification of Amenable C^*-Algebras*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2001.

30. H. Lin, *Classification of simple C^*-algebras with tracial topological rank zero*, Duke. Math. J., **125** (2005), 91–119.

31. H. Lin, *Classification of simple C^*-algebras with tracial rank one*, J. Funct. Anal., **254** (2008), 396–440.

32. H. Lin, *Asymptotic unitary equivalence and classification of simple amenable C^*-algebras*, Invent. Math., **183** (2011), 385–450.

33. S. Liu and X. Fang, *K-theory for certain extension algebras of purely infinite simple C*-algebras*, Proc. Indian Acad. Sci.(Math. Sci.), **120** (2010), 409–415.

34. S. Liu and X. Fang, *Extension algebras of Cuntz algebras (II)*, Bull. Aust. Math. Soc., **80** (2009), 83–90.

35. H. Matui and Y. Sato, *Strict compairison and $\mathcal{Z}$-stable of nuclear* C^**-algebras*, Acta Math., **209** (2012), 179–196.

36. H. Osaka and N. C. Phillips, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergodic Theory Dynam. Systems, **26** (2006), 1579–1621.

37. E. Ortega, F. Perera, and M. Rørdam, *The Corona Factorization Property and refinement monoids*, Trans. Amer. Math. Soc., **363** (2009), 4505–4525.

38. E. Ortega, M. Rørdam, and H. Thiel, *The Cuntz semigroup and comparison of open projections*, J. Funct. Anal., **260** (2011), 3474–3493.

39. C. Peligrad and L. Zsido, *Open projections of C^*-algebras: comparison and regularity*, Operator Theoretical Methods, Timisoara, 1998, Theta Foundation, Bucharest, 2000.

40. F. Perera and A. Toms, *Recasting the Elliott conjecture*, Math. Ann., **338** (2007), 669–702.

41. N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*, Amer. J. Math., **133** (2011), 581–636.

42. N. C. Phillips, *The tracial Rokhlin property is generic*, arXiv: 1209.3859v1.

43. L. Robert, *Nuclear dimension and n-comparison*, arXiv: 1002. 2180v2.

44. L. Robert and A. Tikuisis, *Nuclear dimension and $\mathcal{Z}$-stability of non-simple C^*-algebras*, Trans. Amer. Math. Soc., **369** (2017), 4631–4670.

45. M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF algebra*, J. Funct. Anal., **107** (1992), 255–269.

46. M. Rørdam, *Classification of certain infinite simple C^*-algebras*, J. Funct. Anal., **131** (1995), 415–458.

47. M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras*, Internat. J. Math. **15** (2004), 1065–1084.

48. M. Rørdam and W. Winter, *The Jiang-Su algebra revisited*, J. Reine Angew. Math., **642** (2010), 129–155.

49. Y. Sato, *Trace spaces of simple nuclear C^*-algebras with finite-dimensional extreme boundary*, arXiv: 1209. 3000.

50. Y. Sato, W. White, and W. Winter, *Nuclear dimension and $\mathcal{Z}$-stability*, Invent. Math., **202** (2015), 893–921.

51. A. Toms, *Flat dimension growth for C^*-algebras*, J. Funct. Anal., **238** (2006), 678–708.

52. A. Toms, *On the classification problem for nuclear C^*-algebras*, Ann. of Math., **167** (2008), 1059–1074.

53. A. Toms, and W. Winter, *Simple C^*-algebras with locally finite decomposition rank*, J. Funct. Anal., **243** (2007), 394–425.

54. A. Toms, W. White, and W. Winter, *$\mathcal{Z}$-stability and finite dimensional tracial boundaries*, Int. Math. Res. Not. IMRN, **10** (2015), 2702–2727.
55. A. Tikuisis, S. White, and W. Winter, *Quasidiagonality of nuclear C^*-algebras*, Ann. of Math., **185** (2017), 229–284.
56. W. Winter, *Decomposition rank and $\mathcal{Z}$-stability*, Invent. Math., **179** (2010), 229–301.
57. W. Winter, *Covering dimension for nuclear C^*-algebras*, J. Funct. Anal., **199** (2003), 535–556.
58. W. Winter, *Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras*, Invent. Math., **187** (2012), 259–342.
59. W. Winter and J. Zacharias, *The nuclear dimension of a C^*-algebra*, Adv. Math., **224** (2010), 461–498.

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4
e-mail: elliott@math.toronto.edu

Department of Mathematics, Shanghai Maritime University, Shanghai, China 201306
e-mail: fanqingzhai@fudan.edu.cn,qzfan@shmtu.edu.cn

Department of Mathematics, Tongji University, Shangha, China 200092
e-mail: xfang@mail.tongji.edu.cn