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Uniqueness of the Index Map in Real K-theory

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Abstract. The index map in topological K-theory for real Banach algebra extensions is a natural transformation from the first K-theory of the quotient to the zeroth K-theory of the ideal. We show that any such natural transformation is an integer multiple of the index map.

Résumé. L'application index dans la K-théorie des extensions d'algèbres de Banach réelles est une transformation naturelle entre le premier K-groupe de l'algèbre quotient et le zéroième K-groupe de l'idéal. On démontre qu'une telle transformation naturelle doit être un multiple intégral de l'application index.

Elliott claims in [1] that the group of natural transformations $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ for extensions of real Banach algebras is $\mathbb{Z} \oplus \mathbb{Z}/2$. Here we correct this result: the group of natural transformations is $\mathbb{Z}$, just as in the complex case, and this group is generated by the index map. The extra summand $\mathbb{Z}/2$ appears in [1] because the quotient A/J is assumed unital there. This does make a difference in the real case because the unit element generates a homomorphism $\mathbb{R} \to A/J$ and $\mathrm{K}_1(\mathbb{R}) \cong \mathbb{Z}/2$.

A *natural transformation* $\mathrm{K}_1(A/J) \Rightarrow \mathrm{K}_0(J)$ for extensions of real Banach algebras associates to each extension

$$J \rightarrowtail A \twoheadrightarrow A/J$$

of Banach algebras over $\mathbb{R}$ a map $\Phi_A \colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ such that the diagram

$$\begin{array}{ccc} \mathrm{K}_1(A_1/J_1) & \xrightarrow{\Phi_{A_1}} & \mathrm{K}_0(J_1) \\ \downarrow & & \downarrow \\ \mathrm{K}_1(A_2/J_2) & \xrightarrow{\Phi_{A_2}} & \mathrm{K}_0(J_2) \end{array}$$

commutes for any morphism of extensions, that is, each commuting diagram

$$\begin{array}{ccccc} J_1 & \rightarrowtail & A_1 & \twoheadrightarrow & A_1/J_1 \\ \downarrow & & \downarrow & & \downarrow \\ J_2 & \rightarrowtail & A_2 & \twoheadrightarrow & A_2/J_2. \end{array}$$

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The index map is such a natural transformation. We are going to show that any other natural transformation is an integer multiple of the index map.

Let B^+ for a Banach algebra B denote the algebra obtained by adjoining a unit element to B, turned into a Banach algebra by a suitable norm. A class in $\mathrm{K}_1(A/J)$ is represented by an invertible element $w \in \mathbb{M}_n\big((A/J)^+\big)$ for some $n \geq 1$ such that $w - 1_n \in \mathbb{M}_n(A/J)$; here 1_n denotes the unit element of $\mathbb{M}_n\big((A/J)^+\big)$. We wish to turn w into an algebra homomorphism into $\mathbb{M}_n(A/J)$. The domain of this map depends on an extra parameter $\lambda \in [1, \infty)$, which must be an upper bound for the norms of $w^{\pm 1}$. Define

$$\|f\|_\lambda := \sum_{n\in\mathbb{Z}} |f(n)|\lambda^{|n|} < \infty$$

for $f \colon \mathbb{Z} \to \mathbb{R}$ and let $\ell^1_\lambda(\mathbb{Z}) := \{f \colon \mathbb{Z} \to \mathbb{R} : \|f\|_\lambda < \infty\}$. The convolution

$$(f_1 * f_2)(n) := \sum_{j\in\mathbb{Z}} f_1(j) f_2(n-j)$$

and the norm $\|\cdot\|_\lambda$ turn $\ell^1_\lambda(\mathbb{Z})$ into a unital Banach algebra over $\mathbb{R}$. Let B be another unital Banach algebra. Any unital homomorphism $\psi \colon \ell^1_\lambda(\mathbb{Z}) \to B$ is of the form $\psi_w(f) := \sum_{n\in\mathbb{Z}} f(n) w^n$ for some invertible element $w \in B$. Conversely, an element w induces a bounded homomorphism ψ_w as above if and only if the spectral radii of both w and w^{-1} are at most λ.

So an invertible element $w \in \mathbb{M}_n\big((A/J)^+\big)$ generates a bounded homomorphism $\psi_w \colon \ell^1_\lambda(\mathbb{Z}) \to \mathbb{M}_n\big((A/J)^+\big)$ for sufficiently large λ. Let

$$Q_\lambda := \left\{ f \in \ell^1_\lambda(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} f(n) = 0 \right\}.$$

This is a 1-codimensional closed ideal in $\ell^1_\lambda(\mathbb{Z})$ with $Q_\lambda^+ \cong \ell^1_\lambda(\mathbb{Z})$. The homomorphism $\psi_w \colon \ell^1_\lambda(\mathbb{Z}) \to \mathbb{M}_n\big((A/J)^+\big)$ maps Q_λ to $\mathbb{M}_n(A/J)$ if and only if $w - 1_n \in A/J$. Thus a class in $\mathrm{K}_1(A/J)$ is represented by a bounded homomorphism $Q_\lambda \to \mathbb{M}_n(A/J)$ for some $n \geq 1$ and not by a bounded homomorphism on $\ell^1_\lambda(\mathbb{Z})$. The latter algebra is used in [1].

The split extension $Q_\lambda \rightarrowtail \ell^1_\lambda(\mathbb{Z}) \twoheadrightarrow \mathbb{R}$ yields an isomorphism

$$\mathrm{K}_1(\ell^1_\lambda(\mathbb{Z})) \cong \mathrm{K}_1(Q_\lambda) \oplus \mathrm{K}_1(\mathbb{R}) \cong \mathrm{K}_1(Q_\lambda) \oplus \mathbb{Z}/2.$$

It is shown in [1] that $\mathrm{K}_1(\ell^1_\lambda(\mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. In fact, the computations there show that $\mathrm{K}_1(Q_\lambda) \cong \mathbb{Z}$, generated by the class of the invertible element $\delta_1 \in Q_\lambda^+$. Now we sketch how the same arguments as in [1], but applied to Q_λ instead of $\ell^1_\lambda(\mathbb{Z})$, show that any natural transformation $\{\Phi_A \colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)\}$ is a multiple of the index map.

Let $\tilde{E}_\lambda$ be the universal unital Banach algebra over $\mathbb{R}$ generated by two elements x, y with $|x|, |y| \leq \lambda$. That is, we equip the free algebra on two generators with the largest submultiplicative norm where the generators have norm at most λ. Let $E_\lambda \subseteq \tilde{E}_\lambda$ be the kernel of the character $\tilde{E}_\lambda \to \mathbb{R}$ that maps $x, y \mapsto 1$. This is a 1-codimensional ideal in $\tilde{E}_\lambda$ with $\tilde{E}_\lambda \cong E_\lambda^+$. There is a canonical homomorphism $\tilde{E}_\lambda \to \ell^1_\lambda(\mathbb{Z})$, mapping $x \mapsto w$, $y \mapsto w^{-1}$. This is surjective with a bounded linear section

$$\ell^1_\lambda(\mathbb{Z}) \to \tilde{E}_\lambda, \qquad f \mapsto f(0)\cdot 1 + \sum_{n\geq 1} f(n)x^n + f(-n)y^n.$$

It restricts to a surjective homomorphism $p_\lambda\colon E_\lambda \twoheadrightarrow Q_\lambda$. Let I_λ be the kernel of p_λ. Let $w \in \mathbb{M}_n\big((A/J)^+\big)$ for some $n \geq 1$ be an invertible element with $w - 1_n \in \mathbb{M}_n(A/J)$. Choose $X', Y' \in \mathbb{M}_n(A)$ that lift $w - 1_n$ and $w^{-1} - 1_n$, respectively, and let $X = X' + 1_n$, $Y = Y' + 1_n$. Choose $\lambda \in [1, \infty)$ with $|X|, |Y| \leq \lambda$. Then there is a contractive homomorphism $\tilde{E}_\lambda \to \mathbb{M}_n(A^+)$ mapping $x \mapsto X$, $y \mapsto Y$. This maps the ideal E_λ to $\mathbb{M}_n(A)$ by construction of X, Y. So we get a morphism of Banach algebra extensions

$$\begin{array}{ccccc}
\mathbb{M}_n(J) & \rightarrowtail & \mathbb{M}_n(A) & \twoheadrightarrow & \mathbb{M}_n(A/J) \\
\uparrow{\scriptstyle \psi'_w} & & \uparrow & & \uparrow{\scriptstyle \psi_w} \\
I_\lambda & \rightarrowtail & E_\lambda & \twoheadrightarrow & Q_\lambda
\end{array}$$

The Banach algebra E_λ is contractible. The contracting homotopy is defined by restricting the maps $\tilde{E}_\lambda \to \tilde{E}_\lambda$, $x \mapsto t\cdot 1 + (1-t)\cdot x$, $y \mapsto t \cdot 1 + (1-t)\cdot y$. Hence $\mathrm{K}_j(E_\lambda) = 0$ for $j = 0, 1$ and so the index map is an isomorphism

$$\mathbb{Z} \cong \mathrm{K}_1(Q_\lambda) \xrightarrow{\cong} \mathrm{K}_0(I_\lambda).$$

The element w gives classes both in $\mathrm{K}_1(A)$ and $\mathrm{K}_1(\mathbb{M}_n(A))$, and $\Phi_A([w]) = \Phi_{\mathbb{M}_n(A)}([w])$. This follows from the naturality of Φ applied to the natural corner embedding $A \hookrightarrow \mathbb{M}_n(A)$, which induces isomorphisms $\mathrm{K}_1(A/J) \cong \mathrm{K}_1(\mathbb{M}_n(A/J))$ and $\mathrm{K}_0(J) \cong \mathrm{K}_0(\mathbb{M}_n(J))$. Now $[w] = (\psi_w)_*[\delta_1]$ in $\mathrm{K}_1(\mathbb{M}_n(A))$, where $[\delta_1] \in \mathrm{K}_1(Q_\lambda)$ is the class of the invertible element $\delta_1 \in \ell^1_\lambda(\mathbb{Z}) \cong Q_\lambda^+$. Hence $\Phi_A[w] = (\psi'_w)_*(\Phi_{E_\lambda}[\delta_1])$. So the natural transformation Φ is determined by $\Phi_{E_\lambda}[\delta_1] \in \mathrm{K}_0(I_\lambda) \cong \mathbb{Z}$. These elements are the same for all λ because there are canonical homomorphisms $E_{\lambda'} \to E_\lambda$ for $\lambda' \geq \lambda$ mapping generators to generators, which give morphisms of extensions. The index map is known to be a natural transformation. Being an isomorphism $\mathrm{K}_1(Q_\lambda) \xrightarrow{\cong} \mathrm{K}_0(I_\lambda)$, it maps the generator $[\delta_1]$ of $\mathrm{K}_1(Q_\lambda)$ to a generator of $\mathrm{K}_0(I_\lambda)$. Therefore, the natural transformation with $\Phi_{E_\lambda}[\delta_1] = n$ is equal to n times the index map.

REFERENCES

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