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Uniqueness of the Index Map in Banach Algebra K-theory, II

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Dedicated to the memory of Richard V. Kadison

Abstract. It is shown that the index map in the theory of real Banach algebras is unique as a natural transformation, up to an integral multiple, and modulo a (unique) two-torsion "ghost" map arising from the order-two K_1-group of the Banach algebra $\mathbb{R}$ (of real numbers). (In the earlier paper [3] this was shown for complex Banach algebras, of course without the "ghost" map, but in way—using Bott periodicity to pass to the opposite parity—that is not available for real Banach algebras. The present approach yields a new proof in the complex case.)

Résumé. On démontre que l'application index dans la K-théorie des algèbres de Banach réelles (ou complexes) est essentiellment unique.

1. In [3], it was shown that the index map for complex Banach algebras is unique, up to an integral multiple, as a natural map from the K_1-group of the quotient of a Banach algebra by a closed two-sided ideal to the K_0-group of the ideal.

The purpose of the present note is to extend this result to real Banach algebras—with a slight caveat—see below—, and incidentally to provide perhaps a more perspicuous, if not shorter, proof in the complex case. (In particular, in the complex case the full strength of Bott periodicity is not needed, but simply the special case that states that the K_1-group of the (complex) C^*-algebra $C(\mathbb{T})$ is the integers. Perhaps surprisingly, in the real case, in addition to this fact, it is also necessary to use one of the main parts of the proof of Bott periodicity—in the complex case!)

Recall that by naturality of the map is meant as a transformation between the two functors

$$(A, J) \mapsto K_1(A/J) \text{ and } (A, J) \mapsto K_0(J),$$

where maps between extensions (A, J) and (A', J') are commutative diagrams

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(of contractive homomorphisms)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & A/J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J' & \to & A' & \to & A'/J' & \to & 0. \end{array}$$

As is well known, the index map itself is such a natural transformation.

Theorem. *The index map for real Banach algebras is unique as a natural transformation, up to an integral multiple, and up to the possible addition to it of an actually unique map, of order two. (In the complex case the statement is the same except for the map of order two, which is no longer present.)*

2. **Remark.** Note that, since the class of complex Banach algebras is only a subclass of the class of real Banach algebras—and similarly with the class of ideals—, the uniqueness of the index map as a natural transformation, up to an integral multiple, for complex Banach algebras is not an immediate corollary of (the real case of) Theorem 1—for one thing, one would have to check that the unique order two map in the general real case vanishes in the complex case. Inspection of the proof (in the real case), though, shows that it is valid for the subclass of complex algebras. (Roughly speaking, one has to consider the algebra $l^1(\mathbb{Z})$ with complex coefficients rather than real; but in fact this is necessary as well just to deal with the real case.)

In other words, one does obtain an alternative proof of the result of [3], in the complex case. This is desirable—not just because of the desirability of avoiding Bott periodicity (to the extent possible), but also because the application of Bott periodicity in [3] to transfer the problem from the even index map to the odd one ($\mathrm{K}_0(A/J) \to \mathrm{K}_1(J)$) is perhaps somewhat subtle. Indeed, for the sake of completeness, some additional explanation of this transformation of the problem (in [3]) seems appropriate.

In view of the fact (Bott periodicity) that the operation of double suspension naturally transforms the functor K_* isomorphically onto the functor K_{*+2}, it is natural to anticipate that this natural transformation must be compatible with a given natural transformation $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ defined on extensions (as considered above). In other words, first considering the mapping $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ and then passing by means of the isomorphism $\mathrm{K}_* \to \mathrm{K}_{*+2}$ to a mapping $\mathrm{K}_3(A/J) \to \mathrm{K}_2(J)$ should give the same result (the same mapping) as first passing to the doubly suspended extension

$$0 \to \mathrm{S}^2 J \to \mathrm{S}^2 A \to \mathrm{S}^2 A/\mathrm{S}^2 J \,(= \mathrm{S}^2(A/J)) \to 0$$

and then considering the mapping

$$\mathrm{K}_3(A/J) = \mathrm{K}_1(\mathrm{S}^2(A/J)) = \mathrm{K}_1(\mathrm{S}^2 A/\mathrm{S}^2 J) \to \mathrm{K}_0(\mathrm{S}^2 J) = \mathrm{K}_2(J).$$

Roughly speaking, the two natural transformations should commute. (Or the first intertwine the given map with its double suspension.) Put even more simply, any natural map $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ is equal to its double suspension.

In fact, this is true. To see this, recall that for any Banach algebra D, the Bott periodicity isomorphism $\mathrm{K}_*(D) \to \mathrm{K}_*(\mathrm{C}_0(\mathbb{R}^2) \otimes D)$ (supremum norm on $\mathrm{C}_0(\mathbb{R}^2) \otimes D = \mathrm{C}_0(\mathbb{R}^2, D)$) from $\mathrm{K}_*(D)$ to K_* of the double suspension, $\mathrm{K}_*(\mathrm{S}^2 D)$, is the difference of the K_*-maps of the two fundamental algebra homomorphisms from D into the larger algebra $\mathrm{M}_2(\mathrm{C}_0(\mathbb{R}^2)^\sim) \otimes D = \mathrm{M}_2(\mathrm{C}(\mathrm{S}^2)) \otimes D$, namely, $D \to \mathrm{e}_B \otimes D$ where e_B is the Bott projection in $\mathrm{M}_2(\mathrm{C}(\mathrm{S}^2))$, and $D \to \mathrm{e}_{11} \otimes D$, where $\mathrm{e}_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Note that these two algebra homomorphisms are equal in the canonical quotient D corresponding to the point at infinity in $\mathbb{R}^2$, and so the difference of the corresponding K_*-maps maps into K_* of the image of the ideal $\mathrm{C}_0(\mathbb{R}^2) \otimes A/J$, and that the quotient D lifts (canonically), so that (by exactness) the map of $\mathrm{K}_1(\mathrm{C}_0(\mathbb{R}^2) \otimes D)$ into $\mathrm{K}_1(\mathrm{C}_0(\mathbb{R}^2)^\sim \otimes D)$ is an embedding. By naturality, the K_*-map of each of these two homomorphisms $D \to \mathrm{M}_2(\mathrm{C}_0(\mathbb{R}^2)^\sim) \otimes D$, applied to an extension $0 \to J \to A \to A/J \to 0$, intertwines the given map $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ with the given map $\mathrm{K}_1(\mathrm{M}_2(\mathrm{C}_0(\mathbb{R}^2)^\sim) \otimes A/J) \to \mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^2)^\sim \otimes J)$. Hence, the difference of these two K_*-maps, namely, the Bott periodicity map $\mathrm{K}_*(D) \to \mathrm{K}_*(\mathrm{S}^2 D)$, intertwines the given map $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ with the given map at the level of double suspensions, $\mathrm{K}_1(\mathrm{S}^2(A/J)) \to \mathrm{K}_0(\mathrm{S}^2 J)$, as asserted.

In [3], it is proved that any natural map $\mathrm{K}_0(A/J) \to \mathrm{K}_1(J)$ is equal to an integral multiple of a standard one, the odd index map—which is equal to and even (often) defined as the suspension of the even index map, via this natural isomorphism of K_* and $\mathrm{K}_{*+2} = \mathrm{K}_{*+1}\mathrm{S}$ (S the suspension operator). In particular, the suspension of a given natural map $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$, a natural map $\mathrm{K}_0(A/J) \to \mathrm{K}_1(J)$, must be an integral multiple of the suspension of the (even) index map. This is then true for the double suspensions, $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$, which as shown in the preceding paragraph are equal to the maps themselves.

3. As in [3], the most convenient way to proceed would seem to be to prove the following statement, which clarifies the situation since it does not even mention the index map. Note that, here, the natural transformations under consideration are from $\mathrm{K}_1(A/J)$ to $\mathrm{K}_0(J)$, not with the opposite parity as in [3]. (Presumably, a (non-zero) natural transformation with the opposite parity does not exist in the real case.)

Theorem. *The additive group of natural transformations between the two functors, from the category of extensions of real (respectively, complex) Banach algebras (with contractive homomorphisms)*

$$0 \to J \to A \to A/J \to 0$$

to the category of abelian groups,

$$(A, J) \mapsto \mathrm{K}_1(A/J) \text{ and } (A, J) \mapsto \mathrm{K}_0(J),$$

is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *(respectively, the group* $\mathbb{Z}$*).*

Theorem 1 follows immediately, once the index map is shown not to be a multiple (two or more times) of any other natural map and not to have order two. (This last is immediate, since the index map in the complex case is, by definition, a special case of what it is in the real case, and by the statement of the theorem, in the complex case, there is no possibility for the index map to have order two. The first statement also follows by observing the complex case, and recalling that the index of a Fredholm operator on an infinite-dimensional Hilbert space can be an arbitrary integer.)

4. **Proof of Theorem 3.** Let us consider simultaneously the cases of real and complex scalars. There will be a difference only at one point.

In analogy with the proof of Theorem 2 of [3], consider the universal (real or complex) Banach algebra, say B (or $B_{\mathbb{R}}$ or $B_{\mathbb{C}}$ if there is any risk of confusion) generated by two elements of norm one (in the category of real or complex Banach algebras with contractive homomorphisms). On replacing these two elements by scalar multiples, as in [3] (the case of a single generator), B becomes the universal (real or complex) Banach algebra generated by two elements with arbitrarily specified norm.

With w an invertible element of the quotient of a given (real or complex) Banach algebra A (which we may assume to be unital for the purposes of the theorem—recall that adjoining a unit does not change the K_1-group) by a given closed two-sided ideal J, or, rather, an invertible element of a matrix algebra over A—which, on changing notation we may suppose is A itself—, viewing B as the universal Banach algebra generated by two elements with norm $2\max(\|w\|, \|w^{-1}\|)$, consider the (unique) contractive Banach algebra map from B into A mapping those elements into elements of the preimages of w and w^{-1} of norm at most $2\max(\|w\|, \|w^{-1}\|)$.

For any $\lambda \geq 1$, consider the universal unital (real or complex) Banach algebra C_λ (or $C_{\lambda,\mathbb{R}}$ or $C_{\lambda,\mathbb{C}}$ to avoid confusion) generated by two elements of norm (at most) λ, one of which is the inverse of the other. C_λ is commutative. In the case $\lambda = 1$, C_λ is just the familiar Banach algebra $\mathrm{l}^1(\mathbb{Z})$ (more precisely, $\mathrm{l}^1_{\mathbb{R}}(\mathbb{Z})$ or $\mathrm{l}^1_{\mathbb{C}}(\mathbb{Z})$) of absolutely summable two-sided sequences of (real or complex) scalars, with convolution as product, with the standard generators 1_1 and 1_{-1}.

In the general case, C_λ is the Banach algebra $\mathrm{l}^1_\lambda(\mathbb{Z})$ (more precisely, if necessary, $\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})$ or $\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$) of two-sided sequences of (real or complex) scalars $(a_n)_{n\in\mathbb{Z}}$ such that

$$(a_n \lambda^{|n|})_{n\in\mathbb{Z}} \in \mathrm{l}^1(\mathbb{Z}),$$

with norm

$$\begin{aligned} \|(a_n)_{n\in\mathbb{Z}}\|_\lambda &= \|(a_n\lambda^{|n|})_{n\in\mathbb{Z}}\|_1 \\ &= \sum_{n\in\mathbb{Z}} |a_n|\lambda^{|n|}, \end{aligned}$$

again with convolution (making $l^1_\lambda(\mathbb{Z})$, which is contained in $l^1(\mathbb{Z})$, a subalgebra), and again with the standard generators 1_1 and 1_{-1}. The normed algebra inequality—and in particular closure under multiplication—is seen as follows:

$$\begin{aligned}
\|(a_m)_{m\in\mathbb{Z}} * (b_n)_{n\in\mathbb{Z}}\|_\lambda &= \|(\sum_{m+n=r} a_m b_n)_{r\in\mathbb{Z}}\|_\lambda \\
&= \sum_{r\in\mathbb{Z}} |\sum_{m+n=r} a_m b_n|\lambda^{|r|} \\
&\leq \sum_{r\in\mathbb{Z}} \sum_{m+n=r} |a_m||b_n|\lambda^{|r|} \\
&= \sum_{m,n\in\mathbb{Z}} |a_m||b_n|\lambda^{|m+n|} \\
&\leq \sum_{m,n\in\mathbb{Z}} |a_m||b_n|\lambda^{|m|+|n|} \\
&= \sum_{m\in\mathbb{Z}} |a_m|\lambda^{|m|} \sum_{n\in\mathbb{Z}} |b_n|\lambda^{|n|} \\
&= \|(a_m)_{m\in\mathbb{Z}}\|_\lambda \|(b_n)_{n\in\mathbb{Z}}\|_\lambda .
\end{aligned}$$

By Theorem 5, below, $K_1(C_{\lambda,\mathbb{R}})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and is generated by the K_1-classes of either of the two canonical generators, 1_1 and $1_{-1} = 1_1^{-1}$ in the sequence description above, together with the element $-1 = -1_0$, while $K_1(C_{\lambda,\mathbb{C}})$ is isomorphic to $\mathbb{Z}$, with generator the class of 1_1 $(= 1_{-1}^{-1})$.

Now, with $\lambda = 2\max(\|w\|, \|w^{-1}\|)$, with w an invertible element of the quotient Banach algebra A/J as above, consider the unique contractive Banach algebra map $\varphi_\lambda : B \to C_\lambda$ taking the two canonical generators of B, renormalized as above to have norm λ, into the two canonical generators $1_{\pm 1}$ of C_λ. It is important to note that this map is surjective. Indeed, if according to the description above of C_λ one is given an element $a = (a_n)_{n\in\mathbb{Z}}$ of C_λ, so that $\|(a_n)_{n\in\mathbb{Z}}\| = \sum_{n\in\mathbb{Z}} |a_n|\lambda^{|n|} < +\infty$, then if $x_\pm$ are the canonical generators of B normalized to have norm λ, with $\varphi_\lambda(x_\pm) = 1_{\pm 1} \in C_\lambda$, then the image under φ_λ of the element

$$\sum_{n<0} a_n x_-^n + a_0 x_- x_+ + \sum_{n>0} a_n x_+^n$$

of B (which has norm at most $\sum_{n\in\mathbb{Z}} |a_n|\lambda^{|n|} + a_0(\lambda^2 - 1) = \|(a_n)_{n\in\mathbb{Z}}\|_\lambda + a_0(\lambda^2 - 1) < +\infty$) is equal to $a \in C_\lambda$.

One thus has a short exact sequence

$$0 \to \ker\varphi_\lambda \to B \stackrel{\varphi_\lambda}{\to} C_\lambda \to 0.$$

This now maps into the given short exact sequence

$$0 \to J \to A \to A/J \to 0,$$

by means of a commutative diagram, with vertical maps taking the canonical generators $1_{\pm 1}$ of C_λ into the given invertible element $w \in A/J$ and its inverse, and the canonical generators $x_\pm$ of B of norm $\lambda = 2\max(\|w\|, \|w^{-1}\|)$ into the chosen elements of A of norm at most λ in the preimages of w and w^{-1}—and $\ker\varphi_\lambda$ mapping into J by commutativity.

Since the Banach algebra B is contractible (the identity map connected to the zero map by a continuous path of (contractive) endomorphisms), so that $\mathrm{K}_*(B) = 0$, by the long exact sequence of real K-theory (valid of course also in the complex case), the index map $\mathrm{K}_1(C_\lambda) \to \mathrm{K}_0(\ker\varphi_\lambda)$ is an isomorphism. By the identification of $\mathrm{K}_1(C_\lambda) = \mathrm{K}_1(\mathrm{l}^1_\lambda(\mathbb{Z}))$ in Theorem 5 ($\mathrm{K}_1(C_{\lambda,\mathbb{R}}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, in the real case, and $\mathrm{K}_1(C_{\lambda,\mathbb{C}}) = \mathbb{Z}$, in the complex case, with the class of 1_1 generating the direct summand $\mathbb{Z}$—cf. above), any other map $\mathrm{K}_1(C_\lambda) \to \mathrm{K}_0(\ker\varphi_\lambda)$, restricted (in the real case) to the direct summand $\mathbb{Z}$ ($= \mathbb{Z}[1_1]_1$) must be an integral multiple of this—modulo (in the real case) the map taking this direct summand onto the direct summand of $\mathrm{K}_0(\ker\varphi_\lambda)$ (in the real case) isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Since the generator of $\mathrm{K}_1(C_\lambda)$ corresponding to $1_{+1} \in C_\lambda$ maps into the K_1-class of $w \in A/J$, it follows by naturality that the given map on the class of w is this same integer multiple of the index map (modulo, in the real case, the canonical, in general order-two, map $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ which, as we shall see below, is defined in generality—by varying λ—in the real case according to the criterion of naturality applied to the (unique) map of order two $\mathrm{K}_1(C_{\lambda,\mathbb{R}}) \supset \mathbb{Z}[1_1]_1 \to \mathbb{Z}/2\mathbb{Z} \subset \mathrm{K}_0(\ker\varphi_\lambda)$).

Applying this argument, which required only the inequality $\lambda \geq 2\max(\|w\|, \|w^{-1}\|)$ (not equality), with arbitrarily large $\lambda \geq 1$, adjusted to be at least $2\max(\|w\|, \|w^{-1}\|)$ for a given invertible w in A/J, or in a matrix algebra over A/J—or, also for the purpose of constructing the order-two natural transformation $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ in the case of real scalars, for finitely many (two, at least) such invertible elements—, we see that the given natural map $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ must always—on any w—agree with some integral multiple of the index map (modulo the order-two map in the real case)—a priori conceivably depending on the particular value of λ used, but of course by consistency a unique integer on a given w whenever the index map is non-zero on that w, and indeed on finitely many such ws, and so a single integer for a whole extension (for which the index map is non-zero), whence by the direct sum technique of [3] a single integer overall.

5. **Theorem.** *(i) For each $\lambda \geq 1$, the complex Banach algebra*

$$\mathrm{l}^1_\lambda(\mathbb{Z}) = \mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}) := \left\{ (a_n)_{n\in\mathbb{Z}}; \sum_{n\in\mathbb{Z}} |a_n|\lambda^{|n|} < +\infty \right\},$$

with norm (see calculation above)

$$\|(a_n)_{n\in\mathbb{Z}}\|_{1,\lambda} = \sum_{n\in\mathbb{Z}} |a_n|\lambda^{|n|},$$

has $K_!$-group isomorphic to $\mathbb{Z}$, with generator the class of $1_1 \in \mathrm{l}^1_\lambda(\mathbb{Z})$.

(ii) For each $\lambda \geq 1$, the real Banach algebra

$$\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}) := \{(a_n)_{n\in\mathbb{Z}}; a_n \in \mathbb{R}, \sum |a_n|\lambda^{|n|} < +\infty\}$$

with norm

$$\|(a_n)_{n\in\mathbb{Z}}\|_{1,\lambda} = \sum_{n\in\mathbb{Z}} |a_n|\lambda^{|n|},$$

has K_1-group isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, with generators the classes of $1_1 \in \mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})$ *and* $-1_0 = -1 \in \mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})$ *(the second one of order two).*

Proof. Ad (i). The spectrum of the complex Banach algebra $\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$ is the annulus

$$\mathrm{annulus}(\lambda) := \{z \in \mathbb{C}; \lambda^{-1} \leq |z| \leq \lambda\} \subset \mathbb{C}.$$

By Theorem 7.15 of [4] (the case of a Banach algebra—see also [2]), the Gelfand transform gives rise to an isomorphism from $\mathrm{K}_1(\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$ to $\mathrm{K}_1(\mathrm{C}(\mathrm{annulus}(\lambda)))$. By homotopy equivalence of $\mathrm{annulus}(\lambda)$ with $\mathrm{annulus}(1) = \mathbb{T}$, and (a special case of) Bott periodicity, $\mathrm{K}_1(\mathrm{C}(\mathrm{annulus}(\lambda)))$ is isomorphic to the group $\mathbb{Z}$, with generator the class of the inclusion map $\mathrm{annulus}(\lambda) \hookrightarrow \mathbb{C}$. Since the Gelfand transform carries the element 1_1 of $\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$ into the inclusion map $\mathrm{annulus}(\lambda) \hookrightarrow \mathbb{C}$ (with respect to the canonical identification of $\mathrm{annulus}(\lambda)$ with the spectrum of $\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$), the class of this element generates the group $\mathrm{K}_1(\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$, as asserted.

Ad (ii). This statement is a consequence of the first statement. The canonical inclusion $\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}) \to \mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$ is surjective at the level of K_1 by (i), and the canonical map $\mathrm{l}^1_{\lambda,\mathbb{R}} \to \mathbb{R}$, $(a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} a_n$, is clearly surjective at the level of K_1 (as $\mathrm{K}_1(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ is generated by the class of $-1 \in \mathbb{R}$), and so it remains to show that this canonical surjective map

$$\mathrm{K}_1(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})) \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

is injective.

Suppose that u is an invertible element of the matrix algebra $\mathrm{M}_k(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$ which maps to $1 = 1_k \in \mathrm{M}_k(\mathbb{R})$ by evaluation at the character $1 \in \mathrm{annulus}(\lambda)$, and thus to $0 \in \mathrm{K}_1(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$, and maps to $0 \in \mathrm{K}_1(\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})) = \mathbb{Z}$. Let us analyze what we can say about u following the row and column reduction argument in the proof of surjectivity in Bott periodicity (see [1], [5]), after first approximating u by a trigonometric polynomial, i.e., a finite sequence in $\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})$ (a truncation of u will do) which we may still denote by $u = (u_n)_{n\in\mathbb{Z}}$, where $u_n \in \mathbb{R}$, $n \in \mathbb{Z}$, and now $u_n = 0$ for almost all n. Viewing u as the (finite) sum $u = \sum u_n z^n$, where z is just a convenient notation for $1_1 \in \mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})$ (to recall the calculations in [1] and [5]), and choosing m large enough that $v := z^m u$ is a genuine polynomial in z (only positive powers), expand (i.e., dilate) v to the $(2k \times 2k)$ matrix

$$\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

and apply elementary row and column operations (one of each) to remove the highest power of z if this is the second or higher. Repeat this procedure until the highest power of z is the first, i.e., the polynomial is linear, in $\mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$,

$$az + (1 - a),$$

where $a \in \mathrm{M}_r(\mathbb{R})$. (Note that the procedure does not affect the property that the image with respect to the character $1 \in \mathrm{annulus}(\lambda)$ is the identity matrix, denoted now by $1 \in \mathrm{M}_r(\mathbb{R})$. It is also important to note that the property that all the coefficients u_n, $n \in \mathbb{Z}$, of u, and so those of v, and so those of $az+(1-a)$, are real is also preserved by the matrix dilation and row-and-column operation procedure. Of course, it is critical that this new invertible element, $az+(1-a)$, is path connected (through invertibles) to the invertible element v.

Since the element $az + (1 - a)$ of $\mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$ is still invertible, the spectrum of the matrix $a \in \mathrm{M}_r(\mathbb{R})$ (as in [1] and [5]), now considered as a complex matrix, must be disjoint from the line $\{\frac{1}{2} + iy; y \in \mathbb{R}\} \subset \mathbb{C}$ (consider the homomorphisms $\mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})) \to \mathrm{M}_r(\mathbb{C})$ corresponding to the characters $\gamma \in \mathbb{T} \subseteq \mathrm{annulus}(\lambda)$ of $\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$—cf. below). Applying the holomorphic functional calculus to the matrix $a \in \mathrm{M}_r(\mathbb{R}) \subset \mathrm{M}_r(\mathbb{C})$, with respect to the holomorphic function equal to zero to the left of the vertical line in question (through $\frac{1}{2} \in \mathbb{R}$), and equal to one on the right of this line, we obtain an idempotent e, belonging in the first instance to $\mathrm{M}_r(\mathbb{C})$, connected to a (see below) by a continuous path of elements $(a_t)_{0 \le t \le 1}$ with $a_0 = a$ and $a_1 = e$, such that the spectrum of each a_t is disjoint from the line in question.

As a matter of fact, the invertibility of $az+(1-a) \in \mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})) \subset \mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$ implies invertibility of the matrix $a\gamma + (1 - a) \in \mathrm{M}_r(\mathbb{C})$ for all $\gamma \in \mathrm{annulus}(\lambda)$, when the characters of $\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$ corresponding to points of this annulus are extended to homomorphisms $\mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})) \to \mathrm{M}_r(\mathbb{C})$, and is in fact equivalent to this. To see this, note first that an element of $\mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$ is invertible if and only if it is invertible in $\mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$—indeed, an element of any real (unital) algebra is invertible if and only if it is invertible in the complexification. (If x has inverse $y + iz$ then $xy + ixz = yx + izx$, whence $xy = yx = 1$.) Next, an element of $\mathrm{M}_r(\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$ is invertible if and only if its determinant is invertible as an element of the commutative algebra $\mathrm{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$, and of course this is true as well for an element of $\mathrm{M}_r(\mathbb{C})$, as it is, in fact, for an element of a matrix algebra over any commutative ring with unit. Finally, by standard (Gelfand) commutative complex Banach algebra theory, an element is invertible if and only if its value on any character is invertible, i.e., is a non-zero complex number. It remains to note that for any homomorphism between commutative (unital) rings, the canonical extension to a homomorphism between matrix rings commutes with the operation of taking the determinant.

Now let us calculate, very much as in the proof of Bott periodicity ([1], [5]), what it means in terms of the spectrum of the matrix $a \in \mathrm{M}_r(\mathbb{C})$ that the matrix $a\gamma + (1 - a) = (\gamma - 1)a + 1$ is invertible for all $\gamma \in \mathrm{annulus}(\lambda)$. The spectrum of $(\gamma - 1)a + 1$ is of course $(\gamma - 1)\mathrm{Sp}(a) + 1$, and saying that the point $0 \in \mathbb{C}$ is excluded from this for all $\gamma \in \text{annulus}(\lambda)$ just says that $\text{Sp}(a)$ is disjoint from the set $\{-1/(\gamma-1); \gamma \in \text{annulus}(\lambda)\}$, or, in other words, that it is contained in the union of two disjoint open discs of radius $\lambda/(\lambda^2-1)$, symmetrically situated with respect to the x-axis, and also with respect to the point $\frac{1}{2} \in \mathbb{C}$, with boundary points $1(1+\lambda)$ and $\lambda/(1+\lambda)$, on either side of this point—thus, one containing the point $0 \in \mathbb{C}$ and the other the point $1 \in \mathbb{C}$.

The point is that, not only is the spectrum of a contained in the union of these two discs (an open neighbourhood of the set $\{0,1\}$), but also, the spectrum of a_t is contained in this open set for each $t \in [0,1]$, where $a_t = f_t(a)$ with f_t the holomorphic function on this open set agreeing with the identity function when $t = 0$, and equal to 0 and 1 on the two discs respectively when $t = 1$, in particular 0 at 0 and 1 at 1, and equal to the weighted average $(1-t)f_0 + tf_1$ of f_0 and f_1 for t in between 0 and 1. (The function f_0 takes this open set onto itself, f_1 takes it onto the subset $\{0,1\}$, and f_t takes it onto the subset obtained by shrinking each disc by the factor $1-t$, the first about the point 0 and the second about the point 1.) The spectral mapping theorem for the holomorphic functional calculus in a complex Banach algebra says that the spectrum of $a_t = f_t(a)$ is equal to the set $f_t(\text{Sp}(a))$, and so a subset of the union of the two discs. Hence, by the equivalence pointed out above, the matrix $a_t\gamma + (1-a_t)$, with $\gamma \in \mathbb{T}$, is invertible in $\text{M}_r(\mathbb{C})$ for every $t \in [0,1]$, and, again as observed above, it follows that the element $a_t z + (1-a_t)$ of $\text{M}_r(\text{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$ (our colloquial expression for $a_t 1_1 + (1-a_t)1_0$) is invertible for every $t \in [0,1]$, and so $t \mapsto a_t z + (1-a_t)$ is a continuous path of invertible elements connecting a to 1 in the complex Banach algebra $\text{M}_r(\text{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$.

The crucial step—allowing us to conclude the proof—is now to observe that, by symmetry (with respect to complex conjugation), since a is a real matrix (i.e., $a \in \text{M}_r(\mathbb{R}) \subset \text{M}_r(\mathbb{C})$), also a_t is a real matrix for every $t \in [0,1]$—and so $a_t z + (1-a_t)$ belongs to the real Banach algebra, $\text{M}_r(\text{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$ for every t, and, in particular, is an invertible element of this Banach algebra, and so the invertible element $az + (1-a)$ of $\text{M}_r(\text{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z})$ is connected by a continuous path of invertible elements in this algebra to the invertible element $a_1 z + (1-a_1)$, where $a_1 \in \text{M}_r(\mathbb{R})$ is an idempotent matrix.

(The symmetry argument is that, as $\bar{a} = a$ and $\overline{f_t(z)} = f_t(\bar{z})$ for each $t \in [0,1]$, the complex line integral

$$f_t(a) = \frac{1}{2\pi i} \int_\Gamma f_t(z)(z-a)^{-1} dz,$$

where Γ is a curve equal—except for orientation—to its complex conjugate, is invariant under complex conjugation.)

The original invertible element $u = z^{-m}v$ now gives rise to the same class in $\text{K}_1(\text{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$ as $a_1 z + (1-a_1)$. Since the class of $a_1 z + (1-a_1)$ in K_1 of the complex Banach algebra $\text{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z})$ (which is $\mathbb{Z}$, generated by the class of $z = 1_1$) is just the rank of the idempotent matrix a_1, and so the K_1-class of v is also this rank—and since the class of u in $\text{K}_1(\text{l}^1_{\lambda,\mathbb{C}}(\mathbb{Z}))$ is by assumption zero—, it follows that the rank of a_1 is m.

Since $u = z^{-m}v$, u has the same class in $\mathrm{K}_1(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$ as $a_1z+(1_r-a_1)+a_2z^{-1} \in \mathrm{M}_{r+m}(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$, where a_2 is the idempotent 1_m orthogonal to 1_r in $\mathrm{M}_{r+m}(\mathbb{R})$—with $\mathrm{M}_r(\mathbb{R})$ viewed as the upper left-hand $r \times r$ corner of $\mathrm{M}_{r+m}(\mathbb{R})$. We are thus reduced to looking at the $2m \times 2m$ matrix $a_1z+a_2z^{-1}$ in $\mathrm{M}_{2m}(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$, where a_1 and a_2 are two orthogonal equivalent idempotents (of rank m) with sum $1 = 1_{2m}$; in block matrix form the matrix $\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$, which since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$ can be rotated continuously to $\left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is path connected through invertibles to $\left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{M}_{2m}(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$. This shows that u is trivial in $\mathrm{K}_1(\mathrm{l}^1_{\lambda,\mathbb{R}}(\mathbb{Z}))$, as asserted.

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