

# LUSTERNIK-SCHNIRELMANN CATEGORY IN COMMUTATIVE ALGEBRA AND THE HOMOTOPY LIE ALGEBRA

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**ABSTRACT.** Here we continue the development of Lusternik-Schnirelmann category in local commutative algebra. In particular, we present a version of Félix-Halperin’s mapping theorem which is valid in any characteristic. Then we briefly discuss some consequences for the behaviour of the homotopy Lie algebra of a local homomorphism. This is a short announcement of some of the results in the author’s thesis.

**RÉSUMÉ.** Nous continuons le développement de la catégorie de Lusternik-Schnirelmann en algèbre commutative locale. En particulier, nous présentons une version du théorème de Félix-Halperin qui est valable pour toute caractéristique. Nous tirerons ensuite brièvement des conséquences sur le comportement de l’algèbre de Lie d’homotopie d’un homomorphisme local. Ceci constitue une courte annonce de certains des résultats contenus dans la thèse de l’auteur.

In the early 1980s Avramov and Roos began to uncover deep connections between local commutative algebra and rational homotopy theory, centring around the homotopy Lie algebra. This has led to a great deal of collaboration and progress in both areas. In this vein, we report here on some work towards the development of Lusternik-Schnirelmann (L.S.) category in local commutative algebra. The main result is a positive characteristic version of Félix-Halperin’s mapping theorem.

In [9] Lusternik and Schnirelmann defined the category of a space  $X$  to be the least integer  $\text{cat}(X) = m$  such that  $X$  can be covered by  $m + 1$  open sets, each of which is contractible in  $X$ . If no such  $m$  exists then  $\text{cat}(X)$  is infinite. This invariant has since enjoyed an important place in rational homotopy theory. It was Félix and Halperin who showed how L.S. category can be computed algebraically for rational spaces in terms of their Sullivan models [6]. The definition given below is a direct transcription of their characterisation into our context.

**Minimal Models and the Homotopy Lie Algebra** Let  $Q$  be a local ring (a commutative, Noetherian ring with exactly one maximal ideal) and let  $X$  be

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a graded set of variables in strictly positive degrees. We denote by  $Q[X]$  the free strictly graded commutative  $Q$  algebra on  $X$ . This is the exterior algebra over  $Q$  on odd elements of  $X$  tensored with the polynomial algebra on even elements of  $X$ . A dg algebra over  $Q$  is called *semi-free* if it has the form  $Q[X]$  after forgetting its differential  $\partial : Q[X]_i \rightarrow Q[X]_{i-1}$ . These are non-linear analogues of complexes of free modules; in particular, they can be used to resolve local rings. Note that a semi-free dg algebra is necessarily local, that is,  $A$  has exactly one graded maximal ideal  $\mathfrak{m}_A$ . A *minimal model* is a semi-free dg algebra  $A$  over a regular local ring  $Q$  which satisfies the minimality condition  $\partial(\mathfrak{m}_A) \subseteq \mathfrak{m}_A^2$ . For intuition, one should keep in mind that regular local rings behave homologically as if they were free.

These dg algebras are analogous to those used by Sullivan [11] to model spaces in rational homotopy theory. It was Avramov who imported minimal models into local commutative algebra through the following construction. Let  $R$  be a local ring with completion  $\widehat{R}$ . By the Cohen structure theorem there is a regular local ring  $Q$  with the same embedding dimension as  $R$  and a surjective homomorphism  $Q \rightarrow \widehat{R}$ . From here one can use a construction of Wolffhardt [12] to build a minimal model  $A = Q[X]$  and a quasi-isomorphism  $A \rightarrow \widehat{R}$  of  $Q$  algebras. We call  $A$  a minimal model for  $R$ . Be warned that in general  $A$  is not uniquely determined by  $R$ .

If  $A$  is a minimal model then the indecomposables  $\mathfrak{m}_A/\mathfrak{m}_A^2$  naturally form graded vector space over  $k = A/\mathfrak{m}_A$ . We denote

$$\pi^*(A) = (\varepsilon(\mathfrak{m}_A/\mathfrak{m}_A^2))^\vee.$$

Here  $\varepsilon$  is the shift which acts as  $(\varepsilon V)_i = V_{i-1}$  and  $(-)^\vee$  is the usual  $k$ -linear duality. Remarkably, the differential of  $A$  naturally endows  $\pi^*(A)$  with the structure of a graded Lie algebra over  $k$ . This ultimately goes back to Chevalley-Eilenberg. It is due to Quillen [10, Appendix B] in characteristic zero and Avramov [2, Theorem 4.2] in positive characteristic. In analogy with rational homotopy theory  $\pi^*(A)$  is called the homotopy Lie algebra of  $A$ . The next theorem generalises a classical result of Avramov [2, Theorem 4.2].

**THEOREM 1** ([5, Theorem 15]). *There is a natural isomorphism  $U\pi^*(A) \cong \text{Ext}_A^*(k, k)$ .*

The homotopy Lie algebra of a local ring  $R$  is by definition  $\pi^*(R) = \pi^*(A)$  where  $A$  is any minimal model for  $R$ . It follows from the theorem that  $U\pi^*(R) \cong \text{Ext}_R^*(k, k)$  (note that [2, Theorem 4.2] deals with  $\pi^{>1}(R)$ ). This remarkable fact that  $\text{Ext}_R^*(k, k)$  is always a universal enveloping algebra is the culmination of work of Milnor-Moore, André, Sjödén, Levin and Schoeller.

**L.S. Category** Let  $A$  be a minimal model. The *category* of  $A$  is the least integer  $\text{cat}(A) = m$  such that the quotient  $A \rightarrow A/\mathfrak{m}_A^{m+1}$  factors as a split monomorphism  $A \hookrightarrow B$  into some strictly-commutative, local dg algebra  $B$ , followed by a surjective quasi-isomorphism  $B \rightarrow A/\mathfrak{m}_A^{m+1}$ . If no such  $m$  exists

then  $A$  has infinite category. Rather than dwelling on the meaning of this we refer the reader to [6] or [5] and move on to discussing its use in commutative algebra.

To define category for local rings it is useful to first introduce an operation on minimal models. If  $i$  is a natural number the  $i$ -connected cover of  $A$  is the minimal model  $A^{(i)} = A/([\mathfrak{m}_A]_{<i})$  obtained from  $A$  by factoring by the ideal generated by elements in  $\mathfrak{m}_A$  of degree less than  $i$ . The terminology is borrowed from topology, because there is a canonical isomorphism  $\pi^*(A^{(i)}) \cong \pi^{>i}(A)$ .

Finally, if  $A$  is a minimal model for a local ring  $R$  then by definition

$$\text{cat}_i(R) = \text{cat}(A^{(i)}).$$

So to a local ring we have a sequence of invariants, each of which is either infinite or a natural number. These numbers do not depend on the choice of minimal model (see [5, Theorem 17]). It will follow immediately from the mapping theorem that  $\text{cat}_0(R) \geq \text{cat}_1(R) \geq \text{cat}_2(R) \cdots$ . According to the following proposition  $\text{cat}_0(R)$  is typically infinite, while  $\text{cat}_i(R)$  is always finite for  $i > 0$ . This is one reason for taking connected covers in the above definition.

PROPOSITION 1 ([5, Propositions 7, 9 and 10]). *For a local ring  $R$  we have  $\text{cat}_0(R) = \sup\{n : \mathfrak{m}^n \neq 0\}$  and  $\text{cat}_1(R) \leq \text{edim}(R) - \text{depth}(R)$ . The inequality is an equality if and only if  $R$  is a complete intersection. Moreover*

1.  $\text{cat}_0(R) = 0$  if and only if  $R$  is a field;
2.  $\text{cat}_1(R) = 0$  if and only if  $R$  is regular;
3.  $\text{cat}_2(R) = 0$  if and only if  $R$  is a complete intersection;
4.  $\text{cat}_1(R) \leq 1$  if and only if  $R$  is Golod;
5.  $\text{cat}_i(R) \leq 1$  if and only if  $R$  is generalised Golod of level  $i$ .

Part 3 can be strengthened considerably using results of Assmus and Gulliksen. The reader can consult [3] for discussion on Golod and generalised Golod rings.

In [6] Félix and Halperin prove the *mapping theorem*, an indispensable tool for computing L.S. category in rational homotopy theory. The corresponding fact in local commutative algebra (the next theorem) has been missing for some time, because the proof of the classical mapping theorem uses good properties of the category of dg commutative algebras over  $\mathbb{Q}$  which are not available in our context.

THEOREM 2 ([5, Theorem 20]). *If  $A \rightarrow B$  is a surjective homomorphism of minimal models then  $\text{cat } A \geq \text{cat } B$ .*

This result is fundamental in being able to work freely with L.S. category in commutative algebra. There are also potential applications to homotopy theory at large primes, after the work of Anick [1]. One easy application is the following result, which closely resembles the geometric formulation [6, Theorem I] of the original Mapping Theorem. After Levin [8], homomorphisms satisfying the hypothesis are called *large*.

COROLLARY 1. *Let  $\phi : R \rightarrow S$  be a local homomorphism inducing an isomorphism on residue fields. If the induced map  $\phi^* : \pi^*(S) \rightarrow \pi^*(R)$  is injective then  $\text{cat}_i(R) \geq \text{cat}_i(S)$  for all  $i$ .*

Finally, we briefly discuss the relative version of the above construction. Let  $\phi : R \rightarrow S$  be a local homomorphism. Using a relative version of the Cohen structure theorem [4] one can associate a minimal model  $A$  to  $\phi$ . By these means one obtains a homotopy Lie algebra  $\pi^*(\phi) = \pi^*(A)$  and a sequence of invariants  $\text{cat}_i(\phi) = \text{cat}(A^{(i)})$ . Just as in Proposition 1 the values of  $\text{cat}_i(\phi)$  uncover important information about  $\phi$ . However, the invariants  $\text{cat}_i(\phi)$  can fail to be finite for any  $i$ . For this reason, the Lie algebra  $\pi^*(\phi)$  is often not as well behaved as its absolute counterpart  $\pi^*(R)$ .

This is ultimately the main motivation for studying L.S. category in commutative algebra. As in rational homotopy theory, L.S. category provides a flexible finiteness condition which implies that the homotopy Lie algebra has good properties (see [6, 7]). For example, it follows from the results in [7] that  $\pi^*(\phi)$  has a finite dimensional centre if  $\text{cat}_i(\phi)$  is finite for any  $i$ . In another direction, to a homomorphism  $\phi$  with  $\text{cat}_2(\phi) < \infty$  one can associate a long exact sequence of homotopy Lie algebras generalising the one given by Avramov [2, theorem 2.7] under the stronger assumption that  $\phi$  has finite flat dimension. It is for these reasons that we want methods (such as the mapping theorem) for establishing finiteness of  $\text{cat}_i(\phi)$ .

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