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The Rieffel Projection Via Groupoids

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Abstract. An elementary groupoid construction is shown to underlie Rieffel's Hilbert module construction of a non-trivial projection in the irrational rotation C*-algebra.

Résumé. Une construction élémentaire de groupoïde se révèle à la base de la construction de Rieffel à module de Hilbert d'un projecteur non-trivial dans la C*-algèbre d'une rotation irrationnelle.

1. In [6], Rieffel constructed a canonical finitely generated—in fact, singly generated—projective module over the irrational rotation C*-algebra (the unique C*-algebra generated by two unitaries with commutator equal to a given complex number of absolute value one, of infinite order in the circle group $\mathbb{T}$). He showed that no multiple of this module was free, and it was shown later by Pimsner and Voiculescu in [4] that this module together with the free modules generates the K_0-group. This construction, though canonical, and elegant, was technically somewhat intricate. The corresponding projection that this singly generated projective module gave rise to was, surprisingly, very easy to describe, but only in a non-canonical way.

It is possible to make the construction of the projection canonical, at the cost of its no longer being in the algebra itself, by using Rieffel's module in a natural way to construct a larger C*-algebra, simple and unital and containing the given one as a corner, and such that the projection in the opposite corner yields the same K_0-class as the Rieffel projection. The rotation algebra corner is Morita equivalent to the larger algebra. This construction, although canonical, relies on the full panoply of the Rieffel (Hilbert module) construction.

In the present note, a simple groupoid construction of the larger C*-algebra is given, based on the groupoid construction of the rotation algebra ([5], [3], [1], [8]). Namely, the rotation algebra groupoid is embedded as what might be called a corner of a larger groupoid, the cut-down by a closed and open (clopen) subset of the larger object space, and the two complementary projections referred to are those arising from this clopen set decomposiiton of the larger object space.

This groupoid approach may be viewed as a groupoid construction of Rieffel's module. (Rieffel's module is constructed starting with the real line. As we shall show, the real line, viewed, as a space of arrows, in a natural way, is a left module in a natural sense over the rotation algebra groupoid.)

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2. Recall that a topological group acting on a topological space, continuously with respect to both variables (i.e., the map group $\times$ space $\to$ space jointly continuous), gives rise to a topological groupoid in a natural way (sometimes called the transformation group, or transformation groupoid), namely, with objects the points in the space and arrows the pairs consisting of a point—the tail of the arrow—and a group element, and the product of two arrows being the arrow with the same tail as the first arrow, and the product of the two group elements (the first by the second). (Both the product and the inverse operations for arrows are continuous.) (See Example 1.2a of [5] or Example 2.1.15 of [8]).

A transformation group is an étale groupoid, i.e., some open neighbourhood of arrows about each arrow is parametrized topologically by the set of tails of these arrows, which is an open subset of the space of objects (see [5], or page 1 of [8]), if, and (as is easily seen) only if, the topological group is discrete.

In [5], Renault described an étale groupoid which in the natural way (as the universal enveloping C*-algebra of the *-algebra of continuous (complex-valued) functions with compact support, with convolution (involving only finite sums) as the product, and the natural *-operation) gives rise to the rotation C*-algebra (either rational or irrational).

Renault's groupoid is a transformation group, namely, the group of integers acting on the circle by means of the rotation in question. In this note, we shall embed this groupoid as a subgroupoid of a larger groupoid, étale but not a transformation group, and in such a way that there is a natural inclusion of the C*-algebra of the smaller groupoid in that of the larger one—giving rise to a Morita equivalence. The complement of the unit of the smaller C*-algebra is the second generator of the common K_0-group—and so, although it is not shown to be equivalent to a projection in the rotation algebra itself (this is not even true in the commutative case), it may perhaps still be referred to (roughly) as the Rieffel projection.

The Rieffel Hilbert C*-module (over the rotation C*-algebra) will be exhibited as the enveloping Hilbert C*-module of a purely groupoid module construction, in which the groupoid acting will be that of Renault, and the groupoid module object space (in the appropriate sense) will be just the real line—very much reminiscent of the Rieffel module!

3. The main new notion (not after all genuinely new, as it appears in the Yoneda concretization of an abstract (small) category—[9]) is that of the action of a groupoid, or indeed of just a category, on a set, which, for this purpose should be viewed as a set of arrows. A (small) category then acts on the left when concatenation of an arrow in the category with an arrow in the set, head of the former to tail of the latter, yields another arrow in the set—and this operation is associative as in a group action (or module—a ring acting on an abelian group), and also continuous with respect to given topologies on the category and the set. A right action would be similar, and also a two-sided one (different categories perhaps acting on the two sides) with the natural conditions of associativity and continuity imposed.

Equivalently, the set on which the category acts should just be a subset of some larger category—also containing a second category if this is to act on the right. (The tails and heads of the arrows acted on should of course be objects in the two respective subcategories.)

The example to be considered at the moment involves a groupoid, not just a category, and the larger category containing the set of arrows acted upon is also a groupoid, as is a natural subcategory acting on this set of arrows on the right. The larger category being a groupoid, the set of arrows acted on has a a natural double, or dual, the set of inverses of these arrows, and this is important for the construction of Rieffel's Hilbert C*-module.

The object spaces of the two subgroupoids exhaust the object space of the larger one, decomposing it into the union of two clopen subsets (the first yielding the unit of the original algebra, the rotation C*-algebra, and the second the unit of a closely related rotation C*-algebra, and since the C*-algebra of the large groupoid (all these groupoids are étale in the sense of Renault [5], and so give rise in an elementary way to C*-algebras) is simple, the second clopen set gives rise to a K_0-class of the rotation algebra, equal as we shall see to that of the Rieffel projection (with canonical trace equal to the rotation, as a fraction of 2π).

The key to the construction is the Kronecker flow, derived from the flow in the plane, $\mathbb{R}^2$, in the direction θ, say—i.e. with flow lines of slope θ—, by passing to the quotient by the subgroup $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ (the action of which by translation of course commutes with the flow, which therefore indeed passes to the quotient). This flow on the torus—the well-known Kronecker flow—determines a topological groupoid with object space the torus—and arrows determined by ordered pairs of points lying on the same flow line, with the first point being the tail of the arrow and the second the head. The topology on the arrows must also take the length and direction of the arrow—with or against the flow—into account (and with this in mind it is enough to keep track of the first point of the pair—this together with the length and direction of an arrow determining both the arrows themselves and also the topology). This topology is locally compact and Hausdorff, but not étale. (See [1].)

The subgroupoid with object space the image of the y-axis (and containing all arrows between two points in this image) is in fact the groupoid of Renault giving rise to the rotation algebra corresponding to the rotation $2\pi\theta$, as is seen when this image ($\mathbb{R}/\mathbb{Z}$) is viewed as the unit circle. (See [1].) This subgroupoid is closed (although not open), and so locally compact and Hausdorff. It is in fact étale. (Of course, when we refer to the rotation $2\pi\theta$ we mean $2\pi\theta'$ where $\theta' = \theta$ (modulo $\mathbb{Z}$) and $0 < \theta' \leq 1$; similarly, instead of $2\pi/\theta$, we mean $2\pi\theta''$ where $\theta'' = \theta^{-1}$ (modulo $\mathbb{Z}$) and $0 < \theta'' \leq 1$; this is not actually necessary, but we do exclude the cases $\theta = 0$ (modulo Z) or $\pm\infty$.

The analogous subgroupoid corresponding to the x-axis is in the same way seen to be Renault's groupoid giving rise to the rotation algebra corresponding to the rotation $2\pi/\theta$. Note that the two object spaces have a point (the origin) in common. They form a figure eight.

The first step in the construction of the desired groupoid, containing the two rotation groupoids (for the rotations $2\pi\theta$ and $2\pi/\theta$) as clopen subsets, is to compute the subgroupoid of the Kronecker one obtained by cutting down to this combined object space, so far a (connected) figure eight. (The crucial step will be to break this in two—to disconnect the two circles.) It remains to compute the arrows going between the two circles, and by symmetry it is enough to look a those going from the first to the second.

The arrows from the first circle to the second are in bijective and bicontinuous correspondence with the arrows from the y-axis to the x-axis (either in the flow direction or opposite to it, as the case may be). The parameter for these arrows is just the signed length—positive in the flow direction and negative otherwise—the arrow of zero length being also an object (a slight singularity that prevents the subgroupoid in question from being étale—but this is unimportant as we do not need to consider its C*-algebra, and in any case the convolution of functions with compact support would still involve just finite sums and so the construction of the C*-algebra would be the same).

Strikingly, this space of arrows is just a copy of the real line (the base space for Rieffel's Hilbert module constructed in [6]). The arrows from the x-axis to the y-axis form a second line with the arrow of zero length common to both. This is, as is not hard to see, still true after mapping to the torus.

Let us now disconnect the two circles, and at the same time disconnect the whole groupoid into the disjoint union of the two rotation groupoids (each a disjoint union of circles parametrized naturally by the integers) and two copies of the real line—all four of these subsets being clopen.

The operation is very simple: the single arrow common to all four subsets in question—the two subgroupoids and the two lines—should be replaced by four—so that it now appears as arrows of zero length in each of the four subsets as before, but these are now disjoint. Thus, the single object, common to both circles, is replaced by two objects, one in each circle in the same place as before, and two arrows between these two objects, one in each direction, again each fitting naturally into the one-parameter family it belonged to before—either arrows from the first circle to the second or arrows from the second circle to the first. The new groupoid is still locally compact and Hausdorff, and now étale.

4. The functional analytic (vector space and Hilbert module) version of the Rieffel module is now a simple exercise—convolution of continuous complex-valued functions of compact support—either all at once on the large (étale) groupoid, and completing along with the natural involution to obtain a C*-algebra, simple by 4.3.6 of [8] (together with 4.1.5 and 4.1.6 of [8]), since the isomorphism classes are dense in the space of objects, and with the two rotation algebras as opposite corners and obtaining the Rieffel module as the off-diagonal subspace—the elements going between the two corners (lower right to upper left)—or else convolving functions on the upper left corner groupoid with functions on the real line of arrows from the first circle to the second, acted on on the left by that groupoid. The algebra-valued inner product is obtained by convolving functions on one line with functions on the other (the dual, or opposite, module), and completing as does Rieffel, one obtains his Hilbert module.

The unit of the C*-algebra from the second groupoid gives rise to a K_0-class in the rotation algebra arising from the first groupoid, and, as we shall now check, a simple calculation shows that it has trace θ (modulo $\mathbb{Z}$) when the trace is normalized on the unit of the first groupoid algebra (the upper left corner), and therefore by [4] (Appendix) at least if θ is irrational (in fact, in the rational case, too) is a second generator of K_0 (just as is the Rieffel projection).

The calculation consists of identifying the invariant measure on the space of objects of the groupoid, that arises from the trace on the C*-algebra. Inside each circle, it must be a multiple of Lebesgue measure, and the question is to determine the ratio of the normalizations. Inspecting the Kronecker flow, or already its pre-image in the plane, one notes that the ratio of the densities of flow lines crossing the x- and y-axes is just the slope, θ. This is, of course, the ratio of the measures of the two circles. Thus, our proposed variant of the Rieffel projection indeed has trace θ. (Only if $\theta < 1$ can this projection be equivalent to a projection in the original algebra.)

5. A related question (posed to us by L. Robert) is whether the present construction helps to show that the Rieffel Hilbert module (just as a module) is finitely generated and projective. This can be seen from its description as the upper right off-diagonal part of a simple unital C*-algebra, the upper left corner of which is the given rotation algebra. Since the lower right-hand corner is, by simplicity, majorized by a finite sum of projections equivalent to the upper-left corner projection, in a matrix algebra over the algebra, one sees the module in question exhibited as a direct summand of a finite direct sum of free modules.

6. It would be interesting to see groupoid versions of the finitely generated projective modules, called Heisenberg modules, constructed by Rieffel in [7], over a higher dimensional non-commutative torus.

In this setting, a groupoid model for the C*-algebra itself, analogous to that of Renault in the two-dimensional case, would seem yet to be proposed. (In the case all rotation numbers are rational, equivalently ([2]), the range of the trace on K_0 consists of rational numbers, this can be done because (see [2]) the C*-algebra is a tensor product of rotation algebras.)

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