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A Positive-Definite Energy Functional for Axially Symmetric Maxwell's Equations on Kerr-de Sitter Black Hole Spacetimes

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Abstract. We prove that there exists a phase space of canonical variables, for the initial value problem for axially symmetric Maxwell fields with compactly supported initial data and propagating in Kerr-de Sitter black hole spacetimes, such that their motion is restricted to the level sets of a positive-definite Hamiltonian, despite the ergo-region.

Résumé. On démontre qu'il existe un espace de phase de variables canoniques, pour le problème des valeurs initials pour les champs de Maxwell symétriques à donneés initiales de support compact et à propagation dans les espaces-temps de trou noir Kerr-de Sitter, tel que leur motion est restrainte aux ensembles de niveau d'une hamiltonienne de type positif, en dépit de l'ergo-région.

1. Kerr-de Sitter Black Holes Consider the Kerr-de Sitter family of black holes $(\bar{M}, \bar{g})$:

$$\bar{g} = -\frac{\Delta}{\Sigma}\left(\frac{dt - a\sin^2\theta d\phi}{1+\frac{\Lambda}{3}a^2}\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{\Sigma}{\Pi}d\theta^2$$

$$+\frac{\sin^2\theta(1+\frac{\Lambda}{3}a^2\cos^2\theta)}{\Sigma}\left(\frac{adt-(r^2+a^2)d\phi}{1+\frac{\Lambda}{3}a^2}\right)^2 \tag{1}$$

where

$$\Delta = r^2 - 2mr + a^2 - \frac{\Lambda r^2}{3}(r^2+a^2), \tag{2a}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{2b}$$

$$\Pi = 1 + \frac{\Lambda}{3}a^2\cos^2\theta, \tag{2c}$$

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$\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, $|a| < m$. The quartic polynomial function $\Delta(r)$ is such that it admits precisely one negative root and three distinct positive roots $\{r_\pm, r_c\}, r_+ < r_c$, which correspond to the cases of physical interest. In this work we shall restrict to $a \neq 0$ and the regular region $r_+ < r < r_c$. The Kerr-de Sitter family is a solution of Einstein's equations in 3+1 dimensions with a positive cosmological constant:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_{\bar{g}} + \Lambda\bar{g}_{\mu\nu} = 0, \quad \Lambda > 0 \quad (\bar{M}, \bar{g}), \tag{3}$$

which reduces to the Schwarzschild-de Sitter family if $a = 0$ and de Sitter if a and $m = 0$. The question of stability of the 3+1 de Sitter spacetime has been resolved by Friedrich in a series of landmark works [18–20]. Existence and stability of even dimensional de Sitter spacetimes in higher dimensions was proved in [1]. In a remarkable recent breakthrough, Hintz and Vasy have resolved the *nonlinear* stability of the Kerr-de Sitter black holes for small angular-momentum [25] (see also [24]).

The evolution of their methods, developed from Melrose's $b-$calculus, can be found in the list of references therein. In this context, there were preceding results on the stability of the Kerr-de Sitter family for small angular-momentum for various model problems. The local energy decay of the wave equation on Schwarzschild-de Sitter is studied in [6]. Asymptotics and resonances of linear waves on the Kerr-de Sitter metric are studied in [12, 13]. The asymptotic behaviour of the Klein-Gordon equation on the Kerr-de Sitter metric is studied in [21]. Global boundedness for linear waves on Schwarzschild-de Sitter and Kerr-de Sitter cosmologies is proved in [37]. A partial proof of *nonlinear* stability of Schwarzschild-de Sitter cosmologies is discussed in [38]. The decay of Maxwell's equations on Schwarzschild-de Sitter metric was proved recently in [28].

The case $\Lambda = 0$ in (1) and (3) corresponds to the Kerr family of black holes, the stability of which is being pursued in a long-standing program that began soon after their discovery. In a remarkable recent development, the linear stability of Schwarzschild black hole spacetimes has been resolved in [8] using the Teukolsky variables, by carrying forward the classic works in [32, 35, 41, 42]. The stability of Schwarzschild using metric coefficients has been resolved in [26] and [27]. A Morawetz estimate for the linearized gravity on Schwarzschild was proved in [4]. The decay of Maxwell's equations on the Schwarzschild metric was proved in [5]. Model problems for *nonlinear* stability of Schwarzschild are considered in [29, 30, 33].

In contrast with Schwarzschild and Schwarzschild-de Sitter ($a = 0$), an important obstacle for Kerr and Kerr-de Sitter ($a \neq 0$) is that the energy of even the linear wave equation is not necessarily positive-definite. This is caused by the ergo-region, which always surrounds a Kerr black hole with non-vanishing angular-momentum. A variety of techniques are introduced to cope with this issue for fields propagating on Kerr with small angular-momentum [2, 3, 10, 31, 40].

This leads us naturally to the question of stability of the Kerr-de Sitter family for large, but sub-extremal, angular-momentum ($|a| < m$). It may be noted that the lack of positivity of energy (and the related superradiance effect) makes proving the decay even more subtle for large $|a|$. The extraction of energy from a sub-extremal Kerr black hole using a linear wave equation is discussed in [17, 39], which is equivalent to the Penrose process [7]. The decay of the linear wave equation on a sub-extremal Kerr for fixed azimuthal modes is proved in [15,16] using spectral methods [14]. Extending these works, the decay for general solutions of the linear wave equation is proved in the intricate and remarkable work [11]. However, little is known about the global behaviour of nonscalar and coupled fields propagating on Kerr or Kerr-de Sitter for large $|a|$.

The special case of axially symmetric linear waves, propagating on the Kerr-de Sitter spacetimes, admits a fortuitous simplification as the energy from the energy-momentum tensor is immediately positive-definite and is thus directly amenable to Morawetz and decay estimates (see, e.g., [2,9] for the Kerr counterpart). The problem becomes much more subtle even for the axially symmetric coupled vector fields (e.g. Maxwell's equations) propagating on Kerr-de Sitter. Indeed the problem of positivity of total energy of axially symmetric Maxwell's equations on Kerr spacetimes has been an open problem for decades where, in principle, counter-examples for positivity of energy density can be constructed. This has recently been resolved for the full range of sub-extremal Kerr black holes in [23] and separately in [34].

The subject of this paper is to prove equivalent results for Kerr-de Sitter ($|a| < m$). Analogous to [23], these results also hold for the fully coupled axially symmetric Einstein-Maxwell perturbations of the Kerr-Newman-de Sitter spacetimes, which shall be discussed rigorously in a separate article. Importantly, in the pure Maxwell problem, the positive-definite energy we construct is naturally associated to *gauge-invariant* quantities. In addition, without the several technicalities of the fully coupled Einstein-Maxwell problem, the pure Maxwell problem is more transparent.

Following [23], we shall use the Hamiltonian formulation as it provides a mechanism to construct a gauge-invariant notion of mass-energy for the perturbative theory of $(\bar{M}, \bar{g})$ for the full $|a| < m$. Consider a Maxwell 2-form F and a vector potential A defined on $(\bar{M}, \bar{g})$, such that $F := dA$; then Maxwell's equations are the critical points of the variational principle:

$$S_M[F] := -\frac{1}{4} \int \|F\|_{\bar{g}}^2 \, \bar{\mu}_{\bar{g}} \tag{4}$$

for compactly supported variations. If we perform the ADM decomposition

$$(\bar{M}, \bar{g}) = (\bar{\Sigma}, \bar{q}) \times \mathbb{R}$$

of the metric $\bar{g}$ and the vector potential A,

$$\bar{g} = -\bar{N}^2 dt^2 + \bar{q}_{ij}(dx^i + \bar{N}^i dt) \otimes (dx^j + \bar{N}^j dt) \tag{5}$$

$$A = A_0 dt + A_i dx^i, \quad i, j = 1, 2, 3, \tag{6}$$

and define

$$\mathfrak{B}^i := \frac{1}{2}\epsilon^{ijk}(\partial_j A_k - \partial_k A_j),$$

the ADM variational principle is defined as

$$I_{ADM}[A_i, \mathfrak{E}^i] := \int \Big(A_i \partial_t \mathfrak{E}^i - \frac{1}{2}\bar{N}\bar{\mu}_{\bar{q}}^{-1}\bar{q}_{ij}(\mathfrak{E}^i\mathfrak{E}^j + \mathfrak{B}^i\mathfrak{B}^j) + \epsilon_{ijk}\bar{N}^i\mathfrak{E}^j\mathfrak{B}^k$$
$$- A_0\partial_i\mathfrak{E}^i\Big) d^4x \tag{7}$$

for the phase space X^{Max},

$$X^{\mathrm{Max}} := \{(A_i, \mathfrak{E}^i), i = 1, 2, 3\},$$

which results in the Maxwell field equations

$$\partial_t A_i = -\bar{N}\bar{\mu}_{\bar{q}}^{-1}\bar{q}_{ij}\mathfrak{E}^j - \epsilon_{ijk}\bar{N}^j\mathfrak{B}^k + \partial_i A_0, \tag{8a}$$

$$\partial_t \mathfrak{E}^i = -\partial_\ell(\bar{N}\bar{\mu}_{\bar{q}}^{-1}\bar{q}_{kj}\mathfrak{B}^j\epsilon^{k\ell i}) + \partial_\ell(\bar{N}^\ell\mathfrak{E}^i - \bar{N}^i\mathfrak{E}^\ell), \tag{8b}$$

where $\bar{\mu}_{\bar{q}}$ is the square root of the metric determinant of $(\bar{\Sigma}, \bar{q})$. The $\mathfrak{E}^i$ field can be concisely represented in terms of the F tensor as

$$\mathfrak{E}^i = \frac{1}{2}\epsilon^{ijk}\,{}^*F_{jk}. \tag{9}$$

The Maxwell constraint equations are

$$\partial_i\mathfrak{E}^i = 0, \quad i = 1, 2, 3. \tag{10}$$

2. Dynamics with a Positive-Definite Hamiltonian Let $(\bar{M}, \bar{g})$ be the Kerr-de Sitter spacetime represented in (1), where $\Delta(r)$'s positive roots are $r_\pm$ and r_c with $(r_\pm < r_c)$. If we consider the ADM decomposition of (1),

$$\bar{M} = \mathbb{R} \times \bar{\Sigma},$$

where $\bar{\Sigma}$ is Riemannian. The group $SO(2)$ acts on $(\bar{\Sigma}, \bar{q})$ in such a way that ∂_ϕ is the associated Killing vector field. We define

$$\Sigma := \bar{\Sigma}/SO(2). \tag{11}$$

The fixed point set of the $SO(2)$ action on $\bar{\Sigma}$ is a union of two disjoint sets, which we represent together as Γ ('the axes') for brevity. It may be noted that the fixed point set Γ corresponds to $\|\partial_\phi\|_{\bar{g}} = 0$ and also a boundary of Σ. Finally, we define a Lorentzian manifold with boundary M such that

$$M := \bar{M}/SO(2) = \Sigma \times \mathbb{R}.$$

PROPOSITION 2.1. *Suppose* $(\bar{M}, \bar{g})$ *is a Kerr-de Sitter spacetime with* Δ *as in* (1). *Then the following statements hold.*

1. *The metric* $\bar{g}$ *can be represented in Weyl-Papapetrou form:*

$$\bar{g} = e^{-2\gamma}\mathbf{g} + e^{2\gamma}\Phi^2, \tag{12}$$

where $\Phi = d\phi + \mathcal{A}_\nu dx^\nu, \nu = 0, 1, 2$, $\mathbf{g}$ *is the Lorentzian metric of* M.

2. *There exists an auxiliary (scalar) potential* ω,

$$\omega \,:\, (M, \mathbf{g}) \to \mathbb{R}$$

such that (γ, ω)*satisfies the 'shifted' wave maps equation:*

$$\square_{\mathbf{g}}\gamma + \frac{1}{2}e^{-4\gamma}\mathbf{g}^{\alpha\beta}\partial_\alpha\omega\partial_\beta\omega + \Lambda e^{-2\gamma} = 0 \tag{13a}$$

$$\square_{\mathbf{g}}\omega - 4e^{-4\gamma}\mathbf{g}^{\alpha\beta}\partial_\alpha\omega\partial_\beta\gamma = 0, \quad on \quad (M, \mathbf{g}) \setminus \Gamma, \tag{13b}$$

we shall refer to ω *as the gravitational twist potential.*

3. *There exists a 3+1 decomposition of* $(\bar{M}, \bar{g})$ *such that it is smoothly foliated by 3D Riemannian maximal hypersurfaces.*

PROOF. For proofs of statements 1 and 2 above, see [22]; the system (13) is coupled to 2+1 Einstein equations. The fact that the expansion parameter Λ decouples from the Einstein equations and appears as the forcing term of (13a) will play a crucial role in our problem. For the convenience of the reader, we shall provide the construction of the scalar potential ω below. The Einstein field equations (3) imply that the 1-form G such that

$$\mathcal{F}_{\mu\nu} = e^{-4\gamma}\varepsilon_{\mu\nu\delta}\mathbf{g}^{\delta\alpha}G_\alpha, \tag{14}$$

where ε is the volume form of the metric $\mathbf{g}$ and $\mathcal{F}_{\mu\nu} := \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu, \mu, \nu, \alpha, \delta = 0, 1, 2$, is closed. Therefore, by the Poincaré Lemma, $G = d\omega$, where ω is the gravitational twist potential.

For (3), consider the ADM decompositon of (1),

$$\bar{g} = -\bar{N}^2dt^2 + \bar{q}_{ij}(dx^i + \bar{N}^idt) \otimes (dx^j + \bar{N}^jdt), \quad i, j = 1, 2, 3. \tag{15}$$

Subsequently, if $\nabla(\bar{q})$ is the (intrinsic) covariant derivative of $(\bar{\Sigma}, \bar{q})$, then it follows that

$$\nabla_i(\bar{q})\bar{N}^i \equiv 0, \quad (\bar{\Sigma}_t, \bar{q}_t), \quad \forall t \in \mathbb{R}, \tag{16}$$

which holds for all t in view of the t-translational symmetry of $(\bar{M}, \bar{g})$ in (1). Furthermore, consider the ADM decomposition of $(M, \mathbf{g})$,

$$\mathbf{g} = -N^2dt^2 + q_{ab}(dx^a + N^adt) \otimes (dx^b + N^bdt), \quad a, b = 1, 2. \tag{17}$$

We also have

$$\nabla_a(q)N^a \equiv 0, \quad (\Sigma_t, q_t), \quad \forall t \in \mathbb{R}. \tag{18}$$

□

It follows from the definition (14) of ω that

$$\partial_a \mathcal{A}_0 + Ne^{-4\gamma}\epsilon_{ab}\bar{\mu}_q q^{bc}\partial_c \omega = 0. \tag{19}$$

The explicit expression of Kerr-de Sitter spacetime in the 'Weyl-Papapetrou' form is as follows:

$$\begin{aligned} \bar{g}_{KdS} = &- e^{-2\gamma}\frac{\Pi\Delta \sin^2\theta}{(1+\frac{\Lambda}{3}a^2)^4}dt^2 + e^{-2\gamma}\left(\frac{\Sigma e^{2\gamma}}{\Delta}dr^2 + \frac{\Sigma e^{2\gamma}}{\Pi}d\theta^2\right) \\ &+ e^{2\gamma}\left(d\phi + \frac{a(-2mr - \frac{\Lambda}{3}a^2(r^2+a^2)(1+\cos^2\theta))}{-a^2\sin^2\theta\Delta + \Pi(r^2+a^2)^2}dt\right)^2. \end{aligned} \tag{20}$$

Reading off various components of the Weyl-Papapetrou form, we have

$$e^{2\gamma} = \frac{\sin^2\theta(-a^2\sin^2\theta\Delta + \Pi(r^2+a^2)^2)}{\Sigma(1+\frac{\Lambda}{3}a^2)^2}, \tag{21a}$$

$$N = \frac{(\Pi\Delta)^{\frac{1}{2}}\sin\theta}{(1+\frac{\Lambda}{3}a^2)^2}, \tag{21b}$$

$$\mathcal{A}_0 = \frac{a(-2mr - \frac{\Lambda}{3}a^2(r^2+a^2)(1+\cos^2\theta))}{-a^2\sin^2\theta\Delta + \Pi(r^2+a^2)^2}, \tag{21c}$$

$$\bar{\mu}_q^{-1}q_{ab}dx^a \otimes dx^b = \left(\frac{\Pi}{\Delta}\right)^{\frac{1}{2}}dr^2 + \left(\frac{\Delta}{\Pi}\right)^{\frac{1}{2}}d\theta^2, \tag{21d}$$

where $\bar{\mu}_q$ is the square root of the metric determinant of (Σ, q). We are interested in the initial value problem of Maxwell's equations (8) with axial symmetry. We assume that the axially symmetric F tensor is derivable from an axially symmetric vector potential A.

In view of the fact that the Kerr-de Sitter spacetime is also axially symmetric, let us construct a new phase space X. Firstly, consider a twist potential

$$\lambda : (M, \mathbf{g}) \to \mathbb{R}$$

such that $\lambda := A_\phi$, so that $\mathfrak{B}^a = \epsilon^{ab}\partial_b\lambda$ and $\partial_a\mathfrak{B}^a = 0, a, b = 1, 2$. It follows from the Maxwell constraint equations $\partial_a\mathfrak{E}^a = 0$ and the Poincaré Lemma that there exists a twist potential

$$\eta : (M, \mathbf{g}) \to \mathbb{R}$$

such that $\mathfrak{E}^a = \epsilon^{ab}\partial_b\eta$. Likewise, it follows from the variational principle (7) that the conjugate momenta u and v defined as

$$u := \mathfrak{B}^\phi, \quad v := -\mathfrak{E}^\phi \tag{22}$$

form the dynamical canonical pairs with η and λ respectively. Thus we define the phase space X as

$$X := \{(\lambda, v), (\eta, u)\}.$$

Maxwell's equations (8), can be transformed into the phase space X and locally represented as

$$\partial_t \eta = N e^{2\gamma} \bar{\mu}_q^{-1} u, \quad \partial_t \lambda = N e^{2\gamma} \bar{\mu}_q^{-1} v, \tag{23a}$$

$$\partial_t u = \partial_b (N \bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \eta) + N \bar{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \lambda, \tag{23b}$$

$$\partial_t v = \partial_b (N \bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \lambda) - N \bar{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \eta. \tag{23c}$$

For the initial value problem of (23), we assume that the axially symmetric F tensor has smooth and compactly supported initial data in a $t = t_0$ initial data slice $(\bar{\Sigma}_0, \bar{q}_0)$. Define initial data in X as

$$ID := \{(\lambda_0, u_0), (\eta_0, v_0)\}, \quad (\Sigma_0, q_0). \tag{24}$$

In this work, we shall assume that the initial data is compactly supported, strictly within the interior of Σ, i.e., $Supp(ID) \subset \Sigma$. As a consequence, in the computations the boundary terms vanish at both the horizons. It may be noted that the global propagation of regularity of the Maxwell field F in the *domain of outer communications* of the Kerr-de Sitter metric $(\bar{M}, \bar{g})$ is standard. As a consequence, we have the following prescribed behaviour on the axes Γ of the quotient space (Σ, q) :

$$\partial_{\vec{s}} \lambda = 0, \, \partial_{\vec{s}} \eta = 0, \quad \text{on} \quad \Gamma, \tag{25a}$$

$$\partial_{\vec{n}} \lambda = 0, \, \partial_{\vec{n}} \eta = 0, \quad \text{on} \quad \Gamma, \tag{25b}$$

$$\partial_t \lambda = 0, \, \partial_t \eta = 0, \quad \text{on} \quad \Gamma, \quad \forall \, t \in \mathbb{R}, \tag{25c}$$

where $\partial_{\vec{s}}$ and $\partial_{\vec{n}}$ are the derivatives tangential and normal to the axes Γ respectively. In view of (25) it may be noted that

$$\mathfrak{E}^i = 0, \quad \mathfrak{B}^i = 0, \quad \text{on} \quad \Gamma.$$

It follows from (25) that one can choose λ, η such that they are (uniformly) 0 along Γ. In principle, our Hamiltonian framework allows for the Coulomb type conserved charges; however, the behaviour at the axes is chosen only for convenience in functional analysis arguments. We now state the main theorem of the paper.

THEOREM 2.2. *Suppose F is the electromagnetic Faraday tensor with $\mathcal{L}_{\partial_\phi} F \equiv 0$, $\mathcal{L}_{\partial_\phi} A \equiv 0$, propagating on the Kerr-de Sitter black holes* (1) *with $|a| < m$ and further suppose that $F \in C^\infty(\bar{\Sigma}_0, \bar{q}_0)$ with $Supp(ID) \subset \Sigma_0$. Then the following statements hold for the initial value problem of F on Kerr-de Sitter $(\bar{M}, \bar{g})$*

1. *There exists a positive-definite Hamiltonian H^{Alt} for the dynamics of the canonical pairs in the phase space $X = \{(\lambda, u), (\eta, v)\}$ i.e.,*

$$D_u \cdot H^{Alt} = \partial_t \eta, \quad D_\eta \cdot H^{Alt} = -\partial_t u, \tag{26a}$$

$$D_v \cdot H^{Alt} = \partial_t \lambda, \quad D_\lambda \cdot H^{Alt} = -\partial_t v, \tag{26b}$$

where D is the (variational) directional derivative in the phase-space X.

2. *There exits a divergence-free spacetime vector density J such that its flux through t-constant hypersurfaces is positive-definite.*
3. *There exists a canonical transformation $U : (X, H^{Alt}) \to (\underline{X}, H^{Reg})$ to a 'regularized' phase space*

$$\underline{X} := \{(\underline{\lambda}, \underline{u}), (\underline{\eta}, \underline{v})\}$$

where,

$$\underline{\lambda} := e^{-\gamma}\lambda, \quad \underline{\eta} := e^{-\gamma}\eta, \quad \underline{u} := e^{\gamma}u, \quad \underline{v} := e^{\gamma}v,$$

such that the corresponding Hamiltonian H^{Reg} is positive-definite.

PROOF. The ADM Hamiltonian energy, re-expressed in the phase space X and using (19), consecutively transforms as follows:

$$\begin{aligned} H = & \int_\Sigma \Big(\frac{1}{2} N \bar{\mu}_q^{-1} e^{2\gamma}(u^2 + v^2) + \frac{1}{2} N \bar{\mu}_q q^{ab} e^{-2\gamma}(\partial_a \eta \partial_b \eta + \partial_a \lambda \partial_b) \\ & - \mathcal{A}_0 \epsilon^{ab} \partial_a \eta \partial_b \lambda \Big) d^2x \\ = & \int_\Sigma \Big(\frac{1}{2} N \bar{\mu}_q^{-1} e^{2\gamma}(u^2 + v^2) + \frac{1}{2} N \bar{\mu}_q q^{ab} e^{-2\gamma}(\partial_a \eta \partial_b \eta + \partial_a \lambda \partial_b \lambda) \\ & + N e^{-4\gamma} \bar{\mu}_q q^{ab} \partial_a \omega \partial_b \eta \lambda \Big) d^2x \end{aligned} \tag{27}$$

where the $-v\epsilon^{ab}\partial_b \eta + u \epsilon^{ab} \partial_b \lambda$ terms drop out. Now consider the quantity

$$\begin{aligned} I := & \frac{1}{4} N e^{-2\gamma} \bar{\mu}_q q^{ab} \left((\partial_a \lambda - 2\lambda \partial_a \gamma)(\partial_b \lambda - 2\lambda \partial_b \gamma) + (\partial_a \eta - 2\eta \partial_a \gamma)(\partial_b \eta - 2\eta \partial_b \gamma) \right) \\ & + \frac{1}{4} N e^{-2\gamma} \bar{\mu}_q q^{ab} \big((\partial_a \eta + \lambda e^{-2\gamma} \partial_a \omega)(\partial_b \eta + \lambda e^{-2\gamma} \partial_b \omega) \\ & + (\partial_a \lambda - \eta e^{-2\gamma} \partial_a \omega)(\partial_b \lambda - \eta e^{-2\gamma} \partial_b \omega) \big) - \frac{1}{2} N e^{-2\gamma} \bar{\mu}_q q^{ab} (\partial_a \lambda \partial_b \lambda + \partial_a \eta \partial_b \eta). \end{aligned}$$

We have,

$$\begin{aligned} I = & N e^{-2\gamma} \bar{\mu}_q q^{ab} (\partial_a \gamma \partial_b \gamma + \frac{1}{4} e^{-4\gamma} \partial_a \omega \partial_b \omega)(\lambda^2 + \eta^2) \\ & - N e^{-2\gamma} \bar{\mu}_q q^{ab} (\lambda \partial_a \gamma \partial_b \lambda + \eta \partial_a \gamma \partial_b \eta - \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega \partial_b \eta + \frac{1}{2} \eta e^{-2\gamma} \partial_a \omega \partial_b \lambda). \end{aligned} \tag{28}$$

Recall the wave map system satisfied by (γ, ω):

$$\partial_b(N\bar{\mu}_q q^{ab}\partial_a\gamma) + \frac{1}{2}N\bar{\mu}_q e^{-4\gamma}q^{ab}\partial_a\omega\partial_b\omega + N\bar{\mu}_q\Lambda e^{-2\gamma} = 0, \tag{29a}$$

$$\partial_b(N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega) = 0. \tag{29b}$$

Now consider the quantity:

$$II := \frac{1}{2}\partial_b\left(-N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\eta\lambda - N\bar{\mu}_q q^{ab}e^{-2\gamma}\partial_a\gamma(\eta^2+\lambda^2)\right). \tag{30}$$

We have

$$\begin{aligned}
II = &-\frac{1}{2}\big(N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\partial_b\eta\lambda + \lambda\eta\partial_b(N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega) \\
&+ N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\partial_b\lambda\eta\big) \\
&- \frac{1}{2}\big(e^{-2\gamma}\partial_b(N\bar{\mu}_q q^{ab}\partial_a\gamma)(\lambda^2+\eta^2) - 2e^{-2\gamma}N\bar{\mu}_q q^{ab}\partial_a\gamma\partial_b\gamma(\lambda^2+\eta^2) \\
&- 2e^{-2\gamma}N\bar{\mu}_q q^{ab}\partial_a\gamma(\lambda\partial_b\lambda + \eta\partial_b\eta)\big) \\
= &-\frac{1}{2}\left(N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\partial_b\eta\lambda + N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\partial_b\lambda\eta\right) \\
&- \frac{1}{2}(e^{-2\gamma}(-\frac{1}{2}N\bar{\mu}_q e^{-4\gamma}q^{ab}\partial_a\omega\partial_b\omega - N\bar{\mu}_q\Lambda e^{-2\gamma}) \\
&- 2e^{-2\gamma}N\bar{\mu}_q q^{ab}\partial_a\gamma\partial_b\gamma)(\lambda^2+\eta^2) - e^{-2\gamma}N\bar{\mu}_q q^{ab}\partial_a\gamma(\lambda\partial_b\lambda + \eta\partial_b\eta).
\end{aligned}$$

Consequently,

$$I - II = -\frac{1}{2}(\lambda^2+\eta^2)\Lambda e^{-4\gamma}N\bar{\mu}_q + N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\partial_b\eta\lambda, \tag{31}$$

as the $\partial\omega\partial\eta$ term occurs in both I and II with opposite signs. Therefore, using I–II, we can transform the Hamiltonian energy (27) into the following manifestly positive form:

$$\begin{aligned}
H^{\mathrm{Alt}} := \int_\Sigma \Big(&\frac{1}{2}Ne^{2\gamma}\bar{\mu}_q^{-1}(u^2+v^2) + \frac{1}{2}\Lambda N\bar{\mu}_q e^{-4\gamma}(\lambda^2+\eta^2) \\
&+ \frac{1}{4}Ne^{-2\gamma}\bar{\mu}_q q^{ab}\big((\partial_a\lambda - 2\lambda\partial_a\gamma)(\partial_b\lambda - 2\lambda\partial_b\gamma) \\
&+ (\partial_a\eta - 2\eta\partial_a\gamma)(\partial_b\eta - 2\eta\partial_b\gamma)\big) \\
&+ \frac{1}{4}Ne^{-2\gamma}\bar{\mu}_q q^{ab}\big((\partial_a\eta + \lambda e^{-2\gamma}\partial_a\omega)(\partial_b\eta + \lambda e^{-2\gamma}\partial_b\omega) \\
&+ (\partial_a\lambda - \eta e^{-2\gamma}\partial_a\omega)(\partial_b\lambda - \eta e^{-2\gamma}\partial_b\omega)\big)\Big)d^2x.
\end{aligned} \tag{32}$$

The aforementioned transformation of the original ADM Hamiltonian into a positive form in (32) is motivated by the construction of the Robinson's identity [36], but now adapted to our problem. Crucially, we need to prove that the Hamiltonian structure of the equations is retained.

Consider a (variational) 1-parameter flow of a generic phase point P in the phase space X, parametrized by s. We shall denote the components of the variation at P with respect to this flow as

$$u' := D\cdot u(P), \quad v' := D\cdot v(P), \quad \lambda' := D\cdot \lambda(P), \quad \eta' := D\cdot \eta(P). \tag{33}$$

Consider $D_u \cdot H^{\mathrm{Alt}}$ and $D_v \cdot H^{\mathrm{Alt}}$; we have

$$D_u \cdot H^{\mathrm{Alt}} = Ne^{2\gamma}\bar{\mu}_q^{-1} u \quad \text{and} \quad D_v \cdot H^{\mathrm{Alt}} = Ne^{2\gamma}\bar{\mu}_q^{-1} v,$$

respectively. The quantities $D_\lambda \cdot H^{\mathrm{Alt}}$ and $D_\eta \cdot H^{\mathrm{Alt}}$ are more difficult. From (32), $D_\eta \cdot H^{\mathrm{Alt}}$ and $D_\lambda \cdot H^{\mathrm{Alt}}$ have the following types of terms:

- 1st order $\partial\eta\partial\eta'$ and $\partial\lambda\partial\lambda'$:

$$Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_a\eta\partial_b\eta' = \partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_a\eta\eta') - \partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_b\eta)\eta'$$

and

$$Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_a\lambda\partial_b\lambda' = \partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_a\lambda\lambda') - \partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_a\lambda)\lambda';$$

- 1st order $\partial\eta'\partial\omega$ and $\partial\lambda'\partial\omega$:

$$\frac{1}{2}\lambda Ne^{-4\gamma}\bar{\mu}_q q^{ab}\partial_a\eta'\partial_b\omega = \partial_b(\frac{1}{2}\lambda Ne^{-4\gamma}\bar{\mu}_q q^{ab}\partial_a\omega\eta') - \partial_b(\frac{1}{2}\lambda Ne^{-4\gamma}\bar{\mu}_q q^{ab}\partial_b\omega)\eta'$$

and

$$-\frac{1}{2}Ne^{-4\gamma}\bar{\mu}_q q^{ab}\eta\partial_a\lambda'\partial_b\omega = -\frac{1}{2}\partial_b(\eta Ne^{-4\gamma}\bar{\mu}_q q^{ab}\partial_a\omega\lambda') + \frac{1}{2}\partial_b(\eta Ne^{-4\gamma}\bar{\mu}_q q^{ab}\partial_a\omega)\lambda';$$

- mixed type $\eta'\partial\eta$, $\eta\partial\eta'$ and $\lambda'\partial\lambda$, $\lambda\partial\lambda'$:

$$\begin{aligned}&-Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_a\eta\partial_b\gamma\eta' - Ne^{-2\gamma}\bar{\mu}_q q^{ab}\eta\partial_a\eta'\partial_b\gamma\\ &= -Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_a\eta\partial_b\gamma\eta' - \partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\eta\partial_a\gamma\eta') + \partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\eta\partial_b\gamma)\eta'\end{aligned}$$

and

$$\begin{aligned}&-Ne^{-2\gamma}\bar{\mu}_q q^{ab}\lambda'\partial_a\lambda\partial_b\gamma - Ne^{-2\gamma}\bar{\mu}_q q^{ab}\lambda\partial_a\lambda'\partial_b\gamma\\ =&-Ne^{-2\gamma}\bar{\mu}_q q^{ab}\lambda'\partial_a\lambda\partial_b\gamma - \partial_b(e^{-2\gamma}N\bar{\mu}_q q^{ab}\lambda\partial_b\gamma\lambda') + \partial_b(e^{-2\gamma}N\bar{\mu}_q q^{ab}\lambda\partial_b\gamma)\lambda';\end{aligned}$$

- 0th order:

$$2(N\bar{\mu}_q e^{-2\gamma} q^{ab}\partial_a\gamma\partial_b\gamma + \frac{1}{4}N\bar{\mu}_q e^{-6\gamma} q^{ab}\partial_a\omega\partial_b\omega + \frac{1}{2}\Lambda N\bar{\mu}_q e^{-4\gamma})\eta\eta' \tag{34}$$

and

$$2(N\bar{\mu}_q e^{-2\gamma} q^{ab}\partial_a\gamma\partial_b\gamma + \frac{1}{4}N\bar{\mu}_q e^{-6\gamma} q^{ab}\partial_a\omega\partial_b\omega + \frac{1}{2}\Lambda N\bar{\mu}_q e^{-4\gamma})\lambda\lambda'. \tag{35}$$

Combining all the above, while using the system (29) again, we recover the full set of field equations:

$$\begin{aligned} D_\eta \cdot H^{\text{ Alt}} &= -\,\partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_b\eta) - N\bar{\mu}_q q^{ab} e^{-4\gamma}\partial_a\omega\partial_b\lambda \\ &= -\,\partial_t u \end{aligned} \tag{36}$$

and

$$\begin{aligned} D_\lambda \cdot H^{\text{Alt}} &= -\,\partial_b(Ne^{-2\gamma}\bar{\mu}_q q^{ab}\partial_b\lambda) + N\bar{\mu}_q q^{ab} e^{-4\gamma}\partial_a\omega\partial_b\eta \\ &= -\,\partial_t v. \end{aligned} \tag{37}$$

Now let us turn to Part 2. Define the energy density $\mathcal{E}^{\text{Alt}}$ of the Hamiltonian H^{Alt} such that

$$H^{\text{Alt}} = \int_\Sigma \mathcal{E}^{\text{Alt}}\, d^2x. \tag{38}$$

Now define $\bar{v} = N\bar{\mu}_q^{-1}v$ and $\bar{u} := N\bar{\mu}_q^{-1}u$. We shall construct the divergence-free vector density from the time derivative of the density $\mathcal{E}^{\text{Alt}}$ of the Hamiltonian H^{Alt}. The purpose of calculating the time derivative of the energy density is to obtain a pure spatial divergence. Therefore, we collect the terms with $\bar{v}$ and $\bar{u}$ and their spatial derivatives $(\partial_a\bar{v}, \partial_a\bar{u})$ separately, both of which occur. It turns out that these terms combine to form a pure patial divergence, if we use the background field equations. Explicitly, we have the following terms in $\frac{\partial}{\partial t}\mathcal{E}^{\text{Alt}}$:

$$\partial_a\bar{v}(\frac{1}{2}N\bar{\mu}_q q^{ab}(2\partial_b\lambda - \eta e^{-2\gamma}\partial_b\omega - 2\lambda\partial_b\lambda)) \tag{39}$$

and

$$\partial_a\bar{u}(\frac{1}{2}N\bar{\mu}_q q^{ab}(2\partial_b\eta + \lambda e^{-2\gamma}\partial_b\omega - 2\eta\partial_b\gamma)), \tag{40}$$

where the terms with $\bar{v}$ and $\bar{u}$ are

$$\begin{aligned}&\bar{u}(\frac{1}{2}Ne^{-2\gamma}\bar{\mu}_q q^{ab}(-\partial_a\omega(\partial_b\lambda-\eta e^{-2\gamma}\partial_b\omega)+2e^{2\gamma}\partial_a\gamma(\partial_b\eta+\lambda e^{-2\gamma}\partial_b\omega)))\\ &+\bar{u}e^{2\gamma}(\partial_b(N\bar{\mu}_q q^{ab}e^{-2\gamma}\partial_b\eta)+N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\partial_b\lambda+N\lambda e^{-4\gamma}\bar{\mu}_q\Lambda)\end{aligned}\tag{41}$$

and

$$\begin{aligned}&\bar{v}(\frac{1}{2}Ne^{-2\gamma}\bar{\mu}_q q^{ab}(\partial_a\omega(\partial_b\eta+\lambda e^{-2\gamma}\partial_b\omega)+2e^{2\gamma}\partial_a\gamma(\partial_b\lambda-\eta e^{-2\gamma}\partial_b\omega)))\\ &+\bar{v}e^{2\gamma}(\partial_b(N\bar{\mu}_q q^{ab}e^{-2\gamma}\partial_a\lambda)-N\bar{\mu}_q q^{ab}e^{-4\gamma}\partial_a\omega\partial_b\eta+N\eta e^{-4\gamma}\bar{\mu}_q\Lambda)\end{aligned}\tag{42}$$

which, in view of the system (29), can be transformed to

$$\bar{v}\partial_b(N\bar{\mu}_q q^{ab}(\partial_a\eta-\frac{1}{2}\eta e^{-2\gamma}\partial_a\omega-\lambda\partial_b\gamma))\tag{43}$$

and

$$\bar{u}\partial_b(N\bar{\mu}_q q^{ab}(\partial_a\eta+\frac{1}{2}\lambda e^{-2\gamma}\partial_a\omega-\eta\partial_a\gamma)),\tag{44}$$

respectively. Therefore, the time derivative of $\mathcal{E}^{\mathrm{Alt}}$ can be transformed into a pure spatial divergence:

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{E}^{\mathrm{Alt}}=&\frac{\partial}{\partial x^b}\big(\bar{u}N\bar{\mu}_q q^{ab}(\partial_a\eta+\frac{1}{2}\lambda e^{-2\gamma}\partial_a\omega-\eta\partial_a\gamma)\\ &+\bar{v}N\bar{\mu}_q q^{ab}(\partial_a\lambda-\frac{1}{2}\eta e^{-2\gamma}\partial_a\omega-\lambda\partial_a\gamma)\big)\\ =&\frac{\partial}{\partial x^b}\big(uN^2q^{ab}(\partial_a\eta+\frac{1}{2}\lambda e^{-2\gamma}\partial_a\omega-\eta\partial_a\gamma)\\ &+vN^2q^{ab}(\partial_a\lambda-\frac{1}{2}\eta e^{-2\gamma}\partial_a\omega-\lambda\partial_a\gamma)\big)\end{aligned}\tag{45}$$

which can be transformed into a divergence-free vector density

$$J:=J^t\partial_t+J^b\partial_b\tag{46}$$

where

$$\begin{aligned}J^t&:=\mathcal{E}^{\mathrm{Alt}}\\ J^b&:=-uN^2q^{ab}(\partial_a\eta+\frac{1}{2}\lambda e^{-2\gamma}\partial_a\omega-\eta\partial_a\gamma)-vN^2q^{ab}(\partial_a\lambda-\frac{1}{2}\eta e^{-2\gamma}\partial_a\omega-\lambda\partial_a\gamma).\end{aligned}$$

For Part 3, consider the regularized phase space $\underline{X} := \{(\underline{\lambda}, \underline{v}), (\underline{\eta}, \underline{u})\}$

$$\underline{\gamma} := e^{-\gamma}\gamma, \quad \underline{\eta} := e^{-\gamma}\eta, \quad \underline{u} := e^{\gamma}u, \quad \underline{v} := e^{\gamma}v$$

To construct a regularized Hamiltonian H^{Reg}, we shall use further identities:

$$\begin{aligned}
&\frac{1}{4}Ne^{-2\gamma}\bar{\mu}_q q^{ab}\big((\partial_a\lambda - 2\lambda\partial_a\gamma)(\partial_b\lambda - 2\lambda\partial_b\gamma) + (\partial_a\eta - 2\eta\partial_a\gamma)(\partial_b\eta - 2\eta\partial_b\gamma)\big)\\
&\quad + \frac{1}{4}Ne^{-2\gamma}\bar{\mu}_q q^{ab}\big((\partial_a\eta + \lambda e^{-2\gamma}\partial_a\omega)(\partial_b\eta + \lambda e^{-2\gamma}\partial_b\omega)\\
&\quad + (\partial_a\lambda - \eta e^{-2\gamma}\partial_a\omega)(\partial_b\lambda - \eta e^{-2\gamma}\partial_b\omega)\big)\\
= &\quad \frac{1}{4}N\bar{\mu}_q q^{ab}((\partial_a\underline{\lambda} - \underline{\lambda}\partial_a\gamma)(\partial_b\underline{\lambda} - \underline{\lambda}\partial_b\gamma) + (\partial_b\underline{\eta} - \underline{\eta}\partial_a\gamma)(\partial_b\underline{\eta} - \underline{\eta}\partial_b\gamma))\\
&\quad + \frac{1}{4}N\bar{\mu}_q q^{ab}((\partial_a\underline{\eta} + \underline{\eta}\partial_a\gamma + \underline{\lambda}e^{-2\gamma}\partial_a\omega)(\partial_b\underline{\eta} + \underline{\eta}\partial_b\gamma + \underline{\lambda}e^{-2\gamma}\partial_b\omega))\\
&\quad + \frac{1}{4}N\bar{\mu}_q q^{ab}((\partial_a\underline{\lambda} + \underline{\lambda}\partial_a\gamma - \underline{\eta}e^{-2\gamma}\partial_a\omega)(\partial_b\underline{\lambda} + \underline{\lambda}\partial_b\gamma - \underline{\eta}e^{-2\gamma}\partial_b\omega))\\
= &\quad \frac{1}{2}N\bar{\mu}_q q^{ab}((\partial_a\underline{\lambda} - \frac{1}{2}\underline{\eta}e^{-2\gamma}\partial_a\omega)(\partial_b\underline{\lambda} - \frac{1}{2}\underline{\eta}e^{-2\gamma}\partial_b\omega)\\
&\quad + (\partial_a\underline{\eta} + \frac{1}{2}\underline{\lambda}e^{-2\gamma}\partial_a\omega)(\partial_b\underline{\eta} + \frac{1}{2}\underline{\lambda}e^{-2\gamma}\partial_b\omega))\\
&\quad + \frac{1}{2}N\bar{\mu}_q q^{ab}(\partial_a\gamma\partial_b\gamma + \frac{1}{4}e^{-4\gamma}\partial_a\omega\partial_b\omega)(\underline{\lambda}^2 + \underline{\eta}^2),
\end{aligned}$$

where the $\partial\omega\partial\gamma$ terms cancel. Thus, the energy H^{Alt} can be transformed into

$$\begin{aligned}
H^{\mathrm{Reg}} := \int \Big(&\frac{1}{2}N\bar{\mu}_q^{-1}(\underline{u}^2 + \underline{v}^2) + \frac{1}{2}N\Lambda\bar{\mu}_q e^{-2\gamma}(\underline{\lambda}^2 + \underline{\eta}^2)\\
&+ \frac{1}{2}N\bar{\mu}_q q^{ab}(\partial_a\gamma\partial_b\gamma + \frac{1}{4}e^{-4\gamma}\partial_a\omega\partial_b\omega)(\underline{\lambda}^2 + \underline{\eta}^2)\\
&+ \frac{1}{2}N\bar{\mu}_q q^{ab}((\partial_a\underline{\lambda} - \frac{1}{2}\underline{\eta}e^{-2\gamma}\partial_a\omega)(\partial_b\underline{\lambda} - \frac{1}{2}\underline{\eta}e^{-2\gamma}\partial_b\omega)\\
&+ (\partial_a\underline{\eta} + \frac{1}{2}\underline{\lambda}e^{-2\gamma}\partial_a\omega)(\partial_b\underline{\eta} + \frac{1}{2}\underline{\lambda}e^{-2\gamma}\partial_b\omega))\Big)d^2x.
\end{aligned} \tag{47}$$

Analogously to the calculations for H^{Alt}, we recover the field equations for the regularized phase space $\underline{X}$:

$$\underline{D_u} \cdot H^{\mathrm{Reg}} = \partial_t\underline{\eta}, \quad \underline{D_\eta} \cdot H^{\mathrm{Reg}} = -\partial_t\underline{u}, \tag{48a}$$

$$\underline{D_v} \cdot H^{\mathrm{Reg}} = \partial_t\underline{\lambda}, \quad \underline{D_\lambda} \cdot H^{\mathrm{Reg}} = -\partial_t\underline{v}, \tag{48b}$$

where $\underline{D}$ is the usual (variational) directional derivative in $\underline{X}$. Likewise, if we define the energy density $\mathcal{E}^{\mathrm{Reg}}$ such that

$$H^{\mathrm{Reg}} = \int_{\Sigma} \mathcal{E}^{\mathrm{Reg}} \, d^2x \tag{49}$$

and time differentiate, using the system (48) we get:

(50)
$$\frac{\partial}{\partial t}\mathcal{E}^{\mathrm{Reg}} = \frac{\partial}{\partial x^b}(N^2 q^{ab}\underline{u}(\partial_a \underline{\eta} + \frac{1}{2}\underline{\lambda}e^{-2\gamma}\partial_a\omega) + N^2 q^{ab}\underline{v}(\partial_a\underline{\lambda} - \frac{1}{2}\underline{\eta}e^{-2\gamma}\partial_a\omega)),$$

which results in a vector field density

$$J^{\mathrm{Reg}} := (J^{\mathrm{Reg}})^t\partial_t + (J^{\mathrm{Reg}})^b\partial_b, \tag{51}$$

where

$$\begin{aligned}(J^{\mathrm{Reg}})^t =& \mathcal{E}^{\mathrm{Reg}},\\ (J^{\mathrm{Reg}})^b =& -N^2 q^{ab}\underline{u}(\partial_a \underline{\eta} + \frac{1}{2}\underline{\lambda}e^{-2\gamma}\partial_a\omega) - N^2 q^{ab}\underline{v}(\partial_a\underline{\lambda} - \frac{1}{2}\underline{\eta}e^{-2\gamma}\partial_a\omega),\end{aligned} \tag{52}$$

which is also divergence free. The advantage of recasting in the $(\underline{X}, H^{\mathrm{Reg}})$ framework is that it has better behaviour on the axes and the horizon than (X, H^{Alt}). □

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