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A Note on the Lipschitz Selection

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Abstract. We present an alternate proof of the passage from the finiteness principle for metric trees to the construction of the core in the C. Fefferman and Shvartsman finiteness theorem for Lipschitz selection problems.

Résumé. On présente une preuve alternative du passage du principe de la finitude pour les arbres métriques jusqu'à la construction du noyau dans le théorème de finitude de C. Fefferman et Shvartsman pour la sélection des problèmes Lipschitz.

1. **Introduction** Let $(Y, \|\cdot\|)$ be a Banach space, $\mathcal{K}_m(Y)$ be the family of all nonempty compact convex subsets $K \subset Y$ of (covering) dimension at most m. Let $(\mathcal{M}, \rho)$ be a pseudometric space (i.e., $\rho : \mathcal{M} \times \mathcal{M} \to [0, \infty]$ satisfies $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ for all $x, y, z \in \mathcal{M}$) and $F : \mathcal{M} \to \mathcal{K}_m(Y)$. For a subset $\mathcal{M}' \subset \mathcal{M}$ a Lipschitz map $f : \mathcal{M}' \to Y$ such that $f(x) \in F(x)$ for all $x \in \mathcal{M}'$ is called a Lipschitz selection of $F|_{\mathcal{M}'}$. Recall that $f : \mathcal{M}' \to Y$ is Lipschitz if there is $C \in \mathbb{R}_+$ such that

$$\|f(x) - f(y)\| \le C\rho(x, y) \quad \text{for all} \quad x, y \in \mathcal{M}'.$$

The least C here is called the Lipschitz seminorm of f and is denoted by $\|f\|_{\mathrm{Lip}(\mathcal{M}', Y)}$.

The following statement is the main result of C. Fefferman and Shvartsman paper [2]. In its formulation, $N(m, Y)$ is equal to $2^{\min(m+1, \dim Y)}$ if $\dim Y < \infty$ and 2^{m+1} otherwise.

Theorem ([2], Theorem 1.2). *Fix $m \ge 1$. Let $(\mathcal{M}, \rho)$ be a pseudometric space, and let $F : \mathcal{M} \to \mathcal{K}_m(Y)$ for a Banach space Y. Let λ be a positive real number.*

Suppose that for every $\mathcal{M}' \subset \mathcal{M}$ consisting of at most $N(m, Y)$ points, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'}$ with Lipschitz seminorm $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}', Y)} \le \lambda$. Then F has a Lipschitz selection f with Lipschitz seminorm $\|f\|_{\mathrm{Lip}(\mathcal{M}, Y)} \le \gamma\lambda$.

Here, γ depends only on m.

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The proof goes along the following lines. First the authors prove the result for $(\mathcal{M},\rho)$ being the vertex set of a finite metric tree with $N(m,Y)$ and γ replaced by some $k^\sharp = k^\sharp(m) \in \mathbb{N}$ and $\gamma_0 = \gamma_0(m) \in \mathbb{R}_+$ (both depending on m only), see [2, Cor. 4.16]. (The proof uses the fact that the *Nagata dimension* of a metric tree is 1 which allows to carry over arguments in [3] from $\mathbb{R}^n$ to an arbitrary metric tree.) Then to pass from finite metric trees to arbitrary metric spaces the authors prove the following result. In its formulation $d_H(A,B)$ stands for the Hausdorff distance between $A,B \in \mathcal{K}_m(Y)$ and $k^\sharp$ and γ_0 are the constants from [2, Cor. 4.16].

THEOREM ([2], Theorem 5.2). *Let $(\mathcal{M},\rho)$ be a metric space, and let $F : \mathcal{M} \to \mathcal{K}_m(Y)$ for a Banach space Y. Let λ be a positive real number.*
Suppose that for every subset $\mathcal{M}' \to \mathcal{M}$ consisting of at most $k^\sharp$ points, the restriction $F|_{\mathcal{M}'}$ has a Lipschitz selection $f_{\mathcal{M}'}$ with Lipschitz seminorm

$$\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',Y)} \le \lambda.$$

Then there exists $G : \mathcal{M} \to \mathcal{K}_m(Y)$ (called the core of F) such that $G(x) \subset F(x)$ and $d_H(G(x),G(y)) \le \gamma_0\lambda\rho(x,y)$ for all $x,y \in \mathcal{M}$.

The main result of [2] for arbitrary metric spaces is then obtained straightforwardly from the previous one by applications of the Shvartsman selection theorem [2, Th. 1.6] (see also the references therein) and another Shvartsman's theorem [4]. Finally, the result for pseudometric spaces follows easily from the one for metric spaces, cf. [2, Prop. 6.1].

In this note we give an alternate short proof of Theorem 5.2 of [2] (the passage from the finiteness principle for metric trees to the construction of the core).

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2. Proof of Theorem 5.2 of [2]

2.1. We assume an acquaintance of the readers with basic facts of metric geometry (see, e.g., [1, Ch. 3] for the references).

Let (X,d) be a connected and locally simply connected metric length space. Let $p : X_\Gamma \to X$ be a connected regular covering of X with the deck transformation group Γ. Using the lifting property for paths we equip X_Γ with the length metric d_Γ pulled back from X so that $p : (X_\Gamma, d_\Gamma) \to (X,d)$ is a local isometry and the deck transformation group Γ acts discretely on X_Γ by isometries: $\Gamma \times X_\Gamma \ni (\gamma,x) \mapsto \gamma x \in X_\Gamma$. Moreover,

$$d(x,y) = \inf_{\gamma\in\Gamma} d_\Gamma(\tilde{x},\gamma\tilde{y}) \quad \text{for all } x,y \in X,\ \tilde{x} \in p^{-1}(x), \tilde{y} \in p^{-1}(y). \tag{2.1}$$

Theorem 5.2 of [2] is a consequence of the following observation.

Proposition 2.1. *Let $X' \subset X$ be a subspace and $F : X' \to \mathcal{K}_m(Y)$ for a Banach space Y. Suppose the pullback $p^*F := F \circ p : X'_\Gamma \to \mathcal{K}_m(Y)$, $X'_\Gamma := p^{-1}(X')$, of F has a Lipschitz selection with Lipschitz seminorm $\le c$. Then there exists $G : X' \to \mathcal{K}_m(Y)$ such that $G(x) \subset F(x)$ and $d_H(G(x), G(y)) \le cd(x,y)$ for all $x, y \in X'$.*

Proof. Let f be a Lipschitz selection of p^*F with Lipschitz seminorm $\le c$. For each $x \in X'$ let $G(x)$ be the closed convex hull of the set $f(p^{-1}(x)) \subset F(x)$. Due to (2.1)

$$\inf_{\gamma\in\Gamma} \|f(\gamma\tilde{y}) - f(\tilde{x})\| \le c \inf_{\gamma\in\Gamma} d_\Gamma(\tilde{x}, \gamma\tilde{y}) = cd(x,y)$$
$$\text{for all } x, y \in X',\ \tilde{x} \in p^{-1}(x), \tilde{y} \in p^{-1}(y).$$

This implies that for every $\varepsilon > 0$ the Minkowski sum $G(x) + \bar{B}_{(c+\varepsilon)d(x,y)}$ contains set $\{f(\gamma\tilde{y})\}_{\gamma\in\Gamma} = f(p^{-1}(y))$. (Here $\bar{B}_r \subset Y$ stands for the closed ball of radius r centered at $0 \in Y$.) As the former is a closed convex subset of Y (because $G(x)$ is compact), the sum contains $G(y)$ as well. Similarly, $G(x) \subset G(y) + \bar{B}_{(c+\varepsilon)d(x,y)}$. This shows that $d_H(G(x), G(y)) \le cd(x,y)$. □

2.2. Proof of Theorem 5.2 of [2]. We apply [2, Cor. 4.16] (the version of the main theorem [2, Th. 1.2] for vertex sets of finite metric trees). The corollary is proved for finite metric trees. A standard compactness argument involving the Tikhonov theorem (cf. the construction in the proof of Proposition 6.1 of [2]) extends the corollary to an arbitrary metric tree. In this general form we use it in the subsequent proof.

Let $(\mathcal{M}, \rho)$ be a metric space. We embed it isometrically into the *complete* metric graph $X := (\mathcal{M}, E)$ with the vertex set $\mathcal{M}$ and the set of edges E consisting of the intervals joining distinct points in $\mathcal{M}$. If $[x, y] \in E$ its length equals $\rho(x,y)$. Then in a natural way we define the path metric d on X. By definition X is a one-dimensional simplicial complex, so its fundamental group $\Gamma := \pi_1(X)$ is a free group and the universal covering $p : X_\Gamma \to X$ is a metric tree with the path metric d_Γ pulled back from X and with the vertex set $\mathcal{M}_\Gamma = p^{-1}(\mathcal{M})$.

If $F : \mathcal{M} \to \mathcal{K}_m(Y)$ satisfies conditions of Theorem 5.2, its pullback $p^*F : \mathcal{M}_\Gamma \to \mathcal{K}_m(Y)$ meets these conditions as well. (Indeed, if $S \subset \mathcal{M}_\Gamma$ is a subset consisting of at most $k^\sharp$ points and $f_{p(S)}$ is a Lipschitz selection of $F|_{p(S)}$ with Lipschitz seminorm $\le \lambda$, then its pullback $p^* f_{p(S)}$ is a Lipschitz selection of $p^*F|_S$ with Lipschitz seminorm $\le \lambda$ as $d(p(x), p(y)) \le d_\Gamma(x,y)$ for all $x, y \in X_\Gamma$.) Thus due to the extension of [2, Cor. 4.16] there is a Lipschitz selection of p^*F with Lipschitz seminorm $\le \gamma_0\lambda$. Applying Proposition 2.1 in this setting we get the required core $G : \mathcal{M} \to \mathcal{K}_m(Y)$ of Theorem 5.2. □

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