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On Properties of Geometric Preduals of $\mathbf{C}^{k,\omega}$ Spaces

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Abstract. Let $C_b^{k,\omega}(\mathbb{R}^n)$ be the Banach space of C^k functions on $\mathbb{R}^n$ bounded together with all derivatives of order $\le k$ and with derivatives of order k having moduli of continuity majorated by $c\cdot\omega$, $c\in\mathbb{R}_+$, for some $\omega\in C(\mathbb{R}_+)$. Let $C_b^{k,\omega}(S):=C_b^{k,\omega}(\mathbb{R}^n)|_S$ be the trace space to a closed subset $S\subset\mathbb{R}^n$. The geometric predual $G_b^{k,\omega}(S)$ of $C_b^{k,\omega}(S)$ is the minimal closed subspace of the dual $\left(C_b^{k,\omega}(\mathbb{R}^n)\right)^*$ containing evaluation functionals of points in S. We study geometric properties of spaces $G_b^{k,\omega}(S)$ and their relations to the classical Whitney problems on the characterization of trace spaces of C^k functions on $\mathbb{R}^n$.

Résumé. Soit $C_b^{k,\omega}(\mathbb{R}^n)$ l'espace de Banach des fonctions C^k sur $\mathbb{R}^n$ bornées avec toutes leurs dérivées d'ordre jusqu'à k et avec les dérivées d'ordre k ayant des modules de continuité majorés par $c\cdot\omega$, $c\in\mathbb{R}_+$, pour quelque $\omega\in C(\mathbb{R}_+)$. Soit $C_b^{k,\omega}(S):=C_b^{k,\omega}(\mathbb{R}^n)|_S$ l'espace de trace à un fermé $S\subset\mathbb{R}^n$. Le predual géométrique $G_b^{k,\omega}(S)$ de $C_b^{k,\omega}(S)$ est le sous-espace minimal fermé du dual $\left(C_b^{k,\omega}(\mathbb{R}^n)\right)^*$ contenant les fonctionnelles d'évaluation aux points de S. Nous étudions les propriétés géométriques des espaces $G_b^{k,\omega}(S)$ et leur relation avec les problèmes classiques de Whitney sur la caractérisation des espaces de trace des fonctions C^k sur $\mathbb{R}^n$

1. Formulation of Main Results

1.1. Geometric preduals of $\mathbf{C}^{k,\omega}$ *spaces* In what follows we use the standard notation of Differential Analysis. In particular, $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$ denotes a multi-index and $|\alpha|:=\sum_{i=1}^n\alpha_i$. Also, for $x=(x_1,\dots,x_n)\in\mathbb{R}^n$,

$$x^\alpha:=\prod_{i=1}^n x_i^{\alpha_i}\quad\text{and}\quad D^\alpha:=\prod_{i=1}^n D_i^{\alpha_i},\quad\text{where}\quad D_i:=\frac{\partial}{\partial x_i}. \tag{1.1}$$

Let ω be a nonnegative function on $(0,\infty)$ (referred to as *modulus of continuity*) satisfying the following conditions:

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(i) $\omega(t)$ and $\dfrac{t}{\omega(t)}$ are nondecreasing functions on $(0,\infty)$;

(ii) $\lim\limits_{t\to 0^+} \omega(t) = 0$.

DEFINITION 1.1. $C_b^{k,\omega}(\mathbb{R}^n)$ is the Banach subspace of functions $f \in C^k(\mathbb{R}^n)$ with norm

$$\|f\|_{C_b^{k,\omega}(\mathbb{R}^n)} := \max\left(\|f\|_{C_b^k(\mathbb{R}^n)}, |f|_{C_b^{k,\omega}(\mathbb{R}^n)}\right), \tag{1.2}$$

where

$$\|f\|_{C_b^k(\mathbb{R}^n)} := \max_{|\alpha|\le k}\left\{\sup_{x\in\mathbb{R}^n} |D^\alpha f(x)|\right\} \tag{1.3}$$

and

$$|f|_{C_b^{k,\omega}(\mathbb{R}^n)} := \max_{|\alpha|=k}\left\{\sup_{x,y\in\mathbb{R}^n,\, x\ne y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x-y\|)}\right\}. \tag{1.4}$$

Here $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$.

If $S \subset \mathbb{R}^n$ is a closed subset, then by $C_b^{k,\omega}(S)$ we denote the trace space of functions $g \in C_b^{k,\omega}(\mathbb{R}^n)|_S$ equipped with the quotient norm

$$\|g\|_{C_b^{k,\omega}(S)} := \inf\{\|\tilde g\|_{C_b^{k,\omega}(\mathbb{R}^n)} \,:\, \tilde g \in C_b^{k,\omega}(\mathbb{R}^n),\ \tilde g|_S = g\}. \tag{1.5}$$

Let $\left(C_b^{k,\omega}(\mathbb{R}^n)\right)^*$ be the dual of $C_b^{k,\omega}(\mathbb{R}^n)$. Clearly, each evaluation functional δ_x^0 at $x \in \mathbb{R}^n$ (i.e., $\delta_x^0(f) := f(x)$, $f \in C_b^{k,\omega}(\mathbb{R}^n)$) belongs to $\left(C_b^{k,\omega}(\mathbb{R}^n)\right)^*$ and has norm one. By $G_b^{k,\omega}(S) \subset \left(C_b^{k,\omega}(\mathbb{R}^n)\right)^*$ we denote the minimal closed subspace containing all δ_x^0, $x \in S$.

THEOREM 1.2. *The restriction map to the set* $\{\delta_s^0 : s \in S\} \subset G_b^{k,\omega}(S)$ *determines an isometric isomorphism between the dual of* $G_b^{k,\omega}(S)$ *and* $C_b^{k,\omega}(S)$.

In what follows, $G_b^{k,\omega}(S)$ will be referred to as the *geometric predual* of $C_b^{k,\omega}(S)$. In the present paper we study some properties of these spaces. The subject is closely related to the classical Whitney problems, see [23, 24], asking about the characterization of trace spaces of C^k functions on $\mathbb{R}^n$ (see survey [12] and book [2] and references therein for recent developments in the area). Some of the main results of the theory can be reformulated in terms of certain geometric characteristics of spaces $G_b^{k,\omega}(S)$, see Sections 1.2 and 1.3.

1.2. Finiteness Principle For $C_b^{k,\omega}(S)$ this principle was conjectured by Yu. Brudnyi and P. Shvartsman in the 1980s in the following form (cf. [12, p. 210]).

> *Finiteness Principle.* To decide whether a given $f : S \to \mathbb{R}$, $S \subset \mathbb{R}^n$, extends to a function $F \in C_b^{k,\omega}(\mathbb{R}^n)$, it is enough to look at all restrictions $f|_{S'}$, where $S' \subset S$ is an arbitrary d-element subset. Here d is an integer constant depending only on k and n.

More precisely, if $f|_{S'}$ extends to a function $F^{S'} \in C_b^{k,\omega}(\mathbb{R}^n)$ of norm at most 1 (for each $S' \subset S$ with at most d elements), then f extends to a function $F \in C_b^{k,\omega}(\mathbb{R}^n)$, whose norm is bounded by a constant depending only on k and n.

For $k = 0$ (the Lipschitz case) the McShane extension theorem [20] implies the Finiteness Principle with the optimal constant $d = 2$. Also, for $n = 1$ the Finiteness Principle is valid with the optimal constant $d = k + 2$. The result is essentially due to Merrien [19].

In the multidimensional case the Finiteness Principle was proved by Yu. Brudnyi and Shvartsman for $k = 1$ with the optimal constant $d = 3 \cdot 2^{n-1}$, see [5]. In the early 2000s the Finiteness Principle was proved by C. Fefferman for all k and n for regular moduli of continuity ω (i.e., $\omega(1) = 1$), see [10]. The upper bound for the constant d in the Fefferman proof was reduced later to $d = 2^{\binom{k+n}{k}}$ by Bierstone and P. Milman [4] and independently and by a different method by Shvartsman [22]. The obtained results (and the Finiteness Principle in general) admit the following reformulation in terms of geometric characteristics of closed unit balls $B_b^{k,\omega}(S)$ of $G_b^{k,\omega}(S)$. Specifically, let $B_b^{k,\omega}(S;m) \subset B_b^{k,\omega}(S)$, $m \in \mathbb{N}$, be the balanced closed convex hull of the union of all finite-dimensional balls $B_b^{k,\omega}(S') \subset G_b^{k,\omega}(S')$, $S' \subset S$, $\operatorname{card} S' \le m$.

Theorem 1.3. *There exist constants $d \in \mathbb{N}$ and $c \in (1, \infty)$ such that*

$$B_b^{k,\omega}(S;d) \subset B_b^{k,\omega}(S) \subset c \cdot B_b^{k,\omega}(S;d).$$

Here for $k = 0$, $d = 2$ (- optimal) and $c = 1$, for $n = 1$, $d = k + 2$ (- optimal) and c depends on k only, for $k = 1$, $d = 3 \cdot 2^{n-1}$ (- optimal) and c depends on k and n only, and for $k \ge 2$, $d = 2^{\binom{k+n}{k}}$ and $c = \frac{\tilde{c}}{\omega(1)}$, where $\tilde{c}$ depends on k and n only.

1.3. Complementability of spaces $\mathbf{G}_b^{k,\omega}(S)$ We begin with a result describing bounded linear operators on $G_b^{k,\omega}(\mathbb{R}^n)$. To this end, for a Banach space X by $C_b^{k,\omega}(\mathbb{R}^n; X)$ we denote the Banach space of X-valued C^k functions on $\mathbb{R}^n$ with norm defined similarly to that of Definition 1.1 with absolute values replaced by norms $\|\cdot\|_X$ in X. Let $\mathcal{L}(X_1; X_2)$ stand for the Banach space of bounded linear operators between Banach spaces X_1 and X_2 equipped with the operator norm.

Theorem 1.4. *The restriction map to the set $\{\delta_x^0 : x \in \mathbb{R}^n\} \subset G_b^{k,\omega}(\mathbb{R}^n)$ determines an isometric isomorphism between $\mathcal{L}\big(G_b^{k,\omega}(\mathbb{R}^n); X\big)$ and $C_b^{k,\omega}(\mathbb{R}^n; X)$.*

Let $q_S : C_b^{k,\omega}(\mathbb{R}^n) \to C_b^{k,\omega}(S)$ be the quotient map induced by the restriction of functions on $\mathbb{R}^n$ to S. A right inverse $T \in \mathcal{L}(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$ for q_S (i.e., $q_S \circ T = \mathrm{id}$) is called a *linear extension operator*. The set of such operators is denoted by $Ext(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$.

DEFINITION 1.5. An operator $T \in Ext(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$ has depth $d \in \mathbb{N}$ if for all $x \in \mathbb{R}^n$ and $f \in C_b^{k,\omega}(S)$,

$$(Tf)(x) = \sum_{i=1}^{d} \lambda_i^x \cdot f(y_i^x), \tag{1.6}$$

where $y_i^x \in S$ and λ_i^x depend only on x.

Linear extension operators of finite depth exist. For $k = 0$ (the Lipschitz case) the Whitney-Glaeser linear extension operators $C_b^{0,\omega}(S) \to C_b^{0,\omega}(\mathbb{R}^n)$, see [15], have depth d depending on n only and norms bounded by a constant depending on n only. In the 1990s bounded linear extension operators $C_b^{1,\omega}(S) \to C_b^{1,\omega}(\mathbb{R}^n)$ of depth d depending on n only with norms bounded by a constant depending on n only were constructed by Yu. Brudnyi and Shvartsman [6]. Recently bounded linear extensions operators of depth d depending on k and n only were constructed by Luli [18] for all spaces $C_b^{k,\omega}(S)$; their norms are bounded by $\frac{C}{\omega(1)}$, where $C \in (1, \infty)$ is a constant depending on k and n only. (Earlier such extension operators were constructed for finite sets S by C. Fefferman [11, Th. 8].)

In the following result we identify $(G_b^{k,\omega}(S))^*$ with $C_b^{k,\omega}(S)$ by means of the isometric isomorphism of Theorem 1.2.

THEOREM 1.6. *For each $T \in Ext(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$ of finite depth there exists a bounded linear projection $P : G_b^{k,\omega}(\mathbb{R}^n) \to G_b^{k,\omega}(S)$ whose adjoint $P^* = T$.*

REMARK 1.7. It is easily seen that if T has depth d and is defined by (1.6), then

$$p(x) := P(\delta_x^0) = \sum_{i=1}^{d} \lambda_i^x \cdot \delta_{y_i^x}^0 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Moreover, $p \in C_b^{k,\omega}(\mathbb{R}^n; G_b^{k,\omega}(S))$ and has norm equal to $\|T\|$ by Theorem 1.4.

1.4. Approximation property Recall that a Banach space X is said to have the *approximation property*, if, for every compact set $K \subset X$ and every $\varepsilon > 0$, there exists an operator $T : X \to X$ of finite rank so that $\|Tx - x\| \le \varepsilon$ for every $x \in K$.

Although it is strongly believed that the class of spaces with the approximation property includes practically all spaces which appear naturally in analysis, it is not known yet even for the space H^∞ of bounded holomorphic functions on the open unit disk. The first example of a space which fails to have the approximation property was constructed by Enflo [9]. Since Enflo's work several other examples of such spaces were constructed, for the references see, e.g., [17].

A Banach space has the λ-*approximation property*, $1 \le \lambda < \infty$, if it has the approximation property with the approximating finite rank operators of norm $\le \lambda$. A Banach space is said to have the *bounded approximation property*, if it has the λ-approximation property for some λ. If $\lambda = 1$, then the space is said to have the *metric approximation property*.

Every Banach spaces with a basis has the bounded approximation property. Also, it is known that the approximation property does not imply the bounded approximation property, see [13]. It was established by Pełczyński [21] that a separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a separable Banach space with a basis.

Next, for Banach spaces X, Y by $\mathcal{F}(X,Y) \subset \mathcal{L}(X,Y)$ we denote the subspace of linear bounded operators of finite rank $X \to Y$. Let us consider the trace mapping V from the projective tensor product $Y^*\hat{\otimes}_\pi X \to \mathcal{F}(X,Y)^*$ defined by

$$(Vu)(T) = \mathrm{trace}(Tu), \quad \text{where} \quad u \in Y^*\hat{\otimes}_\pi X,\ T \in \mathcal{F}(X,Y),$$

that is, if $u = \sum_{n=1}^\infty y_n^* \otimes x_n$, then $(Vu)(T) = \sum_{n=1}^\infty y_n^*(Tx_n)$.

It is easy to see that $\|Vu\| \le \|u\|_\pi$. The λ-bounded approximation property of X is equivalent to the fact that $\|u\|_\pi \le \lambda \|Vu\|$ for all Banach spaces Y. This well-known result (see, e.g., [7, page 193]) is essentially due to Grothendieck [14].

Our result concerning spaces $G_b^{k,\omega}(S)$ reads as follows.

Theorem 1.8. (1) *Spaces $G_b^{k,\omega}(\mathbb{R}^n)$ have the λ-approximation property with $\lambda = \lambda(k,n,\omega) := 1 + C \cdot \lim_{t\to\infty} \frac{1}{\omega(t)}$, where C depends on k and n only.*

(2) *All the other spaces $G_b^{k,\omega}(S)$ have the λ-approximation property with $\lambda = C' \cdot \lambda(1,n,\omega)$, where C' is a constant depending on n only, if $k = 0,1$, and with $\lambda = \frac{C'' \cdot \lambda(k,n,\omega)}{\omega(1)}$, where C'' is a constant depending on k and n only, if $k \ge 2$.*

If $\lim_{t\to\infty} \omega(t) = \infty$, then (1) implies that the corresponding space $G_b^{k,\omega}(\mathbb{R}^n)$ has the metric approximation property. In case $\lim_{t\to\infty}\omega(t) < \infty$, one can define the new modulus of continuity $\widetilde{\omega}$ (cf. properties (i) and (ii) in its definition) by the formula

$$\widetilde{\omega}(t) = \max\{\omega(t), t\}, \quad t \in (0,\infty).$$

It is easily seen that spaces $C_b^{k,\omega}(\mathbb{R}^n)$ and $C_b^{k,\widetilde{\omega}}(\mathbb{R}^n)$ are isomorphic. Thus $G_b^{k,\omega}(\mathbb{R}^n)$ is isomorphic to space $G_b^{k,\widetilde{\omega}}(\mathbb{R}^n)$ having the metric approximation property. However, the distortion of this isomorphism depends on ω. So, in general, it is not clear whether $G_b^{k,\omega}(\mathbb{R}^n)$ itself has the metric approximation property.

In fact, in some cases spaces $G_b^{k,\omega}(S)$ still have the metric approximation property. E.g., by the classical result of Grothendieck [14, Ch. I], separable dual spaces with the approximation property have the metric approximation property. The class of such spaces $G_b^{k,\omega}(S)$ is studied in the next section.

REMARK 1.9. It is not known, even for the case $k=0$, whether all spaces $C_b^{k,\omega}(\mathbb{R}^n)$ have the approximation property (for some results in this direction for $k=0$ see, e.g., [16]).

At the end of this section we formulate a result describing the structure of operators in $\mathcal{L}(G_b^{k,\omega}(\mathbb{R}^n);X)$, where X is a separable Banach space with the λ-approximation property. In particular, it can be applied to $X=G_b^{k,\omega}(S)$ and $\lambda:=\lambda(S,k,n,\omega)$ the constant of the approximation property for $G_b^{k,\omega}(S)$ of Theorem 1.8 (2).

THEOREM 1.10. *There exists the family of norm one vectors* $\{v_j\}_{j\in\mathbb{N}}\subset X$ *and given* $H\in\mathcal{L}(G_b^{k,\omega}(\mathbb{R}^n);X)$ *the family of functions* $\{h_j\}_{j\in\mathbb{N}}\subset C_b^{k,\omega}(\mathbb{R}^n)$ *of norms* $\le 32\cdot\lambda^2\cdot\|H\|$ *such that for all* $x\in\mathbb{R}^n$, $\alpha\in\mathbb{Z}_+^n$, $|\alpha|\le k$,

$$H(\delta_x^\alpha)=\sum_{j=1}^{\infty}D^\alpha h_j(x)\cdot v_j \tag{1.7}$$

(convergence in X*).*

REMARK 1.11. If $X=G_b^{k,\omega}(S)$ and $H\in\mathcal{L}(G_b^{k,\omega}(\mathbb{R}^n);G_b^{k,\omega}(S))$ is a projection onto $G_b^{k,\omega}(S)$, then in addition to (1.7) we have

$$\delta_x^0=\sum_{j=1}^{\infty}h_j(x)\cdot v_j \quad \text{for all} \quad x\in S. \tag{1.8}$$

In this case, the adjoint H^* of H belongs to $Ext(C_b^{k,\omega}(S);C_b^{k,\omega}(\mathbb{R}^n))$ and for all $x\in\mathbb{R}^n$, $\alpha\in\mathbb{Z}_+^n$, $|\alpha|\le k$, the extension H^*f of $f\in C_b^{k,\omega}(S)$ satisfies

$$D^\alpha(H^*f)(x):=\sum_{j=1}^{\infty}D^\alpha h_j(x)\cdot f(v_j). \tag{1.9}$$

1.5. Preduals of $\mathbf{G}_b^{k,\omega}(S)$ *spaces* Let $C_0^{k,\omega}(\mathbb{R}^n)$ be the subspace of functions $f\in C_b^{k,\omega}(\mathbb{R}^n)$ such that

(i) for all $\alpha\in\mathbb{Z}_+^n$, $0\le|\alpha|\le k$,

$$\lim_{\|x\|\to\infty}D^\alpha f(x)=0;$$

(ii) for all $\alpha\in\mathbb{Z}_+^n$, $|\alpha|=k$,

$$\lim_{\|x-y\|\to 0}\frac{D^\alpha f(x)-D^\alpha f(y)}{\omega(\|x-y\|)}=0.$$

It is easily seen that $C_0^{k,\omega}(\mathbb{R}^n)$ equipped with the norm induced from $C_b^{k,\omega}(\mathbb{R}^n)$ is a Banach space. By $C_0^{k,\omega}(S)$ we denote the trace of $C_0^{k,\omega}(\mathbb{R}^n)$ to a closed subset $S \subset \mathbb{R}^n$ equipped with the trace norm.

If $\lim_{t\to 0^+} \frac{t}{\omega(t)} > 0$ (see condition (i) for ω in Section 1.1), then clearly, the corresponding space $C_0^{k,\omega}(\mathbb{R}^n)$ is trivial. Thus we may naturally assume that ω satisfies the condition

$$\lim_{t\to 0^+} \frac{t}{\omega(t)} = 0. \tag{1.10}$$

In the sequel, the weak* topology of $C_b^{k,\omega}(S)$ is defined by means of functionals in $G_b^{k,\omega}(S) \subset \big(C_b^{k,\omega}(\mathbb{R}^n)\big)^*$.

Theorem 1.12. *Suppose ω satisfies condition* (1.10).

(1) *Space $\big(C_0^{k,\omega}(\mathbb{R}^n)\big)^*$ is isomorphic to $G_b^{k,\omega}(\mathbb{R}^n)$, isometrically if $\lim\limits_{t\to\infty} \omega(t) = \infty$.*
(2) *If there exists a weak* continuous operator $T \in Ext(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$ such that $T\big(C_0^{k,\omega}(S)\big) \subset C_0^{k,\omega}(\mathbb{R}^n)$, then $\big(C_0^{k,\omega}(S)\big)^*$ is isomorphic to $G_b^{k,\omega}(S)$.*

From the first part of the theorem we obtain (for ω satisfying (1.10)):

Corollary 1.13. *The space of C^∞ functions with compact supports on $\mathbb{R}^n$ is dense in $C_0^{k,\omega}(\mathbb{R}^n)$. In particular, all spaces $C_0^{k,\omega}(S)$ are separable.*

It is not clear whether the condition of the second part of the theorem is valid for all spaces $C_b^{k,\omega}(S)$ with ω subject to (1.10). Here we describe a class of sets S satisfying this condition. By $\mathcal{P}_{k,n}$ we denote the space of real polynomials on $\mathbb{R}^n$ of degree k, and by $Q_r(x) \subset \mathbb{R}^n$ the closed cube centered at x of sidelength $2r$.

Definition 1.14. A point x of a subset $S \subset \mathbb{R}^n$ is said to be weak k-Markov if

$$\varliminf_{r\to 0} \left\{ \sup_{p\in\mathcal{P}_{k,n}\setminus 0} \left(\frac{\sup_{Q_r(x)} |p|}{\sup_{Q_r(x)\cap S} |p|} \right) \right\} < \infty.$$

A closed set $S \subset \mathbb{R}^n$ is said to be weak k-Markov if it contains a dense subset of weak k-Markov points.

The class of weak k-Markov sets, denoted by $\mathrm{Mar}_k^*(\mathbb{R}^n)$, was introduced and studied by Yu. Brudnyi and the author, see [1, 3]. It contains, in particular, the closure of any open set, the Ahlfors p-regular compact subsets of $\mathbb{R}^n$ with $p > n-1$, a wide class of fractals of arbitrary positive Hausdorff measure, direct products $\prod_{j=1}^l S_j$, where $S_j \in \mathrm{Mar}_k^*(\mathbb{R}^{n_j})$, $1 \le j \le l$, $n = \sum_{j=1}^l n_j$, and closures of unions of any combination of such sets. Solutions of the Whitney problems (see Sections 1.2 and 1.3 above) for sets in $\mathrm{Mar}_k^*(\mathbb{R}^n)$ are relatively simple, see [1].

We prove the following result.

THEOREM 1.15. *Let $S' \in \mathrm{Mar}_k^*(\mathbb{R}^n)$ and ω satisfy* (1.10). *Suppose $H : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map such that*

(a) the entries of its Jacobian matrix belong to $C_b^{k-1,\omega_o}(\mathbb{R}^n)$, where ω_o satisfies

$$\lim_{t\to 0^+} \frac{\omega_o(t)}{\omega(t)} = 0; \tag{1.11}$$

(b) the map $H|_{S'} : S' \to S =: H(S')$ is a proper retraction.[1]

Then the condition of Theorem 1.12 *holds for $C_b^{k,\omega}(S)$. Thus $G_b^{k,\omega}(S)$ is isomorphic to $\big(C_0^{k,\omega}(S)\big)^*$ and so $G_b^{k,\omega}(S)$ and $C_0^{k,\omega}(S)$ have the metric approximation property.*

REMARK 1.16. (1) In addition to weak k-Markov sets $S \subset \mathbb{R}^n$, Theorem 1.15 is valid, e.g., for a compact subset S of a C^{k+1}-manifold $M \subset \mathbb{R}^n$ such that the base of the topology of S consists of relatively open subsets of Hausdorff dimension $> \dim M - 1$. Indeed, in this case there exist tubular open neighbourhoods $U_M \subset V_M \subset \mathbb{R}^n$ of M such that $\mathrm{cl}(U_M) \subset V_M$ together with a C^{k+1} retraction $r : U_M \to M$. Then, due to the hypothesis for S, the base of topology of $S' := r^{-1}(S) \cap \mathrm{cl}(U_M)$ consists of relatively open subsets of Hausdorff dimension $> n - 1$ and so $S' \in \mathrm{Mar}_p^*(\mathbb{R}^n)$ for all $p \in \mathbb{N}$, see, e.g., [3, page 536]. Moreover, it is easily seen that $r|_{S'}$ is the restriction to S' of a map $H \in C_b^{k+1}(\mathbb{R}^n; \mathbb{R}^n)$. Decreasing V_M, if necessary, we may assume that S' is compact, and so the triple (H, S', S) satisfies the hypothesis of the theorem.

(2) Under conditions of Theorem 1.15, $C_b^{k,\omega}(S)$ is isomorphic to the second dual of $C_0^{k,\omega}(S)$.

Proofs of results of the paper are presented in arXiv:1607.04824.

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[1] I.e., $S \subset S'$ and $H|_{S'}(x) = x$ for all $x \in S$, and for each compact $K \subset S$ its preimage $(H|_{S'})^{-1}(K)$ is compact.

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