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Grauert and Ramspott Type Theorems on the Maximal Ideal Space of $\mathbf{H}^\infty$

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Abstract. The classical Grauert and Ramspott theorems constitute the foundation of the Oka principle on Stein spaces. In this paper we establish analogous results on the maximal ideal space $M(H^\infty)$ of the Banach algebra H^∞ of bounded holomorphic functions on the open unit disk $\mathbb{D} \subset \mathbb{C}$. We illustrate our results by some examples and applications to the theory of operator-valued H^∞ functions.

Résumé. Les théorèmes classiques de Grauert et Ramspott constituent la base du principe d'Oka par rapport aux espaces Stein. Dans cet article, nous démontrons des résultats analogues sur l'espace idéal maximal $M(H^\infty)$ de l'algèbre de Banach H^∞ des fonctions holomorphes bornées sur une disque d'unité ouverte $\mathbb{D} \subset \mathbb{C}$. Nous présentons nos résultats avec des exemples et des applications à la théorie des fonctions H^∞ évaluées par l'opérateur.

1. Introduction Let H^∞ be the Banach algebra of bounded holomorphic functions in the open unit disk $\mathbb{D} \subset \mathbb{C}$ equipped with pointwise multiplication and supremum norm. In this paper, following our earlier work [8], [9], [10], we investigate further the relationship between certain analytic and topological objects on the *maximal ideal space* of H^∞. The subject is intertwined with the area of the so-called *Oka Principle* which in a broad sense means that on Stein spaces (closed complex subvarieties of complex coordinate spaces) cohomologically formulated analytic problems have only topological obstructions (for recent advances in the theory see, e.g., survey [14]). This principle can be transferred, to some extent, to the theory of commutative Banach algebras to reveal (via the Gelfand transform) some connections between algebraic structure of a Banach algebra and topological properties of its maximal ideal space. (The most general results there are due to Novodvorski, Taylor and Raeburn, see, e.g., [29] and references therein.)

Recall that for a unital commutative complex Banach algebra A the maximal ideal space $M(A)$ is the set of all nonzero homomorphisms $A \to \mathbb{C}$. Since norm of each $\varphi \in M(A)$ is at least one, $M(A)$ is a subset of the closed unit ball of the dual

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space A^*. It is a compact Hausdorff space in the weak* topology induced by A^* (called the *Gelfand topology*). Let $C(M(A))$ be the Banach algebra of continuous complex-valued functions on $M(A)$ equipped with supremum norm. An element $a \in A$ can be thought of as a function in $C(M(A))$ via the *Gelfand transform* $\hat{}: A \to C(M(A))$, $\hat{a}(\varphi) := \varphi(a)$. Map $\hat{}$ is a nonincreasing-norm morphism of Banach algebras. Algebra A is called *uniform* if the Gelfand transform is an isometry (as for H^∞).

For H^∞ evaluation at a point of $\mathbb{D}$ is an element of $M(H^\infty)$, so $\mathbb{D}$ is naturally embedded into $M(H^\infty)$ as an open subset. The famous Carleson corona theorem [12] asserts that $\mathbb{D}$ is dense in $M(H^\infty)$. In general, if a Stein space X is embedded as an open dense subset into a normal topological space $\bar{X}$, it is natural to introduce an analog of the complex analytic structure on $\bar{X}$ regarding it as a ringed space with the structure sheaf $\mathcal{O}_{\bar{X}}$ of germs of complex-valued continuous functions on open subsets U of $\bar{X}$ whose restrictions to $U \cap \bar{X}$ are holomorphic. Then one can ask whether analogs of classical results of complex analysis (including the Oka Principle) are valid on $(\bar{X}, \mathcal{O}_{\bar{X}})$. Some results in this direction for $\bar{X}$ being a fiberwise compactification of an unbranched covering X of a Stein manifold have been obtained in [4], [5]. Another example, $X = \mathbb{D}$ and $\bar{X} = M(H^\infty)$, has been the main subject of papers [8], [9], [10] where the Stein-like theory analogous to the classical complex function theory on Stein spaces (see, e.g., [17]) has been developed and then applied to the celebrated Sz.-Nagy operator corona problem [32]. In the present paper we continue this line of research and establish (in the framework of the Oka Principle on $M(H^\infty)$) the Grauert and Ramspott type theorems (see [18], [30]) for holomorphic principal bundles on $M(H^\infty)$ and holomorphic maps of $M(H^\infty)$ into complex homogeneous spaces. Then we formulate some applications of the obtained results, in particular, to the theory of operator-valued H^∞ functions.

2. Oka Principle for Principal Bundles

2.1. Maximal ideal spaces of algebras $\mathbf{H}_I^\infty$ We work in a more general setting of (uniform) Banach algebras $H_I^\infty := \mathbb{C} + I$, where $I \subset H^\infty$ is a closed ideal. Such algebras arise naturally in the theory of bounded holomorphic functions in balls and polydisks, see [2], [3]. They were also studied in the framework of the theory of univariate H^∞ functions, see [15], [23].

The corona theorem for algebra H_I^∞ can be derived from the Carleson corona theorem, see [23, Th. 1.6]. It states that for a n-tuple of functions $f_1, \dots, f_n \in H_I^\infty$, $n \in \mathbb{N}$, satisfying the corona condition

$$\sum_{j=1}^{n} |f_j(z)| \geq \delta > 0 \quad \text{for all} \quad z \in \mathbb{D}, \tag{2.1}$$

there exist functions $g_1, \dots, g_n \in H_I^\infty$ such that

$$\sum_{j=1}^{n} f_j g_j = 1 \quad \text{on} \quad \mathbb{D}. \tag{2.2}$$

From here using some basic results due to Suárez [33] and Treil [39] on the structure of $M(H^\infty)$ one obtains the following topological description of $M\big(H_I^\infty\big)$.

For an ideal $I \subset H^\infty$, we define

$$\operatorname{hull}(I) := \{x \in M(H^\infty) \,:\, \hat{f}(x) = 0 \quad \forall f \in I\}. \tag{2.3}$$

PROPOSITION 2.1. (a) *There is a continuous surjective map* $Q_Z : M(H^\infty) \to M(H_I^\infty)$*,* $Z := \operatorname{hull}(I)$*, sending* Z *to a point and one-to-one outside of* Z*.*
(b) *Covering dimension* $\dim M(H_I^\infty) = 2$*.*
(c) *Čech cohomology group* $H^2(M(H_I^\infty), \mathbb{Z}) = 0$*.*

(Recall that for a normal space X, $\dim X \le n$ if every finite open cover of X can be refined by an open cover whose order $\le n+1$. If $\dim X \le n$ and the statement $\dim X \le n-1$ is false, we say that $\dim X = n$.)

REMARK 2.2. (1) Part (a) of the proposition says that $M(H_I^\infty)$ is homeomorphic to the (Alexandroff) one-point compactification of space $M(H^\infty) \setminus Z$ and $\mathbb{D} \setminus Z$ is an open dense subset of $M(H_I^\infty)$.
(2) Proposition 2.1 implies that algebra H_I^∞ is *projective free* (i.e., every projective H_I^∞-module is free), see, e.g., [7, Cor. 1.4].

A closed subset $Z \subset M(H^\infty)$ such that $Z = \operatorname{hull}(I)$ for an ideal $I \subset H^\infty$ is called a *hull*. For a hull Z by $\mathscr{A}_Z$ we denote the partially ordered by inclusion set of all algebras H_I^∞ for which $\operatorname{hull}(I) = Z$. E.g., if $Z = \varnothing$, then $\mathscr{A}_Z = \{H^\infty\}$. But, in general, set $\mathscr{A}_Z$ may be even uncountable.

EXAMPLE 2.3. Let v be a (unbounded) holomorphic function on $\mathbb{D}$ such that $u := e^v \in H^\infty$ is an inner function. For each $\alpha \in (0,1)$ we define the inner function $u_\alpha := e^{\alpha v} \in H^\infty$. Let $I(u_\alpha) \subset H^\infty$ be the principal ideal generated by u_α. Then $\operatorname{hull}(I(u_\alpha)) = \operatorname{hull}(I(u_1)) =: Z$ is a nonempty compact subset of $M(H^\infty) \setminus \mathbb{D}$. Moreover, all ideals $I(u_\alpha)$ are closed and $I(u_\alpha) \subsetneq I(u_\beta)$ for $\beta < \alpha$. Thus, the corresponding set $\mathscr{A}_Z$ contains a subset of the cardinality of the continuum.

Each set $\mathscr{A}_Z$ contains the unique maximal subalgebra $H_{I(Z)}^\infty$, where $I(Z) := \{f \in H^\infty \,:\, \hat{f}(x) = 0 \quad \forall x \in Z\}$. For instance, if Z is a single point, then $I(Z) \subset H^\infty$ is a maximal ideal and $H_{I(Z)}^\infty = H^\infty$. In turn, if Z is the zero locus of $\hat{b}$, where b is a Blaschke product with simple zeros, then $H_{I(Z)}^\infty = H_{I(b)}^\infty$.

CONVENTION. In what follows we assume that map Q_Z of Proposition 2.1 satisfies $Q_Z|_{M(H^\infty)\setminus Z} = \mathrm{id}$. Then maximal ideal spaces of algebras in $\mathscr{A}_Z$ coincide with $M(H_{I(Z)}^\infty)$. This particular space will be denoted by $M(\mathscr{A}_Z)$.

2.2. Oka principle for holomorphic principal bundles on $\mathbf{M}(\mathscr{A}_{\mathbf{Z}})$. Let $U \subset M(\mathscr{A}_Z)$ be an open subset and X be a complex Banach manifold (i.e., a complex manifold modelled on a complex Banach space).

A continuous map $f : U \to X$ is said to be *holomorphic* (written as $f \in \mathcal{O}(U, X)$) if restriction $f|_{U\cap(\mathbb{D}\setminus Z)}$ is a holomorphic map of complex manifolds.
For $X = \mathbb{C}$ we set $\mathcal{O}(U) := \mathcal{O}(U, \mathbb{C})$.

Using [8, Prop. 1.3] one obtains the following description of X-valued holomorphic maps on $U \cap (\mathbb{D} \setminus Z)$ having continuous extensions to U.

PROPOSITION 2.4. *A map $f \in \mathcal{O}(U \cap (\mathbb{D} \setminus Z), X)$ extends to a map in $\mathcal{O}(U, X)$ if and only if there exist open covers $(U_\alpha)_{\alpha\in A}$ of U and $(V_\beta)_{\beta\in B}$ of X and a map $\tau : A \to B$ such that*

(a) for each $\beta \in B$ holomorphic functions on V_β separate points and

$$f(U_\alpha \cap (\mathbb{D} \setminus Z)) \Subset V_{\tau(\alpha)} \quad \text{for all} \quad \alpha \in A;$$

(b)

$$\bigcap_{W\subset Q_Z^{-1}(U)\,:\,\mathring{W}\cap Z\neq\varnothing} \overline{f\big(W \cap (\mathbb{D} \setminus Z)} \neq \varnothing.$$

(Here for topological spaces S, Y the implication $S \Subset Y$ means that the closure $\bar{S}$ of S in Y is compact and $\mathring{S}$ stands for the interior of S.)

REMARK 2.5. (1) Condition (a) implies that map $Q_Z^* f$ $(:= f \circ Q_Z)$ extends continuously to $M(H^\infty)$ and then (b) guarantees that this extension is constant on Z. If $Z = \overline{Z \cap \mathbb{D}}$, then instead of (b) one can assume that $Q_Z^* f$ extends to a continuous map on $\mathbb{D}$ taking the same value on Z.
(2) If X is a complex submanifold of a complex Banach space, $f \in \mathcal{O}(\mathbb{D} \setminus Z, X)$ extends to a map in $\mathcal{O}(M(\mathscr{A}_Z), X)$ iff $f(\mathbb{D} \setminus Z) \Subset X$ and for each $g \in \mathcal{O}(X)$ the Gelfand transform $\widehat{g \circ f}$ is constant on Z. (Note that $g \circ f \in \mathcal{O}(\mathbb{D} \setminus Z)$ extends to a function in H^∞ by the Riemann extension theorem.) Here the first condition implies that $Q_Z^* f$ extends continuously to $M(H^\infty)$ by [8, Prop. 1.3], and the second one that this extension is constant on Z.

Let U be an open subset of $M(\mathscr{A}_Z)$. A topological principal G-bundle $\pi : P \to U$ with fibre a complex Banach Lie group is called *holomorphic* if it is defined on an open cover $(U_i)_{i\in I}$ of U by a cocycle $\{g_{ij} \in \mathcal{O}(U_i \cap U_j, G)\}_{i,j\in I}$. In this case, $P|_{U\cap(\mathbb{D}\setminus Z)}$ is a holomorphic principal G-bundle on $U \cap (\mathbb{D} \setminus Z)$ in the usual sense.

Recall that P is defined as the quotient space of disjoint union $\sqcup_{i\in I} U_i \times G$ by the equivalence relation:

$$U_j \times G \ni u \times g \sim u \times g g_{ij}(u) \in U_i \times G. \tag{2.4}$$

The projection $\pi : P \to U$ is induced by natural projections $U_i \times G \to U_i$, $i \in I$.

A bundle isomorphism $\varphi : (P_1, G, \pi_1) \to (P_2, G, \pi_2)$ of holomorphic principal G-bundles on U is called *holomorphic* if $\varphi|_{U\cap(\mathbb{D}\setminus Z)} : P_1|_{U\cap(\mathbb{D}\setminus Z)} \to P_2|_{U\cap(\mathbb{D}\setminus Z)}$ is a biholomorphic map of complex Banach manifolds.

We say that a holomorphic principal G-bundle (P, G, π) on U is *trivial* if it is holomorphically isomorphic to the trivial bundle $U \times G$. (For basic facts of the theory of bundles, see, e.g., [21].)

For a complex Banach Lie group G by G_0 we denote the connected component containing unit $1_G \in G$. Then G_0 is a clopen normal subgroup of G. By $q : G \to G/G_0 =: C(G)$ we denote the continuous quotient homomorphism onto the discrete group of connected components of G. Let $\pi : P \to U\,(\subset M(\mathscr{A}_Z))$ be a holomorphic principal G-bundle defined on an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of U by a cocycle $g = \{g_{ij} \in \mathcal{O}(U_i \cap U_j, G)\}_{i,j\in I}$. By $P_{C(G)}$ we denote the principal bundle on U with discrete fibre $C(G)$ defined on cover $\mathfrak{U}$ by locally constant cocycle $q(g) = \big\{q(g_{ij}) \in C(U_i \cap U_j, C(G))\big\}_{i,j\in I}$. Let $K \subset U$ be a compact subset.

The next result used in the applications in particular characterizes trivial holomorphic bundles on $M(\mathscr{A}_Z)$.

THEOREM 2.6. *P is trivial over a neighbourhood of K if and only if the associated bundle $P_{C(G)}$ is topologically trivial over K.*

COROLLARY 2.7. *P is trivial over a neighbourhood of K in one of the following cases:*

(1) *Group G is connected;*
(2) *$U = M(H^\infty)$ and images of all maps $q(g_{ij})$ belong to a finite subgroup of $C(G)$ (e.g., this is true if G has finitely many connected components);*
(3) *P is trivial over K in the category of topological principal G-bundles.*

In turn, not all principal bundles on $M(\mathscr{A}_Z)$ are trivial:

PROPOSITION 2.8. *Let G be a complex Banach Lie group such that $C(G)$ has a nontorsion element. Then there exists a nontrivial holomorphic principal G-bundle on $M(\mathscr{A}_Z)$.*

Let $\mathscr{P}_G^{\mathcal{O}}$ and $\mathscr{P}_G^{C}$ be the sets of isomorphism classes of holomorphic and topological principal G-bundles on $M(\mathscr{A}_Z)$, respectively.

The following two results (analogous to the classical results of Grauert [18] and Bungart [11]) show that the natural map $i : \mathscr{P}_G^{\mathcal{O}} \to \mathscr{P}_G^{C}$ induced by inclusion of the category of holomorphic principal bundles on $M(\mathscr{A}_Z)$ to the category of topological ones is a bijection. These constitute the Oka Principle for holomorphic principal bundles on $M(\mathscr{A}_Z)$.

THEOREM 2.9 (Injectivity of i). *If two holomorphic principal G-bundles on $M(\mathscr{A}_Z)$ are isomorphic as topological bundles, they are holomorphically isomorphic.*

Theorem 2.10 (Surjectivity of i). *Each topological principal G-bundle on $M(\mathscr{A}_Z)$ is isomorphic to a holomorphic one.*

Let $A \in \mathscr{A}_Z$, i.e., $A = H^\infty_I$ for some closed ideal $I \subset H^\infty$ such that $\operatorname{hull}(I) = Z$. Recall that we assumed that $M(A)$ coincides with $M(\mathscr{A}_Z)$.

Let D be the set of all finite subsets of A directed by inclusion. If $\alpha = \{f_1, \dots, f_n\} \in D$ we let A_α be the unital closed subalgebra of A generated by α. By $M(A_\alpha)$ we denote the maximal ideal space of A_α. It is naturally identified with the polynomially convex hull of the image of $F_\alpha : M(\mathscr{A}_Z) \to \mathbb{C}^n$, $F_\alpha(x) := (\hat{f}_1(x), \dots, \hat{f}_n(x))$ (here $\hat{}$ is the Gelfand transform for algebra $H^\infty_{I(Z)}$). If $\alpha, \beta \in D$ with $\alpha \supseteq \beta$, then linear map $F^\alpha_\beta : \mathbb{C}^{\#\alpha} \to \mathbb{C}^{\#\beta}$, $\mathbb{C}^{\#\alpha} \ni (z_1, \dots, z_{\#\alpha}) \mapsto (z_1, \dots, z_{\#\beta}) \in \mathbb{C}^{\#\beta}$, sends $M(A_\alpha)$ to $M(A_\beta)$. Thus we obtain the inverse system of compacta $\{M(A_\alpha), F^\alpha_\beta\}$ whose limit is naturally identified with $M(\mathscr{A}_Z)$ and the limit projections coincide with maps F_α (see, e.g., [31] for details).

Let P_i be holomorphic principal G-bundles defined on open neighbourhoods $O_i \Subset \mathbb{C}^{\#\alpha}$ of $M(A_\alpha)$, $i = 1, 2$. We say that P_1 and P_2 are isomorphic if they are holomorphically isomorphic on an open neighbourhood $O \subset O_1 \cap O_2$ of $M(A_\alpha)$. By $(\mathscr{P}^{\mathcal{O}}_G)_\alpha$ we denote the set of isomorphism classes of holomorphic principal G-bundles defined on neighbourhoods of $M(A_\alpha)$. Projections F^α_β induce maps $(\mathscr{F}^\alpha_\beta)^* : (\mathscr{P}^{\mathcal{O}}_G)_\beta \to (\mathscr{P}^{\mathcal{O}}_G)_\alpha$ assigning to the isomorphism class of a bundle P the isomorphism class of its pullback $(F^\alpha_\beta)^* P$. Thus we obtain the direct system of sets $\{(\mathscr{P}^{\mathcal{O}}_G)_\beta, (\mathscr{F}^\alpha_\beta)^*\}$. Similarly, limit projection F_α induces a map $\mathscr{F}^*_\alpha : (\mathscr{P}^{\mathcal{O}}_G)_\alpha \to \mathscr{P}^{\mathcal{O}}_G$ assigning to the isomorphism class of a bundle P the isomorphism class of its pullback $F^*_\alpha P$. Since $\mathscr{F}^*_\beta = \mathscr{F}^*_\alpha \circ (\mathscr{F}^\alpha_\beta)^*$ for all $\alpha \supseteq \beta$ in D, the family of maps $\{\mathscr{F}^*_\alpha\}_{\alpha \in D}$ induces a map $\mathscr{F}_A$ of the direct limit $\varinjlim (\mathscr{P}^{\mathcal{O}}_G)_\alpha$ to $\mathscr{P}^{\mathcal{O}}_G$.

Theorem 2.11. *Map $\mathscr{F}_A : \varinjlim (\mathscr{P}^{\mathcal{O}}_G)_\alpha \to \mathscr{P}^{\mathcal{O}}_G$ is a bijection.*

In particular,

$$\mathscr{P}^{\mathcal{O}}_G = \bigcup_{\alpha \in D} \mathscr{F}^*_\alpha((\mathscr{P}^{\mathcal{O}}_G)_\alpha).$$

The statement consists of two parts:

(1) (*Surjectivity of $\mathscr{F}_A$*). For each holomorphic principal G-bundle P on $M(\mathscr{A}_Z)$ there exist $\alpha \in D$ and a holomorphic principal G-bundle $\tilde{P}$ defined on a neighbourhood of $M(A_\alpha)$ such that bundles P and $F^*_\alpha \tilde{P}$ are holomorphically isomorphic.

(2) (*Injectivity of $\mathscr{F}_A$*). If holomorphic principal G-bundles P_1, P_2 defined on a neighbourhood of $M(A_\beta)$ are such that $F^*_\beta P_1$ and $F^*_\beta P_2$ are holomorphically isomorphic bundles, then there exist $\alpha \supseteq \beta$ and a neighbourhood U of $M(A_\alpha)$ such that bundles $(F^\alpha_\beta)^* P_1$ and $(F^\alpha_\beta)^* P_2$ are defined on U and holomorphically isomorphic.

3. Oka Principle for Maps to Complex Homogeneous Spaces

3.1. Holomorphic maps from $\mathbf{M}(\mathscr{A}_Z)$ *to Banach homogeneous spaces* Let $K \subset M(\mathscr{A}_Z)$ be a compact subset and X a complex Banach manifold. By $\mathcal{O}(K,X) \subset C(K,X)$ we denote the set of continuous maps holomorphic on neighbourhoods of K and by $\mathcal{A}(K,X)$ the closure of $\mathcal{O}(K,X)$ in the topology of uniform convergence of $C(K,X)$. If G is a complex Banach Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp_G : \mathfrak{g} \to G$, set $\mathcal{A}(K,G)$ is a complex Banach Lie group with respect to pointwise product of maps with Lie algebra $\mathcal{A}(K,\mathfrak{g})$ and the exponential map being the composition of $\exp_G$ with elements of $\mathcal{A}(K,\mathfrak{g})$.

The following result interesting in its own right is used in the proofs.

THEOREM 3.1. *If G is simply connected, then $\mathcal{A}(K,G)$ is path-connected.*

To formulate further results we invoke the definition of a complex Banach homogeneous space (see, e.g., [29, Sec. 1]).

Suppose a complex Banach Lie group G with unit e acts *holomorphically* and *transitively* on a complex Banach manifold X, i.e., there is a holomorphic map $(g,p) \mapsto g \cdot p$ of $G \times X$ onto X satisfying $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ and $e \cdot p = p$ for all $g_1, g_2 \in G$, $p \in X$, and for each pair $p, q \in X$ there is some $g \in G$ with $g \cdot p = q$. For $p \in X$ consider holomorphic surjective map $\pi^p : G \to X$, $\pi^p(g) := g \cdot p$, $g \in G$. Let $G_e\,(\cong \mathfrak{g})$ and X_p be tangent spaces of G at e and X at p. By $d\pi^p_e : G_e \to X_p$ we denote the differential of π^p at e.

X is called a *Banach homogeneous space* under the action of G if there is some $p \in X$ such that $\ker d\pi^p_e$ is a complemented subspace of G_e and $d\pi^p_e : G_e \to X_p$ is surjective.

By $[M(\mathscr{A}_Z), X]$ we denote the set of homotopy classes of continuous maps from $M(\mathscr{A}_Z)$ to X. Maps $f_0, f_1 \in \mathcal{O}(M(\mathscr{A}_Z), X)$ are said to be homotopic in $\mathcal{O}(M(\mathscr{A}_Z), X)$ if there is a homotopy $F : M(\mathscr{A}_Z) \times [0,1] \to X$ connecting them such that $f_t := F(t,\cdot) \in \mathcal{O}(M(\mathscr{A}_Z), X)$ for all $t \in [0,1]$. This homotopy relation is an equivalence relation with the set of equivalence classes $[M(\mathscr{A}_Z), X]_{\mathcal{O}}$ consisting of path-connected components of $\mathcal{O}(M(\mathscr{A}_Z), X)$. Next, embedding $\mathcal{O}(M(\mathscr{A}_Z), X) \hookrightarrow C(M(\mathscr{A}_Z), X)$ induces a map $\mathscr{O} : [M(\mathscr{A}_Z), X]_{\mathcal{O}} \to [M(\mathscr{A}_Z), X]$.

THEOREM 3.2. *Let X be a complex Banach homogeneous space. Then $\mathscr{O}$ is a bijection.*

The statement consists of two parts:

(1) (*Injectivity of $\mathscr{O}$*). If two maps in $\mathcal{O}(M(\mathscr{A}_Z), X)$ are homotopic in $C(M(\mathscr{A}_Z), X)$, they are homotopic in $\mathcal{O}(M(\mathscr{A}_Z), X)$.
(2) (*Surjectivity of $\mathscr{O}$*). Every map in $C(M(\mathscr{A}_Z), X)$ is homotopic to a map in $\mathcal{O}(M(\mathscr{A}_Z), X)$.

Next, recall that a path-connected topological space X is *n-simple* if for each $x \in X$ the fundamental group $\pi_1(X, x_0)$ acts trivially on the n-homotopy group

$\pi_n(X,x)$ (see, e.g., [19, Ch. IV.16] for the corresponding definitions and results). For instance, X is n-simple if group $\pi_n(X)=0$ and 1-simple if and only if group $\pi_1(X)$ is abelian. Also, it is worth noting that every path-connected topological group is n-simple for all n.

COROLLARY 3.3. *Let X be a complex Banach homogeneous space n-simple for all $n\le 2$. Then there is a natural one-to-one correspondence between elements of $[M(\mathscr{A}_Z),X]_{\mathcal{O}}$ and the Čech cohomology group $H^1(M(\mathscr{A}_Z),\pi_1(X))$. In particular, if X is simply connected, space $\mathcal{O}(M(\mathscr{A}_Z),X)$ is path-connected.*

REMARK 3.4. (1) The correspondence of the corollary is described in [20].
(2) Group $H^1(M(\mathscr{A}_Z),\mathbb{Z})$ is always nontrivial.

Recall that for each $A\in\mathscr{A}_Z$ space $M(\mathscr{A}_Z)$ can be presented as the inverse limit of the inverse limit system $\{M(A_\alpha),F_\beta^\alpha\}$, $\alpha\in D$, see Section 2.2. Two maps $M(A_\alpha)\to X$ holomorphic in a neighbourhood U of $M(A_\alpha)$ are said to be *homotopic in* $\mathcal{O}(M(A_\alpha),X)$ if their restrictions to a neighbourhood $V\subset U$ of $M(A_\alpha)$ can be joined by a path in $\mathcal{O}(V,X)$. This homotopy relation is an equivalence relation with the set of equivalence classes denoted by $[M(A_\alpha),X]_{\mathcal{O}}$.

Projections F_β^α induce maps $(\mathfrak{F}_\beta^\alpha)^*:[A_\beta,X]_{\mathcal{O}}\to[A_\alpha,X]_{\mathcal{O}}$ assigning to the homotopy class of a map $f\in\mathcal{O}(M(A_\beta),X)$ the homotopy class of its pullback $(F_\beta^\alpha)^*f$. Thus we obtain the direct system of sets $\big\{[M(A_\beta),X]_{\mathcal{O}},(\mathfrak{F}_\beta^\alpha)^*\big\}$. Similarly, limit projection F_α induces a map $\mathfrak{F}_\alpha^*:[M(A_\alpha),X]_{\mathcal{O}}\to[M(\mathscr{A}_Z),X]$ assigning to the homotopy class of $f\in\mathcal{O}(M(A_\alpha),X)$ the homotopy class of its pullback F_α^*f. Since $\mathfrak{F}_\beta^*=\mathfrak{F}_\alpha^*\circ(\mathfrak{F}_\beta^\alpha)^*$ for all $\alpha\supseteq\beta$ in D, the family of maps $\{\mathfrak{F}_\alpha^*\}_{\alpha\in D}$ induces a map $\mathfrak{F}_A$ of the direct limit $\varinjlim[M(A_\alpha),X]_{\mathcal{O}}$ to $[M(\mathscr{A}_Z),X]_{\mathcal{O}}$.

THEOREM 3.5. *Let X be a complex Banach homogeneous space. Then $\mathfrak{F}_A$ is a bijection.*

In particular,

$$[M(\mathscr{A}_Z),X]_{\mathcal{O}}=\bigcup_{\alpha\in D}\mathfrak{F}_\alpha^*\big([M(A_\alpha),X]_{\mathcal{O}}\big).$$

To prove the result we establish the following:

(1) (*Surjectivity of* $\mathfrak{F}_A$). For each holomorphic map $f:M(\mathscr{A}_Z)\to X$ there exist $\alpha\in D$ and a holomorphic map into X defined on a neighbourhood of $M(A_\alpha)$ such that maps f and F_α^*f are homotopic in $\mathcal{O}(M(\mathscr{A}_Z),X)$.
(2) (*Injectivity of* $\mathfrak{F}_A$). If holomorphic maps f_1,f_2 into X defined on a neighbourhood of $M(A_\beta)$ are such that $F_\beta^*f_1$ and $F_\beta^*f_2$ are homotopic in $\mathcal{O}(M(\mathscr{A}_Z),X)$, then there exist $\alpha\supseteq\beta$ and a neighbourhood U of $M(A_\alpha)$ such that maps $(F_\beta^\alpha)^*f_1$ and $(F_\beta^\alpha)^*f_2$ are defined on U and homotopic in $\mathcal{O}(U,X)$.

Let X be a complex Banach homogeneous manifold under the action of a complex Banach Lie group G. It is known, see, e.g., [29, Prop. 1.4], that the stabilizer of a point $p \in X$, $G(p) := \{g \in G : \pi^p(g) = p\}$, is a closed complex Banach Lie subgroup of G and stabilizers of different points are conjugate in G by inner automorphisms.

The following result will be used in applications.

THEOREM 3.6. *Let X be a complex Banach homogeneous space under the action of a complex Banach Lie group G. Assume that group $G(p) \subset G$, $p \in X$, is connected. Then for every $f \in \mathcal{O}(M(\mathscr{A}_Z), X)$ and each $p \in X$ there is a map $\tilde{f}_p \in \mathcal{O}(M(\mathscr{A}_Z), G)$ such that $f(x) = \tilde{f}_p(x) \cdot p$ for all $x \in M(\mathscr{A}_Z)$.*

3.2. Nonlinear approximation and interpolation problems A compact subset $K \subset M(\mathscr{A}_Z)$ is called *holomorphically convex* if for any $x \notin K$ there is $f \in \mathcal{O}(M(\mathscr{A}_Z))$ such that

$$\max_K |f| < |f(x)|.$$

Note that for a natural number l any subset of $M(\mathscr{A}_Z)$ of the form

$$\{x \in M(\mathscr{A}_Z) : \max_{1 \le j \le l} |f_j(x)| \le 1, \ f_j \in \mathcal{O}(M(\mathscr{A}_Z)), \ 1 \le j \le l\}$$

is holomorphically convex and each holomorphically convex $K \subset M(\mathscr{A}_Z)$ is intersection of such subsets.

Let U be an open neighbourhood of a holomorphically convex set $K \subset M(\mathscr{A}_Z)$ and X be a complex Banach homogeneous space. The following result is a nonlinear analog of the Runge approximation theorem.

THEOREM 3.7. *(1) Suppose a map in $\mathcal{O}(U, X)$ can be uniformly approximated on K by maps in $C(M(\mathscr{A}_Z), X)$. Then it can be uniformly approximated on K by maps in $\mathcal{O}(M(\mathscr{A}_Z), X)$.*
(2) If X is simply connected, then each map in $\mathcal{O}(U, X)$ can be uniformly approximated on K by maps in $\mathcal{O}(M(\mathscr{A}_Z), X)$.

Let $Z \subset M(H^\infty)$ be a hull and U be an open neighbourhood of Z.

THEOREM 3.8. *Let X be a complex Banach homogeneous space and $f \in \mathcal{O}(U, X)$.*

(1) *Restriction $f|_Z \in C(Z, X)$ extends to a map in $\mathcal{O}(M(H^\infty), X)$ iff it extends to a map in $C(M(H^\infty), X)$.*
(2) *$f|_Z$ extends to a map in $\mathcal{O}(M(H^\infty), X)$ if X is n-simple for all $n \le 2$.*

REMARK 3.9. (1) The quantitative version of the second part of Theorem 3.8 for $X = \mathbb{C}$ and Z the zero locus of the Gelfand transform of a Blaschke product was proved by Carleson [12]. For $X = \mathbb{C}^*$ the result of part (2) follows from Treil's theorem [39] via its cohomological interpretation due to Suárez [33, Th. 1.3] and the Arens-Royden theorem [1], [31]. For X a complex Banach space the results of second parts of Theorems 3.7, 3.8 were established by the author [8, Th. 1.7, 1.9].

(2) Relation between spaces $\mathcal{O}(U,X)$ and $\mathcal{O}(U\cap\mathbb{D},X)$ is given by Proposition 2.4:

($\circ$) *a map* $f\in\mathcal{O}(U\cap\mathbb{D},X)$ *extends to a map in* $\mathcal{O}(U,X)$ *iff there exist open covers* $(U_\alpha)_{\alpha\in A}$ *of* U *and* $(V_\beta)_{\beta\in B}$ *of* X *and a map* $\tau:A\to B$ *such that for each* $\beta\in B$ *holomorphic functions on* V_β *separate points and* $f(U_\alpha\cap\mathbb{D})\Subset V_{\tau(\alpha)}$ *for all* $\alpha\in A$.

If X *is a complex submanifold of a complex Banach space, then it suffices to assume that* $f(U\cap\mathbb{D})\Subset X$*, see [8, Prop. 1.3].*

Example 3.10. Let $F\subset\{z\in\mathbb{C}:|z|=1\}$ be a closed subset of Lebesgue measure zero. By $I_F\subset H^\infty$ we denote the ideal generated by all functions of the *disk-algebra* $A(\mathbb{D})\,(=H^\infty\cap C(\bar{\mathbb{D}}))$ equal zero on F. According to the Rudin-Carleson theorem for each $f\in C(F)$ there exists a function $\tilde f\in A(\mathbb{D})$ such that $\tilde f|_F=f$ and $\|\tilde f\|_{A(\mathbb{D})}=\|f\|_{C(F)}$. The map transposed to embedding $A(\mathbb{D})\hookrightarrow H^\infty$ induces a continuous surjection of the maximal ideal spaces $p:M(H^\infty)\to\bar{\mathbb{D}}\,(=M(A(\mathbb{D}))$ such that $Z:=\mathrm{hull}(I_F)=p^{-1}(F)$. Let $O\subset\bar{\mathbb{D}}$ be a relatively open subset containing F. Then $U:=p^{-1}(O)$ is an open neighbourhood of Z and $U\cap\mathbb{D}=O\cap\mathbb{D}$. Theorem 3.8 implies the following nonlinear version of the Rudin-Carleson theorem: *Let* G *be a connected complex Banach Lie group and* $f\in\mathcal{O}(U\cap\mathbb{D},G)$ *satisfy* ($\circ$)*. Then there exists a map* $\tilde f\in\mathcal{O}(\mathbb{D},G)$ *satisfying* ($\circ$) *such that* $\lim_{z\to x}\tilde f(z)f(z)^{-1}=1_G$ *for all* $x\in F$*; here* $1_G\in G$ *is the unit.*

4. Applications and Examples

4.1. Holomorphic maps of $\mathbf{M}(\mathscr{A}_Z)$ *into flag manifolds and complex tori* For basic results of the Lie group theory, see, e.g., [27].

Recall that the maximal connected solvable Lie subgroup B of a connected (finite-dimensional) complex Lie group G is called a *Borel subgroup*. A Lie subgroup $P\subset G$ containing some Borel subgroup is called *parabolic*. The set of orbits $X=G/P$ under the action of a parabolic subgroup $P\subset G$ on G by right multiplications has the natural structure of a complex homogeneous space and is called the *flag manifold*. Since P contains the radical of G, one may assume that in the definition of X group G is semisimple. Each flag manifold is simply connected and thus by Corollary 3.3 and Theorem 3.6 space $\mathcal{O}(M(\mathscr{A}_Z),X)$ *is path-connected and consists of all maps of the form* $f\cdot p_0$*, where* $f\in\mathcal{O}(M(\mathscr{A}_Z),G)$ *and* p_0 *is the orbit of the unit of* G. Since G admits a faithful linear representation, we also obtain (see Remark 2.5(2)) that $g\in\mathcal{O}(\mathbb{D}\setminus Z,X)$ *extends to a map in* $\mathcal{O}(M(\mathscr{A}_Z),X)$ *iff* $g=f\cdot p_0$ *for some* $f\in\mathcal{O}(\mathbb{D}\setminus Z,G)$ *such that* $f(\mathbb{D}\setminus Z)\Subset G$ *and for every* $h\in\mathcal{O}(G)$ *the Gelfand transform* $\widehat{h\circ f}$ *is constant on* Z.

Example 4.1. The complex *flag manifold* $\mathbf{F}(d_1,\dots,d_k)$ is the space of all flags of type $(d_1,\dots,d_k)$ in $\mathbb{C}^n$, $n:=d_k$, i.e., of increasing sequences of subspaces

$$\{0\}=V_0\subset V_1\subset V_2\subset\cdots\subset V_k=\mathbb{C}^n\qquad\text{with}\qquad \dim_{\mathbb{C}}V_i=d_i.$$

It has the natural structure of a simply connected compact complex homogeneous space under the action of group $GL_n(\mathbb{C})$. The stabilizer of a flag is the connected subgroup of $GL_n(C)$ isomorphic to the group of nonsingular block upper triangular matrices with dimensions of blocks $d_i - d_{i-1}$ with $d_0 := 0$. Thus space $\mathcal{O}(M(\mathscr{A}_Z), \mathbf{F}(d_1, \dots, d_k))$ is path-connected and consists of all maps of the form $f \cdot p_0$, where $f \in \mathcal{O}(M(\mathscr{A}_Z), GL_n(\mathbb{C}))$ and p_0 is the flag of subspaces $V_i^0 := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_1 = z_2 = \cdots = z_{n-d_i} = 0\}$, $0 \le i \le k-1$, $V_k := \mathbb{C}^n$. Here the image of V_i^0 under $f(x)$, $x \in M(\mathscr{A}_Z)$, is the subspace of $\mathbb{C}^n$ generated by the last d_i column vectors of matrix $f(x)$.

For example, in the case of $\mathbf{F}(1,n)$, the $(n-1)$-dimensional complex projective space, we obtain that space $O(M(\mathscr{A}_Z), \mathbf{F}(1,n))$ consists of all maps f of the form $f(x) = [f_1(x) : f_2(x) : \cdots : f_n(x)]$, $x \in M(\mathscr{A}_Z)$, where $f_i \in \mathcal{O}(M(\mathscr{A}_Z))$ and satisfy the corona condition on $M(\mathscr{A}_Z)$ (cf. (2.1)). (Note that the column vector composed of such f_i extends automatically to an invertible matrix with entries in $\mathcal{O}(M(\mathscr{A}_Z))$ because of projective freeness of this algebra, see Remark 2.2(2).)

Let $\mathbb{CT}^n$ be the complex torus obtained as the quotient of $\mathbb{C}^n$ by a lattice $\Gamma\,(\cong \mathbb{Z}^{2n})$ (acting on $\mathbb{C}^n$ by translations). By $\pi : \mathbb{C}^n \to \mathbb{CT}^n$ we denote the holomorphic projection map. In the next result we describe the structure of space $\mathcal{O}(M(\mathscr{A}_Z), \mathbb{CT}^n)$.

Recall that a holomorphic function f on $\mathbb{D}$ belongs to the BMOA space if it has boundary values a.e. on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ satisfying

$$\sup_I \frac{1}{|I|} \int_I |f(\theta) - f_I|\, d\theta < \infty, \tag{4.1}$$

where $I \Subset \mathbb{T}$ is an open arc of arclength $|I|$ and $f_I := \frac{1}{|I|}\int_I f(\theta) d\theta$.

THEOREM 4.2. *A map $f \in \mathcal{O}(\mathbb{D}, \mathbb{CT}^n)$ extends continuously to $M(H^\infty)$ iff it is factorized as $f = \pi \circ \tilde{f}$ for some $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n) \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ with all $\tilde{f}_i \in BMOA$.*

REMARK 4.3. (1) Let $Q_Z : M(H^\infty) \to M(\mathscr{A}_Z)$ be the continuous map transposed to embedding $H_I^\infty \hookrightarrow H^\infty$. Clearly, $f \in \mathcal{O}(\mathbb{D} \setminus Z, \mathbb{CT}^n)$ extends continuously to $M(\mathscr{A}_Z)$ iff $Q_Z^* f$ extends to a map $C(M(H^\infty), \mathbb{CT}^n)$ constant on Z.

(2) It follows from the proof of Theorem 3.2 that if $f, g \in \mathcal{O}(M(\mathscr{A}_Z), \mathbb{CT}^n)$ are homotopic, then there is a map $h \in \mathcal{O}(M(\mathscr{A}_Z), \mathbb{C}^n)$ such that $f(x) = g(x) + (\pi \circ h)(x)$ for all $x \in M(\mathscr{A}_Z)$. (Here $+$ stands for addition on $\mathbb{CT}^n$ induced from that on $\mathbb{C}^n$.) Space $\mathcal{O}(M(\mathscr{A}_Z), \mathbb{CT}^n)$ is an abelian complex Banach Lie group under the pointwise addition of maps. Thus, according to Corollary 3.3, $H^1(M(\mathscr{A}_Z), \mathbb{Z}^{2n})$ is naturally isomorphic to the quotient of group $\mathcal{O}(M(\mathscr{A}_Z), \mathbb{CT}^n)$ by the connected component of its unit $\pi\big(\mathcal{O}(M(\mathscr{A}_Z), \mathbb{C}^n)\big)$.

4.2. Structure of spaces of $\mathbf{H}^\infty$ *idempotents* Let $\mathfrak{A}$ be a complex Banach algebra with unit $1_{\mathfrak{A}}$. By $\operatorname{id}\mathfrak{A} = \{a \in \mathfrak{A} : a^2 = a\}$ we denote the set of idempotents of $\mathfrak{A}$.

It is a closed complex Banach submanifold of $\mathfrak{A}$ which is a discrete union of connected complex Banach homogeneous spaces, see [29, Cor. 1.7]. Specifically, let $\mathfrak{A}_0^{-1}$ be the connected component containing the unit $1_{\mathfrak{A}}$ of the complex Banach Lie group $\mathfrak{A}^{-1}$ of invertible elements of $\mathfrak{A}$. Then each connected component of $\mathrm{id}\,\mathfrak{A}$ is a complex Banach homogeneous space under the action $\mathfrak{A}_0^{-1}\times \mathrm{id}\,\mathfrak{A}\to \mathrm{id}\,\mathfrak{A}$ by similarity transformations $(a,p)\mapsto apa^{-1}$.

Analogously for unital complex Banach algebras $C(M(\mathscr{A}_Z),\mathfrak{A})$ and $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A})$ their sets of idempotents $\mathrm{id}\,C(M(\mathscr{A}_Z),\mathfrak{A})$ and $\mathrm{id}\,\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A})$ coincide with $C(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ and $\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ (because $M(\mathscr{A}_Z)$ is connected) and are discrete unions of connected complex Banach homogeneous spaces. Each connected component of $C(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ or $\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ is the complex homogeneous space under the action of the complex Banach Lie group $C(M(\mathscr{A}_Z),\mathfrak{A})_0^{-1}$ or $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A})_0^{-1}$ by similarity transformations.

THEOREM 4.4. (a) *Embedding* $\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})\hookrightarrow C(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ *induces bijection between connected components of these complex Banach manifolds.*

(b) *If the stabilizer* $\mathfrak{A}_0^{-1}(p)\subset\mathfrak{A}_0^{-1}$ *of a point* $p\in\mathrm{id}\,\mathfrak{A}$ *is connected, then for each* $f\in\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ *whose image belongs to the connected component containing* p *there is* $g\in\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_0^{-1})$ *such that* $f=gpg^{-1}$.

(c) *If* $A\in\mathscr{A}_Z$, *then for each* $f\in\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ *there are* $\alpha\in D$, *a map* $h\in\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ *uniformly approximated by maps of the algebraic tensor product* $\widehat{A}_\alpha\otimes\mathfrak{A}$ *and a map* $g\in\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_0^{-1})$ *such that* $gfg^{-1}=h$.
Here $\widehat{A}_\alpha$ *is the image of* A_α *under the Gelfand transform* $\hat{}: A\to C(M(\mathscr{A}_Z))$.

REMARK 4.5. (1) If $\mathfrak{A}_0^{-1}$ is simply connected and the hypothesis of part (b) holds for all $p\in\mathrm{id}\,\mathfrak{A}$, then, since in this case $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_0^{-1})$ coincides with $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A})_0^{-1}$ (cf. Theorem 3.1), embedding $\mathrm{id}\,\mathfrak{A}\hookrightarrow\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$ assigning to each p the map of the constant value p induces bijection between sets of connected components of $\mathrm{id}\,\mathfrak{A}$ and $\mathcal{O}(M(\mathscr{A}_Z),\mathrm{id}\,\mathfrak{A})$.

(2) Part (c) follows from Theorem 3.5 and the fact that $F_\alpha^*\big(\mathcal{O}(M(A_\alpha),\mathfrak{A})\big)$ coincides with $A_\alpha\otimes_\varepsilon\mathfrak{A}$, the injective tensor product of these algebras.

(3) Let $H_Z^\infty(\mathfrak{A})$ be the Banach algebra of bounded maps $F\in\mathcal{O}(\mathbb{D},\mathfrak{A})$ equipped with pointwise product and addition with norm $\|F\|:=\sup_{z\in\mathbb{D}}\|F(z)\|_{\mathfrak{A}}$ such that for each $\varphi\in\mathfrak{A}^*$ the Gelfand transform of $\varphi\circ F\in H^\infty$ is constant on Z. Let $H_{Z\,\mathrm{comp}}^\infty(\mathfrak{A})\subset H_Z^\infty(\mathfrak{A})$ be the closed subalgebra of maps with relatively compact images. (If Z is empty or a single point we omit $_Z$ in the indices.) Each map in $H_{Z\,\mathrm{comp}}^\infty(\mathfrak{A})$ admits a continuous extension to $M(H^\infty)$ constant on Z, see, e.g., [8, Prop. 1.3]. This implies that algebras $H_{Z\,\mathrm{comp}}^\infty(\mathfrak{A})$ and $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A})$ are isometrically isomorphic. Thus Theorem 4.4 describes the structure of set $\mathrm{id}\,H_{Z\,\mathrm{comp}}^\infty(\mathfrak{A})$.

EXAMPLE 4.6. Let $L(X)$ be the Banach algebra of bounded linear operators on a complex Banach space X equipped with the operator norm. By $I_X\in L(X)$ we denote the identity operator and by $GL(X)\subset L(X)$ the set of invertible bounded linear operators on X. Clearly, $L(X)^{-1}:=GL(X)$. By $GL_0(X)\subset$

$GL(X)$ we denote the connected component of I_X. Each $P \in \mathrm{id}\, L(X)$ determines a direct sum decomposition $X = X_0 \oplus X_1$, where $X_0 := \ker(P)$ and $X_1 := \ker(I_X - P)$. It is easily seen that the stabilizer $GL_0(X)(P) \subset GL_0(X)$ consists of all operators $B \in GL_0(X)$ such that $B(X_k) \subset X_k$, $k = 1, 2$. In particular, restrictions of operators in $GL_0(X)(P)$ to X_k determine a monomorphism of complex Banach Lie groups $S_P : GL_0(X)(P) \to GL(X_1) \oplus GL(X_2)$. Moreover, S_P is an isomorphism if $GL(X_i)$, $i = 1, 2$ are connected. Now, Theorem 4.4 leads to the following statement:

(1) *Suppose $P \in \mathrm{id}\, L(X)$ is such that groups $GL(X_1)$ and $GL(X_2)$ are connected. Then for each $F \in \mathcal{O}(M(\mathscr{A}_Z), \mathrm{id}\, L(X))$ whose image belongs to the connected component containing P there is $G \in \mathcal{O}(M(\mathscr{A}_Z), GL_0(X))$ such that $GFG^{-1} = P$.*

In particular, the result is valid for X being one of the spaces: a finite-dimensional space, a Hilbert space, c_0 or ℓ^p, $1 \le p \le \infty$. Indeed, $GL(X)$ is connected if $\dim_{\mathbb{C}} X < \infty$ and contractible for other listed above spaces, see, e.g., [24] and references therein. Moreover, each subspace of a Hilbert space is Hilbert, and each infinite-dimensional complemented subspace of X being either c_0 or ℓ^p, $1 \le p \le \infty$, is isomorphic to X, see [28], [22]. This gives the required conditions.

It is worth noting that there are complex Banach spaces X for which groups $GL(X)$ are not connected, see, e.g., [13].

In turn, part (c) of the theorem implies in this case:

(2) *Let $A \in \mathscr{A}_Z$. For each $F \in \mathcal{O}(M(\mathscr{A}_Z), \mathrm{id}\, L(X))$ there exist a finitely generated unital subalgebra $B \subset A$ and maps $H \in \mathrm{id}\, \widehat{B} \otimes_\varepsilon L(X)$, $G \in \mathcal{O}(M(\mathscr{A}_Z), GL_0(X))$ such that $GFG^{-1} = H$.*

4.3. Extension of operator-valued $\mathbf{H}^\infty$ functions Let $\mathfrak{A}$ be a complex Banach algebra with unit $1_{\mathfrak{A}}$. Let $\mathfrak{A}_l^{-1} = \{a \in \mathfrak{A} : \exists\, b \in \mathfrak{A} \text{ such that } ba = 1_{\mathfrak{A}}\}$ be the set of left-invertible elements of $\mathfrak{A}$. Clearly, $\mathfrak{A}_l^{-1}$ is an open subset of $\mathfrak{A}$ and complex Banach Lie group $\mathfrak{A}_0^{-1}$ acts holomorphically on each connected component of $\mathfrak{A}_l^{-1}$ by left multiplications: $(g, a) \mapsto ga$. In fact, we have

PROPOSITION 4.7. *Each connected component of $\mathfrak{A}_l^{-1}$ is a complex Banach homogeneous space under the action of $\mathfrak{A}_0^{-1}$.*

Next, similarly to Theorem 4.4 the following result holds.

THEOREM 4.8. *(a) If the stabilizer $\mathfrak{A}_0^{-1}(p) \subset \mathfrak{A}_0^{-1}$ of a point $p \in \mathfrak{A}_l^{-1}$ is connected, then for each $f \in \mathcal{O}(M(\mathscr{A}_Z), \mathfrak{A}_l^{-1})$ with image in the connected component containing p there is $g \in \mathcal{O}(M(\mathscr{A}_Z), \mathfrak{A}_0^{-1})$ such that $f = gp$.*

(b) If $A \in \mathscr{A}_Z$, then for each $f \in \mathcal{O}(M(\mathscr{A}_Z), \mathfrak{A}_l^{-1})$ there are $\alpha \in D$, a map $h \in \mathcal{O}(M(\mathscr{A}_Z), \mathfrak{A}_l^{-1})$ uniformly approximated by maps of the algebraic tensor product $\widehat{A}_\alpha \otimes \mathfrak{A}$ and a map $g \in \mathcal{O}(M(\mathscr{A}_Z), \mathfrak{A}_0^{-1})$ such that $gf = h$.

The analog of Theorem 4.4(a) for Banach algebras $C(M(\mathscr{A}_Z), \mathfrak{A})$ and $\mathcal{O}(M(\mathscr{A}_Z), \mathfrak{A})$ is also true.

THEOREM 4.9.
$$C(M(\mathscr{A}_Z),\mathfrak{A})_l^{-1} = C(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1})$$
and
$$\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A})_l^{-1} = \mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1}).$$
Thus every connected component of $C(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1})$ or $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1})$ is a complex Banach homogeneous space under the group action of $C(M(\mathscr{A}_Z),\mathfrak{A})_0^{-1}$ or $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A})_0^{-1}$ by left multiplications, and embedding
$$\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1}) \hookrightarrow C(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1})$$
induces bijection between connected components of these complex Banach manifolds.

REMARK 4.10. (1) If $\mathfrak{A}_0^{-1}$ is simply connected and the hypothesis of part (b) of Theorem 4.8 holds for all $p \in \mathfrak{A}_l^{-1}$, then embedding $\mathfrak{A}_l^{-1} \hookrightarrow \mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1})$ assigning to each p the map of the constant value p induces bijection between sets of connected components of $\mathfrak{A}_l^{-1}$ and $\mathcal{O}(M(\mathscr{A}_Z),\mathfrak{A}_l^{-1})$, (cf. Remark 4.5(1)).
(2) While the first identity of Theorem 4.9 for algebra $C(M(\mathscr{A}_Z),\mathfrak{A})$ is true in general with a compact Hausdorff space substituted for $M(\mathscr{A}_Z)$, the second one was previously unknown even for H^∞, see, e.g., [41]. For $H^\infty_{Z\,\mathrm{comp}}$, see Remark 4.5(3), the second identity of Theorem 4.9 can be reformulated as follows:

($\star$) *A map $F \in H^\infty_{Z\,\mathrm{comp}}(\mathfrak{A})$ has a left inverse $G \in H^\infty_{Z\,\mathrm{comp}}(\mathfrak{A})$ if and only if for every $z \in \mathbb{D}$ there exists a left inverse G_z of $F(z)$ such that* $\sup_{z\in\mathbb{D}} \|G_z\|_{\mathfrak{A}} < \infty$.

This result is related to the classical Sz.-Nagy operator corona problem [32]:

SZ.-NAGY PROBLEM. *Let H_1, H_2 be separable Hilbert spaces and $F \in H^\infty(L(H_1,H_2))$ be such that for some $\delta > 0$ and all $x \in H_1, z \in \mathbb{D}$, $\|F(z)x\| \ge \delta\|x\|$. Does there exist $G \in H^\infty(L(H_2,H_1))$ such that $G(z)F(z) = I_{H_1}$ for all $z \in \mathbb{D}$?*

This problem is of great interest in operator theory (angles between invariant subspaces, unconditionally convergent spectral decompositions), as well as in control theory. It is also related to the study of submodules of H^∞ and to many other subjects of analysis, see [25], [26], [34], [35], [42] and references therein. In general, the answer is known to be negative (see [36], [37], [40]); it is positive as soon as $\dim H_1 < \infty$ or F is a "small" perturbation of a left invertible function $F_0 \in H^\infty(L(H_1,H_2))$ (e.g., if $F - F_0$ belongs to $H^\infty(L(H_1,H_2))$ with values in the class of Hilbert Schmidt operators), see [38], or $F \in H^\infty_{\mathrm{comp}}(L(H_1,H_2))$, see [9, Th. 1.5] and ($\star$) above.

It is worth noting that the proof of statement ($\star$) for $H^\infty_{\mathrm{comp}}(\mathfrak{A})$ would be shorter if we knew that H^∞ has the *Grothendieck approximation property*, cf. [41, Th. 2.2]. However, this long-standing problem remains unsolved (for some developments see, e.g., [6, Th. 9] and [8, Th. 1.21]).

EXAMPLE 4.11. Let $L(X)$ be the Banach algebra of bounded linear operators on a complex Banach space X. Each $A \in L(X)_l^{-1}$ determines complemented subspace $X_1 := \operatorname{ran} A \subset X$ isomorphic to X. Then the stabilizer $GL_0(X)(A) \subset GL_0(X)$ of A consists of all operators $B \in GL_0(X)$ such that $B|_{X_1} = I_{X_1}$. If $X_2 \subset X$ is a complemented subspace to X_1, then each $B \in GL_0(X)(A)$ has a form

$$B = \begin{pmatrix} I_{X_1} & C \\ 0 & D \end{pmatrix}, \quad \text{where} \quad D \in GL(X_2) \ \text{and} \ C \in L(X_2, X_1). \tag{4.2}$$

Thus $GL_0(X)(A)$ is homotopy equivalent to the subgroup of

$$GL(X_2)\,(\cong GL(X/X_1))$$

consisting of all operators D such that $\operatorname{diag}(I_{X_1}, D) \in GL_0(X)$. In particular, this subgroup coincides with $GL(X_2)$ if the latter is connected.

Now, Theorem 4.8 leads to the following statement:

(1) *Suppose $A \in L(X)_l^{-1}$ is such that group $GL(X/X_1)$ is connected. Then for each $F \in \mathcal{O}(M(\mathscr{A}_Z), L(X)_l^{-1})$ whose image belongs to the connected component containing A there is $G \in \mathcal{O}(M(\mathscr{A}_Z), GL_0(X))$ such that $F = GA$.*

Let us identify X with X_1 by means of A and regard $F(x)$, $x \in M(\mathscr{A}_Z)$, and A as operators in $L(X_1, X_1 \oplus X_2)$. Then we obtain

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} I_{X_1} \\ 0 \end{pmatrix},$$

where $F_i \in \mathcal{O}(M(\mathscr{A}_Z), L(X_1, X_i))$, $i = 1, 2$, and $G_{ii} \in \mathcal{O}(M(\mathscr{A}_Z), L(X_i))$, $i = 1, 2$, $G_{ij} \in \mathcal{O}(M(\mathscr{A}_Z), L(X_j, X_i))$, $i, j \in \{1, 2\}$, $i \neq j$. Now identity $F = GA$ implies that

$$F_1 = G_{11}, \quad F_2 = G_{21},$$

that is, G extends F to an invertible holomorphic operator-valued function.

Thus from (1) we obtain the following generalization of [9, Th. 1.4].

($1'$) Suppose $F \in \mathcal{O}(M(\mathscr{A}_Z), L(Y_1, Y_2))$, where Y_i, $i = 1, 2$, are complex Banach spaces is such that for each $z \in \mathbb{D} \setminus Z$ there exists a left inverse G_z of $F(z)$ satisfying

$$\sup_{z \in \mathbb{D} \setminus Z} \|G_z\| < \infty.$$

Let $Y := \ker G_0$. Assume that $GL(Y)$ is connected. Then *there exist maps $H \in \mathcal{O}(M(\mathscr{A}_Z), L(Y_1 \oplus Y, Y_2))$, $G \in \mathcal{O}(M(\mathscr{A}_Z), L(Y_2, Y_1 \oplus Y))$ such that for all $x \in M(\mathscr{A}_Z)$,*

$$H(x)G(x) = I_{Y_2}, \qquad G(x)H(x) = I_{Y_1 \oplus Y} \qquad \text{and} \qquad H(x)|_{Y_1} = F(x).$$

Note that group $GL(Y)$ is connected in the following cases (see, e.g., [10, Cor.] for the references): (1) $\dim_{\mathbb{C}} Y < \infty$; (2) Y_2 is isomorphic to a Hilbert space or c_0 or one of the spaces ℓ^p, $1 \leq p \leq \infty$; (3) Y_2 is isomorphic to one of the spaces $L^p[0,1]$, $1 < p < \infty$, or $C[0,1]$ and Y_1 is not isomorphic to Y_2.

Statement ($1'$) is related to the following

COMPLETION PROBLEM. *Let $F \in H^\infty(L(H_1, H_2))$ satisfy the hypotheses of the Sz.-Nagy problem and $H_3 := \big(F(0)(H_1)\big)^{\perp}$. Do there exist functions $D \in H^\infty(L(H_1 \oplus H_3, H_2))$ and $E \in H^\infty(L(H_2, H_1 \oplus H_3))$ such that*

$$D(z)E(z) = I_{H_2},\ E(z)D(z) = I_{H_1 \oplus H_3} \quad \text{and} \quad D(z)|_{H_1} = F(z) \quad \text{for all} \quad z \in \mathbb{D}.$$

Seemingly much stronger, this problem is equivalent to the Sz.-Nagy problem. This result, known as the Tolokonnikov lemma, is proved in full generality in [38].

Finally, part (b) of Theorem 4.8 implies in our case:

(2) *Let $A \in \mathscr{A}_Z$. For each $F \in \mathcal{O}(M(\mathscr{A}_Z), L(X)_l^{-1})$ there exist a finitely generated unital subalgebra $B \subset A$ and maps $H \in \widehat{B} \otimes L(X)$, $G \in \mathcal{O}(M(\mathscr{A}_Z), GL_0(X))$ such that $GF = H$.*

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